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Some Tauberian conditions on logarithmic density

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Abstract

This article is based on the study on the λ-statistical convergence with respect to the logarithmic density and de la Vallee Poussin mean and generalizes some results of logarithmic λ-statistical convergence and logarithmic (V, λ)-summability theorems. Hardy's and Landau's Tauberian theorems to the statistical convergence, which was introduced by Fast long back in 1951, have been extended by J.A. Fridy and M.K. Khan (Proc. Am. Math. Soc. 128:2347–2355, 2000) in recent years. In this article we try to generalize some Tauberian conditions on logarithmic statistical convergence and logarithmic (V, λ)-statistical convergence, and we find some new results on it.

Keywords: Statistical convergence; λ-convergence; de la Vallee Poussin mean; Logarithmic density

1 Introduction and preliminary concepts

In 1951, Fast [2] and Steinhaus [3] independently introduced the concept of statistical convergence for sequences of real numbers, and since then this concept has been generalized and investigated in different ways by different authors. Likewise summability theory and convergence of sequences have also been studied actively in the area of pure mathematics for the last several decades. Extensive works on the topic are applicable in topology, functional analysis, Fourier analysis, measure theory, applied mathematics, mathematical modeling, computer science, analytic number theory, etc. One may refer to [4–9], etc.

Let $A \subseteq \mathbb{N}$ and $A_n = \{\psi \leq n : \psi \in A\}$. We say that A has natural density, i.e., $\delta(A) = \lim_n \frac{1}{n}|A_n|$, if the limit exists, where $|A_n|$ denotes the cardinality of A_n.

By the concept of statistical convergence, we mean a sequence $\tilde{x} = (x_\psi)$ of real numbers which statistically converges to ℓ if for every $\varepsilon > 0$ the set $A_\varepsilon = \{\psi \in \mathbb{N} : |x_\psi - \ell| \geq \varepsilon\}$ has natural density zero, i.e., for each $\varepsilon > 0$,

$$\lim_n \frac{1}{n}\bigl|\{\psi \leq n : |x_\psi - \ell| \geq \varepsilon\}\bigr| = 0.$$

Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to ∞ such that

$$\lambda_{n+1} \leq \lambda_n + 1 \tag{1}$$

and $\lambda_1 = 0$.

The generalized de la Vallee Poussin mean of a sequence $\tilde{x} = (x_\psi)$ is defined by $T_n(x) = \frac{1}{\lambda_n}\sum_{\psi\in I_n} x_\psi$, where $I_n = [n - \lambda_n + 1, n]$.

Now, a sequence $\tilde{x} = (x_\psi)$ is said to be (V,λ)-summable to ℓ if $T_n(x)$ converges to ℓ, i.e.,

$$\lim_n \frac{1}{\lambda_n}\sum_{\psi\in I_n} |x_\psi - \ell| = 0.$$

Also a sequence $\tilde{x} = (x_\psi)$ is said to be statistically λ-convergent to ℓ if, for every $\varepsilon > 0$,

$$\lim_n \frac{1}{n}\bigl|\bigl\{\psi \le n : \bigl|T_\psi(x) - \ell\bigr| \ge \varepsilon\bigr\}\bigr| = 0.$$

By logarithmic density, we mean $\delta_{\log_n}(E) = \frac{1}{\log_n}\sum_{\psi=1}^{n}\frac{\chi_E(\psi)}{\psi}$ for $E \in \mathbb{N}$, where $\log_n = \sum_{\psi=1}^{n}\frac{1}{\psi} \approx \log n$, $n \in \mathbb{N}$ [8].

A sequence $\tilde{x} = (x_\psi)$ is logarithmic statistically convergent to ℓ if

$$\lim_n \frac{1}{\log_n}\left|\left\{\psi \le n : \frac{1}{\psi}|x_\psi - \ell| \ge \varepsilon\right\}\right| = 0.$$

A sequence $\tilde{x} = (x_\psi)$ is logarithmic (V,λ)-statistically convergent to ℓ if

$$\lim_n \frac{1}{\log_{\lambda_n}}\left|\left\{\psi \in I_n : \frac{1}{\psi}\bigl|T_\psi(x) - \ell\bigr| \ge \varepsilon\right\}\right| = 0,$$

where $\log_{\lambda_n} = \sum_{\psi=1}^{\lambda_n}\frac{1}{\psi} \approx \log\lambda_n$ $(n = 1,2,3,\ldots)$.

Let $\mu_n = \frac{1}{\log_{\lambda_n}}\sum_{\psi\in I_n}\frac{T_\psi(x)}{\psi}$, where $\log_{\lambda_n} = \sum_{\psi=1}^{\lambda_n}\frac{1}{\psi} \approx \log\lambda_n$ $(n = 1,2,3,\ldots)$. A sequence $\tilde{x} = (x_\psi)$ is logarithmic (V,λ)-summable to ℓ if (μ_n) is convergent to ℓ, i.e., $\lim_n \frac{1}{\log_{\lambda_n}}\sum_{\psi\in I_n}\frac{|T_\psi(x)-\ell|}{\psi} = 0$.

A sequence $\tilde{x} = (x_\psi)$ is logarithmic (V,λ)-statistically summable to ℓ if (μ_n) is λ-statistically convergent, i.e.,

$$\lim_n \frac{1}{\lambda_n}\bigl|\bigl\{\psi \in I_n : |\mu_\psi - \ell| \ge \varepsilon\bigr\}\bigr| = 0.$$

We define it as $st_{\log_{\lambda_n}} - \lim_n T_n = \ell$.

Moricz [10] studied the concept of Tauberian conditions for statistical convergence followed from statistical summability $(C,1)$. Braha [11] extended these results using Tauberian conditions for λ-statistical convergence, which was followed from statistical summability (V,λ). Braha [12] also explained the Tauberian theorems for the generalized Norlund–Euler summability method. One may refer to [13–15].

In this paper, we study the Tauberian theorems for logarithmic (V,λ)-statistical convergence which is followed from de la Vallee Poussin mean. We also try to establish some results involving the logarithmic density.

2 Main results

Theorem 2.1 *Let λ be a real-valued sequence defined in (1). Then,*

1. *If $\tilde{x} = (x_\psi)$ is logarithmic (V,λ)-statistically summable to ℓ, then it is logarithmic (V,λ)-statistically convergent to ℓ, provided* $\liminf_n \frac{1}{\lambda_n} > 0$.

2. *If $\tilde{x} = (x_\psi)$ is bounded, then logarithmic (V,λ)-statistical convergence implies logarithmic (V,λ)-statistical summability.*
3. *$\Omega(\log_n,\lambda) \cap \ell_\infty = \Pi(\log_n,\lambda)$, where $\Omega(\log_n,\lambda)$ is the collection of all logarithmic (V,λ)-statistical convergence sequences, ℓ_∞ is the collection of all bounded sequences, and $\Pi(\log_n,\lambda)$ is the collection of all logarithmic (V,λ)-summable sequences.*

Proof (1) Let $\tau_n = \{\psi \in I_n : \frac{1}{\log_{\lambda_n}} \sum_{\psi\in I_n} \frac{1}{\psi}|T_\psi(x) - \ell| \geq \varepsilon\}$.

Since $\tilde{x} = (x_\psi)$ is logarithmic (V,λ)-statistically summable to ℓ, then τ_n is λ-statistically convergent to ℓ, i.e.,

$$\lim_n \frac{1}{\lambda_n}\left|\left\{\psi \in I_n : \frac{1}{\log_{\lambda_n}} \sum_{\psi\in I_n} \frac{1}{\psi}\left|T_\psi(x) - \ell\right| \geq \varepsilon\right\}\right| = 0.$$

Also we can write

$$\begin{aligned} \frac{1}{\log_{\lambda_n}} \sum_{\psi\in I_n} \frac{1}{\psi}\left|T_\psi(x) - \ell\right| &\geq \frac{1}{\log_{\lambda_n}} \sum_{\psi\in I_n, |\frac{T_\psi(x)}{\psi} - \ell| \geq \varepsilon} \frac{1}{\psi}\left|T_\psi(x) - \ell\right| \\ &\geq \frac{1}{\log_{\lambda_n}}\left(\left|\left\{\psi \in I_n : \frac{1}{\psi}\left|T_\psi(x) - \ell\right| \geq \varepsilon\right\}\right|\varepsilon\right), \end{aligned}$$

which implies that

$$\frac{1}{\lambda_n}\left\{\psi \in I_n : \frac{1}{\log_{\lambda_n}} \sum_{\psi\in I_n} \frac{1}{\psi}\left|T_\psi(x) - \ell\right| \geq \varepsilon\right\} \geq \frac{1}{\lambda_n} \frac{(|\{\psi \in I_n : \frac{1}{\psi}|T_\psi(x) - \ell| \geq \varepsilon\}|\varepsilon)}{\log_{\lambda_n}}.$$

Since $\liminf_n \frac{1}{\lambda_n} > 0$ and $\tilde{x} = (x_\psi)$ is logarithmic (V,λ)-statistically summable to ℓ, so by taking $n \to \infty$, we get $\tilde{x} = (x_\psi)$ is logarithmic (V,λ)-statistically convergent to ℓ. This completes the proof. □

Proof (2) Let $\tilde{x} = (x_\psi)$ be bounded and logarithmic (V,λ)-statistically convergent to ℓ. Then there exists $M > 0$ such that $|x_\psi - \ell| \leq M$. Now, for any $\varepsilon > 0$,

$$\begin{aligned} \frac{1}{\log_{\lambda_n}} \sum_{\psi\in I_n} \frac{1}{\psi}\left|T_\psi(x) - \ell\right| &= \frac{1}{\log_{\lambda_n}} \sum_{\psi\in I_n, \psi\notin B(n)} \frac{1}{\psi}\left|T_\psi(x) - \ell\right| \\ &\quad + \frac{1}{\log_{\lambda_n}} \sum_{\psi\in I_n, \psi\in B(n)} \frac{1}{\psi}\left|T_\psi(x) - \ell\right| \\ &= K_1(n) + K_2(n), \end{aligned}$$

where $B(n) = \{\psi \in I_n : \frac{1}{\psi}|T_\psi(x) - \ell| \geq \varepsilon\}$

Now, if $\psi \notin B(n)$, then $K_1(n) < \varepsilon$. For $\psi \in B(n)$, we have

$$K_2(n) \leq \left(\sup\left|T_\psi(x) - \ell\right|\right).\left(\left|B(n)\right|/\log_{\lambda_n}\right) \leq M\left|B(n)\right|/\log_{\lambda_n} \to 0$$

as $n \to \infty$.

Since logarithmic density of $B(n)$ is zero, hence we can say that $\tilde{x} = (x_\psi)$ is logarithmic (V,λ)-statistically summable. This completes the proof. □

Proof Proof of (3) follows from the proof of (1) and (2), so it is omitted here. □

3 Tauberian theorems

Theorem 3.1 *Let* (λ_n) *be a sequence of real numbers and* $st_{\log_{\lambda_n}} - \lim_n \inf \frac{\lambda_{t_n}}{\lambda_n} > 1$ *for all* $t > 1$, *where* t_n *denotes the integral parts of* $[t.n]$ *for every* $n \in \mathbb{N}$, *and let* (T_ψ) *be a sequence of real numbers such that* $st_{\log_{\lambda_n}} - \lim_n T_n = \ell$. *Then* $\tilde{x} = (x_\psi)$ *is* $st_{\log_{\lambda_n}}$*-convergent to* ℓ *iff the following conditions hold*:

$$\inf_{t>1} \limsup_n \frac{1}{\log_{\lambda_n}} \left| \left\{ \psi \in I_n : \frac{1}{\lambda_{t_\psi} - \lambda_\psi} \sum_{j=\psi+1}^{t_\psi} \frac{1}{\psi}(x_j - x_\psi) \le -\varepsilon \right\} \right| = 0$$

and

$$\inf_{0<t<1} \limsup_n \frac{1}{\log_{\lambda_n}} \left| \left\{ \psi \in I_n : \frac{1}{\lambda_\psi - \lambda_{t_\psi}} \sum_{j=t_\psi+1}^{k} \frac{1}{\psi}(x_\psi - x_j) \le -\varepsilon \right\} \right| = 0.$$

Remark Let us suppose that

$$st_{\log_{\lambda_n}} - \lim_n x_n = \ell \quad \text{and} \quad st_{\log_{\lambda_n}} - \lim_n T_n = \ell \tag{2}$$

are satisfied, then for every $t > 1$, the following relation is valid:

$$st_{\log_{\lambda_n}} - \lim_n x_n = \ell \quad \text{implies that} \quad \lim_n \frac{1}{\log_{\lambda_n}} \sum_{\psi=1}^{n} \frac{1}{\psi} |x_\psi - \ell| = 0$$

and

$$st_{\log_{\lambda_n}} - \lim_n T_n = \ell \quad \text{implies that} \quad \lim_n \frac{1}{\log_{\lambda_n}} \sum_{\psi=1}^{n} \frac{1}{\psi} \left| T_\psi(x) - \ell \right| = 0,$$

from which it follows that $st_\lambda - \lim_n \frac{1}{\log(\lambda_{t_\psi} - \lambda_\psi)} \sum_{\psi=n+1}^{t_n} \frac{1}{\psi} x_\psi = 0$ holds for $t > 1$, i.e.,

$$\lim_n \frac{1}{\log_{\lambda_n}} \left| \left\{ \psi \le n : \frac{1}{\lambda_{t_\psi} - \lambda_\psi} \sum_{\psi=1}^{t_n} \frac{|x_\psi|}{\psi} \ge \varepsilon \right\} \right| = 0,$$

and for $0 < t < 1$, we have $st_\lambda - \lim_n \frac{1}{\lambda_\psi - \lambda_{t_\psi}} \sum_{\psi=t_n+1}^{n} \frac{x_\psi}{\psi} = 0$, i.e.,

$$\lim_n \frac{1}{\log_{\lambda_n}} \left| \left\{ \psi \le n : \frac{1}{\lambda_\psi - \lambda_{t_\psi}} \sum_{\psi=t_n+1}^{n} \frac{|x_\psi|}{\psi} \ge \varepsilon \right\} \right| = 0$$

holds.

Lemma 3.1 *For the sequence of real numbers* $\lambda = (\lambda_n)$, (2) *is equivalent to* $st_{\lambda_n} - \lim_n \inf \frac{\lambda_n}{\lambda_{t_n}} > 1$ *for all* $0 < t < 1$ [12].

Lemma 3.2 *If* $st_{\log_{\lambda_n}} - \lim_n x_n = \ell$ *and* $st_{\log_{\lambda_n}} - \lim_n T_n = \ell$ *are satisfied, and let* $\tilde{x} = (x_\psi)$ *be a sequence of complex numbers which is logarithmic* (V,λ)*-statistically convergent to* ℓ*, then for any* $t > 1$,

$$st_{\log_{\lambda_n}} - \lim_n T_{t_n} = \ell, \quad \text{i.e.,} \quad \frac{1}{\log_{\lambda_n}} \left| \left\{ \psi \le n : \sum_{\psi=1}^{n} \frac{1}{\psi} |T_{t_\psi} - \ell| \ge \varepsilon \right\} \right| = 0.$$

Proof Case I: Let us consider that $t > 1$, then from construction of the sequence $\lambda = (\lambda_n)$ we get

$$\lim_n (n - \lambda_n) = \lim_n (t_n - \lambda_{t_n}), \tag{3}$$

and for every $\varepsilon > 0$, we have

$$\begin{aligned} &\left\{ \psi \in I_{t_n} : \frac{1}{\psi} |T_{t_\psi} - \ell| \ge \varepsilon \right\} \\ &\quad \subset \left\{ \psi \in I_n : \frac{1}{\psi} |T_\psi - L| \ge \varepsilon \right\} \cup \left\{ \psi \in I_n : \frac{1}{\log_{\lambda_\psi}} \sum_{j=\psi-\lambda_\psi+1}^{\psi} \frac{x_j}{j} \ne \frac{1}{\log_{\lambda_{t_\psi}}} \sum_{j=t_\psi-\lambda_{t_\psi}+1}^{t_\psi} \frac{x_j}{j} \right\}. \end{aligned}$$

Following Eq. (3), we can say that $st_{\log_\lambda} - \lim T_{t_n} = \ell$.

Case II: Now suppose that $0 < t < 1$. For the definition of $t_n = [t.n]$, for any natural number n, we can conclude that (T_{t_n}) does not appear more than $[1 + t^{-1}]$ times in the sequence (T_n). In fact, if there exist integers ψ, m such that

$$n \le t.\psi < t(\psi + 1) < \cdots < t(\psi + m - 1) < n + 1 \le t(\psi + m),$$

then

$$n + t(m-1) \le t(\psi + m - 1) < n + 1 \quad \Rightarrow \quad m < 1 + \frac{2}{t}.$$

So, we have the following inequality:

$$\begin{aligned} \frac{1}{\log_{\lambda_{t_n}}} \left| \left\{ \psi \in I_{t_n} : \frac{1}{\psi} |T_{t_\psi} - \ell| \ge \varepsilon \right\} \right| &\le \left(1 + \frac{1}{t} \right) \frac{1}{\log_{\lambda_{t_n}}} \left| \left\{ \psi \in I_n : \frac{1}{\psi} |T_\psi - \ell| \ge \varepsilon \right\} \right| \\ &\le 2(1+t) \frac{1}{\log_{\lambda_n}} \left| \left\{ \psi \in I_n : \frac{1}{\psi} |T_\psi - \ell| \ge \varepsilon \right\} \right|, \end{aligned}$$

which gives that $st_{\log_{\lambda_n}} - \lim_n T_{t_n} = \ell$. □

Lemma 3.3 *If* $st_{\log_{\lambda_n}} - \lim_n x_n = \ell$ *and* $st_{\log_{\lambda_n}} - \lim_n T_n = \ell$ *are satisfied and* $\tilde{x} = (x_\psi)$ *is logarithmic* (V,λ)*-statistically convergent to* ℓ*, then we have*

$$\text{(i)} \quad \lim_n \frac{1}{\log_{\lambda_n}} \left| \left\{ \psi \le n : \frac{1}{\lambda_{t_n} - \lambda_n} \sum_{\psi=n+1}^{t_n} \frac{|x_\psi - \ell|}{\psi} \ge \varepsilon \right\} \right| = 0 \quad \textit{for every } t > 1 \tag{4}$$

and

$$\text{(ii)}\quad \lim_n \frac{1}{\log_{\lambda_n}} \left| \left\{ \psi \leq n : \frac{1}{\lambda_n - \lambda_{t_n}} \sum_{\psi=t_n+1}^{n} \frac{1}{\psi} |x_\psi - \ell| \geq \varepsilon \right\} \right| = 0 \quad \textit{for any } 0 < t < 1. \tag{5}$$

Proof (i) Let us suppose that $t > 1$. We get

$$\begin{aligned} \frac{1}{\lambda_{t_n} - \lambda_n} \sum_{j=n+1}^{t_n} \frac{x_j}{j} &= T_n + \frac{\lambda_{t_n}}{\lambda_{t_n} - \lambda_n}(T_{t_n} - T_n) \\ &\quad + \frac{1}{\lambda_{t_n} - \lambda_n} \sum_{j=n-\lambda_n+1}^{t_n} \frac{x_j}{j} - \frac{1}{\lambda_{t_n} - \lambda_n} \sum_{j=t_n-\lambda_{t_n}+1}^{t_n} \frac{x_j}{j} \\ \Rightarrow \quad \frac{1}{\lambda_{t_n} - \lambda_n} \sum_{j=n+1}^{t_n} \frac{x_j}{j} &= T_n + \frac{\lambda_{t_n}}{\lambda_{t_n} - \lambda_n}(T_{t_n} - T_n) \\ &\quad + \frac{1}{\lambda_{t_n} - \lambda_n} \left(\sum_{j=n-\lambda_n+1}^{t_n} \frac{x_j}{j} - \sum_{j=t_n-\lambda_{t_n}+1}^{t_n} \frac{x_j}{j} \right). \end{aligned} \tag{6}$$

From the definition of the sequence (λ_n) and logarithmic density, we obtain

$$st_{\log_{\lambda_n}} - \limsup_n \sum_{j=n-\lambda_n+1}^{t_n} x_j = st_{\log_{\lambda_n}} - \limsup_n \sum_{j=t_n-\lambda_{t_n}+1}^{t_n} x_j. \tag{7}$$

Let us suppose that $st_{\log_{\lambda_n}} - \lim_n \sup \sum_{j=n-\lambda_n+1}^{t_n} x_j = L$, and for every $\varepsilon > 0$, we get

$$\begin{aligned} &\frac{\lim_n \frac{1}{\log_{\lambda_n}} |\{\psi \in I_{t_n} : |\sum_{j=t_\psi-\lambda_{t_\psi}+1}^{t_\psi} \frac{|x_j-\ell|}{j}| \geq \varepsilon\}|}{\lambda_{t_n}} \\ &\quad \leq \frac{\lim_n \frac{1}{\log_{\lambda_n}} |\{\psi \in I_n : |\sum_{j=\psi-\lambda_\psi+1}^{t_\psi} \frac{|x_j-\ell|}{j}| \geq \varepsilon\}|}{\lambda_n} \\ &\qquad + \frac{\lim_n \frac{1}{\log_{\lambda_n}} |\{\psi \in I_n : \sum_{j=t_\psi-\lambda_{t_\psi}+1}^{t_\psi} \frac{x_j}{j} \neq \sum_{j=\psi-\lambda_\psi+1}^{t_\psi} \frac{x_j}{j}\}|}{\lambda_n}, \end{aligned}$$

from which it follows that $st_{\log_{\lambda_n}} - \lim_n \sup \sum_{j=t_n-\lambda_{t_n}+1}^{t_n} x_j = L$.

Also, since $st_\lambda - \lim_n \sup \frac{\lambda_{t_n}}{\lambda_{t_n}-\lambda_n} < \infty$ and $st_\lambda - \lim_n \sup \frac{1}{\lambda_{t_n}-\lambda_n} < \infty$, then we get

$$st_{\log_{\lambda_n}} - \lim_n \frac{1}{\lambda_{t_n} - \lambda_n} \sum_{j=n+1}^{t_n} x_j = \ell.$$

(ii) If $0 < t < 1$, we have

$$\frac{1}{\lambda_n - \lambda_{t_n}} \sum_{j=t_n+1}^{n} \frac{x_j}{j} = T_n + \frac{\lambda_{t_n}}{\lambda_n - \lambda_{t_n}}(T_n - T_{t_n}) + \frac{1}{\lambda_n - \lambda_{t_n}} \sum_{j=n-\lambda_n+1}^{n} \frac{x_j}{j} - \frac{1}{\lambda_{t_n} - \lambda_n} \sum_{j=t_n-\lambda_{t_n}+1}^{n} \frac{x_j}{j}.$$

This completes the proof. □

Following the above procedure, we can get the proof of Theorem 3.1.

Proof of Theorem 3.1 Let us suppose that $st_{\log_\lambda} - \lim_\psi x_\psi = L$ and $st_{\log_\lambda} - \lim_\psi T_\psi = \ell$. For every $t > 1$, we get (by Lemma 3.2)

$$\inf_{t>1} \limsup_n \frac{1}{\log_{\lambda_n}} \left| \left\{ \psi \in I_n : \frac{1}{\lambda_{t_\psi} - \lambda_\psi} \sum_{j=\psi+1}^{t_\psi} \frac{1}{\psi}(x_j - x_\psi) \leq -\varepsilon \right\} \right| = 0.$$

Similarly, if $0 < t < 1$, we obtain (by Lemma 3.2)

$$\inf_{0<t<1} \limsup_n \frac{1}{\log_{\lambda_n}} \left| \left\{ \psi \in I_n : \frac{1}{\lambda_\psi - \lambda_{t_\psi}} \sum_{j=t_\psi+1}^{\psi} \frac{1}{\psi}(x_\psi - x_j) \leq -\varepsilon \right\} \right| = 0.$$

Now assume that $st_{\log_\lambda} - \lim_n T_n = \ell$ and

$$st_{\log_{\lambda_n}} - \liminf_n \frac{\lambda_{t_n}}{\lambda_n} > 1 \quad \text{for all } t > 1, \tag{8}$$

$$\inf_{t>1} \limsup_n \frac{1}{\log_{\lambda_n}} \left| \left\{ \psi \in I_n : \frac{1}{\lambda_{t_\psi} - \lambda_\psi} \sum_{j=\psi+1}^{t_\psi} \frac{1}{\psi}(x_j - x_\psi) \leq -\varepsilon \right\} \right| = 0, \tag{9}$$

and

$$\inf_{0<t<1} \limsup_n \frac{1}{\log_{\lambda_n}} \left| \left\{ \psi \in I_n : \frac{1}{\lambda_\psi - \lambda_{t_\psi}} \sum_{j=t_\psi+1}^{\psi} \frac{1}{\psi}(x_\psi - x_j) \leq -\varepsilon \right\} \right| = 0 \tag{10}$$

are satisfied. We have to prove that $st_{\log_\lambda} - \lim_n x_n = \ell$ or equivalently $st_{\log_\lambda} - \lim_n (T_n - x_n) = 0$.

Case I: If $t > 1$, let us suppose

$$x_n - T_n = \lambda_{t_n} \frac{T_{t_n} - T_n}{\lambda_{t_n} - \lambda_n} - \frac{1}{\lambda_{t_n} - \lambda_n} \sum_{j=n+1}^{t_n} \frac{x_j - x_n}{j}.$$

For any $\varepsilon > 0$, we obtain

$$\begin{aligned} &\{\psi \in I_n : x_n - T_n \geq \varepsilon\} \\ &\quad \subset \left\{ \psi \in I_n : \frac{\lambda_{t_n}}{\lambda_{t_n} - \lambda_n}(T_{t_n} - T_n) \geq \frac{\varepsilon}{2} \right\} \cup \left\{ \psi \in I_n : \frac{1}{\lambda_{t_n} - \lambda_n} \sum_{j=n+1}^{t_n} \frac{x_j - x_\psi}{j} \leq \frac{-\varepsilon}{2} \right\}. \end{aligned}$$

From the above relation (9), it follows that, for any arbitrary $\gamma > 0$, there exists $t > 1$ such that

$$\limsup_n \frac{1}{\log_{\lambda_n}} \left| \left\{ \psi \in I_n : \frac{1}{\lambda_{t_\psi} - \lambda_\psi} \sum_{j=\psi+1}^{t_\psi} \frac{1}{x_j - x_\psi} j \leq -\varepsilon \right\} \right| \leq \gamma.$$

Also following Lemma 3.2 and the relations $st_\lambda - \lim_n \sup \frac{\lambda_{t_n}}{\lambda_{t_n}-\lambda_n} < \infty$ and $st_\lambda - \lim_n \sup \frac{1}{\lambda_{t_n}-\lambda_n} < \infty$, we get

$$\limsup_n \frac{1}{\log_{\lambda_n}} \left| \left\{ \psi \in I_n : \left| \frac{\lambda_{t_n}}{\lambda_{t_n}-\lambda_n} T_{t_n} - T_n \right| \geq \frac{\varepsilon}{2} \right\} \right| = 0.$$

Combining these relations, we have

$$\limsup_n \frac{1}{\log_{\lambda_n}} \left| \left\{ \psi \in I_n : \sum_j \frac{x_n - T_n}{j} \geq \varepsilon \right\} \right| \leq \gamma.$$

Since γ is arbitrary, we conclude that, for every $\varepsilon > 0$,

$$\limsup_n \frac{1}{\log_{\lambda_n}} \left| \left\{ \psi \in I_n : \sum_j \frac{x_n - T_n}{j} \geq \varepsilon \right\} \right| = 0.$$

Case II: If $0 < t < 1$, let us suppose

$$x_n - T_n = \lambda_{t_n} \frac{T_{t_n} - T_n}{\lambda_n - \lambda_{t_n}} - \frac{1}{\lambda_n - \lambda_{t_n}} \sum_{j=t_n+1}^{n} \frac{x_n - x_j}{j}.$$

For any $\varepsilon > 0$,

$$\begin{aligned} &\{\psi \in I_n : x_n - T_n \leq -\varepsilon\} \\ &\subset \left\{ \psi \in I_n : \frac{\lambda_{t_n}}{\lambda_n - \lambda_{t_n}} (T_n - T_{t_n}) \leq -\frac{\varepsilon}{2} \right\} \cup \left\{ \psi \in I_n : \frac{1}{\lambda_n - \lambda_{t_n}} \sum_{j=t_n+1}^{n} \frac{x_\psi - x_j}{j} \leq -\frac{\varepsilon}{2} \right\}. \end{aligned}$$

Proceeding in the same way as above, we get the result as follows:

$$\lim_n \frac{1}{\log_{\lambda_n}} \left| \left\{ \psi \in I_n : \sum_n \frac{x_n - T_n}{n} \geq \varepsilon \right\} \right| = 0.$$

This completes the proof of the theorem. □

Theorem 3.2 *Let* (λ_n) *be a sequence of complex numbers which satisfies the following condition*:

$$st_{\log_{\lambda_n}} - \liminf_n \frac{\lambda_{t_n}}{\lambda_n} > 1 \quad \textit{for all } t > 1,$$

and also consider that $st_{\log_\lambda} - \lim T_\psi = \ell$. *Then* (x_ψ) *is* $st_{\log_\lambda}$*-statistically convergent to the same number* ℓ *if and only if the following two conditions hold*: *for every* $\varepsilon > 0$,

$$\inf_{t>1} \limsup_n \frac{1}{\log_{\lambda_n}} \left| \left\{ \psi \in I_n : \frac{1}{\lambda_{t_\psi} - \lambda_\psi} \sum_{j=\psi+1}^{t_\psi} \frac{x_j - x_\psi}{j} \geq \varepsilon \right\} \right| = 0,$$

and

$$\inf_{0<t<1} \limsup_{n} \frac{1}{l_{\lambda_n}} \left| \left\{ \psi \in I_n : \frac{1}{\lambda_\psi - \lambda_{t_\psi}} \sum_{j=t_\psi+1}^{\psi} \frac{x_\psi - x_j}{j} \geq \varepsilon \right\} \right| = 0.$$

Proof Proofs can be obtained by following Theorem 3.1. □

4 Conclusion

In this paper, the Tauberian conditions under the logarithmic statistical convergence following from (V,λ)-summability are studied. The Tauberian conditions can be further applied in probabilistic normed linear spaces with f-density. They can also be studied in the approximation theorem point of view in more extended forms.

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