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On some Hermite–Hadamard type inequalities for *tgs*-convex functions via generalized fractional integrals

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Abstract

In this research article, we establish some Hermite–Hadamard type inequalities for *tgs*-convex functions via Katugampola fractional integrals and ψ-Riemann–Liouville fractional integrals. Through these results we give some new Hermite–Hadamard type inequalities for *tgs*-convex functions via Riemann–Liouville fractional integrals and classical integrals.

Keywords: Hermite–Hadamard inequality; *tgs*-convex functions; Riemann–Liouville fractional integrals; Katugampola fractional integrals; ψ-Riemann–Liouville fractional integrals

1 Introduction

The convex function and its generalization play an important role in optimization theory and in other field of sciences. These functions have many integral inequalities (see [1, 10, 16]). The Hermite–Hadamard inequality [4, 5] for convex functions $\chi : \mathcal{H} \to \mathbb{R}$ on an interval $\mathcal{H}$ of the real line is defined by

$$\chi\left(\frac{h_1 + h_2}{2}\right) \leq \frac{1}{h_2 - h_1} \int_{h_1}^{h_2} \chi(g)\, dg \leq \frac{\chi(h_1) + \chi(h_2)}{2}, \tag{1}$$

for all $h_1, h_2 \in \mathcal{H}$ with $h_1 < h_2$. Several applications are found by using the Hermite–Hadamard inequality (see [2, 3, 6, 12, 14]).

Fractional calculus [8] has played a key role in different scientific fields due to its long term memory methods. In [15], Sarikaya et al. proved some Hermite–Hadamard type integral inequalities for fractional integrals and also gave some applications. In [10, 11, 13], the authors have established several Hermite–Hadamard type inequalities for new fractional conformable integral operators, Katugampola fractional integrals and ψ-Riemann–Liouville fractional integrals, respectively.

Motivated by Liu et al. [9] and by [11, 13], we prove Hermite–Hadamard type inequalities using ψ-Riemann–Liouville fractional integrals and Katugampola fractional integrals.

2 Preliminaries

In this section, we give some definitions and relevant results essential for this research article.

Definition 2.1 ([18]) Let $\chi : \mathcal{H} \subseteq \mathbb{R} \to \mathbb{R}$ be a nonnegative function. Then χ is called *tgs*-convex, if it satisfies the following inequality:

$$\chi\big(rh_1 + (1-r)h_2\big) \leq r(1-r)\big[\chi(h_1) + \chi(h_2)\big], \tag{2}$$

for all $h_1, h_2 \in \mathcal{H}$ and $r \in [0,1]$.

Definition 2.2 ([8]) Let $\chi \in L[h_1, h_2]$. The right-hand side and left-hand side Riemann–Liouville fractional integrals $J^{\alpha}_{h_1+}\chi$ and $J^{\alpha}_{h_2-}\chi$ of order $\alpha > 0$ with $h_2 > h_1 \geq 0$ are defined by

$$J^{\alpha}_{h_1+}\chi(g) = \frac{1}{\Gamma(\alpha)}\int_{h_1}^{g}(g-t)^{\alpha-1}\chi(t)\,dt, \quad g > h_1$$

and

$$J^{\alpha}_{h_2-}\chi(g) = \frac{1}{\Gamma(\alpha)}\int_{g}^{h_2}(t-g)^{\alpha-1}\chi(t)\,dt, \quad g < h_2,$$

respectively, where $\Gamma(\cdot)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^{\infty} e^{-t}t^{\alpha-1}\,dt$.

Definition 2.3 ([7]) Let $[h_1, h_2] \subset \mathbb{R}$ be a finite interval. Then, the left- and right-side Katugampola fractional integrals of order $\alpha(> 0)$ of $\chi \in X^p_c(h_1, h_2)$ are defined by

$${}^{\rho}I^{\alpha}_{h_1+}\chi(g) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{h_1}^{g}\big(g^{\rho} - t^{\rho}\big)^{\alpha-1}t^{\rho-1}\chi(t)\,dt$$

and

$${}^{\rho}I^{\alpha}_{h_2-}\chi(g) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{g}^{h_2}\big(t^{\rho} - g^{\rho}\big)^{\alpha-1}t^{\rho-1}\chi(t)\,dt,$$

with $h_1 < g < h_2$ and $\rho > 0$. Here $X^p_c(h_1, h_2)$ $(c \in \mathbb{R}, 1 \leq p \leq \infty)$ is the space of those complex valued Lebesgue measurable functions χ on $[h_1, h_2]$ for which $\|\chi\|_{X^p_c} < \infty$, where the norm is defined by

$$\|\chi\|_{X^p_c} = \left(\int_{h_1}^{h_2}\big|t^c\chi(t)\big|^p\frac{dt}{t}\right)^{1/p} < \infty,$$

for $1 \leq p < \infty$, $c \in \mathbb{R}$ and, for the case $p = \infty$,

$$\|\chi\|_{X^{\infty}_c} = \operatorname*{ess\,sup}_{h_1 \leq t \leq h_2}\big[t^c\big|\chi(t)\big|\big].$$

Here ess sup stands for essential supremum.

Definition 2.4 ([8, 17]) Let (h_1, h_2) $(-\infty \leq h_1 < h_2 \leq \infty)$ be a finite or infinite real interval and $\gamma > 0$. Let $\psi(x)$ be an increasing and positive monotone function on $(h_1, h_2]$ with continuous derivative on (h_1, h_2). Then the left- and right-sided ψ-Riemann–Liouville fractional integrals of a function χ with respect to ψ on $[h_1, h_2]$ are defined by

$$\mathcal{I}_{h_1+}^{\gamma:\psi}\chi(g) = \frac{1}{\Gamma(\gamma)}\int_{h_1}^{g}\psi'(z)\big(\psi(g)-\psi(z)\big)^{\gamma-1}\chi(z)\,dz,$$

$$\mathcal{I}_{h_2-}^{\gamma:\psi}\chi(g) = \frac{1}{\Gamma(\gamma)}\int_{g}^{h_2}\psi'(z)\big(\psi(z)-\psi(g)\big)^{\gamma-1}\chi(z)\,dz,$$

respectively.

Liu et al. [9] established Hermite–Hadamard type inequalities via ψ-Riemann–Liouville fractional integrals for convex functions.

Lemma 2.1 ([9]) *Let $\chi : [h_1, h_2] \to \mathbb{R}$ be a differentiable mapping, for $0 \leq h_1 < h_2$, and $\chi \in L_1[h_1, h_2]$. Let $\psi(g)$ be an increasing and positive monotone function on $(h_1, h_2]$, with continuous derivative $\psi'(g)$ on (h_1, h_2) and $\gamma \in (0, 1)$. Then the following equality for fractional integral holds*:

$$\begin{aligned}&\frac{\chi(h_1)+\chi(h_2)}{2} - \frac{\Gamma(\gamma+1)}{2(h_2-h_1)^{\gamma}}\big[\mathcal{I}_{\psi^{-1}(h_1)+}^{\gamma:\psi}(\chi\circ\psi)\big(\psi^{-1}(h_2)\big) \\ &\qquad + \mathcal{I}_{\psi^{-1}(h_2)-}^{\gamma:\psi}(\chi\circ\psi)\big(\psi^{-1}(h_1)\big)\big] \\ &\quad = \frac{1}{2(h_2-h_1)^{\gamma}}\int_{\psi^{-1}(h_1)}^{\psi^{-1}(h_2)}\big[\big(\psi(g)-h_1\big)^{\gamma}-\big(h_2-\psi(g)\big)^{\gamma}\big]\big(\chi'\circ\psi\big)(g)\psi'(g)\,dg. \qquad (3)\end{aligned}$$

Lemma 2.2 ([9]) *Let $\chi : [h_1, h_2] \to \mathbb{R}$ be a differentiable mapping, for $0 \leq h_1 < h_2$, and $\chi \in L_1[h_1, h_2]$. Let $\psi(g)$ be an increasing and positive monotone function on $(h_1, h_2]$, with continuous derivative $\psi'(g)$ on (h_1, h_2) and $\gamma \in (0, 1)$. Then the following equality for fractional integral holds*:

$$\begin{aligned}&\frac{\Gamma(\gamma+1)}{2(h_2-h_1)^{\gamma}}\big[\mathcal{I}_{\psi^{-1}(h_1)+}^{\gamma:\psi}(\chi\circ\psi)\big(\psi^{-1}(h_2)\big) + \mathcal{I}_{\psi^{-1}(h_2)-}^{\gamma:\psi}(\chi\circ\psi)\big(\psi^{-1}(h_1)\big)\big] \\ &\qquad - \chi\left(\frac{h_1+h_2}{2}\right) \\ &\quad = \int_{\psi^{-1}(h_1)}^{\psi^{-1}(h_2)} k\big(\chi'\circ\psi\big)(g)\psi'(g)\,dg \\ &\qquad + \frac{1}{2(h_2-h_1)^{\gamma}}\int_{\psi^{-1}(h_1)}^{\psi^{-1}(h_2)}\big[\big(\psi(g)-h_1\big)^{\gamma}-\big(h_2-\psi(g)\big)^{\gamma}\big]\big(\chi'\circ\psi\big)(g)\psi'(g)\,dg, \qquad (4)\end{aligned}$$

where

$$k = \begin{cases}\frac{1}{2}, & \psi^{-1}(\frac{h_1+h_2}{2}) \leq z \leq \psi^{-1}(h_2), \\ -\frac{1}{2}, & \psi^{-1}(h_1) < z < \psi^{-1}(\frac{h_1+h_2}{2}).\end{cases}$$

3 Inequalities via Katugampola fractional integrals

In this section, we find a Hermite–Hadamard inequality for a *tgs*-convex function via Katugampola fractional integrals.

Theorem 3.1 *Let $\alpha > 0$ and $\rho > 0$. Let $\chi : [h_1^\rho, h_2^\rho] \subset \mathbb{R} \to \mathbb{R}$ be a nonnegative function with $0 \le h_1 < h_2$ and $\chi \in X_c^p(h_1^\rho, h_2^\rho)$. If χ is also a tgs-convex function on $[h_1^\rho, h_2^\rho]$, then the following inequalities hold*:

$$\begin{aligned} &2\chi\left(\frac{h_1^\rho + h_2^\rho}{2}\right) \\ &\quad\le \frac{\rho^\alpha \Gamma(\alpha+1)}{2(h_2^\rho - h_1^\rho)^\alpha}\left[{}^\rho I_{h_1+}^\alpha \chi\left(h_2^\rho\right) + {}^\rho I_{h_2-}^\alpha \chi\left(h_1^\rho\right)\right] \\ &\quad\le \frac{\alpha(\chi(h_1^\rho) + \chi(h_2^\rho))}{\rho(\alpha+1)(\alpha+2)}. \end{aligned} \tag{5}$$

Proof Let $r \in [0,1]$. Consider $x, y \in [h_1, h_2]$, $h_1 \ge 0$, defined by $x^\rho = r^\rho h_1^\rho + (1-r^\rho)h_2^\rho$, $y^\rho = r^\rho h_2^\rho + (1-r^\rho)h_1^\rho$. Since χ is a *tgs*-convex function on $[h_1^\rho, h_2^\rho]$, we have

$$\chi\left(\frac{x^\rho + y^\rho}{2}\right) \le \frac{\chi(x^\rho) + \chi(y^\rho)}{4}.$$

Then we have

$$4\chi\left(\frac{h_1^\rho + h_2^\rho}{2}\right) \le \chi\left(r^\rho h_1^\rho + \left(1-r^\rho\right)h_2^\rho\right) + \chi\left(r^\rho h_2^\rho + \left(1-r^\rho\right)h_1^\rho\right). \tag{6}$$

Multiplying both sides of (6) by $r^{\alpha\rho-1}$, $\alpha > 0$ and then integrating the resulting inequality with respect to r over $[0,1]$, we obtain

$$\begin{aligned} \frac{4}{\alpha\rho}\chi\left(\frac{h_1^\rho + h_2^\rho}{2}\right) &\le \int_0^1 r^{\alpha\rho-1}\chi\left(r^\rho h_1^\rho + \left(1-r^\rho\right)h_2^\rho\right)dr \\ &\quad + \int_0^1 r^{\alpha\rho-1}\chi\left(r^\rho h_2^\rho + \left(1-r^\rho\right)h_1^\rho\right)dr \\ &= \int_{h_2}^{h_1}\left(\frac{h_2^\rho - g^\rho}{h_2^\rho - h_1^\rho}\right)^{\alpha-1}\chi\left(g^\rho\right)\frac{g^{\rho-1}}{h_1^\rho - h_2^\rho}\,dg \\ &\quad + \int_{h_1}^{h_2}\left(\frac{k^\rho - h_1^\rho}{h_2^\rho - h_1^\rho}\right)^{\alpha-1}\chi\left(k^\rho\right)\frac{k^{\rho-1}}{h_2^\rho - h_1^\rho}\,dk \\ &= \frac{\rho^{\alpha-1}\Gamma(\alpha)}{(h_2^\rho - h_1^\rho)^\alpha}\left[{}^\rho I_{h_1+}^\alpha \chi\left(h_2^\rho\right) + {}^\rho I_{h_2-}^\alpha \chi\left(h_1^\rho\right)\right]. \end{aligned} \tag{7}$$

This establishes the first inequality. For the proof of the second inequality in (5), we first observe that, for a *tgs*-convex function χ, we have

$$\chi\left(r^\rho h_1^\rho + \left(1-r^\rho\right)h_2^\rho\right) \le r^\rho\left(1-r^\rho\right)\left(\chi\left(h_1^\rho\right) + \chi\left(h_2^\rho\right)\right)$$

and

$$\chi\left(r^\rho h_2^\rho + \left(1-r^\rho\right)h_1^\rho\right) \le r^\rho\left(1-r^\rho\right)\left(\chi\left(h_1^\rho\right) + \chi\left(h_2^\rho\right)\right).$$

By adding these inequalities, we get

$$\chi\left(r^\rho h_1^\rho + \left(1-r^\rho\right)h_2^\rho\right) + \chi\left(r^\rho h_2^\rho + \left(1-r^\rho\right)h_1^\rho\right) \le 2r^\rho\left(1-r^\rho\right)\left(\chi\left(h_1^\rho\right) + \chi\left(h_2^\rho\right)\right). \tag{8}$$

Multiplying both sides of (8) by $r^{\alpha\rho-1}$, $\alpha > 0$ and then integrating the resulting inequality with respect to r over $[0,1]$, we obtain

$$\frac{\rho^{\alpha-1}\Gamma(\alpha)}{(h_2^\rho - h_1^\rho)^\alpha}\left[{}^\rho I_{h_1+}^\alpha \chi\left(h_2^\rho\right) + {}^\rho I_{h_2-}^\alpha \chi\left(h_1^\rho\right)\right] \le 2\int_0^1 r^{\alpha\rho+\rho-1}\left(1-r^\rho\right)\left(\chi\left(h_1^\rho\right) + \chi\left(h_2^\rho\right)\right) dr. \tag{9}$$

Since

$$\int_0^1 \left(r^{\alpha\rho+\rho-1} - r^{\alpha\rho+2\rho-1}\right) dt = \frac{1}{\rho(\alpha+1)(\alpha+2)},$$

(9) becomes

$$\frac{\rho^{\alpha-1}\Gamma(\alpha)}{(h_2^\rho - h_1^\rho)^\alpha}\left[{}^\rho I_{h_1+}^\alpha \chi\left(h_2^\rho\right) + {}^\rho I_{h_2-}^\alpha \chi\left(h_1^\rho\right)\right] \le \frac{2(\chi(h_1^\rho) + \chi(h_2^\rho))}{\rho(\alpha+1)(\alpha+2)}. \tag{10}$$

Thus (7) and (10) give (5). □

Remark 3.1 (1) By letting $\rho \to 1$ in (5) of Theorem 3.1 we get inequality 3.1 of Theorem 3.1 in [18].

(2) By letting $\rho \to 1$ and $\alpha = 1$ in (5) of Theorem 3.1 we get inequality 2.2 of Theorem 2.1 in [18].

Theorem 3.2 *Let $\alpha > 0$ and $\rho > 0$. Let $\chi : [h_1^\rho, h_2^\rho] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable and non-negative mapping on (h_1^ρ, h_2^ρ) with $0 \le h_1 < h_2$. If $|\chi'|$ is tgs-convex on $[h_1^\rho, h_2^\rho]$, then the following inequality holds*:

$$\begin{aligned}&\left|\frac{\chi(h_1^\rho) + \chi(h_2^\rho)}{2} - \frac{\rho^\alpha\Gamma(\alpha+1)}{2(h_2^\rho - h_1^\rho)^\alpha}\left[{}^\rho I_{h_1+}^\alpha \chi\left(h_2^\rho\right) +^\rho I_{h_2-}^\alpha \chi\left(h_1^\rho\right)\right]\right| \\ &\quad \le \frac{h_2^\rho - h_1^\rho}{(\alpha+2)(\alpha+3)}\left[\left|\chi'\left(h_1^\rho\right)\right| + \left|\chi'\left(h_2^\rho\right)\right|\right].\end{aligned} \tag{11}$$

Proof From (7) one can have

$$\begin{aligned}&\frac{\rho^{\alpha-1}\Gamma(\alpha)}{(h_2^\rho - h_1^\rho)^\alpha}\left[{}^\rho I_{h_1+}^\alpha \chi\left(h_2^\rho\right) +^\rho I_{h_2-}^\alpha \chi\left(h_1^\rho\right)\right] \\ &\quad = \int_0^1 r^{\alpha\rho-1}\chi\left(r^\rho h_1^\rho + \left(1-r^\rho\right)h_2^\rho\right) dr + \int_0^1 r^{\alpha\rho-1}\chi\left(r^\rho h_2^\rho + \left(1-r^\rho\right)h_1^\rho\right) dr.\end{aligned} \tag{12}$$

By integrating by parts, we then get

$$\begin{aligned}&\frac{\chi(h_1^\rho) + \chi(h_2^\rho)}{\alpha\rho} - \frac{\rho^{\alpha-1}\Gamma(\alpha)}{(h_2^\rho - h_1^\rho)^\alpha}\left[{}^\rho I_{h_1+}^\alpha \chi\left(h_2^\rho\right) +^\rho I_{h_2-}^\alpha \chi\left(h_1^\rho\right)\right] \\ &\quad = \frac{h_2^\rho - h_1^\rho}{\alpha}\int_0^1 r^{\rho(\alpha+1)-1}\left[\chi'\left(r^\rho h_2^\rho + \left(1-r^\rho\right)h_1^\rho\right) - \chi'\left(r^\rho h_1^\rho + \left(1-r^\rho\right)h_2^\rho\right)\right] dr.\end{aligned} \tag{13}$$

By using the triangle inequality and the *tgs*-convexity of $|\chi'|$, we obtain

$$\begin{aligned}
&\left|\frac{\chi(h_1^\rho)+\chi(h_2^\rho)}{\alpha\rho}-\frac{\rho^{\alpha-1}\Gamma(\alpha)}{(h_2^\rho-h_1^\rho)^\alpha}\left[{}^\rho I^\alpha_{h_1+}\chi\left(h_2^\rho\right)+{}^\rho I^\alpha_{h_2-}\chi\left(h_1^\rho\right)\right]\right| \\
&\leq \frac{h_2^\rho-h_1^\rho}{\alpha}\int_0^1 r^{\rho(\alpha+1)-1}\left|\chi'\left(r^\rho h_2^\rho+\left(1-r^\rho\right)h_1^\rho\right)-\chi'\left(r^\rho h_1^\rho+\left(1-r^\rho\right)h_2^\rho\right)\right|dr \\
&\leq \frac{h_2^\rho-h_1^\rho}{\alpha}\int_0^1 r^{\rho(\alpha+1)-1}\left[\chi'\left(r^\rho h_2^\rho+\left(1-r^\rho\right)h_1^\rho\right)+\chi'\left(r^\rho h_1^\rho+\left(1-r^\rho\right)h_2^\rho\right)\right]dr \\
&= \frac{2(h_2^\rho-h_1^\rho)}{\alpha}\int_0^1 r^{\rho(\alpha+1)-1}r^\rho\left(1-r^\rho\right)\left[\left|\chi'\left(h_1^\rho\right)\right|+\left|\chi'\left(h_2^\rho\right)\right|\right]dr \\
&= \frac{2(h_2^\rho-h_1^\rho)}{\alpha}\frac{|\chi'(h_1^\rho)|+|\chi'(h_2^\rho)|}{\rho(\alpha+2)(\alpha+3)}.
\end{aligned} \tag{14}$$

Multiplying both sides of the above inequality by $\frac{\alpha\rho}{2}$, we get the required inequality (11). □

Corollary 3.3 *Consider the similar assumptions of Theorem 3.2.*

1. *If* $\rho = 1$, *then*

$$\begin{aligned}
&\left|\frac{\chi(h_1)+\chi(h_2)}{2}-\frac{\Gamma(\alpha+1)}{2(h_2-h_1)^\alpha}\left[J^\alpha_{h_1+}\chi(h_2)+J^\alpha_{h_2-}\chi(h_1)\right]\right| \\
&\leq \frac{h_2-h_1}{(\alpha+2)(\alpha+3)}\left[\left|\chi'(h_1)\right|+\left|\chi'(h_2)\right|\right].
\end{aligned} \tag{15}$$

2. *If* $\rho = \alpha = 1$, *then*

$$\left|\frac{\chi(h_1)+\chi(h_2)}{2}-\frac{1}{h_2-h_1}\int_{h_1}^{h_2}\chi(g)\,dg\right| \leq \frac{h_2-h_1}{12}\left[\left|\chi'(h_1)\right|+\left|\chi'(h_2)\right|\right]. \tag{16}$$

For more results we need the following lemma, also proved in [11].

Lemma 3.1 ([11]) *Let* $\alpha > 0$ *and* $\rho > 0$. *Let* $\chi : [h_1^\rho, h_2^\rho] \subset \mathbb{R}_+ = [0,\infty) \to \mathbb{R}$ *be a differentiable mapping on* (h_1^ρ, h_2^ρ) *with* $0 \leq h_1 < h_2$. *Then the following equality holds if the fractional integrals exist*:

$$\begin{aligned}
&\frac{\chi(h_1^\rho)+\chi(h_2^\rho)}{2}-\frac{\rho^\alpha\Gamma(\alpha+1)}{2(h_2^\rho-h_1^\rho)^\alpha}\left[{}^\rho I^\alpha_{h_1+}\chi\left(h_2^\rho\right)+{}^\rho I^\alpha_{h_2-}\chi\left(h_1^\rho\right)\right] \\
&= \frac{\rho(h_2^\rho-h_1^\rho)}{2}\int_0^1\left[\left(1-r^\rho\right)^\alpha-\left(r^\rho\right)^\alpha\right]r^{\rho-1}\chi'\left(r^\rho h_1^\rho+\left(1-r^\rho\right)h_2^\rho\right)dr.
\end{aligned} \tag{17}$$

Proof By using the similar arguments as in the proof of Lemma 2 in [15]. First consider

$$\begin{aligned}
&\int_0^1\left(1-r^\rho\right)^\alpha r^{\rho-1}\chi'\left(r^\rho h_1^\rho+\left(1-r^\rho\right)h_2^\rho\right)dr \\
&= \frac{(1-r^\rho)^\alpha\chi(r^\rho h_1^\rho+(1-r^\rho)h_2^\rho)}{\rho(h_1^\rho-h_2^\rho)}\bigg|_0^1 \\
&\quad+\frac{\alpha}{h_1^\rho-h_2^\rho}\int_0^1\left(1-r^\rho\right)^{\alpha-1}r^{\rho-1}\chi\left(r^\rho h_1^\rho+\left(1-r^\rho\right)h_2^\rho\right)dr
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\chi(h_2^\rho)}{\rho(h_2^\rho - h_1^\rho)} - \frac{\alpha}{h_2^\rho - h_1^\rho} \int_{h_2}^{h_1} \left(\frac{g^\rho - h_1^\rho}{h_2^\rho - h_1^\rho}\right)^{\alpha-1} \cdot \frac{g^{\rho-1}}{h_1^\rho - h_2^\rho}\, dg \\
&= \frac{\chi(h_2^\rho)}{\rho(h_2^\rho - h_1^\rho)} - \frac{\rho^{\alpha-1}\Gamma(\alpha+1)}{(h_2^\rho - h_1^\rho)^{\alpha+1}} \cdot {}^\rho I_{h_2-}^\alpha \chi(g^\rho)\Big|_{g=h_1}. \qquad (18)
\end{aligned}
$$

Similarly, we can show that

$$
\begin{aligned}
&\int_0^1 r^{\rho\alpha} \cdot r^{\rho-1} \chi'\big(r^\rho h_1^\rho + (1-r^\rho)h_2^\rho\big)\, dr \\
&= -\frac{\chi(h_1^\rho)}{\rho(h_2^\rho - h_1^\rho)} + \frac{\rho^{\alpha-1}\Gamma(\alpha+1)}{(h_2^\rho - h_1^\rho)^{\alpha+1}} \cdot {}^\rho I_{h_1+}^\alpha \chi(g^\rho)\Big|_{g=h_2}. \qquad (19)
\end{aligned}
$$

Thus from (18) and (19) we get (17). □

Theorem 3.4 *Let $\alpha > 0$ and $\rho > 0$. Let $\chi : [h_1^\rho, h_2^\rho] \subset \mathbb{R}_+ \to \mathbb{R}$ be a differentiable and non-negative mapping on (h_1^ρ, h_2^ρ) such that $\chi' \in L_1[h_1, h_2]$ with $0 \le h_1 < h_2$. If $|\chi'|^q$ is tgs-convex on $[h_1^\rho, h_2^\rho]$ for some fixed $q \ge 1$, then the following inequality holds:*

$$
\begin{aligned}
&\left|\frac{\chi(h_1^\rho) + \chi(h_2^\rho)}{2} - \frac{\rho^\alpha \Gamma(\alpha+1)}{2(h_2^\rho - h_1^\rho)^\alpha}\left[{}^\rho I_{h_1+}^\alpha \chi(h_2^\rho) + {}^\rho I_{h_2-}^\alpha \chi(h_1^\rho)\right]\right| \\
&\le \frac{(h_2^\rho - h_1^\rho)}{2}\left(\frac{2}{\alpha+1}\right)^{1-1/q} \\
&\quad \times \left(\left[\beta(2, \alpha+2) + \frac{1}{(\alpha+2)(\alpha+3)}\right]\left[|\chi'(h_1^\rho)|^q + |\chi'(h_2^\rho)|^q\right]\right)^{1/q}. \qquad (20)
\end{aligned}
$$

Proof Using Lemma 3.1 and the power mean inequality and the *tgs*-convexity of $|\chi'|^q$, we obtain

$$
\begin{aligned}
&\left|I_\chi(\alpha, \rho, h_1, h_2)\right| \\
&= \left|\frac{\rho(h_2^\rho - h_1^\rho)}{2}\int_0^1 \left\{(1-r^\rho)^\alpha - (r^\rho)^\alpha\right\} r^{\rho-1}\chi'\big(r^\rho h_1^\rho + (1-r^\rho)h_2^\rho\big)\, dr\right| \\
&\le \frac{\rho(h_2^\rho - h_1^\rho)}{2}\left(\int_0^1 \left|(1-r^\rho)^\alpha - (r^\rho)^\alpha\right| r^{\rho-1}\, dr\right)^{1-1/q} \\
&\quad \times \left(\int_0^1 \left|(1-r^\rho)^\alpha - (r^\rho)^\alpha\right| r^{\rho-1}\left|\chi'\big(r^\rho h_1^\rho + (1-r^\rho)h_2^\rho\big)\right|^q dr\right)^{1/q} \\
&\le \frac{\rho(h_2^\rho - h_1^\rho)}{2}\left(\int_0^1 \left\{(1-r^\rho)^\alpha + (r^\rho)^\alpha\right\} r^{\rho-1}\, dr\right)^{1-1/q} \\
&\quad \times \left(\int_0^1 \left\{(1-r^\rho)^\alpha + (r^\rho)^\alpha\right\} r^{\rho-1} r^\rho(1-r^\rho)\left[|\chi'(h_1^\rho)|^q + |\chi'(h_2^\rho)|^q\right] dr\right)^{1/q}. \qquad (21)
\end{aligned}
$$

By using the change of variable $t = r^\rho$, we get

$$
\begin{aligned}
&\int_0^1 \left\{(1-r^\rho)^\alpha + (r^\rho)^\alpha\right\} r^{\rho-1}\, dr \\
&= \int_0^1 (1-r^\rho)^\alpha r^{\rho-1}\, dr + \int_0^1 (r^\rho)^\alpha r^{\rho-1}\, dr
\end{aligned}
$$

$$= \frac{2}{\rho(\alpha+1)}, \tag{22}$$

$$\begin{aligned}
&\int_0^1 \{(1-r^\rho)^\alpha + (r^\rho)^\alpha\} r^{\rho-1} r^\rho (1-r^\rho)\, dr \\
&= \int_0^1 (1-r^\rho)^\alpha r^{\rho-1} r^\rho (1-r^\rho)\, dr + \int_0^1 (r^\rho)^\alpha r^{\rho-1} r^\rho (1-r^\rho)\, dr \\
&= \frac{1}{\rho}\beta(2,\alpha+2) + \frac{1}{\rho(\alpha+2)(\alpha+3)}.
\end{aligned} \tag{23}$$

Hence using (23) and (22) in (21) we get (20). □

Corollary 3.5 *Consider the similar assumptions of Theorem 3.4.*

1. *If $\rho = 1$, then*

$$\begin{aligned}
&\left| \frac{\chi(h_1)+\chi(h_2)}{2} - \frac{\Gamma(\alpha+1)}{2(h_2-h_1)^\alpha}\left[J^\alpha_{h_1+}\chi(h_2) + J^\alpha_{h_2-}\chi(h_1)\right]\right| \\
&\le \frac{(h_2-h_1)}{2}\left(\frac{2}{\alpha+1}\right)^{1-1/q} \\
&\quad\times \left(\left[\beta(2,\alpha+2) + \frac{1}{(\alpha+2)(\alpha+3)}\right]\left[|\chi'(h_1)|^q + |\chi'(h_2)|^q\right]\right)^{1/q}.
\end{aligned} \tag{24}$$

2. *If $\rho = \alpha = 1$, then*

$$\begin{aligned}
&\left| \frac{\chi(h_1)+\chi(h_2)}{2} - \frac{1}{h_2-h_1}\int_{h_1}^{h_2} \chi(g)\, dg\right| \\
&\le \frac{(h_2-h_1)}{2}\left(\frac{2(|\chi'(h_1)|^q + |\chi'(h_2)|^q)}{3}\right)^{1/q}.
\end{aligned} \tag{25}$$

Theorem 3.6 *Let $\alpha > 0$ and $\rho > 0$. Let $\chi : [h_1^\rho, h_2^\rho] \subset \mathbb{R}_+ \to \mathbb{R}$ be a differentiable and non-negative mapping on (h_1^ρ, h_2^ρ) such that $\chi' \in L_1[h_1, h_2]$ with $0 \le h_1 < h_2$. If $|\chi'|^q$ is tgs-convex on $[h_1^\rho, h_2^\rho]$ for some fixed $q \ge 1$, then the following inequality holds*:

$$\begin{aligned}
&\left| \frac{\chi(h_1^\rho)+\chi(h_2^\rho)}{2} - \frac{\rho^\alpha\Gamma(\alpha+1)}{2(h_2^\rho-h_1^\rho)^\alpha}\left[{}^\rho I^\alpha_{h_1+}\chi(h_2^\rho) + {}^\rho I^\alpha_{h_2-}\chi(h_1^\rho)\right]\right| \\
&\le \frac{(h_2^\rho-h_1^\rho)}{2}\left(\left[\beta(2,\alpha+2) + \frac{1}{(\alpha+2)(\alpha+3)}\right]\left[|\chi'(h_1^\rho)|^q + |\chi'(h_2^\rho)|^q\right]\right)^{1/q}.
\end{aligned} \tag{26}$$

Proof Using Lemma 3.1 and the power mean inequality and the *tgs*-convexity of $|\chi'|^q$, we obtain

$$\begin{aligned}
&\left| \frac{\chi(h_1^\rho)+\chi(h_2^\rho)}{2} - \frac{\rho^\alpha\Gamma(\alpha+1)}{2(h_2^\rho-h_1^\rho)^\alpha}\left[{}^\rho I^\alpha_{h_1+}\chi(h_2^\rho) + {}^\rho I^\alpha_{h_2-}\chi(h_1^\rho)\right]\right| \\
&= \left|\frac{\rho(h_2^\rho-h_1^\rho)}{2}\int_0^1 \{(1-r^\rho)^\alpha - (r^\rho)^\alpha\} r^{\rho-1}\chi'(r^\rho h_1^\rho + (1-r^\rho)h_2^\rho)\, dr\right| \\
&\le \frac{\rho(h_2^\rho-h_1^\rho)}{2}\left(\int_0^1 r^{\rho-1}\, dr\right)^{1-1/q}
\end{aligned}$$

$$\times \left(\int_0^1 \left| \left(1-r^\rho\right)^\alpha - \left(r^\rho\right)^\alpha \right| r^{\rho-1} \left| \chi'\left(r^\rho h_1^\rho + \left(1-r^\rho\right)h_2^\rho\right)\right|^q dr \right)^{1/q}$$

$$\le \frac{\rho(h_2^\rho - h_1^\rho)}{2} \left(\frac{1}{\rho}\right)^{1-1/q}$$

$$\times \left(\int_0^1 \left\{ \left(1-r^\rho\right)^\alpha + \left(r^\rho\right)^\alpha \right\} r^{\rho-1} r^\rho \left(1-r^\rho\right) \left[\left|\chi'\left(h_1^\rho\right)\right|^q + \left|\chi'\left(h_2^\rho\right)\right|^q\right] dr \right)^{1/q}. \tag{27}$$

Since by using the change of variable $t = r^\rho$, we get

$$\int_0^1 \left\{ \left(1-r^\rho\right)^\alpha + \left(r^\rho\right)^\alpha \right\} r^{\rho-1} r^\rho \left(1-r^\rho\right) dr$$

$$= \int_0^1 \left(1-r^\rho\right)^\alpha r^{\rho-1} r^\rho \left(1-r^\rho\right) dr + \int_0^1 \left(r^\rho\right)^\alpha r^{\rho-1} r^\rho \left(1-r^\rho\right) dr$$

$$= \frac{1}{\rho}\beta(2, \alpha+2) + \frac{1}{\rho(\alpha+2)(\alpha+3)}. \tag{28}$$

Hence using (28) in (27) we get(26). □

Corollary 3.7 *Consider the similar assumptions of Theorem 3.6. If* $\rho = 1$, *then*

$$\left| \frac{\chi(h_1) + \chi(h_2)}{2} - \frac{\Gamma(\alpha+1)}{2(h_2-h_1)^\alpha} \left[J_{h_1+}^\alpha \chi(h_2) + J_{h_2-}^\alpha \chi(h_1)\right] \right|$$

$$\le \frac{(h_2-h_1)}{2} \left(\left[\beta(2, \alpha+2) + \frac{1}{(\alpha+2)(\alpha+3)}\right] \left[\left|\chi'(h_1)\right|^q + \left|\chi'(h_2)\right|^q\right] \right)^{1/q}. \tag{29}$$

Theorem 3.8 *Let* χ_1, χ_2 *be real valued, symmetric about* $\frac{h_1^\rho + h_2^\rho}{2}$, *nonnegative and tgs-convex functions on* $[h_1^\rho, h_2^\rho]$, *where* $\rho > 0$. *Then, for all* $h_1, h_2 > 0$ *and* $\alpha > 0$, *we have*

$$\frac{\rho^\alpha \, {}^\rho I_{h_1+}^\alpha (\chi_1(h_2^\rho)\chi_2(h_2^\rho))}{(h_2^\rho - h_1^\rho)^\alpha} \le \frac{2\alpha(\alpha+1)[M(h_1^\rho, h_2^\rho) + N(h_1^\rho, h_2^\rho)]}{\Gamma(\alpha+5)} \tag{30}$$

and

$$8\chi_1\left(\frac{h_1^\rho + h_2^\rho}{2}\right)\chi_2\left(\frac{h_1^\rho + h_2^\rho}{2}\right)$$

$$\le \frac{\rho^\alpha \, {}^\rho I_{h_1+}^\alpha (\chi_1(h_2^\rho)\chi_2(h_2^\rho))}{(h_2^\rho - h_1^\rho)^\alpha} + \frac{2\alpha(\alpha+1)[M(h_1^\rho, h_2^\rho) + N(h_1^\rho, h_2^\rho)]}{\Gamma(\alpha+5)}, \tag{31}$$

where $M(h_1^\rho, h_2^\rho) = \chi_1(h_1)\chi_2(h_1) + \chi_1(h_2)\chi_2(h_2)$ *and* $N(h_1^\rho, h_2^\rho) = \chi_1(h_1)\chi_2(h_2) + \chi_1(h_2)\chi_2(h_1)$.

Proof Since χ_1 and χ_2 are *tgs*-convex functions on $[h_1, h_2]$, we can have

$$\chi_1\left(r^\rho h_1^\rho + \left(1-r^\rho\right)h_2^\rho\right) \le r^\rho\left(1-r^\rho\right)\left(\chi_1\left(h_1^\rho\right) + \chi_1\left(h_2^\rho\right)\right)$$

and

$$\chi_2\left(r^\rho h_1^\rho + \left(1-r^\rho\right)h_2^\rho\right) \le r^\rho\left(1-r^\rho\right)\left(\chi_2\left(h_1^\rho\right) + \chi_2\left(h_2^\rho\right)\right),$$

From the above, we obtain

$$\begin{aligned} &\chi_1\big(r^\rho h_1^\rho + (1-r^\rho)h_2^\rho\big)\chi_2\big(r^\rho h_1^\rho + (1-r^\rho)h_2^\rho\big) \\ &\quad\le r^{2\rho}\big(1-r^\rho\big)^2\big(\chi_1(h_1^\rho)+\chi_1(h_2^\rho)\big)\big(\chi_2(h_1^\rho)+\chi_2(h_2^\rho)\big). \end{aligned} \tag{32}$$

Multiplying both sides of (32) by $\frac{r^{\alpha\rho-1}}{\Gamma(\alpha)}$, $\alpha > 0$ and then integrating the resulting inequality with respect to r over $[0,1]$, we obtain

$$\begin{aligned} &\frac{1}{\Gamma(\alpha)}\int_0^1 r^{\alpha\rho-1}\chi_1\big(r^\rho h_1^\rho + (1-r^\rho)h_2^\rho\big)\chi_2\big(r^\rho h_1^\rho + (1-r^\rho)h_2^\rho\big)\,dr \\ &\quad\le \frac{(\chi_1(h_1^\rho)+\chi_1(h_2^\rho))(\chi_2(h_1^\rho)+\chi_2(h_2^\rho))}{\Gamma(\alpha)}\int_0^1 r^{2\rho}\big(1-r^\rho\big)^2\,dr. \end{aligned} \tag{33}$$

By the change of variable $t = r^\rho$, we get

$$\int_0^1 r^{2\rho}\big(1-r^\rho\big)^2\,dr = \frac{2\alpha(\alpha+1)}{\rho\Gamma(\alpha+5)}. \tag{34}$$

Also by letting $x^\rho = r^\rho h_1^\rho + (1-r^\rho)h_2^\rho$, we obtain

$$\begin{aligned} &\frac{1}{\Gamma(\alpha)}\int_0^1 r^{\alpha\rho-1}\chi_1\big(r^\rho h_1^\rho + (1-r^\rho)h_2^\rho\big)\chi_2\big(r^\rho h_1^\rho + (1-r^\rho)h_2^\rho\big)\,dr \\ &\quad= \frac{\rho^{\alpha-1}\ {}^\rho I^\alpha_{h_1+}(\chi_1(h_2^\rho)\chi_2(h_2^\rho))}{(h_2^\rho - h_1^\rho)^\alpha}. \end{aligned} \tag{35}$$

Hence from (33)–(35), we get (30).

Again using the *tgs*-convexity of χ_1 and χ_2 on $[h_1^\rho, h_2^\rho]$, we find

$$\begin{aligned} &\chi_1\left(\frac{h_1^\rho + h_2^\rho}{2}\right)\chi_2\left(\frac{h_1^\rho + h_2^\rho}{2}\right) \\ &\quad\le \chi_1\left(\frac{r^\rho h_1^\rho + (1-r^\rho)h_2^\rho}{2} + \frac{r^\rho h_2^\rho + (1-r^\rho)h_1^\rho}{2}\right) \\ &\qquad\times \chi_2\left(\frac{r^\rho h_1^\rho + (1-r^\rho)h_2^\rho}{2} + \frac{r^\rho h_2^\rho + (1-r^\rho)h_1^\rho}{2}\right) \\ &\quad\le \frac{1}{4}\big[\chi_1\big(r^\rho h_1^\rho + (1-r^\rho)h_2^\rho\big) + \chi_1\big(r^\rho h_2^\rho + (1-r^\rho)h_1^\rho\big)\big] \\ &\qquad\times \frac{1}{4}\big[\chi_2\big(r^\rho h_1^\rho + (1-r^\rho)h_2^\rho\big) + \chi_1\big(r^\rho h_2^\rho + (1-r^\rho)h_1^\rho\big)\big] \\ &\quad= \frac{1}{16}\big[\chi_1\big(r^\rho h_1^\rho + (1-r^\rho)h_2^\rho\big)\chi_2(r^\rho h_1^\rho + \big(1-r^\rho\big)h_2^\rho \\ &\qquad+ \chi_1\big(r^\rho h_2^\rho + (1-r^\rho)h_1^\rho\big)\chi_2(r^\rho h_2^\rho + \big(1-r^\rho\big)h_1^\rho \\ &\qquad+ \chi_1\big(r^\rho h_1^\rho + (1-r^\rho)h_2^\rho\big)\chi_2(r^\rho h_2^\rho + \big(1-r^\rho\big)h_1^\rho \\ &\qquad+ \chi_1\big(r^\rho h_2^\rho + (1-r^\rho)h_1^\rho\big)\chi_2\big(r^\rho h_1^\rho + (1-r^\rho)h_2^\rho\big)\big]. \end{aligned} \tag{36}$$

Multiplying both sides of (36) by $\frac{r^{\alpha\rho-1}}{\Gamma(\alpha)}$, $\alpha > 0$ and then integrating the resulting inequality with respect to r over $[0,1]$, we obtain

$$\begin{aligned}
&\frac{1}{\rho\Gamma(\alpha+1)}\chi_1\left(\frac{h_1^\rho+h_2^\rho}{2}\right)\chi_2\left(\frac{h_1^\rho+h_2^\rho}{2}\right)\\
&\leq \frac{1}{16\Gamma(\alpha)}\biggl[\int_0^1 r^{\alpha\rho-1}\chi_1\bigl(r^\rho h_1^\rho+\bigl(1-r^\rho\bigr)h_2^\rho\bigr)\chi_2(r^\rho h_1^\rho+\bigl(1-r^\rho\bigr)h_2^\rho\,dr\\
&\quad+\int_0^1 r^{\alpha\rho-1}\chi_1\bigl(r^\rho h_2^\rho+\bigl(1-r^\rho\bigr)h_1^\rho\bigr)\chi_2(r^\rho h_2^\rho+\bigl(1-r^\rho\bigr)h_1^\rho\,dr\\
&\quad+\int_0^1 r^{\alpha\rho-1}\chi_1\bigl(r^\rho h_1^\rho+\bigl(1-r^\rho\bigr)h_2^\rho\bigr)\chi_2(r^\rho h_2^\rho+\bigl(1-r^\rho\bigr)h_1^\rho\,dr\\
&\quad+\int_0^1 r^{\alpha\rho-1}\chi_1\bigl(r^\rho h_2^\rho+\bigl(1-r^\rho\bigr)h_1^\rho\bigr)\chi_2\bigl(r^\rho h_1^\rho+\bigl(1-r^\rho\bigr)h_2^\rho\bigr)\,dr\biggr].
\end{aligned}$$

That is,

$$\begin{aligned}
&8\chi_1\left(\frac{h_1^\rho+h_2^\rho}{2}\right)\chi_2\left(\frac{h_1^\rho+h_2^\rho}{2}\right)\\
&\leq \frac{\rho\Gamma(\alpha+1)}{2(h_2^\rho-h_1^\rho)^\alpha}\,{}^\rho I_{h_1+}^\alpha\bigl[\chi_1\bigl(h_2^\rho\bigr)\chi_2\bigl(h_2^\rho\bigr)+\chi_1\bigl(h_2^\rho\bigr)\chi_2\bigl(h_1^\rho\bigr)\bigr]\\
&\quad+{}^\rho I_{h_1+}^\alpha\bigl[\chi_1\bigl(h_1^\rho\bigr)\chi_2\bigl(h_1^\rho\bigr)+\chi_1\bigl(h_1^\rho\bigr)\chi_2\bigl(h_2^\rho\bigr)\bigr].
\end{aligned}$$

After some calculations we get the required inequality (31). □

Remark 3.2 1. By letting $\rho = 1$ in Theorem 3.8 the inequalities (30) and (31) give the inequalities (3.11) and (3.12), respectively, in Theorem 3.2 of [18].

2. By letting $\rho = \alpha = 1$ in Theorem 3.8 the inequality (30) becomes the inequality in Theorem (2.2) of [18].

4 Inequalities via ψ-Riemann–Liouville fractional integrals

First we establish the Hermite–Hadamard inequality via ψ-Riemann–Liouville fractional integrals.

Theorem 4.1 *Let $\chi : [h_1,h_2] \to \mathbb{R}$ be a positive function, for $0 \leq h_1 < h_2$, and $\chi \in L_1[h_1,h_2]$. Let $\psi(z)$ be an increasing and positive monotone function on $(h_1,h_2]$, with continuous derivative $\psi'(z)$ on (h_1,h_2). Let χ be a tgs-convex function, then the following inequalities for a fractional integral hold*:

$$\begin{aligned}
&2\chi\left(\frac{h_1+h_2}{2}\right)\\
&\leq \frac{\Gamma(\gamma+1)}{2(h_2-h_1)^\gamma}\bigl[\mathcal{I}_{\psi^{-1}(h_1)+}^{\gamma:\psi}(\chi\circ\psi)\bigl(\psi^{-1}(h_2)\bigr)+\mathcal{I}_{\psi^{-1}(h_2)-}^{\gamma:\psi}(\chi\circ\psi)\bigl(\psi^{-1}(h_1)\bigr)\bigr]\\
&\leq \frac{\gamma[\chi(h_1)+\chi(h_2)]}{(\gamma+1)(\gamma+2)}.
\end{aligned} \tag{37}$$

Proof Since χ is *tgs*-convex, we have

$$\chi\left(\frac{u+v}{2}\right) \leq \frac{\chi(u)+\chi(v)}{2^2}.$$

Let $u = rh_1 + (1-r)h_2$ and $v = rh_2 + (1-r)h_1$, we get

$$4\chi\left(\frac{h_1+h_2}{2}\right) \leq \chi\big(rh_1+(1-r)h_2\big) + \chi\big(rh_2+(1-r)h_1\big). \tag{38}$$

Multiplying by $r^{\gamma-1}$ on both sides of inequality (38) and then integrating with respect to r over $[0,1]$ imply

$$\frac{4}{\gamma}\chi\left(\frac{h_1+h_2}{2}\right) \leq \int_0^1 r^{\gamma-1}\chi\big(rh_1+(1-r)h_2\big)\,dr + \int_0^1 r^{\gamma-1}\chi\big(rh_2+(1-r)h_1\big)\,dr. \tag{39}$$

Now consider

$$\begin{aligned}
&\frac{\Gamma(\gamma+1)}{2(h_2-h_1)^\gamma}\left[\mathcal{I}^{\gamma:\psi}_{\psi^{-1}(h_1)+}(\chi\circ\psi)\big(\psi^{-1}(h_2)\big) + \mathcal{I}^{\gamma:\psi}_{\psi^{-1}(h_2)-}(\chi\circ\psi)\big(\psi^{-1}(h_1)\big)\right] \\
&= \frac{\Gamma(\gamma+1)}{2(h_2-h_1)^\gamma\Gamma(\gamma)}\left[\int_{\psi^{-1}(h_1)}^{\psi^{-1}(h_2)} \psi'(g)\big(h_2-\psi(g)\big)^{\gamma-1}(\chi\circ\psi)(g)\,dg\right. \\
&\quad \left. + \int_{\psi^{-1}(h_1)}^{\psi^{-1}(h_2)} \psi'(g)\big(\psi(g)-h_1\big)^{\gamma-1}(\chi\circ\psi)(g)\,dg\right] \\
&= \frac{\gamma}{2}\left[\int_{\psi^{-1}(h_1)}^{\psi^{-1}(h_2)} \left(\frac{h_2-\psi(g)}{h_2-h_1}\right)^{\gamma-1}\chi\big(\psi(g)\big)\frac{\psi'(g)}{h_2-h_1}\,dg\right. \\
&\quad \left. + \int_{\psi^{-1}(h_1)}^{\psi^{-1}(h_2)} \left(\frac{\psi(g)-h_1}{h_2-h_1}\right)^{\gamma-1}\chi\big(\psi(g)\big)\frac{\psi'(g)}{h_2-h_1}\,dg\right] \\
&= \frac{\gamma}{2}\left[\int_0^1 r^{\gamma-1}\chi\big(rh_1+(1-r)h_2\big)\,dr + \int_0^1 r^{\gamma-1}\chi\big(rh_2+(1-r)h_1\big)\,dr\right] \\
&\geq 2\chi\left(\frac{h_1+h_2}{2}\right),
\end{aligned} \tag{40}$$

by using (39). Thus first inequality of (37) is proved.

For the next inequality we consider

$$\chi\big(rh_1+(1-r)h_2\big) \leq r(1-r)\big[\chi(h_1)+\chi(h_2)\big]$$

and

$$\chi\big(rh_2+(1-r)h_1\big) \leq r(1-r)\big[\chi(h_2)+\chi(h_1)\big].$$

We add

$$\chi\big(rh_1+(1-r)h_2\big) + \chi\big(rh_2+(1-r)h_1\big) \leq 2r(1-r)\big[\chi(h_1)+\chi(h_2)\big]. \tag{41}$$

Multiplying by $r^{\gamma-1}$ on both sides of inequality (41) and then integrating with respect to r over $[0,1]$ imply

$$\int_0^1 r^{\gamma-1}\chi\big(rh_1+(1-r)h_2\big)\,dr+\int_0^1 r^{\gamma-1}\chi\big(rh_2+(1-r)h_1\big)\,dr \leq \frac{2[\chi(h_1)+\chi(h_2)]}{(\gamma+1)(\gamma+2)}.$$

That is,

$$\frac{\Gamma(\gamma+1)}{(h_2-h_1)^{\gamma}}\big[\mathcal{I}^{\gamma:\psi}_{\psi^{-1}(h_1)+}(\chi\circ\psi)\big(\psi^{-1}(h_2)\big)+\mathcal{I}^{\gamma:\psi}_{\psi^{-1}(h_2)-}(\chi\circ\psi)\big(\psi^{-1}(h_1)\big)\big] \leq \frac{\gamma[\chi(h_1)+\chi(h_2)]}{(\gamma+1)(\gamma+2)}.$$

Hence the proof is completed. □

Remark 4.1 (1) By letting $\psi(g)=g$ in (37) of Theorem 4.1 we get inequality 3.1 of Theorem 3.1 in [18].

(2) By letting $\psi(g)=g$ and $\gamma=1$ in (37) of Theorem 4.1 we get inequality 2.2 of Theorem 2.1 in [18].

For the next two results we use Lemma 2.1 and Lemma 2.2, respectively.

Theorem 4.2 *Let $\chi:[h_1,h_2]\to\mathbb{R}$ be a nonnegative differentiable mapping, for $0\leq h_1<h_2$. Let $\psi(g)$ be an increasing and positive monotone function on $(h_1,h_2]$, with continuous derivative $\psi'(g)$ on (h_1,h_2) and $\gamma\in(0,1)$. If $|\chi'|^q$ is tgs-convex and $q\geq 1$, then the following inequality for fractional integral holds*:

$$\begin{aligned}&\biggl|\frac{\chi(h_1)+\chi(h_2)}{2}-\frac{\Gamma(\gamma+1)}{2(h_2-h_1)^{\gamma}}\big[\mathcal{I}^{\gamma:\psi}_{\psi^{-1}(h_1)+}(\chi\circ\psi)\big(\psi^{-1}(h_2)\big)\\&\quad+\mathcal{I}^{\gamma:\psi}_{\psi^{-1}(h_2)-}(\chi\circ\psi)\big(\psi^{-1}(h_1)\big)\big]\biggr|\\&\quad\leq\frac{h_2-h_1}{2}\left[\frac{2}{\gamma+1}\left(1-\frac{1}{2^{\gamma}}\right)\right]^{\frac{q-1}{q}}\left(\frac{2(|\chi'(h_1)|^q+|\chi(h_2)|^q)}{(\gamma+2)(\gamma+3)}\right)^{\frac{1}{q}}.\end{aligned}\tag{42}$$

Proof First note that, for every $g\in(\psi^{-1}(h_1),\psi^{-1}(h_2))$, we have $h_1<\psi(g)<h_2$. Let $r=\frac{h_2-\psi(g)}{h_2-h_1}$, then we have $\psi(g)=rh_1+(1-r)h_2$. Applying Lemma 2.1 and the *tgs*-convexity of $|\chi'|$, we obtain

$$\begin{aligned}&\biggl|\frac{\chi(h_1)+\chi(h_2)}{2}-\frac{\Gamma(\gamma+1)}{2(h_2-h_1)^{\gamma}}\big[\mathcal{I}^{\gamma:\psi}_{\psi^{-1}(h_1)+}(\chi\circ\psi)\big(\psi^{-1}(h_2)\big)\\&\quad+\mathcal{I}^{\gamma:\psi}_{\psi^{-1}(h_2)-}(\chi\circ\psi)\big(\psi^{-1}(h_1)\big)\big]\biggr|\\&\quad\leq\frac{1}{2(h_2-h_1)^{\gamma}}\int_{\psi^{-1}(h_1)}^{\psi^{-1}(h_2)}\big|\big(\psi(g)-h_1\big)^{\gamma}-\big(h_2-\psi(g)\big)^{\gamma}\big|\big|(\chi'\circ\psi)(g)\big|\,d\psi(g)\\&\quad=\frac{h_2-h_1}{2}\int_0^1\big|(1-r)^{\gamma}-r^{\gamma}\big|\big|\chi'\big(rh_1+(1-r)h_2\big)\big|\,dr\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{h_2-h_1}{2}\int_0^1 \big|(1-r)^\gamma - r^\gamma\big| r(1-r)\big[\big|\chi'(h_2)+|\chi'(h_2)|\big]\,dr \\
&\leq \frac{h_2-h_1}{2}\int_0^1 \big[(1-r)^\gamma + r^\gamma\big] r(1-r)\big[\big|\chi'(h_2)+|\chi'(h_2)|\big]\,dr \\
&= \frac{h_2-h_1}{(\gamma+2)(\gamma+3)}\big[\big|\chi'(h_2)+|\chi'(h_2)|\big].
\end{aligned}
\tag{43}
$$

Since

$$
\int_0^1 \big[(1-r)^\gamma + r^\gamma\big] r(1-r)\,dr = \frac{2}{(\gamma+2)(\gamma+3)},
$$

we get the required inequality (42) for $q=1$.

Now consider the case when $q>1$. Again using Lemma 2.1, the power mean inequality and the s-convexity of $|\chi'|^q$ on $[a_1,a_2]$, we get

$$
\begin{aligned}
&\bigg|\frac{\chi(h_1)+\chi(h_2)}{2} - \frac{\Gamma(\gamma+1)}{2(h_2-h_1)^\gamma}\big[\mathcal{I}^{\gamma:\psi}_{\psi^{-1}(h_1)+}(\chi\circ\psi)\big(\psi^{-1}(h_2)\big) \\
&\quad + \mathcal{I}^{\gamma:\psi}_{\psi^{-1}(h_2)-}(\chi\circ\psi)\big(\psi^{-1}(h_1)\big)\big]\bigg| \\
&\leq \frac{1}{2(h_2-h_1)^\gamma}\int_{\psi^{-1}(h_1)}^{\psi^{-1}(h_2)} \big|\big(\psi(g)-h_1\big)^\gamma - \big(h_2-\psi(g)\big)^\gamma\big|\big|\big(\chi'\circ\psi\big)(g)\big|\,d\psi(g) \\
&= \frac{h_2-h_1}{2}\int_0^1 \big|(1-r)^\gamma - r^\gamma\big|\big|\chi'\big(rh_1+(1-r)h_2\big)\big|\,dr \\
&= \frac{h_2-h_1}{2}\bigg(\int_0^1 \big|(1-r)^\gamma - r^\gamma\big|\,dr\bigg)^{1-\frac{1}{q}} \\
&\quad \times \bigg(\int_0^1 \big|(1-r)^\gamma - r^\gamma\big|\big|\chi'\big(rh_1+(1-r)h_2\big)\big|^q\,dr\bigg)^{\frac{1}{q}} \\
&= \frac{h_2-h_1}{2}\bigg(\int_0^1 \big|(1-r)^\gamma - r^\gamma\big|\,dr\bigg)^{\frac{q-1}{q}} \\
&\quad \times \bigg(\int_0^1 \big[(1-r)^\gamma + r^\gamma\big] r(1-r)\big[\big|\chi'(h_2)\big|^q + \chi'(h_2)|^q\big]\,dr\bigg)^{\frac{1}{q}} \\
&= \frac{h_2-h_1}{2}\bigg[\frac{2}{\gamma+1}\bigg(1-\frac{1}{2^\gamma}\bigg)\bigg]^{\frac{q-1}{q}}\bigg(\frac{2(|\chi'(h_1)|^q + |\chi(h_2)|^q)}{(\gamma+2)(\gamma+3)}\bigg)^{\frac{1}{q}}.
\end{aligned}
\tag{44}
$$

We have

$$
\begin{aligned}
\int_0^1 \big|(1-r)^\gamma - r^\gamma\big|\,dr &= \int_0^{1/2} \big[(1-r)^\gamma - r^\gamma\big]\,dr + \int_{1/2}^1 \big[r^\gamma - (1-r)^\gamma\big]\,dr \\
&= \frac{2}{\gamma+1}\bigg(1-\frac{1}{2^\gamma}\bigg).
\end{aligned}
$$

This completes the proof. □

Corollary 4.3 *Under the similar conditions of Theorem 4.2.*

1. *If $\psi(g)=g$, then we get*

$$\left|\frac{\chi(h_1)+\chi(h_2)}{2}-\frac{\Gamma(\gamma+1)}{2(h_2-h_1)^\gamma}\left[J^\gamma_{h_1+}\chi(h_2)+J^\gamma_{h_2-}\chi(h_1)\right]\right| \leq \frac{h_2-h_1}{2}\left[\frac{2}{\gamma+1}\left(1-\frac{1}{2^\gamma}\right)\right]^{\frac{q-1}{q}}\left(\frac{2(|\chi'(h_1)|^q+|\chi(h_2)|^q)}{(\gamma+2)(\gamma+3)}\right)^{\frac{1}{q}}. \tag{45}$$

2. *If $\psi(g)=g$ and $\gamma=1$, then we get*

$$\left|\frac{\chi(h_1)+\chi(h_2)}{2}-\frac{2}{(h_2-h_1)}\int_{h_1}^{h_2}\chi(g)\,dg\right| \leq \frac{h_2-h_1}{2}\left[\frac{1}{2}\right]^{\frac{q-1}{q}}\left(\frac{|\chi'(h_1)|^q+|\chi(h_2)|^q}{3}\right)^{\frac{1}{q}}. \tag{46}$$

Theorem 4.4 *Let $\chi:[h_1,h_2]\to\mathbb{R}$ be a nonnegative differentiable mapping, for $0\leq h_1<h_2$. Let $\psi(g)$ be an increasing and positive monotone function on $(h_1,h_2]$, with continuous derivative $\psi'(g)$ on (h_1,h_2) and $\gamma\in(0,1)$. If $|\chi'|$ is tgs-convex, then the following inequality for fractional integral holds*:

$$\left|\frac{\Gamma(\gamma+1)}{2(h_2-h_1)^\gamma}\left[\mathcal{I}^{\gamma:\psi}_{\psi^{-1}(h_1)+}(\chi\circ\psi)\left(\psi^{-1}(h_2)\right)+\mathcal{I}^{\gamma:\psi}_{\psi^{-1}(h_2)-}(\chi\circ\psi)\left(\psi^{-1}(h_1)\right)\right] - \chi\left(\frac{h_1+h_2}{2}\right)\right| \leq \frac{\chi(h_2)-\chi(h_1)}{2}+\frac{h_2-h_1}{(\gamma+2)(\gamma+3)}\left(\left|\chi'(h_1)\right|+\left|\chi(h_2)\right|\right). \tag{47}$$

Proof From Lemma 2.2 and the *tgs*-convexity of $|\chi'|$, we have

$$\begin{aligned}
&\left|\frac{\Gamma(\gamma+1)}{2(h_2-h_1)^\gamma}\left[\mathcal{I}^{\gamma:\psi}_{\psi^{-1}(h_1)+}(\chi\circ\psi)\left(\psi^{-1}(h_2)\right)+\mathcal{I}^{\gamma:\psi}_{\psi^{-1}(h_2)-}(\chi\circ\psi)\left(\psi^{-1}(h_1)\right)\right] - \chi\left(\frac{h_1+h_2}{2}\right)\right| \\
&= \left|\int_{\psi^{-1}(h_1)}^{\psi^{-1}(h_2)} k\left(\chi'\circ\psi\right)(g)\psi'(g)\,dg + \frac{1}{2(h_2-h_1)^\gamma}\int_{\psi^{-1}(h_1)}^{\psi^{-1}(h_2)}\left[\left(\psi(g)-h_1\right)^\gamma-\left(h_2-\psi(g)\right)^\gamma\right]\left(\chi'\circ\psi\right)(g)\psi'(g)\,dg\right| \\
&\leq \left|\int_{\psi^{-1}(h_1)}^{\psi^{-1}(h_2)} k\left(\chi'\circ\psi\right)(g)\psi'(g)\,dg\right| + \left|\frac{1}{2(h_2-h_1)^\gamma}\int_{\psi^{-1}(h_1)}^{\psi^{-1}(h_2)}\left[\left(\psi(z)-h_1\right)^\gamma-\left(h_2-\psi(g)\right)^\gamma\right]\left(\chi'\circ\psi\right)(g)\psi'(g)\,dg\right| \\
&:= S_1+S_2,
\end{aligned} \tag{48}$$

where

$$S_1 := \left|\int_{\psi^{-1}(h_1)}^{\psi^{-1}(h_2)} k\left(\chi'\circ\psi\right)(g)\psi'(g)\,dg\right|,$$

$$S_2 := \left| \frac{1}{2(h_2-h_1)^\gamma} \int_{\psi^{-1}(h_1)}^{\psi^{-1}(h_2)} \left[\left(\psi(z)-h_1\right)^\gamma - \left(h_2-\psi(g)\right)^\gamma \right] \left(\chi' \circ \psi\right)(g)\psi'(g)\,dg \right|,$$

and k is defined as in Lemma 2.2. Note that

$$S_1 = \frac{\chi(h_2)-\chi(h_1)}{2}, \tag{49}$$

and from Theorem 4.2 for the case $q = 1$, we have

$$S_2 \le \frac{h_2-h_1}{(\gamma+2)(\gamma+3)}\left(\left(\left|\chi'(h_1)\right| + \left|\chi(h_2)\right|\right)\right). \tag{50}$$

Hence by using (49) and (50) in (48), we get (47). □

Corollary 4.5 *Assume the similar conditions of Theorem 4.4.*

1. *If $\psi(g) = g$, then we get*

$$\left| \frac{\Gamma(\gamma+1)}{2(h_2-h_1)^\gamma}\left[J^\gamma_{h_1+}\chi(h_2) + J^\gamma_{h_2-}\chi(h_1)\right] - \chi\left(\frac{h_1+h_2}{2}\right) \right| \le \frac{\chi(h_2)-\chi(h_1)}{2} + \frac{h_2-h_1}{(\gamma+2)(\gamma+3)}\left(\left(\left|\chi'(h_1)\right| + \left|\chi(h_2)\right|\right)\right). \tag{51}$$

2. *If $\psi(g) = g$ and $\gamma = 1$, then we get*

$$\left| \frac{2}{(h_2-h_1)}\int_{h_1}^{h_2} \chi(g)\,dg - \chi\left(\frac{h_1+h_2}{2}\right) \right| \le \frac{\chi(h_2)-\chi(h_1)}{2} + \frac{h_2-h_1}{6}\left(\left|\chi'(h_1)\right| + \left|\chi(h_2)\right|\right). \tag{52}$$

5 Conclusion

In this paper, we proved in Theorem 3.1 the Hermite–Hadamard inequality for *tgs*-convex functions via Katugampola fractional integrals. From Theorems 3.2–3.6, we established a Hermite–Hadamard type inequality for *tgs*-convex functions via Katugampola fractional integrals. From Corollaries 3.3 and 3.5 we obtained a new Hermite–Hadamard type inequality for *tgs*-convex functions via Riemann–Liouville fractional and classical integrals. Also from Corollary 3.7 we obtained a new Hermite–Hadamard type inequality for *tgs*-convex functions via Riemann–Liouville fractional integrals.

On the other hand, from Theorem 4.1 we obtained the Hermite–Hadamard inequality for *tgs*-convex functions via ψ-Riemann–Liouville fractional integrals. From Theorems 4.2 and 4.4, we established a Hermite–Hadamard type inequality for *tgs*-convex functions via ψ-Riemann–Liouville fractional integrals. From Corollaries 4.3 and 4.5 we obtained a new Hermite–Hadamard type inequality for *tgs*-convex functions via Riemann–Liouville fractional and classical integrals.

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