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Levinson type inequalities for higher order convex functions via Abel–Gontscharoff interpolation

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Abstract

In this paper, Levinson type inequalities are studied for the class of higher order convex functions by using Abel–Gontscharoff interpolation. Cebyšev, Grüss, and Ostrowski-type new bounds are also found for the functionals involving data points of two types.

Keywords: *m*-convex function; Levinson's inequality; Green functions

1 Introduction and preliminaries

The theory of convex functions has encountered a fast advancement. This can be attributed to a few causes: firstly, applications of convex functions are directly involved in modern analysis; secondly, many important inequalities are the results of applications of convex functions, and convex functions are closely related to inequalities (see [1]).

Divided differences are seen to be uncommonly valuable when we are managing limits having assorted diverse of smoothness. In [1, p. 14], the definition of divided difference is given as follows:

mth-order divided difference:

Let a function $f:[\zeta_1,\zeta_2]\to\mathbb{R}$. The mth-order divided difference of a function f at $x_0,\dots,x_m\in[\zeta_1,\zeta_2]$ is defined recursively by

$$\begin{aligned}
&[x_i;f]=f(x_i),\quad i=0,\dots,m,\\
&[x_0,\dots,x_m;f]=\frac{[x_1,\dots,x_m;f]-[x_0,\dots,x_{m-1};f]}{x_m-x_0}.
\end{aligned}\tag{1}$$

It is easy to see that (1) is equivalent to

$$[x_0,\dots,x_m;f]=\sum_{i=0}^{m}\frac{f(x_i)}{q'(x_i)},\quad \text{where } q(x)=\prod_{j=0}^{m}(x-x_j).$$

The following definition of a real-valued convex function is characterized by mth-order divided difference (see [1, p. 15]).

Higher order convex function:

A function $f : [\zeta_1, \zeta_2] \to \mathbb{R}$ is said to be m-convex ($m \geq 0$) if and only if, for all decisions of $(m+1)$ distinct points $x_0, \dots, x_m \in [\zeta_1, \zeta_2]$, $[x_0, \dots, x_m; f] \geq 0$ holds. If this inequality is reversed, then f is said to be m-concave.

Criteria for m-convex functions:

In [1, p. 16], the criterion to examine the m-convexity of a function f is given as follows.

Theorem 1 *If $f^{(m)}$ exists, then f is m-convex if and only if $f^{(m)} \geq 0$.*

In [2] (see also [3, p. 32, Theorem 1]), Ky Fan's inequality is generalized by Levinson for 3-convex functions as follows.

Theorem 2 *Let $f : I = (0, 2\alpha) \to \mathbb{R}$ with $f^{(3)}(t) \geq 0$. Let $x_k \in (0, \alpha)$ and $p_k > 0$. Then*

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) - f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i f(2\alpha - x_i) - f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i (2\alpha - x_i)\right). \tag{2}$$

Functional form of (2) is defined as follows:

$$\begin{aligned} \mathcal{J}_1\big(f(\cdot)\big) = {} & \frac{1}{P_n}\sum_{i=1}^{n} p_i f(2a - x_i) - f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i (2a - x_i)\right) - \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) \\ & + f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right). \end{aligned} \tag{3}$$

Working with the divided differences, assumptions of differentiability on f can be weakened. In [4], Popoviciu noted that (2) is valid on $(0, 2a)$ for 3-convex functions, while in [5] (see also [3, p. 32, Theorem 2]) Bullen gave a different proof of Popoviciu's result and also the converse of (2).

Theorem 3

(a) *Let $f : I = [\zeta_1, \zeta_2] \to \mathbb{R}$ be a 3-convex function and $x_n, y_n \in [\zeta_1, \zeta_2]$ for $n = 1, 2, \dots, k$ such that*

$$\max\{x_1, \dots, x_k\} \leq \min\{y_1, \dots, y_k\}, \quad x_1 + y_1 = \cdots = x_k + y_k \tag{4}$$

and $p_n > 0$, then

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) - f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i f(y_i) - f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i y_i\right). \tag{5}$$

(b) *If f is continuous and $p_n > 0$, (5) holds for all x_k, y_k satisfying (4), then f is 3-convex.*

Functional form of (5) is defined as follows:

$$\mathcal{J}_2\big(f(\cdot)\big) = \frac{1}{P_n}\sum_{i=1}^{n} p_i f(y_i) - f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i y_i\right) - \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) + f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right). \tag{6}$$

Remark 1 It is essential to take note of the fact that under the suppositions of Theorem 2 and Theorem 3, if the function f is 3-convex, then $\mathcal{J}_i(f(\cdot)) \geq 0$ for $i = 1,2$ and $\mathcal{J}_i(f(\cdot)) = 0$ for $f(x) = x$ or $f(x) = x^2$ or f is a constant function.

In [6] (see also [3, p. 32, Theorem 4]), Pečarić weakened assumption (4) and proved that inequality (5) still holds, i.e., the following result holds.

Theorem 4 *Let $f : I = [\zeta_1, \zeta_2] \to \mathbb{R}$ be a 3-convex function, $p_i > 0$, and let $x_i, y_i \in [\zeta_1, \zeta_2]$ such that $x_i + y_i = 2\check{c}$ for $i = 1,\ldots,n$, $x_i + x_{n-i+1} \leq 2\check{c}$, and $\frac{p_i x_i + p_{n-i+1} x_{n-i+1}}{p_i + p_{n-i+1}} \leq \check{c}$. Then (5) holds.*

In [7], Mercer made a notable work by replacing the condition of symmetric distribution of points x_i and y_i with symmetric variances of points x_i and y_i, the second condition is a weaker condition.

Theorem 5 *Let f be a 3-convex function on $[\zeta_1, \zeta_2]$, p_i be positive such that $\sum_{i=1}^{n} p_i = 1$. Also let x_i, y_i satisfy $\max\{x_1,\ldots,x_i\} \leq \min\{y_1,\ldots,y_i\}$ and*

$$\sum_{i=1}^{n} p_i\left(x_i - \sum_{i=1}^{n} p_i x_i\right)^2 = \sum_{i=1}^{n} p_i\left(y_i - \sum_{i=1}^{n} p_i y_i\right)^2, \tag{7}$$

then (5) holds.

In [8], Adeel *et al.* generalized Levinson's inequality for 3-convex function by using two Green functions. In [9], Pečarić *et al.* gave a probabilistic version of Levinson's inequality (2) under Mercer's assumption of equal variances (but for a different number of data points) for the family of 3-convex functions at a point. They showed that this is the largest family of continuous functions for which inequality (2) holds. An operator version of probabilistic Levinson's inequality is discussed in [10] (see also [11]).

On the other hand, the error function $e_F(t)$ can be represented in terms of the Green functions $G_{F,m}(t,s)$ for the boundary value problem

$z^{(m)}(t) = 0$,
$z^{(i)}(a_1) = 0$, $0 \leq i \leq p$,
$z^{(i)}(a_2) = 0$, $p+1 \leq i \leq m-1$:

$$e_F(t) = \int_{\zeta_1}^{\zeta_2} G_{F,m}(t,s) f^{(m)}(s)\,ds, \quad t \in [\zeta_1, \zeta_2],$$

where

$$G_{F,m}(t,s) = \frac{1}{(m-1)!}\begin{cases} \sum_{i=0}^{p} \binom{m-1}{i}(t-\zeta_1)^i(\zeta_1 - s)^{m-i-1}, & \zeta_1 \leq s \leq t; \\ -\sum_{i=p+1}^{m-p} \binom{m-1}{i}(t-\zeta_1)^i(\zeta_1 - s)^{m-i-1}, & t \leq s \leq \zeta_2. \end{cases} \tag{8}$$

Further $\zeta_1 \leq t, s \leq \zeta_2$, the following inequalities hold:

$$(-1)^{m-p-1} \frac{\partial^i G_{F,m}(t,s)}{\partial s^i} \geq 0, \quad 0 \leq i \leq p, \tag{9}$$

$$(-1)^{m-p} \frac{\partial^i G_{F,m}(t,s)}{\partial s^i} \geq 0, \quad p+1 \leq i \leq m-1. \tag{10}$$

The following result holds in [12].

Theorem 6 *Let $f \in C^m[a,b]$, and let P_F be its 'two-point right focal' interpolating polynomial. Then, for $a \leq \zeta_1 < \zeta_2 \leq b$ and $0 \leq p \leq m-2$, the following holds*:

$$\begin{aligned} f(t) &= P_F(t) + e_F(t) \\ &= \sum_{i=0}^{p} \frac{(t-\zeta_1)^i}{i!} f^{(i)}(\zeta_1) \\ &\quad + \sum_{j=0}^{n-p-2} \left(\sum_{i=0}^{j} \frac{(t-\zeta_1)^{p+1+i}(\zeta_1-\zeta_2)^{j-i}}{(p+1+i)!(j-i)!} \right) f^{(p+1+j)}(\zeta_2) \\ &\quad + \int_{\zeta_1}^{\zeta_2} G_{F,m}(t,s) f^{(m)}(s)\, ds, \end{aligned} \tag{11}$$

where $G_{F,m}$ is the Green function defined by (8).

In [13], Butt *et al.* generalized Popoviciu's inequality via Abel–Gontscharoff interpolating polynomial for higher order convex functions. In the same year in [14], Tasadduq *et al.* used Abel–Gontscharoff-type Green's function for a two-point right focal to a generalized refinement of Jensen's inequality from convex functions to higher order convex functions. The results in [13] and [14] are only for one type of data points. But Levinson-type inequalities studied for the class of 3-convex functions involve two types of data points. In this paper Levinson-type inequalities are generalized via Abel–Gontscharoff interpolating polynomial involving two types of data points.

2 Main results

Motivated by identity (6), we construct the following identities with the help of (8) and (11).

2.1 Bullen-type inequalities for higher order convex functions

First we define the following functional:

$\mathcal{F}$: Let $(p_1, \dots, p_{n_1}) \in \mathbb{R}^{n_1}$ and $(q_1, \dots, q_{m_1}) \in \mathbb{R}^{m_1}$ be such that $\sum_{i=1}^{n_1} p_i = P_{n_1}$, $\sum_{i=1}^{m_1} q_i = Q_{m_1}$, and $x_i, y_i, \frac{1}{P_{n_1}} \sum_{i=1}^{n_1} p_i x_i, \frac{1}{Q_{m_1}} \sum_{i=1}^{m_1} p_i y_i \in I_1$. Then

$$\begin{aligned} \breve{\mathcal{J}}\big(f(\cdot)\big) &= \frac{1}{Q_{m_1}} \sum_{i=1}^{m_1} q_i f(y_i) - f\left(\frac{1}{Q_{m_1}} \sum_{i=1}^{m_1} q_i y_i \right) - \frac{1}{P_{n_1}} \sum_{i=1}^{n_1} p_i f(x_i) \\ &\quad + f\left(\frac{1}{P_{n_1}} \sum_{i=1}^{n_1} p_i x_i \right). \end{aligned} \tag{12}$$

Theorem 7 *Assume $\mathcal{F}$. Let $f : I_1 = [\zeta_1, \zeta_2] \to \mathbb{R}$ be a function such that $f \in C^m[\zeta_1, \zeta_2]$ ($m \geq 3$) and $G_{F,m}$, $\breve{\mathcal{J}}(f(\cdot))$ are defined in (8) and (12) respectively. Then*

$$\breve{\mathcal{J}}\big(f(\cdot)\big) = \breve{\mathcal{J}}\big(P_F(\cdot)\big) + \int_{\zeta_1}^{\zeta_2} \breve{\mathcal{J}}\big(G_{F,m}(\cdot, s)\big) f^{(m)}(s)\, ds. \tag{13}$$

Proof Using Abel–Gontscharoff identity (11) in (12), we have

$$\begin{aligned}
\breve{\mathcal{J}}\big(f(\cdot)\big) &= \frac{1}{Q_{m_1}} \sum_{k=1}^{m_1} q_k \Bigg[\sum_{i=0}^{p} \frac{(y_k - \zeta_1)^i}{i!} f^{(i)}(\zeta_1) \\
&\quad + \sum_{j=0}^{n-p-2} \Bigg(\sum_{i=0}^{j} \frac{(y_k - \zeta_1)^{p+1+i} (\zeta_1 - \zeta_2)^{j-i}}{(p+1+i)!(j-i)!} \Bigg) f^{(p+1+j)}(\zeta_2) \\
&\quad + \int_{\zeta_1}^{\zeta_2} G_{F,m}(y_k, s) f^{(m)}(s)\, ds \Bigg] \\
&\quad - \sum_{i=0}^{p} \frac{(\frac{1}{Q_{m_1}} \sum_{k=1}^{m_1} q_k y_k - \zeta_1)^i}{i!} f^{(i)}(\zeta_1) \\
&\quad - \sum_{j=0}^{n-p-2} \Bigg(\sum_{i=0}^{j} \frac{(\frac{1}{Q_{m_1}} \sum_{k=1}^{m_1} q_k y_k - \zeta_1)^{p+1+i} (\zeta_1 - \zeta_2)^{j-i}}{(p+1+i)!(j-i)!} \Bigg) f^{(p+1+j)}(\zeta_2) \\
&\quad - \int_{\zeta_1}^{\zeta_2} G_{F,m}\Bigg(\frac{1}{Q_{m_1}} \sum_{k=1}^{m_1} q_k y_k, s \Bigg) f^{(m)}(s)\, ds - \frac{1}{P_{n_1}} \sum_{k=1}^{n_1} p_k \Bigg[\sum_{i=0}^{p} \frac{(x_k - \zeta_1)^i}{i!} f^{(i)}(\zeta_1) \\
&\quad + \sum_{j=0}^{n-p-2} \Bigg(\sum_{i=0}^{j} \frac{(x_k - \zeta_1)^{p+1+i} (\zeta_1 - \zeta_2)^{j-i}}{(p+1+i)!(j-i)!} \Bigg) f^{(p+1+j)}(\zeta_2) + \int_{\zeta_1}^{\zeta_2} G_{F,m}(x_k, s) f^{(m)}(s)\, ds \Bigg] \\
&\quad + \sum_{i=0}^{p} \frac{(\frac{1}{P_{n_1}} \sum_{k=1}^{n_1} p_k x_k - \zeta_1)^i}{i!} f^{(i)}(\zeta_1) \\
&\quad + \sum_{j=0}^{n-p-2} \Bigg(\sum_{i=0}^{j} \frac{(\frac{1}{P_{n_1}} \sum_{k=1}^{n_1} p_k x_k - \zeta_1)^{p+1+i} (\zeta_1 - \zeta_2)^{j-i}}{(p+1+i)!(j-i)!} \Bigg) f^{(p+1+j)}(\zeta_2) \\
&\quad + \int_{\zeta_1}^{\zeta_2} G_{F,m}\Bigg(\frac{1}{P_{n_1}} \sum_{k=1}^{n_1} p_k x_k, s \Bigg) f^{(m)}(s)\, ds.
\end{aligned}$$

Using the definition of $\breve{\mathcal{J}}(\cdot)$, we have

$$\begin{aligned}
\breve{\mathcal{J}}\big(f(\cdot)\big) &= \frac{1}{Q_{m_1}} \sum_{k=1}^{m_1} q_k \Bigg[\sum_{i=3}^{p} \frac{(y_k - \zeta_1)^i}{i!} f^{(i)}(\zeta_1) \\
&\quad + \sum_{j=0}^{n-p-2} \Bigg(\sum_{i=3}^{j} \frac{(y_k - \zeta_1)^{p+1+i} (\zeta_1 - \zeta_2)^{j-i}}{(p+1+i)!(j-i)!} \Bigg) f^{(p+1+j)}(\zeta_2) \Bigg] \\
&\quad - \sum_{i=3}^{p} \frac{(\frac{1}{Q_{m_1}} \sum_{k=1}^{m_1} q_k y_k - \zeta_1)^i}{i!} f^{(i)}(\zeta_1)
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{j=0}^{n-p-2}\left(\sum_{i=3}^{j}\frac{(\frac{1}{Q_{m_1}}\sum_{k=1}^{m_1}q_k y_k-\zeta_1)^{p+1+i}(\zeta_1-\zeta_2)^{j-i}}{(p+1+i)!(j-i)!}\right)f^{(p+1+j)}(\zeta_2)\\
&-\frac{1}{P_{n_1}}\sum_{k=1}^{n_1}p_k\left[\sum_{i=3}^{p}\frac{(x_k-\zeta_1)^i}{i!}f^{(i)}(\zeta_1)+\sum_{j=0}^{n-p-2}\left(\sum_{i=3}^{j}\frac{(x_k-\zeta_1)^{p+1+i}(\zeta_1-\zeta_2)^{j-i}}{(p+1+i)!(j-i)!}\right)\right.\\
&\left.\times f^{(p+1+j)}(\zeta_2)\right]+\sum_{i=3}^{p}\frac{(\frac{1}{P_{n_1}}\sum_{k=1}^{n_1}p_k x_k-\zeta_1)^i}{i!}f^{(i)}(\zeta_1)\\
&+\sum_{j=0}^{n-p-2}\left(\sum_{i=3}^{j}\frac{(\frac{1}{P_{n_1}}\sum_{k=1}^{n_1}p_k x_k-\zeta_1)^{p+1+i}(\zeta_1-\zeta_2)^{j-i}}{(p+1+i)!(j-i)!}\right)f^{(p+1+j)}(\zeta_2)\\
&+\int_{\zeta_1}^{\zeta_2}\breve{\mathcal{J}}\big(G_{F,m}(\cdot,s)\big)f^{(m)}(s)\,ds.
\end{aligned}
$$

After some simple calculations,

$$
\begin{aligned}
\breve{\mathcal{J}}\big(f(\cdot)\big)={}&\frac{1}{Q_{m_1}}\sum_{k=1}^{m_1}q_k\big(P_F(y_k-\zeta_1)\big)-P_F\left(\frac{1}{Q_{m_1}}\sum_{k=1}^{m_1}q_k y_k-\zeta_1\right)\\
&-\frac{1}{P_{n_1}}\sum_{k=1}^{n_1}p_k\big(P_F(x_k-\zeta_1)\big)+P_F\left(\frac{1}{P_{n_1}}\sum_{k=1}^{n_1}p_k x_k-\zeta_1\right)\\
&+\int_{\zeta_1}^{\zeta_2}\breve{\mathcal{J}}\big(G_{F,m}(\cdot,s)\big)f^{(m)}(s)\,ds.
\end{aligned}
$$

Again, we use the definition of $\breve{\mathcal{J}}(\cdot)$ to get (13). □

In the next result we have generalizations of Bullen-type inequality for m-convex functions.

Theorem 8 *Assume the conditions of Theorem 7 with*

$$\breve{\mathcal{J}}\big(G_{F,m}(\cdot,s)\big)\geq 0,\quad s\in[\zeta_1,\zeta_2]. \tag{14}$$

If f is m-convex such that $f^{(m-1)}$ is absolutely continuous, then we have

$$\breve{\mathcal{J}}\big(f(\cdot)\big)\geq\breve{\mathcal{J}}\big(P_F(\cdot)\big). \tag{15}$$

Proof Since $f^{(m-1)}$ is absolutely continuous on $[\zeta_1,\zeta_2]$, therefore $f^{(m)}$ exists almost everywhere. By using Theorem 1, we have $f^{(m)}(s)\geq 0$ $(m\geq 3)$ a.e. on $[\zeta_1,\zeta_2]$. Hence we can apply Theorem 7 to get (15). □

If we put $m_1=n_1=n$, $p_i=q_i$ and use positive weights in (12), then $\breve{\mathcal{J}}(\cdot)$ is converted to the functional $\mathcal{J}_2(\cdot)$ defined in (6), also in this case, (13), (14), and (15) become

$$\mathcal{J}_2\big(f(\cdot)\big)=\mathcal{J}_2\big(P_F(\cdot)\big)+\int_{\zeta_1}^{\zeta_2}\mathcal{J}_2\big(G_{F,m}(\cdot,s)\big)f^{(m)}(s)\,ds, \tag{13a}$$

where

$$\mathcal{J}_2(P_F(\cdot)) = \frac{1}{P_n}\sum_{k=1}^{n} p_k\big(P_F(y_k-\zeta_1)\big) - P_F\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k y_k - \zeta_1\right)$$
$$- \frac{1}{P_n}\sum_{k=1}^{n} p_k\big(P_F(x_k-\zeta_1)\big) + P_F\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k x_k - \zeta_1\right),$$
$$\mathcal{J}_2(G_{F,m}(\cdot,s)) = \frac{1}{P_n}\sum_{k=1}^{n} p_k G_{F,m}(y_k,s) - G_{F,m}\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k y_k, s\right)$$
$$- \frac{1}{P_n}\sum_{k=1}^{n} p_k G_{F,m}(x_k,s) + G_{F,m}\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k x_k, s\right),$$
$$\mathcal{J}_2(G_{F,m}(\cdot,s)) \geq 0, \quad s \in [\zeta_1,\zeta_2], \tag{14a}$$

and

$$\mathcal{J}_2(f(\cdot)) \geq \mathcal{J}_2(P_F(\cdot)), \tag{15a}$$

respectively.

In the next result, we give a generalization of Bullen-type inequality for n tuples.

Theorem 9 *Let $f \in C^m[\zeta_1,\zeta_2]$ $(m \geq 3)$, $\mathbf{p} = (p_1,\dots,p_n)$ be a positive n-tuple such that $\sum_{i=1}^{n} p_i = P_n$. Also let $x_i, y_i \in I_1$ such that (4) is valid for $i = 1,\dots,n$. Then for the functional $\mathcal{J}_2(f(\cdot))$ defined in (6), we have the following*:

(i) *If n is even and p is odd or p is even and n is odd, then for every m-convex function f, (15a) holds.*

(ii) *Let inequality (15a) be satisfied. If $P_F(\cdot)$ is 3-convex then (6) is valid.*

Proof (i) By using (9), the following inequality

$$(-1)^{n-p-1}\frac{\partial^3 \mathcal{G}_{F,n}(\cdot,s)}{\partial s^3} \geq 0 \tag{16}$$

holds, therefore it is easy to conclude that if (n = even, p = odd) or (p = even, n = odd), then $\frac{\partial^3 \mathcal{G}_{F,n}(\cdot,s)}{\partial s^3} \geq 0$, or if ($n$ = odd, p = odd) or (p = even, n = even), then $\frac{\partial^3 \mathcal{G}_{F,n}(\cdot,s)}{\partial s^3} \leq 0$. So, for the cases ($n$ = even, p = odd) or (p = even, n = odd), $\mathcal{G}_{F,n}(\cdot,s)$ is 3-convex with respect to the first variable, therefore by following Remark 1, inequality (14a) holds for n tuples. Hence, by Theorem 8, inequality (15a) holds.

(ii) Since $P_F(\cdot)$ is assumed to be 3-convex, therefore using the given conditions and by following Remark 1, the nonnegativity of the R.H.S. of (15a) is immediate, and we have (6) for n-tuples. □

Next we have a generalized form (for real weights) of Levinson-type inequality for $2n$ points given in [6](see also [3]).

$\mathcal{I}$: Let $(p_1,\dots,p_{n_1}) \in \mathbb{R}^{n_1}$, $(q_1,\dots,q_{m_1}) \in \mathbb{R}^{m_1}$ be such that $\sum_{i=1}^{n_1} p_i = P_{n_1}$, $\sum_{i=1}^{m_1} q_i = Q_{m_1}$, $\frac{1}{Q_{m_1}}\sum_{i=1}^{m_1} q_i y_i$, and $\frac{1}{P_{n_1}}\sum_{i=1}^{n_1} p_i x_i \in I_1$. Also let $x_1,\dots,x_{n_1}$ and $y_1,\dots,y_{m_1} \in I_1$ such that $x_i + y_i = 2\breve{c}$, $x_i + x_{n-i+1} \leq 2\breve{c}$ and $\frac{p_i x_i + p_{n-i+1}x_{n-i+1}}{p_i + p_{n-i+1}} \leq \breve{c}$ for $i = 1,\dots,n$. Then (12) holds.

Theorem 10 *Assume $\mathcal{I}$. Let $f : I_1 = [\zeta_1, \zeta_2] \to \mathbb{R}$ be such that $f \in C^m[\zeta_1, \zeta_2]$ $(m \geq 3)$, $G_{F,m}$ and $\breve{\mathcal{J}}(f(\cdot))$ as defined in (8) and (12) respectively. Then identity (13) holds.*

Proof Assume $\mathcal{I}$ in Theorem 7 with the given conditions to get the required result. □

Theorem 11 *Assume $\mathcal{I}$. Let $f : I_1 = [\zeta_1, \zeta_2] \to \mathbb{R}$ be such that $f \in C^m[\zeta_1, \zeta_2]$ $(m \geq 3)$ and $f^{(m-1)}$ is absolutely continuous. Also let $G_{F,m}$ and $\breve{\mathcal{J}}(f(\cdot))$ be defined in (8) and (12) respectively. If (14) is valid, then (15) is also valid.*

Proof Proof is similar to Theorem 8. □

Theorem 12 *Let $f \in C^m[\zeta_1, \zeta_2]$ $(m \geq 3)$, $\mathbf{p} = (p_1, \dots, p_n)$ be a positive n-tuple such that $\sum_{i=1}^n p_i = P_n$. Also let $x_i, y_i \in I_1$ such that $x_i + y_i = 2\breve{c}$, $x_i + x_{n-i+1} \leq 2\breve{c}$ and $\frac{p_i x_i + p_{n-i+1} x_{n-i+1}}{p_i + p_{n-i+1}} \leq \breve{c}$ for $i = 1, \dots, n$. Then, for the functional $\mathcal{J}_2(f(\cdot))$ defined in (6), we have the following:*

(i) *If n is even and p is odd or p is even and n is odd, then for every m-convex function f, (15a) holds.*
(ii) *Let inequality (15a) be satisfied. If $P_F(\cdot)$ is 3-convex, then (6) is valid.*

Proof In Theorem 9, replace condition (4) for x_i and y_i with the condition given in the statement to get the required result. □

In [7], Mercer made a significant improvement by replacing condition (4) of symmetric distribution with the weaker one that the variances of the two sequences are equal.

Corollary 1 *Let $f : I_1 = [\zeta_1, \zeta_2] \to \mathbb{R}$ be such that $f \in C^m[\zeta_1, \zeta_2]$ $(m \geq 3)$, x_i, y_i satisfy (7), and $\max\{x_1, \dots, x_n\} \leq \min\{y_1, \dots, y_n\}$. Also let $(p_1, \dots, p_n) \in \mathbb{R}^n$ such that $\sum_{i=1}^n p_i = P_n$. Then (13a) holds.*

2.2 Generalization of Levinson's inequalities

Motivated by identity (3), we construct the following identities with the help of (8) and (11).

$\mathcal{H}$: Let $f : I_2 = [0, 2a] \to \mathbb{R}$ be a function, $x_1, \dots, x_{n_1} \in (0, a)$, $(p_1, \dots, p_{n_1}) \in \mathbb{R}^{n_1}$, $(q_1, \dots, q_{m_1}) \in \mathbb{R}^{m_1}$ be real numbers such that $\sum_{i=1}^{n_1} p_i = P_{n_1}$ and $\sum_{i=1}^{m} q_i = Q_{m_1}$. Also let x_i, $\frac{1}{Q_{m_1}} \sum_{i=1}^{m_1} q_i(2a - x_i)$ and $\frac{1}{P_{n_1}} \sum_{i=1}^{n_1} p_i \in I_2$. Then

$$\tilde{\mathcal{J}}(f(\cdot)) = \frac{1}{Q_{m_1}} \sum_{i=1}^{m_1} q_i f(2a - x_i) - f\left(\frac{1}{Q_{m_1}} \sum_{i=1}^{m_1} q_i(2a - x_i)\right) - \frac{1}{P_n} \sum_{i=1}^{n_1} p_i f(x_i) + f\left(\frac{1}{P_{n_1}} \sum_{i=1}^{n_1} p_i x_i\right). \tag{17}$$

Theorem 13 *Assume $\mathcal{H}$ and let $f \in C^m[0, 2a]$ $(m \geq 3)$. Also let $G_{F,m}$ and $\tilde{\mathcal{J}}(f(\cdot))$ be defined in (8) and (17) respectively. Then we have*

$$\tilde{\mathcal{J}}(f(\cdot)) = \tilde{\mathcal{J}}(P_F(\cdot)) + \int_{\zeta_1}^{\zeta_2} \tilde{\mathcal{J}}(G_{F,m}(\cdot, s)) f^{(m)}(s)\, ds, \tag{18}$$

where $0 \leq \zeta_1 < \zeta_2 \leq 2a$.

Proof Replace $\mathcal{F}$ with $\mathcal{H}$ and y_i with $2a - x_i$ in Theorem 7, we get the required result. □

Theorem 14 *Assume $\mathcal{H}$. Let $f \in C^m[0,2a]$ $(m \geq 3)$ with $f^{(m-1)}$ be absolutely continuous. Also let $G_{F,m}$ and $\tilde{\mathcal{J}}(f(\cdot))$ be defined in (8) and (17) respectively. If*

$$\tilde{\mathcal{J}}\big(G_{F,m}(\cdot,s)\big) \geq 0, \tag{19}$$

then

$$\tilde{\mathcal{J}}\big(f(\cdot)\big) \geq \tilde{\mathcal{J}}\big(P_F(\cdot)\big), \tag{20}$$

where $0 \leq \zeta_1 < \zeta_2 \leq 2a$.

Proof Replace $\mathcal{F}$, $\check{\mathcal{J}}(f(\cdot))$ and y_i with $\mathcal{H}$, $\tilde{\mathcal{J}}(f(\cdot))$, $2a - x_i$ respectively in Theorem 8 to get the required result. □

If we put $m_1 = n_1 = n$, $p_i = q_i$ and by using positive weights in (17), then $\tilde{\mathcal{J}}(\cdot)$ is converted to the functional $\mathcal{J}_1(\cdot)$ defined in (3). Also in this case, (18), (19), and (20) become

$$\mathcal{J}_1\big(f(\cdot)\big) = \mathcal{J}_1\big(P_F(\cdot)\big) + \int_{\zeta_1}^{\zeta_2} \mathcal{J}_1\big(G_{F,m}(\cdot,s)\big) f^{(m)}(s)\,ds, \tag{18a}$$

where

$$\begin{aligned}\mathcal{J}_1\big(P_F(\cdot)\big) = {} & \frac{1}{P_n}\sum_{k=1}^{n} p_k\big(P_F(2a - x_k - \zeta_1)\big) - P_F\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k(2a - x_k) - \zeta_1\right) \\ & - \frac{1}{P_n}\sum_{k=1}^{n} p_k\big(P_F(x_k - \zeta_1)\big) + P_F\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k x_k - \zeta_1\right)\end{aligned}$$

and

$$\begin{aligned}\mathcal{J}_1\big(G_{F,m}(\cdot,s)\big) = {} & \frac{1}{P_n}\sum_{k=1}^{n} p_k G_{F,m}(2a - x_k, s) - G_{F,m}\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k(2a - x_k), s\right) \\ & - \frac{1}{P_n}\sum_{k=1}^{n} p_k G_{F,m}(x_k, s) + G_{F,m}\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k x_k, s\right),\end{aligned}$$

$$\mathcal{J}_1\big(G_{F,m}(\cdot,s)\big) \geq 0, \quad s \in [\zeta_1,\zeta_2], \tag{19a}$$

$$\mathcal{J}_1\big(f(\cdot)\big) \geq \mathcal{J}_1\big(P_F(\cdot)\big), \tag{20a}$$

respectively.

Theorem 15 *Let $f \in C^m[0,2a]$ $(m \geq 3)$, $\mathbf{p} = (p_1,\dots,p_n)$ be a positive n-tuple such that $\sum_{i=1}^{n} p_i = P_n$. Then, for the functional $\mathcal{J}_1(f(\cdot))$ defined in (3) and for $0 \leq \zeta_1 < \zeta_2 \leq 2a$, we have the following*:

(i) *If n is even and p is odd or p is even and n is odd, then for every m-convex function f, (20a) holds.*

(ii) *Let inequality* (20a) *be satisfied. If the* $P_F(\cdot)$ *is* 3*-convex, the R.H.S of* (20a) *is nonnegative and* (3) *is valid.*

Proof Proof is similar to Theorem 9. □

3 New bounds for Levinson-type inequality

For two Lebesgue integrable functions $f_1, f_2 : [\zeta_1, \zeta_2] \to \mathbb{R}$, we consider the Čebyšev functional

$$\Theta(f_1,f_2) = \frac{1}{\zeta_2 - \zeta_2} \int_{\zeta_1}^{\zeta_2} f_1(t) f_2(t)\, dt - \frac{1}{\zeta_2 - \zeta_2} \int_{\zeta_1}^{\zeta_2} f_1(t)\, dt . \frac{1}{\zeta_2 - \zeta_2} \int_{\zeta_1}^{\zeta_2} f_2(t)\, dt, \tag{21}$$

where the integrals are assumed to exist.

The following two results are given in [15].

Theorem 16 *Let* $f_1 : [\zeta_1, \zeta_2] \to \mathbb{R}$ *be a Lebesgue integrable function and* $f_2 : [\zeta_1, \zeta_2] \to \mathbb{R}$ *be an absolutely continuous function with* $(\cdot, -\zeta_1)(\cdot, -\zeta_2)[f_2']^2 \in L[\zeta_1, \zeta_2]$. *Then we have the inequality*

$$\left|\Theta(f_1,f_2)\right| \le \frac{1}{\sqrt{2}} \left[\Theta(f_1,f_1)\right]^{\frac{1}{2}} \frac{1}{\sqrt{\zeta_2 - \zeta_2}} \left(\int_{\zeta_1}^{\zeta_2} (x - \zeta_1)(\zeta_2 - x) \left[f_2'(x)\right]^2 dx \right)^{\frac{1}{2}}. \tag{22}$$

The constant $\frac{1}{\sqrt{2}}$ *is the best possible.*

Theorem 17 *Let* $f_1 : [\zeta_1, \zeta_2] \to \mathbb{R}$ *be absolutely continuous with* $f_1' \in L_\infty[\zeta_1, \zeta_2]$, *and let* $f_2 : [\zeta_1, \zeta_2] \to \mathbb{R}$ *be monotonic nondecreasing on* $[\zeta_1, \zeta_2]$. *Then we have the inequality*

$$\left|\Theta(f_1,f_2)\right| \le \frac{1}{2(\zeta_2 - \zeta_1)} \|f'\|_\infty \int_{\zeta_1}^{\zeta_2} (x - \zeta_1)(\zeta_2 - x) \left[f_2'(x)\right]^2 df_2(x). \tag{23}$$

The constant $\frac{1}{2}$ *is the best possible.*

To generalize the results given in the previous section for two types of data points, we will consider Theorem 16 and Theorem 17.

Theorem 18 *Assume* $\mathcal{F}$. *Let* $f \in C^m[\zeta_1, \zeta_2]$ $(m \ge 3)$ *and* $f^{(m)}$ *be absolutely continuous with* $(. - \zeta_1)(\zeta_2 - \cdot)[f^{(m+1)}]^2 \in L[\zeta_1, \zeta_2]$. *Also let* $G_{F,m}$ *and* $\breve{\mathcal{J}}(f(\cdot))$ *as defined in* (8) *and* (12) *respectively. Then we have*

$$\breve{\mathcal{J}}\big(f(\cdot)\big) = \breve{\mathcal{J}}\big(P_F(\cdot)\big) + \frac{f^{(m-1)}(\zeta_2) - f^{(m-1)}(\zeta_1)}{(\zeta_2 - \zeta_2)} \times \int_{\zeta_1}^{\zeta_2} \breve{\mathcal{J}}\big(G_{F,m}(\cdot, s)\big) f^{(m)}(s)\, ds + \mathcal{R}_m(\zeta_1, \zeta_2; f), \tag{24}$$

and the remainder $\mathcal{R}_m(\zeta_1,\zeta_2;f)$ *satisfies the bound*

$$\begin{aligned}\left|\mathcal{R}_m(\zeta_1,\zeta_2;f)\right| \le \frac{(\zeta_2-\zeta_2)}{\sqrt{2}}\left[\Theta\left(\breve{\mathcal{J}}\left(G_{F,m}(\cdot,s)\right),\breve{\mathcal{J}}\left(G_{F,m}(\cdot,s)\right)\right)\right]^{\frac{1}{2}}\\ \times\frac{1}{\sqrt{\zeta_2-\zeta_1}}\left(\int_{\zeta_1}^{\zeta_2}(s-\zeta_1)(\zeta_2-s)\left[f^{(m+1)}(s)\right]^2\,ds\right)^{\frac{1}{2}}.\end{aligned}\tag{25}$$

Proof Setting $f_1\mapsto\breve{\mathcal{J}}(G_{F,m}(\cdot,s))$ and $f_2\mapsto f^{(m)}$ in Theorem 16, we have

$$\begin{aligned}&\left|\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\breve{\mathcal{J}}\left(G_{F,m}(\cdot,s)\right)f^{(m)}(s)\,ds-\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\breve{\mathcal{J}}\left(G_{F,m}(\cdot,s)\right)ds\right.\\
&\qquad\left.\times\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}f^{(m)}(s)\,ds\right|\\
&\quad\le\frac{1}{\sqrt{2}}\left[\Theta\left(\breve{\mathcal{J}}\left(G_{F,m}(\cdot,s)\right),\breve{\mathcal{J}}\left(G_{F,m}(\cdot,s)\right)\right)\right]^{\frac{1}{2}}\frac{1}{\sqrt{\zeta_2-\zeta_2}}\left(\int_{\zeta_1}^{\zeta_2}(s-\zeta_1)(\zeta_2-s)\right.\\
&\qquad\left.\times\left[f^{(m+1)}(s)\right]^2\,ds\right)^{\frac{1}{2}},\\
&\left|\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\breve{\mathcal{J}}\left(G_{F,m}(\cdot,s)\right)f^{(m)}(s)\,ds-\frac{f^{(m-1)}(\zeta_2)-f^{(m-1)}(\zeta_1)}{(\zeta_2-\zeta_2)^2}\int_{\zeta_1}^{\zeta_2}\breve{\mathcal{J}}\left(G_{F,m}(\cdot,s)\right)ds\right|\\
&\quad\le\frac{1}{\sqrt{2}}\left[\Theta\left(\breve{\mathcal{J}}\left(G_{F,m}(\cdot,s)\right),\breve{\mathcal{J}}\left(G_{F,m}(\cdot,s)\right)\right)\right]^{\frac{1}{2}}\frac{1}{\sqrt{\zeta_2-\zeta_2}}\left(\int_{\zeta_1}^{\zeta_2}(s-\zeta_1)(\zeta_2-s)\right.\\
&\qquad\left.\times\left[f^{(m+1)}(s)\right]^2\,ds\right)^{\frac{1}{2}}.\end{aligned}$$

Multiplying $(\zeta_2-\zeta_2)$ on both sides of the above inequality and using the estimation (25), we get

$$\begin{aligned}\int_{\zeta_1}^{\zeta_2}\breve{\mathcal{J}}\left(G_{F,m}(\cdot,s)\right)f^{(m)}\,ds=\frac{f^{(m-1)}(\zeta_2)-f^{(m-1)}(\zeta_1)}{(\zeta_2-\zeta_1)}\int_{\zeta_1}^{\zeta_2}\breve{\mathcal{J}}\left(G_{F,m}(\cdot,s)\right)ds\\ +\mathcal{R}_m(\zeta_1,\zeta_2;f).\end{aligned}$$

Using identity (13), we get (24). □

The Grüss-type inequalities can be obtained by using Theorem 17.

Theorem 19 *Assume* $\mathcal{F}$. *Let* $f\in C^m[\zeta_1,\zeta_2]$ $(m\ge 3)$ *with* $f^{(m)}$ *be absolutely continuous and* $f^{(m-1)}\ge 0$ *a.e. on* I_1. *Then identity* (24) *holds, where the remainder satisfies the estimation*

$$\begin{aligned}\left|\mathcal{R}_m(\zeta_1,\zeta_2;f)\right|\le(\zeta_2-\zeta_2)\left\|\breve{\mathcal{J}}\left(G_{F,m}(\cdot,s)\right)'\right\|_\infty\left[\frac{f^{(m-1)}(\zeta_2)+f^{(m-1)}(\zeta_1)}{2}\right.\\ \left.-\frac{f^{(m-1)}(\zeta_2)-f^{(m-1)}(\zeta_1)}{\zeta_2-\zeta_2}\right].\end{aligned}\tag{26}$$

Proof Setting $f_1 \mapsto \breve{\mathcal{J}}(G_{F,m}(\cdot,s))$ and $f_2 \mapsto f^{(m)}$ in Theorem 17, we get

$$\begin{aligned}&\left|\frac{1}{\zeta_2-\zeta_2}\int_{\zeta_1}^{\zeta_2}\breve{\mathcal{J}}\big(G_{F,m}(\cdot,s)\big)f^{(m)}(s)\,ds-\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\breve{\mathcal{J}}\big(G_{F,m}(\cdot,s)\big)\,ds\right.\\&\quad\left.\times\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}f^{(m)}(s)\,ds\right|\\&\leq\frac{1}{2}\big\|\breve{\mathcal{J}}\big(G_{F,m}(\cdot,s)\big)'\big\|_\infty\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}(s-\zeta_1)(\zeta_2-s)\big[f^{(m+1)}(s)\big]^2\,ds.\end{aligned}\tag{27}$$

Since

$$\begin{aligned}&\int_{\zeta_1}^{\zeta_2}(s-\zeta_1)(\zeta_2-s)\big[f^{(m+1)}(s)\big]^2\,ds=\int_{\zeta_1}^{\zeta_2}[2s-\zeta_1-\zeta_2]f^m(s)\,ds\\&\quad=(\zeta_2-\zeta_1)\big[f^{(m-1)}(\zeta_2)+f^{(m-1)}(\zeta_1)\big]-2\big(f^{(m-1)}(\zeta_2)-f^{(m-1)}(\zeta_1)\big),\end{aligned}\tag{28}$$

using (13), (27), and (28), we have (24) with (26). □

Theorem 20 *Assume $\mathcal{F}$. Let $f \in C^m[\zeta_1,\zeta_2]$ $(m \geq 3)$ with $f^{(m-1)}$ be absolutely continuous. Also let $G_{F,m}$ and $\breve{\mathcal{J}}(f(\cdot))$ be as defined in (8) and (12) respectively. Moreover, assume that (p,q) is a pair of conjugate exponents, that is, $1 \leq p,q, \leq \infty$, $\frac{1}{p}+\frac{1}{q}=1$. Let $|f^{(m)}|^p : [\zeta_1,\zeta_2] \to \mathbb{R}$ be a Riemann integrable function. Then*

$$\big|\breve{\mathcal{J}}(f(\cdot))-\breve{\mathcal{J}}\big(P_F(\cdot)\big)\big|\leq\big\|f^{(m)}\big\|_p\left(\int_{\zeta_1}^{\zeta_2}\big|\breve{\mathcal{J}}\big(G_{F,m}(\cdot,s)\big)\,ds\big|^q\right)^{\frac{1}{q}}.$$

Proof For the proof see Theorem 3.5 in [16]. □

Remark 2 Similar work can be done for Levinson's inequality (2), (one type of data points) for higher order-convex functions.

Remark 3 We can give related mean value theorems by using nonnegative functionals (13) and (18), and we can construct the new families of m-exponentially convex functions $(m \geq 3)$ and Cauchy means related to these functionals.

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