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Operators constructed by means of basic sequences and nuclear matrices

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Abstract

In this work, we establish an approach to constructing compact operators between arbitrary infinite-dimensional Banach spaces without a Schauder basis. For this purpose, we use a countable number of basic sequences for the sake of verifying the result of Morrell and Retherford. We also use a nuclear operator, represented as an infinite-dimensional matrix defined over the space ℓ_1 of all absolutely summable sequences. Examples of nuclear operators over the space ℓ_1 are given and used to construct operators over general Banach spaces with specific approximation numbers.

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1 Introduction and basic definitions

Banach spaces, which are separable and reflexive, can exist without a Schauder basis as proved by Enflo in 1973 [11]. However, in 1972, Morrell and Retherford [8] showed that in each infinite-dimensional Banach space and for any sequence of positive numbers, that is, monotonically convergent to zero $(\lambda_i)_{i\in N}$, where $N = \{1, 2, 3, \dots\}$, one can construct a weakly square-summable basic sequence whose norms equal to $(\lambda_i)_{i\in N}$.

In 1977, Makarov and Faried [7] showed how to construct compact operators of the form $\sum_{i\in N} \mu_i f_i \otimes x_i$ between arbitrary infinite-dimensional Banach spaces such that its sequence of approximation numbers has a specific rate of convergence to zero. It was also proved that the operator ideal, whose sequence of approximation numbers are p-summable, is a small ideal; see [4, 10, 11].

In this work, we show how to construct compact operators between arbitrary infinite-dimensional Banach spaces using a countable number of basic sequences and nuclear operators, represented in the form of an infinite-dimensional matrix $(\mu_{ij})_{i,j\in N}$ defined over the space ℓ_1 of all absolutely summable sequences, which verifies

$$\lim_j \mu_{ij} = 0$$

for every $i \in N$. For such double-summation operators, a choice of matrix elements is more convenient than choosing sequence elements in the case of single-summation operators.

Such a construction will help give counterexamples of operators between Banach spaces without a Schauder basis. An upper estimate of the sequence of approximation numbers is given for such double-summation operators. For basic notions and some related results, one can see [1, 6, 9, 13].

The following notations are used throughout this study. The normed space of bounded linear operators from a normed space X into a normed space Y is denoted by $L(X,Y)$, while the dual space of the normed space X is denoted by $X^* = L(X,R)$, where R is the set of real numbers.

Also as mentioned before, the space $\{x = (x_i)_{i=1}^{\infty} : \sum_i |x_i|^p < \infty\}$ of all sequences of real numbers that are p-absolutely summable, is denoted by ℓ_p, which is equipped with the norm $\|x\| = (\sum_{i\in N} |x_i|^p)^{\frac{1}{p}}$. The space $\{x = (x_i)_{i=1}^{\infty} : \lim x_i = 0\}$ of all sequences of real numbers that are convergent to zero, is denoted by c_o, which is equipped with the norm $\|x\| = \sup_{i\in N} |x_i|$.

Definition 1.1 ([12]) A map s, which assigns a unique sequence $\{s_r(T)\}_{r=0}^{\infty}$ of real numbers to every operator $T \in L(X,Y)$, is called an s-number sequence if the following conditions are verified:

1. $\|T\| = s_0(T) \geq s_1(T) \geq \cdots \geq 0$ for $T \in L(X,Y)$.
2. $s_{r+m}(U+V) \leq s_r(U) + s_m(V)$ for $U, V \in L(X,Y)$.
3. $s_r(UTV) \leq \|U\| s_r(T) \|V\|$ for $V \in L(X_0,X)$, $T \in L(X,Y)$ and $U \in L(Y,Y_0)$.
4. $s_r(T) = 0$ if and only if $\operatorname{rank}(T) \leq r$ for $T \in L(X,Y)$.
5. $s_r(I_k) = \begin{cases} 1, & \text{for } r < k; \\ 0, & \text{for } r \geq k, \end{cases}$

where I_k is the identity operator on Euclidean space ℓ_2^k.

As an examples of s-numbers, we mention the approximation numbers $\alpha_r(T)$, Gelfand numbers $c_r(T)$, Kolmogorov numbers $d_r(T)$, and Tikhomirov numbers $d_r^*(T)$, defined by

1. $\alpha_r(T) = \inf\{\|T - A\| : A \in L(X,Y) \text{ and } \operatorname{rank}(A) \leq r\}$. Clearly, we always have $\|T\| = \alpha_0(T) \geq \alpha_1(T) \geq \alpha_2(T) \geq \cdots \geq 0$.
2. $c_r(T) = \alpha_r(J_Y T)$, where J_Y is a metric injection from the space Y into a higher space $\ell^{\infty}(\Lambda)$ of all bounded-real functions for a suitable index set Λ.
3. $$d_r(T) = \inf_{\dim K \leq r} \sup_{\|x\| \leq 1} \inf_{y \in K} \|Tx - y\|,$$
   where $K \subseteq Y$.
4. $d_r^*(T) = d_r(J_Y T)$.

Definition 1.2 ([11]) An operator $T \in L(X,Y)$ is nuclear if and only if it can be represented in the form

$$T(x) = \sum_{i=1}^{\infty} a_i(x) y_i,$$

with $a_1, a_2, \ldots \in X^*$ and $y_1, y_2, \ldots \in Y$, such that

$$\sum_{i=1}^{\infty} \|a_i\| \|y_i\| < \infty.$$

On the class $N(X,Y)$ of all nuclear operators from X into Y, a norm $\nu(T)$ is defined by

$$\nu(T) = \inf\left\{\sum_i \|a_i\|\|y_i\|\right\},$$

where the inf is taken over all possible representations of the operator T.

2 Basic theorems and technical lemmas

It is well known that an infinite matrix defines a linear continuous operator from the space ℓ_1 into itself if its columns are absolutely uniformly-summable; see [3, 4, 10].

Lemma 2.1 ([11], 6.3.6) *An operator $T \in L(\ell_1, \ell_1)$ is nuclear if and only if there is an infinite matrix $(\sigma_{ik})_{i,k\in N}$ such that*

$$T(x) = \left(\sum_{k=1}^{\infty} \sigma_{ik} x_k\right)_{i=1}^{\infty} \quad \text{for } x = (x_k)_{k=1}^{\infty} \in \ell_1$$

and

$$\sum_{i=1}^{\infty} \sup_k |\sigma_{ik}| < \infty.$$

In this case

$$\nu(T) = \sum_{i=1}^{\infty} \sup_k |\sigma_{ik}|.$$

Lemma 2.2 ([3]) *If $(T_i)_{i=1}^{\infty}$ is an absolutely summable sequence of bounded linear operators then*

$$\alpha_n\left(\sum_{i=1}^{\infty} T_i\right) \leq \inf\left\{\sum_{i=1}^{\infty} \alpha_{n_i}(T_i) : \sum_{i=1}^{\infty} n_i = n\right\},$$

where the inf *is taken over all possible representations for*

$$\sum_{i=1}^{\infty} n_i = n.$$

The following is a consequence of Lemma 2 in [2].

Theorem 2.3 *Let $(x_i)_{i=1}^{\infty}$ be a sequence in a Banach space X such that*

$$\sum_{i=1}^{\infty} \bigl|f(x_i)\bigr| < \infty \quad \text{for every } f \in X^*,$$

then the series $\sum_{i=1}^{\infty} \lambda_i x_i$ converges unconditionally in X for every sequence $(\lambda_i)_{i=1}^{\infty} \in c_o$.

Theorem 2.4 (Morrell and Retherford [8]) *Let X be an infinite-dimensional Banach space and let $(\lambda_i)_{i=1}^{\infty} \in c_o$ with $0 < \lambda_i < 1$, then there is a basic sequence $(x_i)_{i=1}^{\infty}$ in X such that $\|x_i\| = \lambda_i$ for all $i = 1, 2, \ldots$ that verifies*

$$\sum_{i=1}^{\infty} \left|f(x_i)\right|^2 \le \|f\|^2 \quad \textit{for every } f \in X^*.$$

Remark 2.5 Theorem 2.4 is valuable in the case of sequences that are slowly convergent to zero $(\lambda_i)_{i=1}^{\infty}$. Indeed, if $(\lambda_i)_{i=1}^{\infty}$ converges rapidly to zero then $\sum_{i=1}^{\infty} \|x_i\| < \infty$ and hence, one can write

$$\sum_{i=1}^{\infty} \left|f(x_i)\right|^2 \le \sum_{i=1}^{\infty} \|f\|^2 \|x_i\|^2 \le C\|f\|^2 \quad \text{for every } f \in X^*.$$

Theorem 2.6 (Dini's theorem [5]) *For a convergent series $\sum_{i=1}^{\infty} a_i$ of positive real numbers, the series*

$$\sum_{i=1}^{\infty} \frac{a_i}{R_i^m} \quad \textit{is} \quad \begin{cases} \textit{convergent} & \textit{for } m < 1; \\ \textit{divergent} & \textit{for } m \ge 1, \end{cases}$$

where $R_i = \sum_{j=i}^{\infty} a_j$ is the remainder of the series $\sum_{i=1}^{\infty} a_i$.

Theorem 2.7 ([7]) *Let X and Y be infinite-dimensional Banach spaces and let $(\lambda_r)_{r=1}^{\infty}$ be a monotonically decreasing sequence of positive real numbers, then there is a completely continuous operator $A \in L(X, Y)$ verifying*

$$2^{-4}\lambda_{3r} \le d_r^*(A) \le \alpha_r(A) \le 8\lambda_r \quad \textit{for every } r \in \{1, 2, \ldots\}.$$

Lemma 2.8 ([3]) *Let $\{\xi_i\}_{i\in N}$ be a bounded family of real numbers and let $K \subseteq N$ be an arbitrary subset of indices, such that* card K *is the number of elements in K. Then*

$$\sup_{\operatorname{card} K = r+1} \inf_{i \in K} \xi_i = \inf_{\operatorname{card} K = r} \sup_{i \notin K} \xi_i.$$

3 Main results

Proposition 3.1 *Let X and Y be infinite-dimensional Banach spaces and let $M = (\mu_{ij})_{i,j\in N}$ be an infinite matrix verifying that*:

1. $\lim_j \mu_{ij} = 0$ *for every $i \in N$.*
2 $\sum_{i=1}^{\infty} \sup_{j=1}^{\infty} |\mu_{ij}| < \infty$.

Let $(f_{ij})_{i,j\in N}$ be a matrix of functionals in X^* and $(z_{ij})_{i,j\in N}$ be a matrix of elements in Y that verifies

$$\sup_{i=1}^{\infty} \sum_{j=1}^{\infty} \left|f_{ij}(x)F(z_{ij})\right| < \infty \tag{1}$$

for every F in Y^* and every x in X. Then the expression

$$T(x) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\mu_{ij}f_{ij}(x)z_{ij}$$

defines a linear continuous operator from X into Y.

Proof Let

$$\lambda_n = \sum_{i\geq n}\sup_{j=1}^{\infty}|\mu_{ij}|,$$

then from Dini's theorem 2.6 we get

$$\sum_{i=1}^{\infty}\frac{\sup_{j=1}^{\infty}|\mu_{ij}|}{\sqrt{\lambda_i}} < \infty.$$

From condition (1) and Theorem 2.3, the formula

$$T_i(x) = \sum_{j=1}^{\infty}\frac{\mu_{ij}}{\sqrt{\lambda_i}}f_{ij}(x)z_{ij} \tag{2}$$

defines a linear continuous operator $T_i \in L(X,Y)$ for every $i = 1,2,\ldots$.

Now we need to prove the unconditional convergence of the series

$$T(x) = \sum_{i=1}^{\infty}\sqrt{\lambda_i}T_i(x).$$

In order to do so, it is enough to apply again Theorem 2.3, noting that $\lambda_n \to 0$ and we only have to verify that

$$\sum_{i=1}^{\infty}\bigl|gT_i(x)\bigr| < \infty, \quad \text{for every } g \in Y^*.$$

In fact,

$$\begin{aligned}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left|\frac{\mu_{ij}}{\sqrt{\lambda_i}}f_{ij}(x)g(z_{ij})\right| &\leq \sum_{i=1}^{\infty}\sup_{j=1}^{\infty}\frac{|\mu_{ij}|}{\sqrt{\lambda_i}}\sum_{j=1}^{\infty}\bigl|f_{ij}(x)g(z_{ij})\bigr| \\ &\leq \sum_{i=1}^{\infty}\sup_{j=1}^{\infty}\frac{|\mu_{ij}|}{\sqrt{\lambda_i}}\left[\sup_{i=1}^{\infty}\sum_{j=1}^{\infty}\bigl|f_{ij}(x)g(z_{ij})\bigr|\right] < \infty.\end{aligned}$$

Then the expression

$$T(x) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\mu_{ij}f_{ij}(x)z_{ij}$$

defines a linear continuous operator from X into Y. □

Remark 3.2 From Theorem 2.4 and for every $i = 1, 2, \dots$, there exist a basic sequence of functionals $\{f_{ij}\}_{j=1}^{\infty}$ in X^* and a basic sequence of elements $\{z_{ij}\}_{j=1}^{\infty}$ in Y such that

$$\sum_{j=1}^{\infty} \left|f_{ij}(x)\right|^2 \le \|x\|^2 \quad \text{for every } x \in X$$

and

$$\sum_{j=1}^{\infty} \left|F(z_{ij})\right|^2 \le \|F\|^2 \quad \text{for every } F \in Y^*.$$

Basic sequences can be found by choosing different convergent to zero sequences $(\lambda_i)_{i=1}^{\infty} \in c_o$, as mentioned in Theorem 2.4, according to their rate of convergence.

As a consequence of Proposition 3.1 and Remark 3.2 we get the following result.

Theorem 3.3 *Let X and Y be Banach spaces and let $\{f_{ij}\}_{j=1}^{\infty}$ and $\{z_{ij}\}_{j=1}^{\infty}$, where $i \in N$, be basic sequences in X^* and Y, respectively. Verifying the following,*

1. $\sum_{j=1}^{\infty} |f_{ij}(x)|^2 < \|x\|^2$ *for every $x \in X$*, and $i \in N$.
2. $\sum_{j=1}^{\infty} |F(z_{ij})|^2 < \|F\|^2$ *for every $F \in Y^*$ and $i \in N$, then every nuclear operator*

$$M = \{\mu_{ij}\} : \ell_1 \to \ell_1, \quad \textit{with } \lim_j \mu_{ij} = 0,$$

defines an operator $T : X \to Y$ of the form

$$T(x) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \mu_{ij} f_{ij}(x) z_{ij}.$$

Proof The proof follows directly from Proposition 3.1 and Remark 3.2. □

Theorem 3.4 *Let X and Y be infinite-dimensional Banach spaces and let $\{\mu_i\}_{i=1}^{\infty}$ be a sequence of real numbers that is convergent to zero and $\{f_i\}_{i=1}^{\infty}$, $\{z_i\}_{i=1}^{\infty}$ be sequences in X^* and Y, respectively. Verifying that*

$$\sum_{i=1}^{\infty} \left|f_i(x)\right|^2 \le \|x\|^2 \quad \textit{for every } x \in X,$$

and

$$\sum_{i=1}^{\infty} \left|F(z_i)\right|^2 \le \|F\|^2 \quad \textit{for every } F \in Y^*.$$

Then for the operator

$$T = \sum_{i=1}^{\infty} \mu_i f_i \otimes z_i$$

we have

$$\alpha_n(T) \leq \inf_{\operatorname{card} K \leq n} \sup_{i \notin K} |\mu_i|,$$

where K is any subset of the index set N with $\operatorname{card} K \leq n$.

Proof For every operator $T \in L(X, Y)$ and every subset of indices $K \subset N$ with $\operatorname{card} K \leq n$, we define a finite rank operator

$$A_K = \sum_{i \in K} \mu_i f_i \otimes z_i$$

with $\operatorname{rank}(A_K) \leq n$. From the definition of approximation numbers we get

$$\begin{aligned}
\alpha_n(T) \leq \|T - A_K\| &= \left\| \sum_{i \notin K} \mu_i f_i \otimes z_i \right\| \\
&= \sup_{\|x\|=1} \sup_{\|F\|=1} \left| \sum_{i \notin K} \mu_i f_i(x) F(z_i) \right| \\
&\leq \sup_{\|x\|=1} \sup_{\|F\|=1} \sum_{i \notin K} \left| \mu_i f_i(x) F(z_i) \right| \\
&\leq \sup_{i \notin K} |\mu_i| \sup_{\|x\|=1} \sup_{\|F\|=1} \sum_{i \notin K} \left| f_i(x) F(z_i) \right| \\
&\leq \sup_{i \notin K} |\mu_i|.
\end{aligned}$$

Since this relation is true for every index subset K with $\operatorname{card} K \leq n$,

$$\alpha_n(T) \leq \inf_{\operatorname{card} K \leq n} \sup_{i \notin K} |\mu_i|. \qquad \square$$

Remark 3.5 As a consequence of Theorem 3.4 and by using Lemma 2.8, we can get the following similar result:

$$\alpha_n(T) \leq \sup_{\operatorname{card} K = n+1} \inf_{i \in K} |\mu_i|.$$

Theorem 3.6 *Let X and Y be infinite-dimensional Banach spaces and let $(\mu_{ij})_{i,j \in N}$ be an infinite matrix with linearly independent rows such that conditions of Proposition 3.1 are verified, and let $\{f_{ij}\}_{j=1}^{\infty}$, $\{z_{ij}\}_{j=1}^{\infty}$ for $i = 1, 2, \ldots$, be sequences in X^* and Y, respectively, such that conditions of Theorem 3.4 are fulfilled for all $i = 1, 2, \ldots$. Then for the operator*

$$T = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \mu_{ij} f_{ij} \otimes z_{ij}$$

we have

$$\alpha_n(T) \leq \inf_{\Sigma n_i = n} \sum_{i=1}^{\infty} \left\{ \inf_{\operatorname{card} K \leq n_i} \sup_{j \notin K} |\mu_{ij}| \right\}, \tag{3}$$

where K is a subset of the index set N with $\operatorname{card} K \leq n_i$.

Proof From Lemma 2.2, Theorem 3.4 and by using the same operator T_i defined by Eq. (2) throughout the proof of Proposition 3.1, we get

$$\alpha_n(T) = \alpha_n\left(\sum_{i=1}^{\infty} T_i\right) \leq \sum_{i=1}^{\infty} \alpha_{n_i}(T_i) \leq \sum_{i=1}^{\infty} \inf_{\operatorname{card} K \leq n_i} \sup_{j \notin K} |\mu_{ij}|.$$

This relation is true for every $\Sigma n_i = n$, then we get the proof.

In the following, we are going to give two examples of nuclear operators over ℓ_1 and use them to construct operators over general Banach spaces with specific approximation numbers. □

Example 3.7 Consider the operator $A \in L(c_0, \ell_1)$ such that $A = (a_{ij})_{i,j=1}^{\infty}$, where

$$a_{ij} = 0 \quad \text{for } i \neq j,$$
$$a_{ii} = \frac{1}{2^k(k+1)^2} \quad \text{for } 2^k \leq i < 2^{k+1}.$$

Also, consider $B \in L(\ell_1, c_0)$, such that

$$B = \begin{pmatrix} B_0 & 0 & 0 & \cdots \\ 0 & B_1 & 0 & \cdots \\ 0 & 0 & B_2 & \cdots \\ . & . & . & \\ . & . & . & \\ . & . & . & \end{pmatrix},$$

where

$$B_0 = (1),$$
$$B_k = \begin{pmatrix} B_{k-1} & B_{k-1} \\ B_{k-1} & -B_{k-1} \end{pmatrix} \quad \text{is a } 2^k \times 2^k \text{ matrix for } k = 1, 2, 3, \ldots.$$

Thus we have $D = AB \in L(\ell_1, \ell_1)$, such that

$$D = \begin{pmatrix} D_0 & 0 & 0 & \cdots \\ 0 & D_1 & 0 & \cdots \\ 0 & 0 & D_2 & \cdots \\ . & . & . & \\ . & . & . & \\ . & . & . & \end{pmatrix},$$

where

$$D_0 = (1),$$
$$D_k = \frac{k^2}{2(1+k)^2}\begin{pmatrix} D_{k-1} & D_{k-1} \\ D_{k-1} & -D_{k-1} \end{pmatrix} \quad \text{is a } 2^k \times 2^k \text{ matrix for } k = 1, 2, 3, \ldots.$$

Let $D = (\mu_{ij})_{i,j=1}^{\infty}$, then this operator has the following properties:

1.
$$\sum_{i=1}^{\infty} |\mu_{ii}| = 1 + \left(\frac{1}{8} + \frac{1}{8}\right) + \left(\frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36}\right) + \left(\frac{1}{128} + \frac{1}{128} + \cdots\right) + \cdots$$
$$= \sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}.$$

2.
$$\nu(D) = \sum_{i=1}^{\infty} \sup_j |\mu_{ij}| = \frac{\pi^2}{6} < \infty,$$
then by using Lemma 2.1 D is a nuclear operator.

3. $\mathrm{Trac}(D) = 1 + (\frac{1}{8} - \frac{1}{8}) + (\frac{1}{36} - \frac{1}{36} + \frac{1}{36} - \frac{1}{36}) + (\frac{1}{128} - \frac{1}{128} + \cdots) + \cdots = 1.$
4. $D = (\mu_{ij})_{i,j=1}^{\infty}$ is having linearly independent rows.

Now, for $D = (\mu_{ij})_{i,j=1}^{\infty}$ and by using Proposition 3.1 and Theorem 3.6 one can construct an operator $T \in L(X, Y)$ for any Banach spaces X, Y of the form

$$T = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \mu_{ij} f_{ij} \otimes z_{ij},$$

where $\{f_{ij}\}_{i,j=1}^{\infty}$, $\{z_{ij}\}_{i,j=1}^{\infty}$, are basic sequences in X^* and Y, respectively, such that conditions of Theorem 3.4 are fulfilled for all $i = 1, 2, \ldots$.

Now by applying Eq. (3), one can get

$$\alpha_n(T) \leq \frac{\pi^2}{6} - \sum_{i=1}^{k+1} \frac{1}{i^2} \quad \text{for } n = 1, 2, 3, \ldots \text{ where } 2^k \leq n < 2^{k+1}.$$

Hence, we have

$$\lim_{n\to\infty} \alpha_n(T) \leq \frac{\pi^2}{6} - \sum_{i=1}^{\infty} \frac{1}{i^2} = 0,$$

which is consistent with the properties of the approximation numbers.

By applying Eq. (3) in the case of $n = 0$, we get

$$\alpha_0(T) = \|T\| \leq 1 + \left(\frac{1}{8} + \frac{1}{8}\right) + \left(\frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36}\right) + \left(\frac{1}{128} + \frac{1}{128} + \cdots\right) + \cdots$$
$$= \sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}.$$

Example 3.8 Consider the operator $J \in L(\ell_1, \ell_1)$ such that $J = (\lambda_{ij})_{i,j=1}^{\infty}$ where $\lambda_{ij} = \frac{ij}{2^{i+j}}$, then this operator has the following properties:

1. $\nu(J) = \sum_{i=1}^{\infty} \sup_j |\lambda_{ij}| = \sum_{i=1}^{\infty} \frac{i}{2^i} \sup_j (\frac{j}{2^j}) = 1 < \infty$, then by using Lemma 2.1 J is a nuclear operator.

2. $J = (\lambda_{ij})_{i,j=1}^{\infty}$ has linearly independent rows.

Now for $J = (\lambda_{ij})_{i,j=1}^{\infty}$ and by using Proposition 3.1 and Theorem 3.6, one can construct an operator $T \in L(X, Y)$ for any Banach spaces X, Y on the form,

$$T = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \lambda_{ij} f_{ij} \otimes z_{ij},$$

where $\{f_{ij}\}_{i,j=1}^{\infty}$ and $\{z_{ij}\}_{i,j=1}^{\infty}$ are basic sequences in X^* and Y, respectively, such that conditions of Theorem 3.4 are fulfilled for all $i = 1, 2, \ldots$.

Applying Eq. (3) yields

$$\alpha_n(T) \le \frac{n+1}{2^n} \quad \text{for } n = 1, 2, 3, \ldots.$$

Thus, we have $(\alpha_n(T))_{n=1}^{\infty} \in \ell_1$ because

$$\sum_{n=1}^{\infty} \alpha_n(T) \le \sum_{n=1}^{\infty} \frac{n+1}{2^n} = 3 < \infty.$$

Applying Eq. (3) in the case of $n = 0$ yields

$$\alpha_0(T) = \|T\| \le \frac{1}{2} \sum_{i=1}^{\infty} \frac{i}{2^i} = \frac{1}{2} \times 2 = 1,$$

noting that this is independent of the selection of $\{f_{ij}\}_{i,j=1}^{\infty}$ and $\{z_{ij}\}_{i,j=1}^{\infty}$.

If we choose $\{f_{ij}\}_{i,j=1}^{\infty}$ and $\{z_{ij}\}_{i,j=1}^{\infty}$ such that

$$\|f_{ij}\| = \|z_{ij}\| = \frac{1}{\sqrt{ij}},$$

then we get

$$\nu(T) \le \sum_{i,j=1}^{\infty} \lambda_{ij} \|f_{ij}\| \|z_{ij}\| = \sum_{i,j=1}^{\infty} \left(\frac{ij}{2^{i+j}}\right)\left(\frac{1}{ij}\right) = 1 < \infty,$$

which means that T, in this case, is a nuclear operator.

4 Conclusion

By using nuclear operators defined over ℓ_1 with particular representation, one can construct compact operators over general Banach spaces with specific approximation numbers. Such compact operators are been constructed using a countable number of basic sequences and nuclear operators. For such nuclear operators, its construction in a matrix form will yield to double-summation operators. This double-summation gives more freedom rather than choosing sequence elements in the case of single-summation operators. Such a construction will help give counterexamples of operators between Banach spaces without a Schauder basis.

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