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New parameterized quantum integral inequalities via η-quasiconvexity

Eze R. Nwaeze[1*] and Ana M. Tameru[2]

*Correspondence:
enwaeze@alasu.edu
[1]Department of Mathematics and Computer Science, Alabama State University, Montgomery, USA
Full list of author information is available at the end of the article

Abstract

We establish new quantum Hermite–Hadamard and midpoint types inequalities via a parameter $\mu \in [0,1]$ for a function F whose $|{}_{\alpha}D_q F|^u$ is η-quasiconvex on $[\alpha,\beta]$ with $u \geq 1$. Results obtained in this paper generalize, sharpen, and extend some results in the literature. For example, see (Noor et al. in Appl. Math. Comput. 251:675–679, 2015; Alp et al. in J. King Saud Univ., Sci. 30:193–203, 2018) and (Kunt et al. in Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. 112:969–992, 2018). By choosing different values of μ, loads of novel estimates can be deduced. We also present some illustrative examples to show how some consequences of our results may be applied to derive more quantum inequalities.

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1 Introduction

Quantum calculus is generally described as the ordinary calculus without limits. The q-calculus and h-calculus are the main branches of the quantum calculus. In this article, we shall discuss within the framework of the q-calculus. Quantum calculus has been found to be useful in many areas of mathematics such as orthogonal polynomials, basic hypergeometric functions, combinatorics, the calculus of variations, mechanics, and the theory of relativity. Analogues of many results in the classical calculus have been established in the q-calculus sense. We start by presenting some of the recently published results in this direction. But before that, the following definitions are needed in the sequel:

A function $F : [\alpha,\beta] \subset \mathbb{R} \to \mathbb{R}$ is termed quasiconvex if, for all $x, y \in [\alpha,\beta]$ and $\tau \in [0,1]$, we have

$$F\bigl(\tau x + (1-\tau)y\bigr) \leq \max\bigl\{F(x), F(y)\bigr\}.$$

All convex functions are also quasiconvex, but not all quasiconvex functions are convex, so quasiconvexity is a generalization of convexity. Quasiconvex functions have applications in mathematical analysis, in mathematical optimization, in game theory, and economics. In 2016, the concept of quasiconvexity was generalized in the following way.

Definition 1 ([6]) A function $F:[\alpha,\beta]\to\mathbb{R}$ is called η-quasiconvex on $[\alpha,\beta]$ with respect to $\eta:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ if

$$F\big(\tau x+(1-\tau)y\big)\le\max\big\{F(y),F(y)+\eta\big(F(x),F(y)\big)\big\}$$

for all $x,y\in[\alpha,\beta]$ and $\tau\in[0,1]$.

In the field of mathematical analysis, many estimates have been established via this generalized convexity and quasiconvexity. For instance, estimates of the Hermite–Hadamard, trapezoid, midpoint, Simpson types have all been obtained for this class of functions. We invite the interested reader to see [3, 5, 7, 11, 18].

Embedded in the proof of their main results, Noor et al. [14], Latif et al. [13], and Alp et al. [2] recently established the following quantum inequalities for the class of quasiconvex functions.

Theorem 2 ([14]) *Let $F:[\alpha,\beta]\to\mathbb{R}$ be a q-differentiable function on (α,β) with ${}_\alpha D_qF$ continuous on $[\alpha,\beta]$ where $0<q<1$. If $|{}_\alpha D_qF|^u$ is quasiconvex on $[\alpha,\beta]$ for $u\ge1$, then the following inequality holds*:

$$\begin{aligned}&\left|\frac{1}{\beta-\alpha}\int_\alpha^\beta F(r)\,{}_\alpha d_qr-\frac{qF(\alpha)+F(\beta)}{1+q}\right|\\&\quad\le(\beta-\alpha)\frac{2q}{(1+q)^3}\big[\max\big\{\big|{}_\alpha D_qF(\alpha)\big|^u,\big|{}_\alpha D_qF(\beta)\big|^u\big\}\big]^{\frac{1}{u}}.\end{aligned}\tag{1}$$

Theorem 3 ([13]) *Let $F:[\alpha,\beta]\to\mathbb{R}$ be a q-differentiable function on (α,β) with ${}_\alpha D_qF$ continuous on $[\alpha,\beta]$ where $0<q<1$. If $|{}_\alpha D_qF|^u$ is quasiconvex on $[\alpha,\beta]$ for $u>1$ with $\frac{1}{u}+\frac{1}{v}=1$, then the following inequality holds*:

$$\begin{aligned}&\left|\frac{1}{\beta-\alpha}\int_\alpha^\beta F(r)\,{}_\alpha d_qr-\frac{qF(\alpha)+F(\beta)}{1+q}\right|\\&\quad\le\frac{q(b-a)}{1+q}\left[\int_0^1\big|1-(1+q)\tau\big|^v\,{}_0d_q\tau\right]^{\frac{1}{v}}\big[\max\big\{\big|{}_\alpha D_qF(\alpha)\big|^u,\big|{}_\alpha D_qF(\beta)\big|^u\big\}\big]^{\frac{1}{u}}.\end{aligned}$$

Theorem 4 ([2, 12]) *Let $F:[\alpha,\beta]\to\mathbb{R}$ be a q-differentiable function on (α,β) with ${}_\alpha D_qF$ continuous on $[\alpha,\beta]$ where $0<q<1$. If $|{}_\alpha D_qF|^u$ is quasiconvex on $[\alpha,\beta]$ for $u\ge1$, then the following q-midpoint type inequality holds*:

$$\begin{aligned}&\left|F\left(\frac{q\alpha+\beta}{1+q}\right)-\frac{1}{\beta-\alpha}\int_\alpha^\beta F(r)\,{}_\alpha d_qr\right|\\&\quad\le(\beta-\alpha)\frac{2q}{(1+q)^3}\big[\max\big\{\big|{}_\alpha D_qF(\alpha)\big|^u,\big|{}_\alpha D_qF(\beta)\big|^u\big\}\big]^{\frac{1}{u}}.\end{aligned}$$

Theorem 5 ([2, 12]) *Let $F:[\alpha,\beta]\to\mathbb{R}$ be a q-differentiable function on (α,β) with ${}_\alpha D_qF$ continuous on $[\alpha,\beta]$ where $0<q<1$. If $|{}_\alpha D_qF|^u$ is quasiconvex on $[\alpha,\beta]$ for $u>1$ with*

$\frac{1}{u}+\frac{1}{v}=1$, *then the following q-midpoint type inequality holds*:

$$\begin{aligned}
&\left|F\left(\frac{q\alpha+\beta}{1+q}\right)-\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}F(r)\,{}_{\alpha}d_{q}r\right| \\
&\quad\le q(\beta-\alpha)\left[\max\left\{\left|{}_{\alpha}D_{q}F(\alpha)\right|^{u},\left|{}_{\alpha}D_{q}F(\beta)\right|^{u}\right\}\right]^{\frac{1}{u}} \\
&\qquad\times\left[\left(\frac{1}{(1+q)^{v+1}}\frac{1-q}{1-q^{v+1}}\right)^{\frac{1}{v}}\left(\frac{1}{1+q}\right)^{\frac{1}{u}}+\left(\int_{\frac{1}{1+q}}^{1}\left(\frac{1}{q}-\tau\right)^{v}{}_{0}d_{q}r\right)^{\frac{1}{v}}\left(\frac{q}{1+q}\right)^{\frac{1}{v}}\right].
\end{aligned}$$

The goal of this article is to extend Theorems 2–5 to a more general class of functions. We do this by means of a parameter $\mu\in[0,1]$ and obtain results for a function F whose $|{}_{\alpha}D_{q}F|^{u}$ is η-quasiconvex on $[\alpha,\beta]$ for $u\ge 1$. Our first result sharpens Theorem 2 (see Remark 17); whereas Theorems 3–5 are special cases of our theorems (see Remarks 19, 21, and 23). In addition, we apply our results to some special means to get more results in this direction.

This paper is structured as follows: Sect. 2 contains a quick overview of the quantum calculus. The main results are then framed and justified in Sect. 3. Some illustrative examples are then presented in Sect. 4.

2 Preliminaries

In this section, we present some quick overview of the theory of quantum calculus. For an in-depth study of this subject, we invite the interested reader to the book [8]. We start with the following basic definitions.

Definition 6 ([21]) Suppose that $F:[\alpha,\beta]\subset\mathbb{R}\to\mathbb{R}$ is a continuous function and $z\in[\alpha,\beta]$. Then the expression

$$ {}_{\alpha}D_{q}F(z)=\frac{F(z)-F(qz+(1-q)\alpha)}{(1-q)(z-\alpha)},\quad z\neq\alpha,\qquad {}_{\alpha}D_{q}F(\alpha)=\lim_{z\to\alpha}{}_{\alpha}D_{q}F(z) \tag{2}$$

is called the q-derivative on $[\alpha,\beta]$ of the function at z.

We say that F is q-differentiable on $[\alpha,\beta]$ provided ${}_{\alpha}D_{q}F(z)$ exists for all $z\in[\alpha,\beta]$.

Definition 7 ([21]) Let $F:[\alpha,\beta]\to\mathbb{R}$ be a continuous function. Then the q-integral on $[\alpha,\beta]$ is defined as

$$\int_{\alpha}^{z}F(r)\,{}_{\alpha}d_{q}r=(1-q)(z-\alpha)\sum_{k=0}^{\infty}q^{k}F\left(q^{k}z+\left(1-q^{k}\right)\alpha\right) \tag{3}$$

for $z\in[\alpha,\beta]$. Moreover, if $c\in(\alpha,z)$, then the q-integral on $[\alpha,\beta]$ is defined as

$$\int_{c}^{z}F(r)\,{}_{\alpha}d_{q}r=\int_{\alpha}^{z}F(r)\,{}_{\alpha}d_{q}r-\int_{\alpha}^{c}F(r)\,{}_{\alpha}d_{q}r. \tag{4}$$

Remark 8 In view of Definitions 6 and 7, we make the following observations:

1. By taking $\alpha = 0$, the expression in (2) reduces to the well-known q-derivative, $D_qF(z)$, of the function $F(z)$ defined by

$$D_qF(z) = \frac{F(z) - F(qz)}{(1-q)z}.$$

2. Also, if $\alpha = 0$, then (3) amounts to the classical q-integral of a function $F : [0,\infty) \to \mathbb{R}$ defined by

$$\int_0^z F(r)\,{}_0d_qr = (1-q)z\sum_{k=0}^{\infty} q^k F\bigl(q^k z\bigr).$$

Analogues of some known results in the continuous calculus sense are also given in what follows.

Theorem 9 ([4]) *Let $F, G : [\alpha,\beta] \to \mathbb{R}$ be two continuous functions and suppose $F(r) \leq G(r)$ for all $r \in [\alpha,\beta]$. Then*

$$\int_\alpha^z F(r)\,{}_\alpha d_qr \leq \int_\alpha^z G(r)\,{}_\alpha d_qr.$$

Theorem 10 ([21]) *Let $F : [\alpha,\beta] \to \mathbb{R}$ be a continuous function. Then*

$$\begin{aligned}
&{}_\alpha D_q \int_\alpha^z F(r)\,{}_\alpha d_qr = F(z);\\
&\int_c^z {}_\alpha D_qF(r)\,{}_\alpha d_qr = F(z) - F(c), \quad \textit{for } c \in (\alpha, z).
\end{aligned}$$

Theorem 11 ([21]) *Let $F, G : [\alpha,\beta] \to \mathbb{R}$ be continuous functions and $\gamma \in \mathbb{R}$. Then, for $z \in [\alpha,\beta]$ and $c \in (\alpha, z)$, we have*

$$\begin{aligned}
&\int_\alpha^z \bigl[F(r) + G(r)\bigr]\,{}_\alpha d_qr = \int_\alpha^z F(r)\,{}_\alpha d_qr + \int_\alpha^z G(r)\,{}_\alpha d_qr;\\
&\int_\alpha^z \gamma F(r)\,{}_\alpha d_qr = \gamma \int_\alpha^z F(r)\,{}_\alpha d_qr;\\
&\int_c^z F(r)\,{}_\alpha D_qG(r)\,{}_\alpha d_qr = F(z)G(z) - F(c)G(c) - \int_c^z G\bigl(qr + (1-q)\alpha\bigr)\,{}_\alpha D_qF(r)\,{}_\alpha d_qr.
\end{aligned}$$

3 Main results

The succeeding lemmas will be needed in the proof of our theorems.

Lemma 12 ([22]) *Let $F : [\alpha,\beta] \to \mathbb{R}$ be a continuous and q-differentiable function on (α,β) with $0 < q < 1$. If ${}_\alpha D_qF$ is integrable on $[\alpha,\beta]$, then for all $\mu \in [0,1]$ the following identity holds*:

$$\begin{aligned}
&\mu F(\beta) + (1-\mu)F(\alpha) - \frac{1}{\beta-\alpha}\int_\alpha^\beta F(r)\,{}_\alpha d_qr\\
&\quad = (\beta-\alpha)\int_0^1 (q\tau + \mu - 1)\,{}_\alpha D_qF\bigl(\tau\beta + (1-\tau)\alpha\bigr)\,{}_0d_q\tau.
\end{aligned}$$

Lemma 13 ([22]) *Let $F:[\alpha,\beta]\to\mathbb{R}$ be a continuous and q-differentiable function on (α,β) with $0<q<1$. If ${}_{\alpha}D_qF$ is integrable on $[\alpha,\beta]$, then for all $\mu\in[0,1]$ the following identity holds*:

$$\begin{aligned}
&F\big(\mu\beta+(1-\mu)\alpha\big)-\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}F(r)\,{}_{\alpha}d_qr\\
&\quad=(\beta-\alpha)\bigg[\int_0^{\mu}q\tau\,{}_{\alpha}D_qF\big(\tau\beta+(1-\tau)\alpha\big)\,{}_0d_q\tau\\
&\qquad+\int_{\mu}^{1}(q\tau-1)\,{}_{\alpha}D_qF\big(\tau\beta+(1-\tau)\alpha\big)\,{}_0d_q\tau\bigg].
\end{aligned}$$

Lemma 14 ([22]) *Let $\lambda,\mu\in[0,1]$, $k\in[0,\infty)$, and $0<q<1$. Then*

$$\begin{aligned}
&\int_0^1\tau^k\big|q\tau-(1-\lambda\mu)\big|\,{}_0d_q\tau\\
&\quad=\begin{cases}\frac{(1-q)(1-\lambda\mu)}{1-q^{k+1}}-\frac{q(1-q)}{1-q^{k+2}}, & \lambda\mu+q\le 1,\\ \frac{2(1-q)^2(1-\lambda\mu)^{k+2}}{(1-q^{k+1})(1-q^{k+2})}+\frac{q(1-q)}{1-q^{k+2}}-\frac{(1-q)(1-\lambda\mu)}{1-q^{k+1}}, & \lambda\mu+q>1.\end{cases}
\end{aligned}$$

Lemma 15 ([22]) *Let $\lambda,\mu\in[0,1]$, $\theta\in[1,\infty)$, and $0<q<1$. Then*

$$\begin{aligned}
\Omega_q(\lambda;\mu;\theta)&:=\int_0^1\big|q\tau-(1-\lambda\mu)\big|^{\theta}\,{}_0d_q\tau\\
&=\begin{cases}(1-q)\sum_{k=0}^{\infty}q^k(1-\lambda\mu-q^{k+1})^{\theta}, & 0\le\lambda\mu\le 1-q,\\ (1-q)(1-\lambda\mu)^{\theta+1}\sum_{k=0}^{\infty}q^{k-1}(1-q^k)^{\theta} & \\ \quad+(1-q)\sum_{k=0}^{\infty}q^k(q^{k+1}-1+\lambda\mu)^{\theta} & \\ \quad-(1-q)(1-\lambda\mu)^{\theta+1}\sum_{k=0}^{\infty}q^{k-1}(q^k-1)^{\theta}, & 1-q<\lambda\mu\le 1.\end{cases}
\end{aligned}$$

Let f be an η-quasiconvex function on $[\alpha,\beta]$. We shall use the following notation:

$$\mathfrak{Q}_{\alpha}^{\beta}(f;\eta):=\max\big\{f(\alpha),f(\alpha)+\eta\big(f(\beta),f(\alpha)\big)\big\}.$$

Theorem 16 *Let $F:[\alpha,\beta]\to\mathbb{R}$ be a q-differentiable function on (α,β) with ${}_{\alpha}D_qF$ continuous on $[\alpha,\beta]$ where $0<q<1$. If $|{}_{\alpha}D_qF|^u$ is η-quasiconvex on $[\alpha,\beta]$ for $u\ge 1$, then, for all $\mu\in[0,1]$, the following inequality holds*:

$$\begin{aligned}
&\left|\mu F(\beta)+(1-\mu)F(\alpha)-\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}F(r)\,{}_{\alpha}d_qr\right|\\
&\quad\le\begin{cases}\frac{(1-\mu-\mu q)(\beta-\alpha)}{1+q}\big[\mathfrak{Q}_{\alpha}^{\beta}(|{}_{\alpha}D_qF|^u;\eta)\big]^{\frac{1}{u}}, & 0\le\mu\le 1-q,\\ \frac{(2\mu^2+\mu(q-3)+1)(\beta-\alpha)}{1+q}\big[\mathfrak{Q}_{\alpha}^{\beta}(|{}_{\alpha}D_qF|^u;\eta)\big]^{\frac{1}{u}}, & 1-q<\mu\le 1.\end{cases}
\end{aligned}\tag{5}$$

Proof The η-quasiconvexity of $|{}_{\alpha}D_qF|^u$ on $[\alpha,\beta]$ implies that, for all $\tau\in[0,1]$, one has:

$$\begin{aligned}
&\big|{}_{\alpha}D_qF\big(\tau\beta+(1-\tau)\alpha\big)\big|^u\\
&\quad\le\max\big\{\big|{}_{\alpha}D_qF(\alpha)\big|^u,\big|{}_{\alpha}D_qF(\alpha)\big|^u+\eta\big(\big|{}_{\alpha}D_qF(\beta)\big|^u,\big|{}_{\alpha}D_qF(\alpha)\big|^u\big)\big\}\\
&\quad=:\mathfrak{Q}_{\alpha}^{\beta}\big(|{}_{\alpha}D_qF|^u;\eta\big).
\end{aligned}\tag{6}$$

Taking the absolute values of both sides of Lemma 12 and then using (6) together with Hölder's inequality gives

$$\begin{aligned}
&\left|\mu F(\beta) + (1-\mu)F(\alpha) - \frac{1}{\beta-\alpha}\int_\alpha^\beta F(r)\,{}_\alpha d_q r\right| \\
&\quad\le (\beta-\alpha)\int_0^1 |q\tau+\mu-1|\left|{}_\alpha D_q F\big(\tau\beta+(1-\tau)\alpha\big)\right|\,{}_0 d_q\tau \\
&\quad= (\beta-\alpha)\int_0^1 |q\tau+\mu-1|^{\frac{u-1}{u}}|q\tau+\mu-1|^{\frac{1}{u}}\left|{}_\alpha D_q F\big(\tau\beta+(1-\tau)\alpha\big)\right|\,{}_0 d_q\tau \\
&\quad\le (\beta-\alpha)\Bigg[\left(\int_0^1 |q\tau+\mu-1|\,{}_0 d_q\tau\right)^{1-\frac{1}{u}} \\
&\qquad\times\left(\int_0^1 |q\tau+\mu-1|\left|{}_\alpha D_q F\big(\tau\beta+(1-\tau)\alpha\big)\right|^u\,{}_0 d_q\tau\right)^{\frac{1}{u}}\Bigg] \\
&\quad\le (\beta-\alpha)\Bigg[\left(\int_0^1 |q\tau+\mu-1|\,{}_0 d_q\tau\right)^{1-\frac{1}{u}} \\
&\qquad\times\left(\int_0^1 |q\tau+\mu-1|\,{}_0 d_q\tau\right)^{\frac{1}{u}}\big(\mho_\alpha^\beta\big(|{}_\alpha D_q F|^u;\eta\big)\big)^{\frac{1}{u}}\Bigg] \\
&\quad= (\beta-\alpha)\int_0^1 |q\tau+\mu-1|\,{}_0 d_q\tau\big(\mho_\alpha^\beta\big(|{}_\alpha D_q F|^u;\eta\big)\big)^{\frac{1}{u}}.
\end{aligned}$$

Now, putting $k = 0$ and $\lambda = 1$ in Lemma 14, we get

$$\Omega_q(1;\mu;1) := \int_0^1 |q\tau+\mu-1|\,{}_0 d_q\tau = \begin{cases} \frac{1-\mu-\mu q}{1+q}, & 0 \le \mu \le 1-q, \\ \frac{2\mu^2+\mu(q-3)+1}{1+q}, & 1-q < \mu \le 1. \end{cases}$$

Hence, that completes the proof. □

Remark 17 Let $\eta(x,y) = x-y$ and $\mu = \frac{1}{1+q}$. Then $\frac{1}{1+q} > 1-q$ and (5) boils down to

$$\begin{aligned}
&\left|\frac{F(\beta)+qF(\alpha)}{1+q} - \frac{1}{\beta-\alpha}\int_\alpha^\beta F(r)\,{}_\alpha d_q r\right| \\
&\quad\le (\beta-\alpha)\frac{2q^2}{(1+q)^3}\big[\max\big\{\left|{}_\alpha D_q F(\alpha)\right|^u, \left|{}_\alpha D_q F(\beta)\right|^u\big\}\big]^{\frac{1}{u}}.
\end{aligned} \tag{7}$$

Clearly, $2q^2 < 2q$. Therefore, the new inequality (7) sharpens (1) and thus, provides a better estimate. If, in addition, we let $q \to 1^-$, we get from (7)

$$\left|\frac{F(\beta)+F(\alpha)}{2} - \frac{1}{\beta-\alpha}\int_\alpha^\beta F(r)\,dr\right| \le \frac{\beta-\alpha}{4}\big[\max\big\{\left|F'(\alpha)\right|^u, \left|F'(\beta)\right|^u\big\}\big]^{\frac{1}{u}}. \tag{8}$$

Inequality (8) is already known in the literature. See [1, Theorem 6].

Theorem 18 *Let $F : [\alpha,\beta] \to \mathbb{R}$ be a q-differentiable function on (α,β) with ${}_\alpha D_q F$ continuous on $[\alpha,\beta]$ where $0 < q < 1$. If $|{}_\alpha D_q F|^u$ is η-quasiconvex on $[\alpha,\beta]$ for $u > 1$ with $\frac{1}{u} + \frac{1}{v} = 1$,* *then for all* $\mu \in [0,1]$ *the following inequality holds*:

$$\begin{aligned}&\left|\mu F(\beta)+(1-\mu)F(\alpha)-\frac{1}{\beta-\alpha}\int_\alpha^\beta F(r)\,{}_\alpha d_q r\right| \\ &\quad\le(\beta-\alpha)\bigl[\Omega_q(1;\mu;\nu)\bigr]^{\frac{1}{\nu}}\bigl(\mathfrak{Q}_\alpha^\beta\bigl(|{}_\alpha D_q F|^u;\eta\bigr)\bigr)^{\frac{1}{u}},\end{aligned} \tag{9}$$

where $\Omega_q(1;\mu;\nu)$ *is defined in Lemma* 15.

Proof Using Lemma 12, (6), an Hölder's inequality with the conjugate pair (u,ν), we get

$$\begin{aligned}&\left|\mu F(\beta)+(1-\mu)F(\alpha)-\frac{1}{\beta-\alpha}\int_\alpha^\beta F(r)\,{}_\alpha d_q r\right| \\ &\quad\le(\beta-\alpha)\int_0^1|q\tau+\mu-1|\bigl|{}_\alpha D_q F\bigl(\tau\beta+(1-\tau)\alpha\bigr)\bigr|\,{}_0 d_q\tau \\ &\quad\le(\beta-\alpha)\left[\left(\int_0^1|q\tau+\mu-1|^\nu\,{}_0 d_q\tau\right)^{\frac{1}{\nu}}\left(\int_0^1\bigl|{}_\alpha D_q F\bigl(\tau\beta+(1-\tau)\alpha\bigr)\bigr|^u\,{}_0 d_q\tau\right)^{\frac{1}{u}}\right] \\ &\quad\le(\beta-\alpha)\left(\int_0^1|q\tau+\mu-1|^\nu\,{}_0 d_q\tau\right)^{\frac{1}{\nu}}\bigl(\mathfrak{Q}_\alpha^\beta\bigl(|{}_\alpha D_q F|^u;\eta\bigr)\bigr)^{\frac{1}{u}} \\ &\quad=(\beta-\alpha)\bigl[\Omega_q(1;\mu;\nu)\bigr]^{\frac{1}{\nu}}\bigl(\mathfrak{Q}_\alpha^\beta\bigl(|{}_\alpha D_q F|^u;\eta\bigr)\bigr)^{\frac{1}{u}}.\end{aligned}$$

This completes the proof. □

Remark 19 If we take $\eta(x,y)=x-y$ and $\mu=\frac{1}{1+q}$ in Theorem 18, then we regain Theorem 3.

By taking $\mu=\frac{1}{2}$ and $\eta(x,y)=x-y$ in (9), we deduce the following:

$$\begin{aligned}&\left|\frac{F(\beta)+F(\alpha)}{2}-\frac{1}{\beta-\alpha}\int_\alpha^\beta F(r)\,{}_\alpha d_q r\right| \\ &\quad\le(\beta-\alpha)\left[\Omega_q\left(1;\frac{1}{2};\nu\right)\right]^{\frac{1}{\nu}}\bigl[\max\bigl\{\bigl|{}_\alpha D_q F(\alpha)\bigr|^u,\bigl|{}_\alpha D_q F(\beta)\bigr|^u\bigr\}\bigr]^{\frac{1}{u}}.\end{aligned}$$

Next, we present a generalization of Theorems 4 and 5 involving a parameter.

Theorem 20 *Let* $F:[\alpha,\beta]\to\mathbb{R}$ *be a q-differentiable function on* (α,β) *with* ${}_\alpha D_q F$ *continuous on* $[\alpha,\beta]$ *where* $0<q<1$. *If* $|{}_\alpha D_q F|^u$ *is η-quasiconvex on* $[\alpha,\beta]$ *for* $u\ge1$, *then for all* $\mu\in[0,1]$ *the following inequality holds*:

$$\begin{aligned}&\left|F\bigl(\mu\beta+(1-\mu)\alpha\bigr)-\frac{1}{\beta-\alpha}\int_\alpha^\beta F(r)\,{}_\alpha d_q r\right| \\ &\quad\le\frac{(\beta-\alpha)(2q\mu^2-(1+q)\mu+1)}{1+q}\bigl[\mathfrak{Q}_\alpha^\beta\bigl(|{}_\alpha D_q F|^u;\eta\bigr)\bigr]^{\frac{1}{u}}.\end{aligned} \tag{10}$$

Proof We get, by taking the absolute values of both sides of Lemma 13 and then using Hölder's inequality and the η-quasiconvexity of $|{}_\alpha D_q F|^u$ on $[\alpha,\beta]$, the following estimates:

$$
\begin{aligned}
&\left|F\bigl(\mu\beta+(1-\mu)\alpha\bigr)-\frac{1}{\beta-\alpha}\int_\alpha^\beta F(r)\,{}_\alpha d_q r\right| \\
&\quad\le(\beta-\alpha)\biggl[\int_0^\mu q\tau\bigl|{}_\alpha D_q F\bigl(\tau\beta+(1-\tau)\alpha\bigr)\bigr|\,{}_0d_q\tau \\
&\qquad+\int_\mu^1|q\tau-1|\bigl|{}_\alpha D_q F\bigl(\tau\beta+(1-\tau)\alpha\bigr)\bigr|\,{}_0d_q\tau\biggr] \\
&\quad\le(\beta-\alpha)\biggl[q\biggl(\int_0^\mu\tau\,{}_0d_q\tau\biggr)^{1-\frac{1}{u}}\biggl(\int_0^\mu\tau\bigl|{}_\alpha D_q F\bigl(\tau\beta+(1-\tau)\alpha\bigr)\bigr|^u\,{}_0d_q\tau\biggr)^{\frac{1}{u}} \\
&\qquad+\biggl(\int_\mu^1|q\tau-1|\,{}_0d_q\tau\biggr)^{1-\frac{1}{u}}\biggl(\int_\mu^1|q\tau-1|\bigl|{}_\alpha D_q F\bigl(\tau\beta+(1-\tau)\alpha\bigr)\bigr|^u\,{}_0d_q\tau\biggr)^{\frac{1}{u}}\biggr] \\
&\quad\le(\beta-\alpha)\biggl[q\biggl(\int_0^\mu\tau\,{}_0d_q\tau\biggr)^{1-\frac{1}{u}}\biggl(\int_0^\mu\tau\,{}_0d_q\tau\biggr)^{\frac{1}{u}} \\
&\qquad+\biggl(\int_\mu^1|q\tau-1|\,{}_0d_q\tau\biggr)^{1-\frac{1}{u}}\biggl(\int_\mu^1|q\tau-1|\,{}_0d_q\tau\biggr)^{\frac{1}{u}}\biggr]\bigl[\mathfrak{Q}_\alpha^\beta\bigl(|{}_\alpha D_q F|^u;\eta\bigr)\bigr]^{\frac{1}{u}} \\
&\quad\le(\beta-\alpha)\biggl[q\int_0^\mu\tau\,{}_0d_q\tau+\int_\mu^1|q\tau-1|\,{}_0d_q\tau\biggr]\bigl[\mathfrak{Q}_\alpha^\beta\bigl(|{}_\alpha D_q F|^u;\eta\bigr)\bigr]^{\frac{1}{u}}.
\end{aligned}
\tag{11}
$$

Now, using Definition 3, we get that

$$
\int_0^\mu\tau^p\,{}_0d_q\tau=(1-q)\sum_{k=0}^\infty\mu^{p+1}q^{(p+1)k}=\frac{\mu^{p+1}(1-q)}{1-q^{p+1}}
\tag{12}
$$

for any $p\ge 0$. So, for $p=1$,

$$
\int_0^\mu\tau\,{}_0d_q\tau=\frac{\mu^2}{1+q}.
\tag{13}
$$

Also, using (4) and the fact that $q\tau<1$, we obtain that

$$
\begin{aligned}
\int_\mu^1|q\tau-1|\,{}_0d_q\tau&=\int_\mu^1(1-q\tau)\,{}_0d_q\tau \\
&=\int_0^1(1-q\tau)\,{}_0d_q\tau-\int_0^\mu(1-q\tau)\,{}_0d_q\tau \\
&=\int_0^1 1\,{}_0d_q\tau-q\int_0^1\tau\,{}_0d_q\tau-\int_0^\mu 1\,{}_0d_q\tau+q\int_0^\mu\tau\,{}_0d_q\tau \\
&=1-\frac{q}{q+1}-\mu+\frac{q\mu^2}{1+q} \\
&=\frac{q\mu^2-(1+q)\mu+1}{1+q}.
\end{aligned}
\tag{14}
$$

We get the intended result by combining (11), (13), and (14). □

Remark 21 If we take $\eta(x,y) = x - y$ and $\mu = \frac{1}{1+q}$ in Theorem 20, then we recover Theorem 4.

Substituting $\mu = 0$, $\mu = 1$, and $\mu = \frac{1}{2}$ with $\eta(x,y) = x - y$ in (10), we get, respectively, the following:

$$\left|F(\alpha) - \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} F(r)\,{}_{\alpha}d_q r\right| \leq \frac{\beta-\alpha}{1+q}\left[\max\left\{\left|{}_{\alpha}D_q F(\alpha)\right|^u, \left|{}_{\alpha}D_q F(\beta)\right|^u\right\}\right]^{\frac{1}{u}},$$
$$\left|F(\beta) - \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} F(r)\,{}_{\alpha}d_q r\right| \leq \frac{q(\beta-\alpha)}{1+q}\left[\max\left\{\left|{}_{\alpha}D_q F(\alpha)\right|^u, \left|{}_{\alpha}D_q F(\beta)\right|^u\right\}\right]^{\frac{1}{u}},$$

and

$$\left|F\left(\frac{\alpha+\beta}{2}\right) - \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} F(r)\,{}_{\alpha}d_q r\right| \leq \frac{\beta-\alpha}{2(1+q)}\left[\max\left\{\left|{}_{\alpha}D_q F(\alpha)\right|^u, \left|{}_{\alpha}D_q F(\beta)\right|^u\right\}\right]^{\frac{1}{u}}. \quad (15)$$

Theorem 22 *Let $F : [\alpha,\beta] \to \mathbb{R}$ be a q-differentiable function on (α,β) with ${}_{\alpha}D_q F$ continuous on $[\alpha,\beta]$ where $0 < q < 1$. If $|{}_{\alpha}D_q F|^u$ is η-quasiconvex on $[\alpha,\beta]$ for $u > 1$ with $\frac{1}{u} + \frac{1}{v} = 1$, then for all $\mu \in [0,1]$ the following inequality holds*:

$$\left|F(\mu\beta + (1-\mu)\alpha) - \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} F(r)\,{}_{\alpha}d_q r\right|$$
$$\leq (\beta-\alpha)\left[q\left(\frac{\mu^{v+1}(1-q)}{1-q^{v+1}}\right)^{\frac{1}{v}}\mu^{\frac{1}{u}} + \left(\Theta_q(v;\mu)\right)^{\frac{1}{v}}(1-\mu)^{\frac{1}{u}}\right]\left(\mathfrak{Q}_{\alpha}^{\beta}\left(|{}_{\alpha}D_q F|^u;\eta\right)\right)^{\frac{1}{u}},$$

where $\Theta_q(v;\mu) = \int_{\mu}^{1} |q\tau - 1|^v\,{}_0 d_q\tau$.

Proof Applying, again, Lemma 13 and Hölder's inequality, we obtain

$$\left|F(\mu\beta + (1-\mu)\alpha) - \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} F(r)\,{}_{\alpha}d_q r\right|$$
$$\leq (\beta-\alpha)\left[q\int_0^{\mu} \tau\left|{}_{\alpha}D_q F(\tau\beta + (1-\tau)\alpha)\right|\,{}_0 d_q\tau\right.$$
$$\left.+ \int_{\mu}^{1} |q\tau - 1|\left|{}_{\alpha}D_q F(\tau\beta + (1-\tau)\alpha)\right|\,{}_0 d_q\tau\right]$$
$$\leq (\beta-\alpha)\left[q\left(\int_0^{\mu} \tau^v\,{}_0 d_q\tau\right)^{\frac{1}{v}}\left(\int_0^{\mu}\left|{}_{\alpha}D_q F(\tau\beta + (1-\tau)\alpha)\right|^u\,{}_0 d_q\tau\right)^{\frac{1}{u}}\right.$$
$$\left.+ \left(\int_{\mu}^{1} |q\tau - 1|^v\,{}_0 d_q\tau\right)^{\frac{1}{v}}\left(\int_{\mu}^{1}\left|{}_{\alpha}D_q F(\tau\beta + (1-\tau)\alpha)\right|^u\,{}_0 d_q\tau\right)^{\frac{1}{u}}\right]$$
$$\leq (\beta-\alpha)\left[q\left(\int_0^{\mu} \tau^v\,{}_0 d_q\tau\right)^{\frac{1}{v}}\left(\mathfrak{Q}_{\alpha}^{\beta}\left(|{}_{\alpha}D_q F|^u;\eta\right)\int_0^{\mu} 1\,{}_0 d_q\tau\right)^{\frac{1}{u}}\right.$$
$$\left.+ \left(\int_{\mu}^{1} |q\tau - 1|^v\,{}_0 d_q\tau\right)^{\frac{1}{v}}\left(\mathfrak{Q}_{\alpha}^{\beta}\left(|{}_{\alpha}D_q F|^u;\eta\right)\int_{\mu}^{1} 1\,{}_0 d_q\tau\right)^{\frac{1}{u}}\right]. \quad (16)$$

From relation (4), we deduce that

$$\int_{\mu}^{1} 1\,{}_0d_q\tau = 1-\mu. \tag{17}$$

The desired result is obtained by substituting (12) and (17) into (16). □

Remark 23 Theorem 5 is recaptured by putting $\eta(x,y)=x-y$ and $\mu=\frac{1}{1+q}$ in Theorem 22.

If $\mu=0$ and $\mu=1$ with $\eta(x,y)=x-y$ in Theorem 22, then we get, respectively, the following inequalities:

$$\begin{aligned}&\left|F(\alpha)-\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}F(r)\,{}_\alpha d_q r\right|\\&\quad\leq(\beta-\alpha)\left[(1-q)\sum_{k=0}^{\infty}q^k\left(1-q^{k+1}\right)^{\nu}\right]^{\frac{1}{\nu}}\left[\max\left\{\left|{}_\alpha D_qF(\alpha)\right|^u,\left|{}_\alpha D_qF(\beta)\right|^u\right\}\right]^{\frac{1}{u}}\end{aligned}$$

and

$$\begin{aligned}&\left|F(\beta)-\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}F(r)\,{}_\alpha d_q r\right|\\&\quad\leq q(\beta-\alpha)\left(\frac{1-q}{1-q^{\nu+1}}\right)^{\frac{1}{\nu}}\left[\max\left\{\left|{}_\alpha D_qF(\alpha)\right|^u,\left|{}_\alpha D_qF(\beta)\right|^u\right\}\right]^{\frac{1}{u}}.\end{aligned}$$

4 Application

The following special means of real numbers will be used here.

1. Arithmetic mean:

$$\mathcal{A}(u,\nu)=\frac{u+\nu}{2}.$$

2. Generalized logarithmic mean:

$$\mathcal{L}_m(u,\nu)=\left[\frac{\nu^{m+1}-u^{m+1}}{(m+1)(\nu-u)}\right]^{\frac{1}{m}},\quad m\in\mathbb{N},u\neq\nu.$$

Example 24 Let $0<\alpha<\beta$ and $0<q<1$. Then

$$\begin{aligned}&\left|\frac{\beta^2+q\alpha^2}{1+q}-\frac{(1+q)\beta^2+2q^2\alpha\beta+q(1+q^2)\alpha^2}{(1+q)(1+q+q^2)}\right|\\&\quad\leq(\beta-\alpha)\frac{2q^2}{(1+q)^3}\max\left\{2\alpha,(1+q)\beta+(1-q)\alpha\right\}.\end{aligned}\tag{18}$$

Proof Let $F(x)=x^2$. Then, by the properties of the q-integral, we have

$$\begin{aligned}\int_{\alpha}^{\beta}F(r)\,{}_\alpha d_q r&=\int_{\alpha}^{\beta}r^2\,{}_\alpha d_q r\\&=\int_{\alpha}^{\beta}(r-\alpha+\alpha)^2\,{}_\alpha d_q r\end{aligned}$$

$$
\begin{aligned}
&= \int_\alpha^\beta (r-\alpha)^2 \,{}_\alpha d_q r + 2\alpha \int_\alpha^\beta (r-\alpha) \,{}_\alpha d_q r + \alpha^2 \int_\alpha^\beta 1 \,{}_\alpha d_q r \\
&= \frac{(\beta-\alpha)^3}{1+q+q^2} + 2\alpha \frac{(\beta-\alpha)^2}{1+q} + \alpha^2(\beta-\alpha) \\
&= \frac{(\beta-\alpha)[(1+q)\beta^2 + 2q^2\alpha\beta + q(1+q^2)\alpha^2]}{(1+q)(1+q+q^2)}.
\end{aligned}
$$

Also, for $x \neq \alpha$,

$$
\begin{aligned}
{}_\alpha D_q F(x) &= \frac{x^2 - (qx + (1-q)\alpha)^2}{(1-q)(x-\alpha)} \\
&= \frac{(1+q)x^2 - 2q\alpha x - (1-q)\alpha^2}{x-\alpha} \\
&= (1+q)x + (1-q)\alpha.
\end{aligned}
$$

For $x = \alpha$, we have ${}_\alpha D_q F(\alpha) = \lim_{x\to\alpha}({}_\alpha D_q F(x)) = 2\alpha$. The function $|{}_\alpha D_q F(x)|$ is convex and hence quasiconvex on $[\alpha, \beta]$. The desired inequality is obtained by using (7) with $u = 1$. □

If we let $q \to 1^-$ in (18), we obtain

$$
\left|\mathcal{A}(\alpha^2, \beta^2) - \mathcal{L}_2^2(\alpha, \beta)\right| \le \frac{\beta(\beta-\alpha)}{2}.
$$

Example 25 Let $0 < \alpha < \beta$ and $0 < q < 1$. Then

$$
\begin{aligned}
&\left|\mathcal{A}^2(\alpha, \beta) - \frac{(1+q)\beta^2 + 2q^2\alpha\beta + q(1+q^2)\alpha^2}{(1+q)(1+q+q^2)}\right| \\
&\quad \le \frac{\beta-\alpha}{2(1+q)} \max\{2\alpha, (1+q)\beta + (1-q)\alpha\}. \qquad (19)
\end{aligned}
$$

Proof In this case, we apply (15) to the function $F(x) = x^2$ and proceed as in Example 24. □

If we let $q \to 1^-$, then (19) boils down to

$$
\left|\mathcal{A}^2(\alpha, \beta) - \mathcal{L}_2^2(\alpha, \beta)\right| \le \frac{\beta(\beta-\alpha)}{2}.
$$

5 Conclusion

By introducing a parameter $\mu \in [0, 1]$, we established some quantum inequalities by means of the η-quasiconvexity. Our results sharpen, generalize, and extend some known results as can be seen in Remarks 17, 19, 21, and 23. Some examples are also given to show how new estimates can be obtained from our main results. We anticipate that these novel estimates will stimulate further investigation in this regard. Some recent results concerning quasiconvexity and its generalization can be found in [9, 10, 15–20].

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Author details
[1]Department of Mathematics and Computer Science, Alabama State University, Montgomery, USA. [2]Department of Mathematics, Tuskegee University, Tuskegee, USA.

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