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## RESEARCH

# Existence and regularity of periodic solutions for neutral evolution equations with delays

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## Abstract

The aim of this paper is to study the periodic problem for neutral evolution equation

$$\big(u(t) - G\big(t, u(t - \xi)\big)\big)' + Au(t) = F\big(t, u(t), u(t - \tau)\big), \quad t \in \mathbb{R},$$

in a Banach space $X$, where $A : D(A) \subset X \to X$ is a closed linear operator, and $-A$ generates a compact analytic operator semigroup $T(t)$ ($t \geq 0$). With the aid of the analytic operator semigroup theories and some fixed point theorems, we obtain the existence and uniqueness of periodic mild solution for the neutral evolution equation. The regularity of periodic mild solutions for the evolution equation with delay is studied, and some existence results of the classical and strong solutions are obtained. In the end, we give an example to illustrate the applicability of abstract results. Our works greatly improve and generalize the relevant results of existing literature.

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## 1 Introduction

Let $X$ be a real Banach space with norm $\| \cdot \|$. The purpose of this paper is to discuss the existence and regularity of $\omega$-periodic solutions for the abstract neutral functional differential equation with delays

$$\big(u(t) - G\big(t, u(t - \xi)\big)\big)' + Au(t) = F\big(t, u(t), u(t - \tau)\big), \quad t \in \mathbb{R} \tag{1.1}$$

in $X$, where $A : D(A) \subset X \to X$ is a closed linear operator, and $-A$ generates a compact and exponentially stable analytic operator semigroup $T(t)$ ($t \geq 0$) on $X$, $G$, $F$ are appropriate continuous functions which will be specified later, $\xi$, $\tau$ are positive constants which denote the time delays.

The theory of partial differential equations with delays has extensive physical background and realistic mathematical model, and it has undergone a rapid development in the last fifty years, see [19, 42] and the references therein. During the last few decades, more researchers have given special attention to the study of the equation, in which the delay argument occurs in the derivative of the state variable as well as in the independent variable, so-called neutral differential equations.

Neutral differential equations have many applications. They can model a lot of problems arising from engineering, such as population dynamics, transmission line, immune response, or distribution of albumin in the blood. For instance, in the theory of heat conduction in fading memory material, see [9, 37], the following partial neutral differential equation

$$\frac{d}{dt}\left(u(t,x) + \int_{-\infty}^{t} k_1(t-s)u(s,x)\,ds\right) - c\Delta u(t,x) = \int_{-\infty}^{t} k_2(t-s)\Delta u(s,x)\,ds \qquad (1.2)$$

has been frequently used to describe these phenomena, which has better effects than partial differential equations without neutral item, where $\Omega \subset \mathbb{R}^n$ is a bounded domain with a sufficiently smooth boundary $\partial\Omega$, $(t,x) \in [0,\infty) \times \Omega$ represents the temperature in $x$ at the time $t$, $c$ is a physical constant, $k_1, k_2 : \mathbb{R} \to \mathbb{R}$ are the internal energy and the heat flux relaxation respectively. If the solution $u$ is known on $(-\infty, 0]$, $k_1 \equiv 0$ on $(r, \infty)$, and $k_2 \equiv 0$, then we can transform system (1.2) into the abstract neutral evolution equation (1.1).

In fact, many of partial neutral differential equations can be written as first-order abstract neutral functional differential equations on an appropriate Banach space. There has been an increasing interest in the study of the abstract neutral evolution equations of the form (1.1). The existence and uniqueness of mild solutions to the abstract neutral evolution equations with delay have been considered by many authors in literature. Here we only mention [1–3, 13, 18, 20, 22, 23].

It is noteworthy at this point that the problem concerning periodic solutions of partial neutral functional differential equations has become an important area of investigation. The periodic problems can take into account seasonal fluctuations occurring in the phenomena appearing in the models, and have been studied by some researchers in recent years. Specially, the existence of periodic solutions for the neutral evolution equations has been considered by several authors, see [4–7, 11, 12, 14–17, 21, 24, 43] and the references therein for more comments. We notice that, in many works, the key assumption of prior boundedness is employed and the most important ingredient to prove the existence of periodic solutions is to show that Poincaré's mapping is condensing. Thus, a fixed point theorem can be used to derive periodic solutions. For the delayed evolution equations without neutral item, the existence of periodic solutions has been discussed by more authors, see [8, 26, 30–32, 34–36, 44] and the references therein.

Although there have been many meaningful results on the neutral evolution equation periodic problem in a Banach space, to our knowledge, these results have a relatively large limitation. First of all, the most popular approach is the use of boundedness or ultimate boundedness of solutions and the compactness of Poincaré map realized through some compact embeddings. However, in some concrete applications, it is difficult to choose appropriate initial conditions to guarantee the boundedness of the solution. Secondly, we observe that the most popular condition imposed on the nonlinear term $F$ is its Lipschitz-type condition. In fact, for equations arising in complicated reaction-diffusion processes, the nonlinear function $F$ represents the source of material or population, which depends on time in diversified manners in many contexts. Thus, we may not hope to have the Lipschitz-type condition of $F$. Finally, there are few papers to study the regularity of periodic solutions for the neutral evolution equation with delays.

Motivated by the papers mentioned above, we aim in this work to study the existence and regularity of periodic solutions for the partial neutral functional differential equation (1.1). In this paper, it is worth mentioning that the assumption of prior boundedness of solutions is not necessary. More precisely, the nonlinear term $F$ only satisfies some growth conditions and the functions $G$ and $F$ may not be defined on the whole space $X$. These conditions are much weaker than Lipschitz conditions. The obtained results will improve the main results in [6, 14, 21] and develop the work in [30].

The paper is organized as follows. In Sect. 2, we collect some known notions and results on the fractional powers of the generator of an analytic semigroup and provide preliminary results to be used in theorems stated and proved in the paper. In Sect. 3, we apply the operator semigroup theory to find the periodic mild solutions for Eq. (1.1), and in Sect. 4, we investigate conditions for Eq. (1.1) to have the classical and strong periodic solutions. In the last section, we give an example to illustrate the applicability of abstract results obtained in Sect. 3 and Sect. 4.

## 2 Preliminaries

Throughout this paper, we assume that $X$ is a Banach space with norm $\| \cdot \|$.

Now, we recall some notions and properties of operator semigroups, which are essential for us. Assume that $A : D(A) \subset X \to X$ is a closed linear operator and $-A$ generates a compact analytic operator semigroup $T(t)$ $(t \geq 0)$ on $X$. For the detailed theory of operator semigroups, we refer to [38].

For a general $C_0$-semigroup $T(t)$ $(t \geq 0)$, there exist $M \geq 1$ and $\nu \in \mathbb{R}$ such that $\|T(t)\| \leq Me^{\nu t}$ for all $t \geq 0$ (see [38]). Let

$$\nu_0 = \inf\{\gamma \in \mathbb{R} | \text{ There exists } M \geq 1 \text{ such that } \|T(t)\| \leq Me^{\nu t}, \forall t \geq 0\},$$

then $\nu_0$ is called the growth exponent of the semigroup $T(t)$ $(t \geq 0)$. Specially, if $\nu_0 < 0$, then $T(t)$ $(t \geq 0)$ is called an exponentially stable $C_0$-semigroup. Furthermore, $\nu_0$ can be also obtained by the formula $\nu_0 = \limsup_{t \to +\infty} \frac{\ln \|T(t)\|}{t}$. If $T(t)$ is continuous in the uniform operator topology for every $t > 0$ in $X$, it is well known that $\nu_0$ can also be determined by $\nu_0 = -\inf\{\text{Re}\,\lambda | \lambda \in \sigma(A)\}$ (see [38, 41]). We know that $T(t)$ $(t \geq 0)$ is continuous in the uniform operator topology for $t > 0$ if $T(t)$ $(t \geq 0)$ is a compact semigroup or an analytic semigroup (see [25]).

If $T(t)$ $(t \geq 0)$ is analytic satisfying $0 \in \rho(A)$ ($\rho(A)$ is the resolvent set of $A$), then for any $\alpha > 0$, we can define $A^{-\alpha}$ by

$$A^{-\alpha} := \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} T(t)\,dt.$$

It follows that each $A^{-\alpha}$ is an injective continuous endomorphism of $X$. Hence we can define $A^\alpha := (A^{-\alpha})^{-1}$, which is a closed bijective linear operator on $X$. We define $D(A^\alpha)$ as the domain of the operator $A^\alpha$, obviously, the subspace $D(A^\alpha)$ is dense in $X$. Furthermore, $D(A^\alpha)$ endowed with the norm $\|x\|_\alpha := \|A^\alpha x\|$ for all $x \in D(A^\alpha)$ is a Banach space. Next, we denote $D(A^\alpha)$ by $X_\alpha$ and $C_\alpha := \|A^{-\alpha}\|$. The following properties are well known [38].

**Lemma 2.1** *If $T(t)$ $(t \geq 0)$ is an analytic semigroup with infinitesimal generator $A$ satisfying $0 \in \rho(A)$, then*

    (i)  $D(A^\alpha)$ is a Banach space for $0 \le \alpha \le 1$;

  (ii)  $A^{-\alpha}$ is a bounded linear operator for $0 \le \alpha \le 1$ in $X$;

 (iii)  $T(t) : X \to D(A^\alpha)$ for each $t > 0$;

 (iv)  $A^\alpha T(t)x = T(t)A^\alpha x$ for each $x \in D(A^\alpha)$ and $t \ge 0$;

  (v)  for every $t > 0$, $A^\alpha T(t)$ is bounded in $X$ and there exists $M_\alpha > 0$ such that

$$\left\| A^\alpha T(t) \right\| \le M_\alpha t^{-\alpha};$$

      moreover, if $\alpha \in (0, 1)$, then $M_\alpha = M\Gamma(\alpha)$;

 (vi)  $X_\beta \hookrightarrow X_\alpha$ for $0 \le \alpha \le \beta \le 1$ (with $X_0 = X$ and $X_1 = D(A)$). $X_\beta \hookrightarrow X_\alpha$ is compact whenever the resolvent operator of $A$ is compact.

Observe by Lemma 2.1(iii) and (iv) that the restriction $T_\alpha(t)$ of $T(t)$ to $X_\alpha$ is exactly the part of $T(t)$ in $X_\alpha$. Moreover, for any $x \in X_\alpha$, we have

$$\left\| T_\alpha(t)x \right\|_\alpha = \left\| A^\alpha T(t)x \right\| = \left\| T(t)A^\alpha x \right\| \le \left\| T(t) \right\| \cdot \left\| A^\alpha x \right\| = \left\| T(t) \right\| \|x\|_\alpha$$

and

$$\left\| T_\alpha(t)x - x \right\|_\alpha = \left\| A^\alpha T(t)x - A^\alpha x \right\| = \left\| T(t)A^\alpha x - A^\alpha x \right\| \to 0, \quad t \to 0.$$

It follows that $T_\alpha(t)$ ($t \ge 0$) is a strongly continuous semigroup on $X_\alpha$ and $\|T_\alpha(t)\|_\alpha \le \|T(t)\|$ for all $t \ge 0$. To prove our main results, we need the following lemmas.

**Lemma 2.2** ([33]) *If $T(t)$ ($t \ge 0$) is a compact semigroup in $X$, then $T_\alpha(t)$ ($t \ge 0$) is a compact semigroup in $X_\alpha$.*

**Lemma 2.3** ([10]) *If $X$ is reflexive, then $X_\alpha$ is also reflexive.*

Now, recall some basic facts on abstract linear evolution equations, which are needed to prove our main results.

Let $J$ denote the infinite interval $[0, \infty)$ and $h : J \to X$, consider the initial value problem of the linear evolution equation

$$\begin{cases} u'(t) + Au(t) = h(t), & t \in J, \\ u(0) = x_0. \end{cases} \tag{2.1}$$

It is well known, when $x_0 \in D(A)$ and $h \in C^1(J, X)$, the initial value problem (2.1) has a unique classical solution $u \in C^1(J, X) \cap C(J, X_1)$ expressed by

$$u(t) = T(t)x_0 + \int_0^t T(t - s)h(s)\, ds. \tag{2.2}$$

Generally, for $x_0$ and $h \in C(J, X)$, the function $u$ given by (2.2) belongs to $C(J, X)$ and it is called a mild solution of the linear evolution equation (2.1). A function $u : J \to X$ is called a strong solution of Eq. (2.1) if it is continuous on $J$, differentiable a.e. on $(0, \infty)$, $u' \in L^1_{\text{loc}}(J, X)$, and satisfies Eq. (2.1). Furthermore, we have the following results.

**Lemma 2.4** ([38]) *Let $h \in C([0, a], X)$ $(a > 0)$, $0 \leq \alpha < \beta \leq 1$, $\mu = \beta - \alpha$, $x_0 \in X_\beta$, then the mild solution $u$ of Eq. (2.1) satisfies $u \in c^\mu([0, a], X_\alpha)$.*

**Lemma 2.5** ([38]) *Let $h \in C^\mu([0, a], X)$ $(a > 0)$, $0 < \mu < 1$, $x_0 \in X$. Then the mild solution $u$ of Eq. (2.1) is a classical solution on $[0, a]$.*

Let $C_\omega(\mathbb{R}, X)$ be the Banach space $\{u \in C(\mathbb{R}, X) | u(t) = u(t + \omega), t \in \mathbb{R}\}$ endowed with the norm $\|u\|_C = \max_{t \in [0, \omega]} \|u(t)\|$, and $C_\omega(\mathbb{R}, X_\alpha)$ be the Banach space $\{u \in C(\mathbb{R}, X_\alpha) | u(t) = u(t + \omega), t \in \mathbb{R}\}$ endowed with the norm $\|u\|_{C\alpha} = \max_{t \in [0, \omega]} \|u(t)\|_\alpha$. Clearly, $C_\omega(\mathbb{R}, X_\alpha) \hookrightarrow C_\omega(\mathbb{R}, X)$.

Given $h \in C_\omega(\mathbb{R}, X)$, we consider the existence of an $\omega$-periodic mild solution of the linear evolution equation

$$u'(t) + Au(t) = h(t), \quad t \in \mathbb{R}. \tag{2.3}$$

**Lemma 2.6** ([28, 29]) *If $-A$ generates an exponentially stable $C_0$-semigroup $T(t)$ $(t \geq 0)$ in $X$, then for $h \in C_\omega(\mathbb{R}, X)$, the linear evolution equation (2.3) has a unique $\omega$-periodic mild solution $u$, which can be expressed by*

$$u(t) = \left(I - T(\omega)\right)^{-1} \int_{t-\omega}^t T(t - s)h(s) \, ds := (Ph)(t), \tag{2.4}$$

*and the solution operator $P : C_\omega(\mathbb{R}, X) \to C_\omega(\mathbb{R}, X)$ is a bounded linear operator.*

*Proof* For any $v \in (0, |v_0|)$, there exists $M > 0$ such that

$$\|T(t)\| \leq Me^{-vt} \leq M, \quad t \geq 0.$$

In $X$, define the equivalent norm $|\cdot|$ by

$$|x| = \sup_{t \geq 0} \|e^{vt} T(t)x\|,$$

then $\|x\| \leq |x| \leq M\|x\|$. By $|T(t)|$ we denote the norm of $T(t)$ in $(X, |\cdot|)$, then for $t \geq 0$, it is easy to obtain that $|T(t)| < e^{-vt}$. Hence, $(I - T(\omega))$ has a bounded inverse operator

$$\left(I - T(\omega)\right)^{-1} = \sum_{n=0}^\infty T(n\omega),$$

and its norm satisfies

$$\left|\left(I - T(\omega)\right)^{-1}\right| \leq \frac{1}{1 - |T(\omega)|} \leq \frac{1}{1 - e^{-v\omega}}. \tag{2.5}$$

Set

$$x_0 = \left(I - T(\omega)\right)^{-1} \int_0^\omega T(t - s)h(s) \, ds := Bh, \tag{2.6}$$

then the mild solution $u(t)$ of the linear initial value problem (2.1) given by (2.2) satisfies the periodic boundary condition $u(0) = u(\omega) = x_0$. For $t \in \mathbb{R}^+$, by (2.2) and the properties of the semigroup $T(t)$ $(t \geq 0)$, we have

$$
\begin{aligned}
u(t + \omega) &= T(t + \omega)u(0) + \int_0^{t+\omega} T(t + \omega - s)h(s)\,ds \\
&= T(t)\left( T(\omega)u(0) + \int_0^{\omega} T(\omega - s)h(s)\,ds \right) + \int_0^t T(t - s)h(s - \omega)\,ds \\
&= T(t)u(0) + \int_0^t T(t - s)h(s)\,ds = u(t).
\end{aligned}
$$

Therefore, the $\omega$-periodic extension of $u$ on $\mathbb{R}$ is a unique $\omega$-periodic mild solution of Eq. (2.3). By (2.2) and (2.6), the $\omega$-periodic mild solution can be expressed by

$$
\begin{aligned}
u(t) &= T(t)B(h) + \int_0^t T(t - s)h(s)\,ds \\
&= \left(I - T(\omega)\right)^{-1} \int_{t-\omega}^t T(t - s)h(s)\,ds := (Ph)(t).
\end{aligned}
$$

Evidently, $P : C_\omega(\mathbb{R}, X) \to C_\omega(\mathbb{R}, X)$ is a bounded linear operator. In fact, for every $h \in C_\omega(\mathbb{R}, X)$,

$$
\begin{aligned}
\left\| Qh(t) \right\| &= \left\| \left(I - T(\omega)\right)^{-1} \int_{t-\omega}^t T(t - s)h(s)\,ds \right\| \\
&\leq \left\| \left(I - T(\omega)\right)^{-1} \right\| \cdot \int_{t-\omega}^t \left\| T(t - s) \right\| ds \|h\|_C \\
&\leq CM\omega \|h\|_C,
\end{aligned}
$$

where $C := \|(I - T(\omega))^{-1}\|$, which implies that $P$ is bounded. This completes the proof of Lemma 2.6.  □

To prove our main results, we also need the following lemma.

**Lemma 2.7** ([39]) *Assume that $Q$ is a condensing operator on a Banach space $X$, i.e., $Q$ is continuous and takes bounded sets into bounded sets, and $\alpha(Q(D)) < \alpha(D)$ for every bounded set $D$ of $X$ with $\alpha(D) > 0$. If $Q(\overline{\Omega}) \subset \overline{\Omega}$ for a convex, closed, and bounded set $\overline{\Omega}$ of $X$, then $Q$ has a fixed point in $\overline{\Omega}$ (where $\alpha(\cdot)$ denotes the Kuratowski measure of non-compactness).*

*Remark* 2.8  It is easy to see that, if $Q = Q_1 + Q_2$ with $Q_1$ a completely continuous operator and $Q_2$ a contractive map, then $Q$ is a condensing operator on $X$.

## 3  Existence of mild solution

Now, we are in a position to state and prove our main results of this section.

**Theorem 3.1** *For $\alpha \in [0, 1)$, we assume that $G : \mathbb{R} \times X_\alpha \to X_1$ and $F : \mathbb{R} \times X_\alpha^2 \to X$ are continuous functions, and for every $x, x_0, x_1 \in X_\alpha$, $G(t, x)$, $F(t, x_0, x_1)$ are $\omega$-periodic in $t$. If the following conditions hold*:

(H1) *For any $r > 0$, there is a positive value function $h_r : \mathbb{R} \to \mathbb{R}^+$ such that*

$$\sup_{\|x_0\|_\alpha, \|x_1\|_\alpha < r} \left\| F(t, x_0, x_1) \right\| \leq h_r(t), \quad t \in \mathbb{R},$$

*the function $s \mapsto \frac{h_r(s)}{(t-s)^\alpha}$ belongs to $L_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^+)$ for all $t \in \mathbb{R}$ and there is a positive constant $\gamma > 0$ such that*

$$\liminf_{r \to \infty} \frac{1}{r} \int_{t-\omega}^{t} \frac{h_r(s)}{(t-s)^\alpha} \, ds = \gamma < \infty;$$

(H2) *$G(t, \theta) = \theta$ for $t \in \mathbb{R}$, and there is a constant $L \geq 0$ such that*

$$\left\| AG(t, x) - AG(t, y) \right\| \leq L \|x - y\|_\alpha, \quad t \in \mathbb{R}, x, y \in X_\alpha;$$

(H3) *$CM_\alpha \gamma + C_{1-\alpha} L + CM_\alpha L \frac{\omega^{1-\alpha}}{1-\alpha} < 1$, where $C = \|(I - T(\omega))^{-1}\|$, then Eq. (1.1) has at least one $\omega$-periodic mild solution $u$.*

*Proof* From the assumption, we know that $G(t, u(t - \xi)) \in D(A)$ for every $u \in C_\omega(\mathbb{R}, X_\alpha)$, thus, we can rewrite Eq. (1.1) as follows:

$$\begin{aligned}
\big(u(t) &- G\big(t, u(t - \xi)\big)\big)' + A\big(u(t) - G\big(t, u(t - \xi)\big)\big) \\
&= F\big(t, u(t), u(t - \tau)\big) - AG\big(t, u(t - \xi)\big), \quad t \in \mathbb{R}.
\end{aligned} \tag{3.1}$$

For any $r > 0$, let

$$\overline{\Omega}_r = \big\{ u \in C_\omega(\mathbb{R}, X_\alpha) \,|\, \|u\|_{C\alpha} \leq r \big\}. \tag{3.2}$$

Note that $\overline{\Omega}_r$ is a closed ball in $C_\omega(\mathbb{R}, X_\alpha)$ with center $\theta$ and radius $r$. Moreover, by condition (H2), it follows that

$$\begin{aligned}
\big\| T(t - s)AG\big(s, u(s - \xi)\big) \big\|_\alpha &= \big\| A^\alpha T(t - s)A\big(G\big(s, u(s - \xi)\big) - G(s, \theta)\big) \big\| \\
&\leq \big\| A^\alpha T(t - s) \big\| \cdot \big\| A\big(G\big(s, u(s - \xi)\big) - G(s, \theta)\big) \big\| \\
&\leq \frac{M_\alpha L}{(t - s)^\alpha} \big\| u(s - \xi) \big\|_\alpha,
\end{aligned}$$

which implies that $s \to T(t - s)AG(s, u(s - \xi))$ is integrable on $[t - \omega, t]$ for each $u \in \overline{\Omega}_r$.

Hence, we can define the operator $Q$ on $C_\omega(\mathbb{R}, X_\alpha)$ by

$$\begin{aligned}
Qu(t) := \big(I - T(\omega)\big)^{-1} &\int_{t-\omega}^{t} T(t - s)F\big(s, u(s), u(s - \tau)\big) \, ds + G\big(t, u(t - \xi)\big) \\
&- \big(I - T(\omega)\big)^{-1} \int_{t-\omega}^{t} T(t - s)AG\big(s, u(s - \xi)\big) \, ds, \quad t \in \mathbb{R}.
\end{aligned} \tag{3.3}$$

From Lemma 2.6, it is sufficient to prove that $Q$ has a fixed point.

Now, we show that there is a positive constant $r$ such that $Q(\overline{\Omega}_r) \subset \overline{\Omega}_r$. If this were not the case, then for any $r > 0$, there exist $u_r \in \overline{\Omega}_r$ and $t_r \in \mathbb{R}$ such that $\|Qu_r(t_r)\|_\alpha > r$. Thus, we see by (H1), (H2), and (H3) that

$$
\begin{aligned}
r &< \left\|Qu_r(t_r)\right\|_\alpha \\
&\leq \left\|(I - T(\omega))^{-1} \int_{t_r-\omega}^{t_r} T(t_r - s)F(s, u_r(s), u_r(s - \tau))\,ds\right\|_\alpha + \left\|G(t_r, u_r(t_r - \xi))\right\|_\alpha \\
&\quad + \left\|(I - T(\omega))^{-1} \int_{t_r-\omega}^{t_r} T(t_r - s)AG(s, u_r(s - \xi))\,ds\right\|_\alpha \\
&\leq \left\|(I - T(\omega))^{-1}\right\| \cdot \int_{t_r-\omega}^{t_r} \left\|A^\alpha T(t_r - s)\right\| \cdot \left\|F(s, u_r(s), u_r(s - \tau))\right\| ds \\
&\quad + \left\|A^{\alpha-1}(AG(t_r, u_r(t_r - \xi)) - AG(t_r, \theta))\right\| \\
&\quad + \left\|(I - T(\omega))^{-1}\right\| \cdot \int_{t_r-\omega}^{t_r} \left\|A^\alpha T(t_r - s)\right\| \cdot \left\|AG(s, u_r(s - \xi)) - AG(s, \theta)\right\| ds \\
&\leq CM_\alpha \int_{t_r-\omega}^{t_r} \frac{h_r(s)}{(t_r - s)^\alpha}\,ds + C_{1-\alpha}L\|u_r\|_{C\alpha} + CM_\alpha L \int_{t_r-\omega}^{t_r} \frac{1}{(t_r - s)^\alpha}\,ds\|u_r\|_{C\alpha} \\
&\leq CM_\alpha \int_{t_r-\omega}^{t_r} \frac{h_r(s)}{(t_r - s)^\alpha}\,ds + C_{1-\alpha}Lr + CM_\alpha L \frac{\omega^{1-\alpha}}{1-\alpha}r.
\end{aligned}
$$

Dividing on both sides by $r$ and taking the lower limit as $r \to \infty$, we have

$$
CM_\alpha L\gamma + C_{1-\alpha}L + CM_\alpha L \frac{\omega^{1-\alpha}}{1-\alpha} \geq 1, \tag{3.4}
$$

which contradicts (H3). Hence, there is a positive constant $r$ such that $Q(\overline{\Omega}_r) \subset \overline{\Omega}_r$.

In order to show that the operator $Q$ has a fixed point on $\overline{\Omega}_r$, we introduce the decomposition $Q = Q_1 + Q_2$, where

$$
Q_1 u(t) := (I - T(\omega))^{-1} \int_{t-\omega}^{t} T(t - s)F(s, u(s), u(s - \tau))\,ds, \tag{3.5}
$$

$$
Q_2 u(t) := G(t, u(t - \xi)) - (I - T(\omega))^{-1} \int_{t-\omega}^{t} T(t - s)AG(s, u(s - \xi))\,ds. \tag{3.6}
$$

Then we will prove that $Q_1$ is a compact operator and $Q_2$ is a contraction.

Firstly, we prove that $Q_1$ is a compact operator. Let $\{u_n\} \subset \overline{\Omega}_r$ with $u_n \to u$ in $\overline{\Omega}_r$, then by the continuity of $F$, we have

$$
F(t, u_n(t), u_n(t - \tau)) \to F(t, u(t), u(t - \tau)), \quad n \to \infty,
$$

for each $t \in \mathbb{R}$. Since $\|F(t, u_n(t), u_n(t - \tau)) - F(t, u(t), u(t - \tau))\| \leq 2h_r(t)$ for all $t \in \mathbb{R}$, then the dominated convergence theorem ensures that

$$
\begin{aligned}
&\left\|Q_1 u_n(t) - Q_1 u(t)\right\|_\alpha \\
&= \left\|(I - T(\omega))^{-1} \int_{t-\omega}^{t} T(t - s)F(s, u_n(s), u_n(s - \tau)) - F(s, u(s), u(s - \tau))\,ds\right\|_\alpha
\end{aligned}
$$

$$\leq \left\| (I - T(\omega))^{-1} \right\| \int_{t-\omega}^{t} \left\| A^{\alpha} T(t - s) \right\| \cdot \left\| F(s, u_n(s), u_n(s - \tau)) - F(s, u(s), u(s - \tau)) \right\| ds$$

$$\leq CM_{\alpha} \int_{t-\omega}^{t} \frac{\| F(s, u_n(s), u_n(s - \tau)) - F(s, u(s), u(s - \tau)) \|}{(t - s)^{\alpha}} ds$$

$$\to 0 \quad \text{as } n \to \infty,$$

which implies that $\| Q_1 u_n - Q_1 u \|_{C\alpha} \to 0$ as $n \to \infty$, i.e., $Q_1$ is continuous.

It is easy to see that $Q_1$ maps $\overline{\Omega}_r$ into a bounded set in $C_{\omega}(\mathbb{R}, X_{\alpha})$. Now, we demonstrate that $Q(\overline{\Omega}_r)$ is equicontinuous. For every $u \in \overline{\Omega}_r$, by the periodicity of $u$, we only consider it on $[0, \omega]$. Set $0 \leq t_1 < t_2 \leq \omega$, we get that

$$Q_1 u(t_2) - Q_1 u(t_1)$$

$$= (I - T(\omega))^{-1} \int_{t_2-\omega}^{t_2} T(t_2 - s) F(s, u(s), u(s - \tau)) ds$$

$$- (I - T(\omega))^{-1} \int_{t_1-\omega}^{t_1} T(t_1 - s) F(s, u(s), u(s - \tau)) ds$$

$$= (I - T(\omega))^{-1} \int_{t_2-\omega}^{t_1} (T(t_2 - s) - T(t_1 - s)) F(s, u(s), u(s - \tau)) ds$$

$$- (I - T(\omega))^{-1} \int_{t_1-\omega}^{t_2-\omega} T(t_1 - s) F(s, u(s), u(s - \tau)) ds$$

$$+ (I - T(\omega))^{-1} \int_{t_1}^{t_2} T(t_2 - s) F(s, u(s), u(s - \tau)) ds$$

$$:= I_1 + I_2 + I_3.$$

It is clear that

$$\left\| Q_1 u(t_2) - Q_1 u(t_1) \right\|_{\alpha} \leq \| I_1 \|_{\alpha} + \| I_2 \|_{\alpha} + \| I_3 \|_{\alpha}. \tag{3.7}$$

Thus, we only need to check that $\| I_i \|_{\alpha}$ tends to 0 independently of $u \in \overline{\Omega}_r$ when $t_2 - t_1 \to 0$, $i = 1, 2, 3$. From the continuity of $t \mapsto \| T(t) \|$ for $t > 0$ and condition (H1), we can easily see

$$\| I_1 \|_{\alpha} \leq C \cdot \int_{t_2-\omega}^{t_1} \left\| A^{\alpha} (T(t_2 - s) - T(t_1 - s)) \right\| \cdot \left\| F(s, u(s), u(s - \tau)) \right\| ds$$

$$\leq C \cdot \int_{t_2-\omega}^{t_1} \left\| T\left( \frac{t_2 - s}{2} + \frac{t_2 - t_1}{2} \right) - T\left( \frac{t_1 - s}{2} \right) \right\| \cdot \left\| A^{\alpha} T\left( \frac{t_1 - s}{2} \right) \right\| \cdot h_r(s) ds$$

$$\leq CM_{\alpha} \int_{t_2-\omega}^{t_1} \left\| T\left( \frac{t_2 - s}{2} + \frac{t_2 - t_1}{2} \right) - T\left( \frac{t_1 - s}{2} \right) \right\| \cdot \frac{h_r(s)}{(\frac{t_1-s}{2})^{\alpha}} ds$$

$$\to 0, \quad \text{as } t_2 - t_1 \to 0,$$

$$\| I_2 \|_{\alpha} \leq C \cdot \int_{t_1-\omega}^{t_2-\omega} \left\| AT(t_1 - s) \right\| \cdot \left\| F(s, u(s), u(s - \tau)) \right\| ds$$

$$\leq CM_\alpha \cdot \int_{t_1-\omega}^{t_2-\omega} \frac{h_r(s)}{(t_1-s)^\alpha}\, ds$$

$$\to 0, \quad \text{as } t_2 - t_1 \to 0,$$

$$\|I_3\|_\alpha \leq C \cdot \int_{t_1}^{t_2} \big\|A\big(T(t_2-s)\big)\big\| \cdot \big\|F\big(s,u(s),u(s-\tau)\big)\big\|\, ds$$

$$\leq CM_\alpha \cdot \int_{t_1}^{t_2} \frac{h_r(s)}{(t_2-s)^\alpha}\, ds$$

$$\to 0, \quad \text{as } t_2 - t_1 \to 0.$$

As a result, $\|Q_1 u(t_2) - Q_1 u(t_1)\|_\alpha$ tends to 0 independently of $u \in \overline{\Omega}_r$ as $t_2 - t_1 \to 0$, which means that $Q_1(\overline{\Omega}_r)$ is equicontinuous.

It remains to show that $(Q_1\overline{\Omega}_r)(t)$ is relatively compact in $X_\alpha$ for all $t \in \mathbb{R}$. To do this, we define a set $(Q_\varepsilon \overline{\Omega}_r)(t)$ by

$$(Q_\varepsilon \overline{\Omega}_r)(t) := \big\{(Q_\varepsilon u)(t) | u \in \overline{\Omega}_r, 0 < \varepsilon < \omega, t \in \mathbb{R}\big\}, \tag{3.8}$$

where

$$(Q_\varepsilon u)(t) = \big(I - T(\omega)\big)^{-1} \int_{t-\omega}^{t-\varepsilon} T_\alpha(t-s)F\big(s,u(s),u(s-\tau)\big)\, ds$$

$$= T_\alpha(\varepsilon)\big(I - T(\omega)\big)^{-1} \int_{t-\omega}^{t-\varepsilon} T_\alpha(t-s-\varepsilon)F\big(s,u(s),u(s-\tau)\big)\, ds.$$

From Lemma 2.2, the operator $T_\alpha(\varepsilon)$ is compact in $X_\alpha$, it follows that the set $(Q_\varepsilon \overline{\Omega}_r)(t)$ is relatively compact in $X_\alpha$. For any $u \in \overline{\Omega}_r$ and $t \in \mathbb{R}$, from the following inequality

$$\big\|Q_1 u(t) - Q_\varepsilon u(t)\big\|_\alpha$$

$$\leq C \int_{t-\varepsilon}^{t} \big\|T_\alpha(t-s)F\big(s,u(s),u(s-\tau)\big)\big\|_\alpha\, ds$$

$$\leq C \int_{t-\varepsilon}^{t} \big\|A^\alpha T(t-s)F\big(s,u(s),u(s-\tau)\big)\big\|\, ds$$

$$\leq CM_\alpha \int_{t-\varepsilon}^{t} \frac{h_r(s)}{(t-s)^\alpha}\, ds,$$

one can obtain that the set $(Q_1\overline{\Omega}_r)(t)$ is relatively compact in $X_\alpha$ for all $t \in \mathbb{R}$.

Thus, the Arzela–Ascoli theorem guarantees that $Q_1$ is a compact operator.

Secondly, we prove that $Q_2$ is a contraction. Let $u,v \in \overline{\Omega}_r$, by condition (H2), Lemma 2.1(vi), and Lemma 2.6, we have

$$\big\|Q_2 u(t) - Q_2 v(t)\big\|_\alpha$$

$$= \bigg\|G\big(t,u(t-\xi)\big) - \big(I - T(\omega)\big)^{-1} \int_{t}^{t-\omega} T(t-s)AG\big(s,u(s-\xi)\big)\, ds$$

$$- G\big(t,v(t-\xi)\big) + \big(I - T(\omega)\big)^{-1} \int_{t}^{t-\omega} T(t-s)AG\big(s,v(s-\xi)\big)\bigg\|_\alpha$$

$$\leq \big\|G\big(t,u(t-\xi)\big) - G\big(t,v(t-\xi)\big)\big\|_\alpha$$

$$+ \left\| \left(I - T(\omega)\right)^{-1} \int_t^{t-\omega} T(t-s)A\big(G\big(s, u(s-\xi)\big) - G\big(s, v(s-\xi)\big)\big)\, ds \right\|_\alpha$$

$$\leq \left\| A^{\alpha-1}\big(AG\big(t, u(t-\xi)\big) - AG\big(t, v(t-\xi)\big)\big) \right\|$$

$$+ \left\| \left(I - T(\omega)\right)^{-1} \right\| \cdot \int_t^{t-\omega} \left\| AT(t-s) \right\| \left\| A\big(G\big(s, u(s-\xi)\big) - G\big(s, v(s-\xi)\big)\big) \right\| ds$$

$$\leq C_{1-\alpha} \left\| AG(t, u(t-\xi)) - AG(t, v(t-\delta)) \right\|$$

$$+ CM \cdot \int_t^{t-\omega} \left\| AG\big(s, u(s-\xi)\big) - AG\big(s, v(s-\xi)\big) \right\| ds$$

$$\leq C_{1-\alpha} L \left\| u(t-\xi) - v(t-\xi) \right\|_\alpha + CM_\alpha L \int_t^{t-\omega} \frac{1}{(t-s)^\alpha} \left\| u(s-\xi) - v(s-\xi) \right\|_\alpha ds$$

$$\leq \left( C_{1-\alpha} L + CM_\alpha L \frac{\omega^{1-\alpha}}{1-\alpha} \right) \| u - v \|_{C\alpha};$$

therefore,

$$\| Q_2 u - Q_2 v \|_C \leq \left( C_{1-\alpha} L + CM_\alpha L \frac{\omega^{1-\alpha}}{1-\alpha} \right) \| u - v \|_{C\alpha}. \tag{3.9}$$

Since $CM_\alpha L\gamma + C_{1-\alpha} L + CM_\alpha L\frac{\omega^{1-\alpha}}{1-\alpha} < 1$, so $C_{1-\alpha} L + CM_\alpha L\frac{\omega^{1-\alpha}}{1-\alpha} < 1$, it follows that $Q_2$ is a contraction.

By Lemma 2.7, we know that $Q$ has a fixed point $u \in \overline{\Omega}_r$, that is, Eq. (1.1) has a $\omega$-periodic mild solution. The proof is completed.  □

In condition (H1), if the function $h_r$ is independent of $t$, we can easily obtain a constant $\gamma \geq 0$ satisfying (H3). For example, we replace condition (H1) with the following: (H1′) There are positive constants $a_0$, $a_1$, and $K$ such that

$$\left\| F(t, x_0, x_1) \right\| \leq a_0 \| x_0 \|_\alpha + a_1 \| x_1 \|_\alpha + K$$

for $t \in \mathbb{R}$ and $x_0, x_1 \in X_\alpha$.

In this case, for any $r > 0$ and $x_0, x_1 \in X_\alpha$ with $\| x_0 \|_\alpha, \| x_1 \|_\alpha \leq r$, we have

$$\left\| F(t, x_0, x_1) \right\| \leq r(a_0 + a_1) + K := h_r(t), \quad t \in \mathbb{R},$$

thus,

$$\liminf_{r \to \infty} \frac{1}{r} \int_{t-\omega}^t \frac{h_r(s)}{(t-s)^\alpha}\, ds = (a_0 + a_1) \frac{\omega^{1-\alpha}}{1-\alpha} := \gamma > 0.$$

Therefore, we have the following result.

**Corollary 3.2** *For $\alpha \in [0, 1)$, we assume that $G : \mathbb{R} \times X_\alpha \to X_1$ and $F : \mathbb{R} \times X_\alpha^2 \to X$ are continuous functions, and for every $x, x_0, x_1 \in X_\alpha$, $G(t, x)$, $F(t, x_0, x_1)$ are $\omega$-periodic in $t$. If conditions* (H1′), (H2), *and*

(H3′) $CM_\alpha(a_0 + a_1 + L)\frac{\omega^{1-\alpha}}{1-\alpha} + C_{1-\alpha} L < 1$, *where $C = \| (I - T(\omega))^{-1} \|$,*

*hold, then Eq.* (1.1) *has at least one $\omega$-periodic mild solution $u$.*

Furthermore, we assume that $F$ satisfies the Lipschitz condition, namely

(H1″)  *There are positive constants $a_0$, $a_1$ such that*

$$\left\| F(t,x_0,x_1) - F(t,y_0,y_1) \right\| \le a_0 \|x_0 - y_0\|_\alpha + a_1 \|x_1 - y_1\|_\alpha, \quad t \in \mathbb{R}, x_0, x_1, y_0, y_1 \in X_\alpha,$$

then we can obtain the following result.

**Theorem 3.3** *For $\alpha \in [0,1)$, we assume that $G : \mathbb{R} \times X_\alpha \to X_1$ and $F : \mathbb{R} \times X_\alpha^2 \to X$ are continuous functions, and for every $x, x_0, x_1 \in X_\alpha$, $G(t,x)$, $F(t,x_0,x_1)$ are $\omega$-periodic in $t$. If conditions (H1″), (H2), and (H3′) hold, then Eq. (1.1) has a unique $\omega$-periodic mild solution $u$.*

*Proof* From (H1″) we easily see that (H1′) holds. In fact, for any $t \in \mathbb{R}$ and $x_0, x_1 \in X_\alpha$, by condition (H1″),

$$\begin{aligned}
\left\| F(t,x_0,x_1) \right\| &\le \left\| F(t,x_0,x_1) - F(t,\theta,\theta) \right\| + \left\| F(t,\theta,\theta) \right\| \\
&\le a_0 \|x_0\|_\alpha + a_1 \|x_1\|_\alpha + \left\| F(t,\theta,\theta) \right\|.
\end{aligned}$$

From the continuity and periodicity of $F$, we can choose $K = \max_{t \in [0,\omega]} \|F(t,\theta,\theta)\|$, thus, condition (H1′) holds. Hence, by Corollary 3.2, Eq. (1.1) has $\omega$-periodic mild solutions. Let $u_1, u_2 \in C_\omega(\mathbb{R}, X_\alpha)$ be the $\omega$-periodic mild solutions of Eq. (1.1), then they are the fixed points of the operator $Q$ which is defined by (3.3). Hence,

$$\begin{aligned}
&\left\| Qu_2(t) - Qu_1(t) \right\|_\alpha \\
&\quad \le \left\| (I - T(\omega))^{-1} \int_{t-\omega}^{t} T(t-s)\big(F\big(s,u_2(s),u_2(s-\tau)\big) - F\big(s,u_1(s),u_1(s-\tau)\big)\big)\, ds \right\|_\alpha \\
&\qquad + \left\| G\big(t,u_2(t-\xi)\big) - G\big(t,u_1(t-\xi)\big) \right\|_\alpha \\
&\qquad + \left\| (I - T(\omega))^{-1} \int_{t-\omega}^{t} T(t-s)\big(AG\big(s,u_2(s-\xi)\big) - AG\big(s,u_1(s-\xi)\big)\big)\, ds \right\|_\alpha \\
&\quad \le C \cdot \int_{t-\omega}^{t} \left\| A^\alpha T(t-s) \right\| \cdot \left\| F\big(s,u_2(s),u_2(s-\tau)\big) - F\big(s,u_1(s),u_1(s-\tau)\big) \right\| ds \\
&\qquad + \left\| A^{\alpha-1}\big(AG\big(t,u_2(t-\xi)\big) - AG\big(t,u_1(t-\xi)\big)\big) \right\| \\
&\qquad + C \cdot \int_{t-\omega}^{t} \left\| A^\alpha T(t-s) \right\| \cdot \left\| AG\big(s,u_2(s-\xi)\big) - AG\big(s,u_1(s-\xi)\big) \right\| ds \\
&\quad \le CM_\alpha \cdot \int_{t-\omega}^{t} \frac{1}{(t-s)^\alpha} \big(a_0 \left\| u_2(s) - u_1(s) \right\|_\alpha + a_1 \left\| u_2(s-\tau) - u_1(s-\tau) \right\|_\alpha \big)\, ds \\
&\qquad + C_{1-\alpha} L \left\| u_2(t-\xi) - u_1(t-\xi) \right\|_\alpha \\
&\qquad + CM_\alpha \cdot \int_{t-\omega}^{t} \frac{L}{(t-s)^\alpha} \left\| u_2(t-\xi) - u_1(t-\xi) \right\|_\alpha ds \\
&\quad \le CM_\alpha \frac{\omega^{1-\alpha}}{1-\alpha} (a_0 + a_1) \|u_2 - u_1\|_{C\alpha} + C_{1-\alpha} L \|u_2 - u_1\|_{C\alpha} + CM_\alpha \frac{\omega^{1-\alpha}}{1-\alpha} L \|u_2 - u_1\|_{C\alpha} \\
&\quad = \left( CM_\alpha (a_0 + a_1 + L) \frac{\omega^{1-\alpha}}{1-\alpha} + C_{1-\alpha} L \right) \cdot \|u_2 - u_1\|_{C_\alpha},
\end{aligned}$$

which implies that $\|u_2 - u_1\|_{C\alpha} = \|Qu_2 - Qu_1\|_{C\alpha} \leq (CM\omega(a_0 + a_1 + L) + C_{1-\alpha}L) \cdot \|u_2 - u_1\|_{C\alpha}$. From this and condition (H3′), it follows that $u_2 = u_1$. Thus, Eq. (1.1) has only one $\omega$-periodic mild solution.                                                                    □

## 4  Regularity of mild solutions

In this section, we discuss the regularity properties of the $\omega$-periodic mild solution of Eq. (1.1) and present essential conditions on the nonlinearity $F$ and $G$ to guarantee that Eq. (1.1) has $\omega$-periodic classical and strong solutions.

Now, we are in a position to state and prove the main result of this section.

**Theorem 4.1** *For $\alpha \in [0, 1)$, we assume that $G : \mathbb{R} \times X_\alpha \to X_1$ and $F : \mathbb{R} \times X_\alpha^2 \to X$ are continuous functions, and for every $x, x_0, x_1 \in X_\alpha$, $G(t, x)$, $F(t, x_0, x_1)$ are $\omega$-periodic in $t$. If the following conditions hold*:

(H4)  *There exist $L_1$ and $\mu_1 \in (0, 1)$ such that*

$$\left\| F(t_2, x_0, , x_1) - F(t_1, y_0, y_1) \right\| \leq L_1 \left( |t_2 - t_1|^{\mu_1} + \|x_0 - y_0\|_\alpha + \|x_1 - y_1\|_\alpha \right)$$

*for each $t_1, t_2 \in \mathbb{R}$ and $x_0, x_1, y_0, y_1 \in X_\alpha$;*

(H5)  *$G(t, \theta) = \theta$ for $t \in \mathbb{R}$, there exist $L_2$ and $\mu_2 \in (0, 1)$ such that*

$$\left\| AG(t_2, x) - AG(t_1, y) \right\| \leq L_2 \left( |t_2 - t_1|^{\mu_2} + \|x - y\|_\alpha \right)$$

*for each $t_1, t_2 \in \mathbb{R}$ and $x, y \in X_\alpha$;*

(H6)  *$CM_\alpha(2L_1 + L_2)\frac{\omega^{1-\alpha}}{1-\alpha} + C_{1-\alpha}L_2 < 1$, where $C = \|(I - T(\omega))^{-1}\|$, then Eq. (1.1) has an $\omega$-periodic classical solution.*

*Proof* Let $Q$ be the operator defined by (3.3) in the proof of Theorem 3.1. By the assumptions of Theorem 4.1 and the proof of Theorem 3.1, we know that the operator $Q : C_\omega(\mathbb{R}, X_\alpha) \to C_\omega(\mathbb{R}, X_\alpha)$ is well defined. From conditions (H4) and (H5), for any $u_1, u_2 \in C_\omega(\mathbb{R}, X_\alpha)$, $t \in \mathbb{R}$, similar to the proof of Theorem 3.3, we have

$$\left\| Qu_2(t) - Qu_1(t) \right\|_\alpha \leq \left( CM_\alpha(2L_1 + L_2)\frac{\omega^{1-\alpha}}{1-\alpha} + C_{1-\alpha}L_2 \right) \cdot \|u_2 - u_1\|_{C\alpha},$$

which implies that

$$\|Qu_2 - Qu_1\|_{C\alpha} \leq \left( CM_\alpha(2L_1 + L_2)\frac{\omega^{1-\alpha}}{1-\alpha} + C_{1-\alpha}L_2 \right) \cdot \|u_2 - u_1\|_{C\alpha} < \|u_2 - u_1\|_{C\alpha}. \quad (4.1)$$

Hence, $Q : C_\omega(\mathbb{R}, X_\alpha) \to C_\omega(\mathbb{R}, X_\alpha)$ is a contraction, thus $Q$ has a unique fixed point $u_0 \in C_\omega(\mathbb{R}, X_\alpha)$. By the definition of $Q$, $u_0$ is an $\omega$-periodic mild solution of Eq. (1.1).

Next, we prove that $u_0$ is an $\omega$-periodic classical solution. From the periodicity of $u_0$, we only need prove it on $[0, \omega]$. Let $t \in [0, \omega]$ and $h(t) = F(t, u(t), u(t - \tau)) - AG(t, u(t - \delta))$, then $h \in C([0, \omega], X)$. For $\forall \varepsilon \in (0, \omega)$, since $u_0$ is the $\omega$-periodic mild solution of Eq. (1.1), hence $u_0$ is the mild solution of the initial value problem

$$\begin{cases} (u(t) - G(t, u(t - \delta)))' + A(u(t) - G(t, u(t - \delta))) = h(t), & t \in [\varepsilon, \omega], \\ u(\varepsilon) = u_0(\varepsilon). \end{cases} \quad (4.2)$$

While $u_0(\varepsilon) \in X_\alpha$, from Lemma 2.4 it follows that

$$u_0 \in C^{\mu_3}\big([\varepsilon,\omega], X_{\alpha-\mu_3}\big) \hookrightarrow C^{\mu_3}\big([\varepsilon,\omega], X\big), \quad \mu_3 \in (0,\alpha).$$

On the other hand, from conditions (H4) and (H5), we can deduce $h \in C^\mu([\varepsilon,\omega], X)$, where $\mu = \min\{\mu_1, \mu_2, \mu_3\}$. By Lemma 2.5, we obtain that $u_0$ is a classical solution of Eq. (4.2) and satisfies

$$u_0 \in C^1((\varepsilon,\omega], X) \cap C\big([\varepsilon,\omega], X_1\big).$$

By the arbitrariness of $\varepsilon$, we claim that

$$u_0 \in C^1\big([0,\omega], X\big) \cap C\big([0,\omega], X_1\big).$$

Therefore, $u_0$ is an $\omega$-periodic classical solution of Eq. (1.1) and satisfies

$$u_0 \in C^1_\omega(\mathbb{R}, X) \cap C_\omega(\mathbb{R}, X_1).$$

The proof is completed. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

**Theorem 4.2** *Assume that $X$ is a reflexive Banach space. For $\alpha \in [0,1)$, we assume that $G : \mathbb{R} \times X_\alpha \to X_1$ and $F : \mathbb{R} \times X_\alpha^2 \to X$ are continuous functions, and for every $x, x_0, x_1 \in X_\alpha$, $G(t,x), F(t,x_0,x_1)$ are $\omega$-periodic in $t$. If the conditions*
   (H4$'$)  *There exists a constant $L_1 > 0$ such that*

$$\big\| F(t_2, x_0, x_1) - F(t_1, y_0, y_1) \big\| \leq L_1\big(|t_2 - t_1| + \|x_0 - y_0\|_\alpha + \|x_1 - y_1\|_\alpha\big)$$

   *for any $t_1, t_2 \in \mathbb{R}$ and $x_0, x_1, y_0, y_1 \in X_\alpha$,*
   (H5$'$)  *$G(t,\theta) = \theta$ for $t \in \mathbb{R}$, there exists $L_2$ such that*

$$\big\| AG(t_2, x) - AG(t_1, y) \big\| \leq L_2\big(|t_2 - t_1| + \|x - y\|_\alpha\big)$$

   *for each $t_1, t_2 \in \mathbb{R}$ and $x, y \in X_\alpha$, and* (H6) *hold, then Eq.* (1.1) *has an $\omega$-periodic strong solution $u$.*

*Proof* Let $Q$ be the operator defined by (3.3) in the proof of Theorem 3.1. For a given $r > 0$, let $\overline{\Omega}_r \subset C_\omega(\mathbb{R}, X_\alpha)$ be defined by (3.2). By conditions (H4$'$)–(H6$'$), one can use the same argument as in the proof of Theorem 3.1 to obtain that $(Q\overline{\Omega}_r) \subset \overline{\Omega}_r$.

For this $r$, consider the set

$$\overline{\Omega} = \Big\{ u \in C_\omega(\mathbb{R}, X_\alpha) \mid \|u\|_{C\alpha} \leq r, \big\| u(t_1) - u(t_2) \big\|_\alpha < L^*|t_2 - t_1|, t_1, t_2 \in \mathbb{R} \Big\} \qquad (4.3)$$

for some $L^*$ large enough. It is clear that $\overline{\Omega}$ is a convex closed and nonempty set. We shall prove that $Q$ has a fixed point on $\overline{\Omega}$. Obviously, from the proof of Theorem 3.1, it is sufficient to show that, for any $u \in \overline{\Omega}$,

$$\big\| (Qu)(t_2) - (Qu)(t_1) \big\|_\alpha \leq L^*|t_2 - t_1|, \quad \forall t_1, t_2 \in \mathbb{R}. \qquad (4.4)$$

In fact, by the definition of $Q$, conditions (H4′), (H5′), and (4.3), we have

$$
\begin{aligned}
&\left\| Qu(t_2) - Qu(t_1) \right\|_\alpha \\
&\quad \leq \left\| \left(I - T(\omega)\right)^{-1} \int_{t_2-\omega}^{t_2} T(t_2 - s) F\left(s, u(s), u(s - \tau)\right) ds \right. \\
&\qquad \left. - \left(I - T(\omega)\right)^{-1} \int_{t_1-\omega}^{t_1} T(t_1 - s) F\left(s, u(s), u(s - \tau)\right)) ds \right\|_\alpha \\
&\qquad + \left\| G\left(t_2, u(t_2 - \xi)\right) - G\left(t_1, u(t_1 - \xi)\right) \right\|_\alpha \\
&\qquad + \left\| \left(I - T(\omega)\right)^{-1} \int_{t_2-\omega}^{t_2} T(t_2 - s) AG\left(s, u(s - \xi)\right) ds \right. \\
&\qquad \left. - \left(I - T(\omega)\right)^{-1} \int_{t_1-\omega}^{t_1} T(t_1 - s) AG\left(s, u(s - \xi)\right) ds \right\|_\alpha \\
&\quad \leq \left\| \left(I - T(\omega)\right)^{-1} \int_{0}^{\omega} A^\alpha T(s) \left( F\left(t_2 - s, u(t_2 - s), u(t_2 - s - \tau)\right) \right.\right. \\
&\qquad \left.\left. - F\left(t_1 - s, u(t_1 - s), u(t_1 - s - \tau)\right) \right) ds \right\| \\
&\qquad + C_{1-\alpha} \left\| AG\left(t_2, u(t_2 - \xi)\right) - AG\left(t_1, u(t_1 - \xi)\right) \right\| \\
&\qquad + \left\| \left(I - T(\omega)\right)^{-1} \int_{0}^{\omega} A^\alpha T(s) \left( AG\left(t_2 - s, u(t_2 - s - \xi)\right) \right.\right. \\
&\qquad \left.\left. - AG\left(t_1 - s, u(t_2 - s - \xi)\right) \right) ds \right\| \\
&\quad \leq CM_\alpha \frac{\omega^{1-\alpha}}{1-\alpha} L_1 \left(1 + 2L^*\right) |t_2 - t_1| + C_{1-\alpha} L_2 \left(1 + L^*\right) |t_2 - t_1| \\
&\qquad + CM_\alpha \frac{\omega^{1-\alpha}}{1-\alpha} L_2 \left(1 + L^*\right) |t_2 - t_1| \\
&\quad = \left( CM_\alpha \frac{\omega^{1-\alpha}}{1-\alpha} (L_1 + L_2) + C_{1-\alpha} L_2 + \left( CM_\alpha (2L_1 + L_2) \frac{\omega^{1-\alpha}}{1-\alpha} + C_{1-\alpha} L_2 \right) L^* \right) |t_2 - t_1| \\
&\quad := \left( K_0 + K^* L^* \right) |t_2 - t_1|,
\end{aligned}
$$

where $K_0 = CM_\alpha \frac{\omega^{1-\alpha}}{1-\alpha}(L_1 + L_2) + C_{1-\alpha} L_2$ is a constant independent of $L^*$, and $K^* = (CM_\alpha(2L_1 + L_2)\frac{\omega^{1-\alpha}}{1-\alpha} + C_{1-\alpha}L_2) < 1$. Hence,

$$
\left\| Qu(t_1) - Qu(t_1) \right\|_\alpha \leq L^* |t_2 - t_1| \quad \text{for all } t_2, t_1 \in \mathbb{R}, \tag{4.5}
$$

whenever $L^* \geq \frac{K_0}{1 - K^*}$. Therefore, $Q$ has a fixed point $u$ which is an $\omega$-periodic mild solution of Eq. (1.1).

By the above calculation, we see that, for this $u(\cdot)$, all the following functions:

$$
g(t) = G\left(t, u(t - \xi)\right),
$$

$$
\Phi(t) = \left(I - T(\omega)\right)^{-1} \int_{t-\omega}^{t} T(t - s) F\left(s, u(s), u(s - \tau)\right) ds,
$$

$$
\Psi(t) = \left(I - T(\omega)\right)^{-1} \int_{t-\omega}^{t} T(t - s) AG\left(s, u(x - \xi)\right) ds
$$

are Lipschitz continuous, respectively. Since the $u$ is Lipschitz continuous on $\mathbb{R}$ and the space $X_\alpha$ is reflexive by the assumption and Lemma 2.3, then a result of [27] asserts that $u(\cdot)$ is a.e. differentiable on $\mathbb{R}$ and $u'(\cdot) \in L^1_{\mathrm{loc}}(\mathbb{R}, X_\alpha)$. Furthermore, by a standard argument as Theorem 4.2.4 in [38], we can obtain that

$$
\begin{aligned}
\Phi'(t) = {}&\big(I - T(\omega)\big)^{-1}\bigg( \big(I - T(\omega)\big)F\big(t, u(t), u(t-\tau)\big) \\
&- \int_{t-\omega}^{t} AT(t-s)F\big(s, u(s), u(s-\tau)\big)\,ds \bigg) \\
\Psi'(t) = {}&\big(I - T(\omega)\big)^{-1}\bigg( \big(I - T(\omega)\big)AG\big(t, u(t-\xi)\big) \\
&- \int_{t-\omega}^{t} AT(t-s)AG\big(s, u(x-\xi)\big)\,ds \bigg).
\end{aligned}
$$

Hence, for almost every $t \in \mathbb{R}$,

$$
\begin{aligned}
u'(t) = {}&\Phi'(t) + g'(t) - \Psi'(t) \\
= {}&\big(I - T(\omega)\big)^{-1}\bigg( \big(I - T(\omega)\big)F\big(t, u(t), u(t-\tau)\big) \\
&- \int_{t-\omega}^{t} AT(t-s)F\big(s, u(s), u(s-\tau)\big)\,ds \bigg) + G'\big(t, u(t-\xi)\big) \\
&- \big(I - T(\omega)\big)^{-1}\bigg( \big(I - T(\omega)\big)AG\big(t, u(t-\xi)\big) \\
&- \int_{t-\omega}^{t} AT(t-s)AG\big(s, u(x-\xi)\big)\,ds \bigg) \\
= {}&F\big(t, u(t), u(t-\tau)\big) - AG\big(t, u(t-\xi)\big) + G'\big(t, u(t-\xi)\big) \\
&- A\bigg( \big(I - T(\omega)\big)^{-1}\int_{t-\omega}^{t} T(t-s)F\big(s, u(s), u(s-\tau)\big)\,ds \\
&- \big(I - T(\omega)\big)^{-1}\int_{t-\omega}^{t} T(t-s)AG\big(s, u(x-\xi)\big)\,ds \bigg) \\
= {}&F\big(t, u(t), u(t-\tau)\big) - AG\big(t, u(t-\xi)\big) + G'\big(t, u(t-\xi)\big) \\
&- A\big(u(t) - G\big(t, u(t-\xi)\big)\big),
\end{aligned}
$$

which implies that

$$
\big(u(t) - G\big(t, u(t-\xi)\big)\big)' + Au(t) = F\big(t, u(t), u(t-\tau)\big), \quad \text{a.e } t \in \mathbb{R}. \tag{4.6}
$$

This shows that $u$ is a strong solution for Eq. (1.1) and the proof is completed.　　　□

## 5 Application

In this section, we present one example, which indicates how our abstract results can be applied to concrete problems.

Consider the following parabolic boundary value problem with delays:

$$
\begin{cases}
\frac{\partial}{\partial t}\left(u(x,t) - \int_0^1 g(x,y,t,u(y,t-\xi))\,dy\right) - \frac{\partial^2}{\partial x^2}u(x,t) \\
\quad = f(x,t,u(x,t),u(x,t-\tau)), \quad x \in [0,1], t \in \mathbb{R}, \\
u(0,t) = u(1,t) = 0,
\end{cases}
\tag{5.1}
$$

where $g \in C([0,1] \times [0,1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $f \in C([0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, and $f, g$ are $\omega$-periodic in $t$, and $\xi, \tau$ are positive constants which denote the time delays.

To treat this system in the abstract form (1.1), we choose the space $X = L^2([0,1], \mathbb{R})$, endowed with the $L^2$-norm $\| \cdot \|_{L^2}$. We remark that $X$ is reflexive.

Define operator $A : D(A) \subset X \to X$ by

$$
D(A) := \left\{ u \in X \mid u'', u' \in X, u(0) = u(1) = 0 \right\}, \qquad Au = -\frac{\partial^2 u}{\partial x^2}.
\tag{5.2}
$$

It is well know that $-A$ generates an exponentially stable compact analytic semigroup $T(t)$ ($t \geq 0$) in $X$. In addition, we note that $0 \in \rho(A)$ and that fractional powers of $A$ are well defined. Moreover, $A$ has a discrete spectrum with eigenvalues of the form $n^2\pi^2, n \in \mathbb{N}$, and the associated normalized eigenfunctions are given by $e_n(x) = \sqrt{2}\sin(n\pi x)$ for $x \in [0,1]$, the associated semigroup $T(t)$ ($t \geq 0$) is explicitly given by

$$
T(t)u = \sum_{n=1}^{\infty} e^{-n^2\pi^2 t}(u, e_n)e_n, \quad t \geq 0, u \in X,
\tag{5.3}
$$

where $(\cdot, \cdot)$ is an inner product on $X$, and it is not difficult to verify that $\|T(t)\| \leq e^{-\pi^2 t}$ for all $t \geq 0$. Hence, we take $M = 1$, $M_{\frac{1}{2}} = \Gamma(\frac{1}{2})$ and $\|(I - T(\omega))^{-1}\| \leq \frac{1}{1-e^{-\pi^2\omega}}$. The proof of the following lemma can be found in [40].

**Lemma 5.1** *If $v \in D(A^{\frac{1}{2}})$, then $v$ is absolutely continuous with $v' \in X$ and $\|v'\|_{L^2} = \|A^{\frac{1}{2}}v\|_{L^2}$.*

According to Lemma 5.1, we define the Banach space $X_{\frac{1}{2}} := (D(A^{\frac{1}{2}}), \| \cdot \|_{\frac{1}{2}})$, where $\|v\|_{\frac{1}{2}} = \|A^{\frac{1}{2}}v\|_{L^2}$ for all $v \in X_{\frac{1}{2}}$. Under the above assumptions we discuss the existence and regularity of mild solutions of time $\omega$-periodic solutions of the delays parabolic boundary value problem (5.1).

**Proposition 5.1** *If the following conditions hold*:

(F1) *There are positive constants $a_0, a_1$, and $K$ such that, for every*
$(x, t, v, w) \in [0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$,

$$
\left| f(x, t, v, w) \right| \leq a_0|v| + a_1|w| + K;
$$

(F2) $g : [0,1] \times [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *satisfies the following conditions*:
(i) *There is a positive function $b : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}^+$ such that*

$$
\left| g(x, y, t, v_2) - g(x, y, t, v_1) \right| \leq b(x, y, t)|v_2 - v_1|
$$

for all $x, y \in [0,1]$, $t \in \mathbb{R}$ and $v_1, v_2 \in \mathbb{R}$; moreover, $(x, y, t) \mapsto \frac{\partial^2}{\partial x^2} b(x, y, t)$ is well defined and measurable with

$$l^2 := \max_{t \in [0,\omega]} \int_0^1 \int_0^1 \left( \frac{\partial^2}{\partial x^2} b(x, y, t) \right)^2 dy\, dx < \infty,$$

(ii) $g(0, y, t, v) = g(1, y, t, v) = 0$ for all $t \in \mathbb{R}$, $y \in [0,1]$ and $v \in \mathbb{R}$;

(F3) $\frac{2\omega^{\frac{1}{2}}}{1-e^{-\pi^2 \omega}} \Gamma(\frac{1}{2})(a_0 + a_1 + l) + l < 1$, then the neutral partial differential equation with delays (5.1) has at least one time $\omega$-periodic mild solution.

*Proof* Let $F : \mathbb{R} \times X_{\frac{1}{2}} \times X_{\frac{1}{2}} \to X$ and $G : \mathbb{R} \times X_{\frac{1}{2}} \to X$ be defined by

$$F(t, \phi, \varphi)(x) = f(x, t, \phi(x), \varphi(x)),$$

$$G(t, \phi)(x) = \int_0^1 g(x, y, t, \phi(y))\, dy$$

for $t \in \mathbb{R}$, $\phi, \varphi \in X_{\frac{1}{2}}$ and $x \in [0,1]$. Moreover, if $u : \mathbb{R} \times [0,1] \to \mathbb{R}$, we define $u : \mathbb{R} \to X$ by $u(t)(x) = u(x, t)$ for $x \in [0,1]$, which implies that $u(t - \tau)(\cdot) = u(\cdot, t - \tau)$ and $u(t - \xi)(\cdot) = u(\cdot, t - \xi)$. Therefore, the partial differential equation with delays (5.1) can be rewritten into the abstract evolution equation with delays (1.1).

From the definition of $F$ and assumption (F1), we can get $F : \mathbb{R} \times X_{\frac{1}{2}} \times X_{\frac{1}{2}} \to X$ is a continuous function, and for each $\phi, \varphi \in X_{\frac{1}{2}}$ and $t \in \mathbb{R}$,

$$
\begin{aligned}
\|F(t, \phi, \varphi)\|_{L^2} &= \left( \int_0^1 \left( f(x, t, \phi(x), \varphi(x)) \right)^2 dx \right)^{\frac{1}{2}} \\
&\leq \left( \int_0^1 \left( a_0 |\phi(x)| + a_1 |\varphi(x)| + K \right)^2 dx \right)^{\frac{1}{2}} \\
&\leq a_0 \|\phi\|_{L^2} + a_1 \|\varphi\|_{L^2} + K \\
&\leq a_0 \|\phi\|_{\frac{1}{2}} + a_1 \|\varphi\|_{\frac{1}{2}} + K,
\end{aligned}
$$

thus, condition (H1′) in Sect. 3 holds.

By the definition of $G$ and assumption (F2), we see that $G : \mathbb{R} \times X_{\frac{1}{2}} \to X_1 (X_1 := D(A))$ is continuous and

$$
\begin{aligned}
\|AG(t, \phi) - AG(t, \varphi)\|_{L^2}^2 &\\
&\leq \int_0^1 \int_0^1 \left( \frac{\partial^2}{\partial x^2} b(x, y, t) \cdot |\phi(y) - \varphi(y)| \right)^2 dy\, dx \\
&\leq \int_0^1 \int_0^1 \left( \frac{\partial^2}{\partial x^2} b(x, y, t) \right)^2 dy\, dx \cdot \|\phi - \varphi\|_{L^2}^2 \\
&\leq l^2 \|\phi - \varphi\|_{\frac{1}{2}}^2,
\end{aligned}
$$

for each $t \in \mathbb{R}$ and $\phi, \varphi \in X_{\frac{1}{2}}$. Thus, condition (H2) in Sect. 3 holds.

Finally, by (F3) and Corollary 3.2, it follows that the neutral partial differential equation with delays (5.1) has at least one time $\omega$-periodic mild solution. The proof is completed. □

For showing the existence of classical and strong solutions, the following assumptions are needed:

(F4)  There exist constants $l_1$ and $\mu_1 \in (0, 1]$ such that, for every $t_i, v_i, w_i \in \mathbb{R}$ ($i = 1, 2$) and $x \in [0, 1]$,

$$\left| f(x, t_2, v_2, w_2) - f(x, t_1, v_1, w_1) \right| \leq l_1 \left( |t_2 - t_1|^{\mu_1} + |v_2 - v_1| + |w_2 - w_1| \right),$$

(F5)  $g : [0, 1] \times [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:

  (i)  $(x, y, t, v) \mapsto \frac{\partial^2}{\partial x^2} g(x, y, t, v)$ is well defined and measurable. Moreover, there are constants $l_2 > 0$ and $\mu_2 \in (0, 1]$ such that, for every $t_i, v_i \in \mathbb{R}$ ($i = 1, 2$),

$$\left| \int_0^1 \frac{\partial^2}{\partial x^2} g(x, y, t_2, v_2)\, dy - \int_0^1 \frac{\partial^2}{\partial x^2} g(x, y, t_1, v_1)\, dy \right| \leq l_2 \left( |t_2 - t_1|^{\mu_2} + |v_2 - v_1| \right),$$

  (ii)  $g(0, y, t, v) = g(1, y, t, v) = 0$ for all $t \in \mathbb{R}$, $y \in [0, 1]$ and $v \in \mathbb{R}$,

(F6)  $\frac{2\omega^{\frac{1}{2}}}{1 - e^{-\pi^2 \omega}} \Gamma(\frac{1}{2})(2l_1 + l_2) + l_2 < 1.$

Hence, for every $t_1, t_2 \in \mathbb{R}$ and $\phi_1, \varphi_1, \phi_2, \varphi_2 \in X_{\frac{1}{2}}$, we have

$$\left\| F(t_2, \phi_2, \varphi_2) - F(t_1, \phi_1, \varphi_1) \right\|_{L^2}$$
$$= \left( \int_0^1 \left( f\left(x, t_2, \phi_2(x), \varphi_2(x)\right) - f\left(x, t_1, \phi_1(x), \varphi_1(x)\right) \right)^2 dx \right)^{\frac{1}{2}}$$
$$\leq \left( \int_0^1 l^2 \left( |t_2 - t_1|^{\mu_1} + |\phi_2(x) - \phi_1(x)| + |\varphi_2(x) - \varphi_1(x)| \right)^2 dx \right)^{\frac{1}{2}}$$
$$\leq l_1 \left( |t_2 - t_1|^{\mu_1} + \|\phi_2 - \phi_1\|_{L^2} + \|\varphi_2 - \varphi_1\|_{L^2} \right)$$
$$\leq l_1 \left( |t_2 - t_1|^{\mu_1} + \left\| (\phi_2 - \phi_1) \right\|_{\frac{1}{2}} + \|\varphi_2 - \varphi_1\|_{\frac{1}{2}} \right),$$

and

$$\left\| AG(t_2, \phi_2) - AG(t_1, \phi_1) \right\|_{L^2}$$
$$= \left( \int_0^1 \left( \int_0^1 \frac{\partial^2}{\partial x^2} g\left(x, y, t_2, \phi_2(y)\right) dy - \int_0^1 \frac{\partial^2}{\partial x^2} g\left(x, y, t_1, \phi_1(y)\right) dy \right)^2 dx \right)^{\frac{1}{2}}$$
$$\leq \left( \int_0^1 l_2^2 \left( |t_2 - t_1|^{\mu_2} + |\phi_2(x) - \phi_1(x)| \right)^2 dx \right)^{\frac{1}{2}}$$
$$\leq l_2 \left( |t_2 - t_1|^{\mu_2} + \|\phi_2 - \phi_1\|_{L^2} \right)$$
$$\leq l_2 \left( |t_2 - t_1|^{\mu_2} + \|\phi_2 - \phi_1\|_{\frac{1}{2}} \right),$$

which implies that conditions (H4) and (H5) for $\mu_1, \mu_2 \in (0, 1)$ (or (H4′) and (H5′) for $\mu_1 = \mu_2 = 1$) hold. On the other hand, by condition (F6), we can easily prove that condition (H6) holds in Sect. 4.

Consequently, all the conditions stated in Theorem 4.1 and Theorem 4.2 are satisfied, and we obtain the following interesting results.

**Proposition 5.2**  *If conditions (F4)–(F5) hold for $\mu_1, \mu_2 \in (0, 1)$, then the neutral partial differential equation with delays* (5.1) *has time $\omega$-periodic classical solutions.*

**Proposition 5.3** *If conditions* (F4)–(F5) *hold for* $\mu_1 = \mu_2 = 1$, *then the neutral partial differential equation with delays* (5.1) *has time $\omega$-periodic strong solutions.*

### Availability of data and materials
Not applicable.

### Competing interests
Q. Li and H. Zhang declare that they have no competing interests.

### Authors' contributions
QL and HZ contributed equally and significantly in writing this article. All authors read and approved the finial manuscript.

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