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Triple solutions for a damped impulsive differential equation

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Abstract
In this paper, we aim to obtain the existence of at least three classical solutions for nonlinear impulsive problems neither adding superlinear or local sublinear conditions to nonlinear term at zero nor adding superlinear assumptions to nonlinear term at infinity. Our approach relies on variational methods, more specifically, on a three critical points theorem.

MSC: 34B15; 35B38

Keywords: Triple solutions; Damped term; Impulsive effects

1 Introduction

Recently, a great deal of work has been done in dealing with the existence of solutions for impulsive differential equations, for example, [1–12] and the references therein.

Nieto initially introduced a new approach via variational methods to obtain the existence of a weak solution to linear impulsive problems in [1] by the Lax–Milgram theorem, and also obtained at least one weak solution to nonlinear problems via Mountain Pass Theorem.

Moreover, Nieto in [2], by using the Lax–Milgram theorem studied the existence of a unique weak solution for a differential equations with impulses and with a discontinuous damped term.

In [3], by exploiting critical point theory and variational methods, we obtained the existence of one and infinity many solutions for some impulsive boundary value problems with a parameter when the nonlinear term satisfied the Ambrosetti–Rabinowitz type superlinear condition.

In [4], we considered impulsive differential equations with small non-autonomous perturbations via variational methods and a three critical points theorem. The existence of at least three classical solutions was obtained when the primitive function of nonlinearity $f(t,u)$ satisfied some superlinear conditions at infinity.

In contrast to [3] and [4], the aim of this paper is to deal with the existence of at least three classical solutions of the following damped impulsive differential equations:

$$\begin{cases} -u''(t) + u(t) + g(t)u'(t) = \lambda f(t,u) \quad \text{a.e. } t \in [0,T], \\ \Delta u'(t_i) = I_i(u(t_i)) \quad i = 1,2,\ldots,p, \\ u(0) = u(T) = 0, \end{cases} \tag{1.1}$$

where λ is a parameter, $T > 0, g \in C[0,T], f \in C([0,T]\times\mathbb{R},\mathbb{R})$ and $I_i : \mathbb{R} \to \mathbb{R}$ $(i = 1,2,\dots,p)$ are continuous, $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = T$, $\Delta u'(t_i) = u'(t_i^+) - u'(t_i^-) = \lim_{t\to t_i^+} u'(t) - \lim_{t\to t_i^-} u'(t)$, without assuming asymptotic conditions neither at zero nor at infinity on the nonlinear term. Our main tools are variational methods and a three critical points theorem by Bonanno and Marano.

We need the following assumptions:

(H_1) $I_i(u)$ are nondecreasing, and $I_i(u)u \geq 0$ for any $u \in \mathbb{R}$.

(H_2) There exist positive constants C_0, a, l with $l < 2$, and a function $b \in L^1[0,T]$ such that for all $|u| > C_0$ and $t \in [0,T]$

$$\bigl|F(t,u)\bigr| \leq b(t)\bigl(a + |u|^l\bigr),$$

where $F(t,u) = \int_0^u f(t,s)\,ds$.

It is worth pointing out that conditions (H_2) and (H_4) in Theorem 3.1 of this paper are not local conditions at zero, thus we neither add any superlinear or sublinear assumptions to the nonlinear term at zero nor add superlinear assumptions to the nonlinear term at infinity.

We would also like to mention the work of Candito, Carl and Livrea in [13–15], in which non-coercive differential problems involving a positive parameter λ were investigated. Authors presented novel variational approaches to obtain multiplicity, regularity, and a priori estimates of solutions by assuming certain growth conditions on the nonlinearity prescribed only near zero.

The rest of the paper is organized as follows: In Sect. 2, we give variational structure. In Sect. 3, we formulate and prove our results, and an example is presented to illustrate our results.

2 Variational structure

Our aim is to get the existence of at least three classical solutions for nonlinear impulsive problems by using variational methods. The main tool we used is the following three critical points theorem obtained in [16] or [17].

Theorem 2.1 ([16, 17]) *Let X be a reflexive real Banach space, $\varphi : X \to \mathbb{R}$ a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^*, and let $\phi : X \to \mathbb{R}$ be a sequentially weakly upper semicontinuous, continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that there exist $r \in \mathbb{R}$ and $x_0, \bar{x} \in X$, with $\varphi(x_0) < r < \varphi(\bar{x})$, and $\phi(x_0) = 0$ such that*

(i) $\sup_{\varphi(x)\leq r} \phi(x) < (r - \varphi(x_0))\frac{\phi(\bar{x})}{\varphi(\bar{x})-\varphi(x_0)}$,

(ii) *For each λ in $\Lambda_r := (\frac{\varphi(\bar{x})-\varphi(x_0)}{\phi(\bar{x})}, \frac{r-\varphi(x_0)}{\sup_{\varphi(x)\leq r}\phi(x)})$, the functional $\varphi - \lambda\phi$ is coercive.*

Then, for each $\lambda \in \Lambda_r$ the functional $\varphi - \lambda\phi$ has at least three distinct critical points in X.

Let $\alpha = \min_{t\in[0,T]} e^{G(t)}$, $\beta = \max_{t\in[0,T]} e^{G(t)}$, and $G(t) = -\int_0^t g(s)\,ds$.

Denote by $H := H_0^1(0,T) = \{u : [0,T] \to \mathbb{R} | u \text{ is absolutely continuous}, u' \in L^2(0,T), u(0) = u(T) = 0\}$ the Sobolev space with the inner product and induced norm given by

$$(u,v)_H = \int_0^T e^{G(t)}\bigl(u(t)v(t) + u'(t)v'(t)\bigr)\,dt,$$

$$\|u\|_H = \left(\int_0^T e^{G(t)} \big(u'(t)^2 + u(t)^2 \big) \, dt \right)^{1/2}.$$

It is well known that H is a reflexive Banach space [18, p. 7].

Consider the functional $\varphi(u) - \lambda\phi(u) : H \to \mathbb{R}$, where

$$\varphi(u) = \frac{1}{2}\|u\|_H^2 + \sum_{i=1}^{p} e^{G(t_i)} \int_0^{u(t_i)} I_i(t)\, dt,$$

$$\phi(u) = \int_0^T e^{G(t)} (F\big(t, u(t)\big)\, dt.$$

In view of the continuity of f and I_i ($i = 1, 2, \ldots, p$), we obtain that φ and ϕ are Gateaux differentiable and

$$\begin{aligned}(\varphi - \lambda\phi)'(u)v = {} & \int_0^T e^{G(t)} u'(t)v'(t)\, dt + \int_0^T e^{G(t)} u(t)v(t)\, dt \\ & + \sum_{i=1}^{p} e^{G(t_i)} I_i\big(u(t_i)\big)v(t_i) - \lambda \int_0^T e^{G(t)} f\big(t, u(t)\big)v(t)\, dt,\end{aligned}$$

for any $v \in H$. Hence, a critical point of the functional $\varphi - \lambda\phi$ gives us a weak solution of problem (1.1).

Definition 2.2 We say that $u \in H$ is a weak solution of problem (1.1) if

$$\begin{aligned}& \int_0^T e^{G(t)} u'(t)v'(t)\, dt + \int_0^T e^{G(t)} u(t)v(t)\, dt + \sum_{i=1}^{p} e^{G(t_i)} I_i\big(u(t_i)\big)v(t_i) \\ & - \lambda \int_0^T e^{G(t)} f\big(t, u(t)\big)v(t)\, dt = 0\end{aligned}$$

holds for any $v \in H$.

By the same calculations as in [6], we get the following lemma.

Lemma 2.3 ([6]) *If $u \in H$ is a weak solution of problem (1.1), then u is a classical solution of problem (1.1), and there exists $M > 0$ such that $\|u\|_\infty \le M\|u\|_H$, where $u \in H$, $\|u\|_\infty = \max_{t\in[0,T]} |u(t)|$.*

Definition 2.4 ([18]) Let E be a Banach space. A functional $\varphi : E \to \mathbb{R}$ is said to be sequentially weakly lower semicontinuous if $\liminf_{n\to+\infty} \varphi(x_n) \ge \varphi(x)$ as $x_n \rightharpoonup x$ in E.

Lemma 2.5 *Assume that (H_1) holds. Then the functional φ is sequentially weakly lower semicontinuous, coercive, and its derivative admits a continuous inverse on H^*.*

Proof Firstly, we prove that φ is sequentially weakly lower semicontinuous.

Let $u_n \rightharpoonup u$ in H, then one has $\|u\| \le \liminf_{n\to\infty} \|u_n\|$, and $u_n \rightrightarrows u$ in $[0,T]$ when $n \to \infty$, hence

$$\liminf_{n\to\infty} \varphi(u_n) = \liminf_{n\to\infty} \left(\frac{1}{2}\|u_n\|_H^2 + \sum_{i=1}^{p} e^{G(t_i)} \int_0^{u_n(t_i)} I_i(t)\, dt \right)$$

$$\begin{aligned}&\ge \frac{1}{2}\|u\|_H^2 + \sum_{i=1}^{p} e^{G(t_i)} \int_0^{u(t_i)} I_i(t)\,dt\\&= \varphi(u).\end{aligned}$$

Thus, by Definition 2.4, φ is sequentially weakly lower semicontinuous.

Moreover, condition (H_1) implies that $\sum_{i=1}^{p}\int_0^{u(t_i)} I_i(t)\,dt \ge 0$, thus

$$\varphi(u) = \frac{1}{2}\|u\|_H^2 + \sum_{i=1}^{p} e^{G(t_i)} \int_0^{u(t_i)} I_i(t)\,dt \ge \frac{1}{2}\|u\|_H^2, \tag{2.1}$$

so the functional φ is coercive.

Secondly, we prove that φ' admits a continuous inverse on H^*.

For any $u \in H \setminus \{0\}$, one has

$$\begin{aligned}\langle \varphi'(u), u\rangle &= \int_0^T e^{G(t)} u'(t)u'(t)\,dt + \int_0^T e^{G(t)} u(t)u(t)\,dt\\&\quad + \sum_{i=1}^{p} e^{G(t_i)} I_i\bigl(u(t_i)\bigr)u(t_i)\\&\ge \|u\|_H^2,\end{aligned}$$

thus

$$\liminf_{\|u\|_H\to\infty} \frac{\langle \varphi'(u), u\rangle}{\|u\|_H} = +\infty,$$

and so φ' is coercive.

In view of the assumption that $I_i(u)$ $(i = 1, 2, \ldots, p)$ are nondecreasing, for any $u, v \in H$, one has

$$\begin{aligned}\langle \varphi'(u) - \varphi'(v), u - v\rangle &= \int_0^T e^{G(t)} \bigl(u'(t) - v'(t)\bigr)\bigl(u'(t) - v'(t)\bigr)\,dt\\&\quad + \int_0^T e^{G(t)} \bigl(u(t) - v(t)\bigr)\bigl(u(t) - v(t)\bigr)\,dt\\&\quad + \sum_{i=1}^{p} e^{G(t_i)} I_i\bigl(u(t_i) - v(t_i)\bigr)\bigl(u(t_i) - v(t_i)\bigr)\\&\ge \|u - v\|_H^2,\end{aligned}$$

thus we have that φ' is uniformly monotone. Taking account of [19, Theorem 26], we get that $(\varphi')^{-1}$ exists and is continuous on H^*. □

Lemma 2.6 *The functional ϕ is a sequentially weakly upper semicontinuous continuously Gâteaux differentiable functional and its derivative is compact.*

Proof Let $\{u_n\} \in H$ satisfying $u_n \rightharpoonup u \in H$, then $u_n \rightrightarrows u$ in $[0,T]$ as $n \to +\infty$. By using the Reverse Fatou Lemma, one has

$$\begin{aligned}\limsup_{n\to\infty} \phi(u_n) &= \limsup_{n\to\infty} \int_0^T e^{G(t)} F\bigl(t,u_n(t)\bigr)\,dt \\ &\le \lim_{n\to\infty} \int_0^T e^{G(t)} \sup F\bigl(t,u_n(t)\bigr)\,dt \\ &= \int_0^T e^{G(t)} F\bigl(t,u(t)\bigr)\,dt = \phi(u).\end{aligned}$$

Thus, ϕ is sequentially weakly upper semicontinuous. Now, we prove that ϕ' is strongly continuous on H. Noting that $f(t,u)$ is continuous on u, $f(t,u_n) \to f(t,u)$ as $n \to +\infty$, and then

$$\lim_{n\to\infty} \int_0^T f\bigl(t,u_n(t)\bigr) v\,dt = \int_0^T f\bigl(t,u(t)\bigr) v\,dt, \quad \forall v \in H.$$

Thus we get that ϕ' is strongly continuous on H, which implies that ϕ' is compact by [19, Proposition 26.2]. In addition, ϕ' is continuous taking into account that it is strongly continuous. □

3 Main results

Without adding any superlinear or sublinear local condition on the potential F at zero, we get the following theorems.

Theorem 3.1 *Suppose that (H_1) and (H_2) hold, and there exist positive constants c, d such that*

(H_3) $\frac{c^2}{M^2} < d^2 \|e^{G(t)}\|_{L^1} + 2\sum_{i=1}^p e^{G(t_i)} \int_0^d I_i(t)\,dt$,

(H_4) $\frac{M^2 \int_0^T \max_{|s|\le c} F(t,s)\,dt}{c^2} < \frac{\int_0^T e^{G(t)} F(t,d)\,dt}{d^2 \|e^{G(t)}\|_{L^1} + 2\sum_{i=1}^p e^{G(t_i)} \int_0^d I_i(t)\,dt}$,

then for any $\lambda \in \Lambda_r := \bigl(\frac{d^2 \|e^{G(t)}\|_{L^1} + 2\sum_{i=1}^p e^{G(t_i)} \int_0^d I_i(t)\,dt}{2\int_0^T e^{G(t)} F(t,d)\,dt}, \frac{c^2}{2M^2 \int_0^T e^{G(t)} \max_{|s|\le c} F(t,s)\,dt}\bigr)$, *problem* (1.1) *has at least three classical solutions.*

Proof For all $t \in [0,T]$, we choose $u_0(t) = 0$, $\bar{u}(t) = d$. Then, obviously, $u_0, \bar{u} \in H$, $\varphi(u_0) = \phi(u_0) = 0$, $\varphi(\bar{u}) = \frac{d^2}{2} \|e^{G(t)}\|_{L^1} + \sum_{i=1}^p e^{G(t_i)} \int_0^d I_i(t)\,dt$, and $\phi(\bar{u}) = \int_0^T e^{G(t)} F(t,d)\,dt$.

Set $r = \frac{c^2}{2M^2}$, by simple calculations, we get

$$\begin{aligned}\bigl(r - \varphi(u_0)\bigr) \frac{\phi(\bar{u})}{\varphi(\bar{u}) - \varphi(u_0)} &= \frac{c^2}{2M^2} \frac{\int_0^T e^{G(t)} F(t,d)\,dt}{\frac{d^2}{2} \|e^{G(t)}\|_{L^1} + \sum_{i=1}^p e^{G(t_i)} \int_0^d I_i(t)\,dt} \\ &= \frac{c^2}{M^2} \frac{\int_0^T e^{G(t)} F(t,d)\,dt}{d^2 \|e^{G(t)}\|_{L^1} + 2\sum_{i=1}^p e^{G(t_i)} \int_0^d I_i(t)\,dt},\end{aligned} \tag{3.1}$$

and combining with (H_3), we obtain $\varphi(u_0) < r < \varphi(\bar{u})$.

For all $u \in H$ satisfying $\varphi(u) \le r$, combining with (2.1), one has $\|u\|_H \le \sqrt{2r}$, thus $\|u\|_\infty \le M\|u\|_H \le M\sqrt{2r} = c$. Therefore,

$$\sup_{\varphi(x)\le r} \phi(x) \le \int_0^T e^{G(t)} \max_{|s|\le c} F(t,s)\,dt < +\infty. \tag{3.2}$$

Taking in account of (3.1), (3.2) and (H_4), we have that condition (i) in Theorem 2.1 is satisfied.

Let $T_n = \{t \in [0,T] : |u(t)| > C_0\}$. Then from (H_2), there is $C_1 > 0$ such that for all $|u| > C_0$ and $t \in [0,T]$,

$$\begin{aligned}
\varphi(u) - \lambda\phi(u) &= \frac{1}{2}\|u\|_H^2 + \sum_{i=1}^{p} e^{G(t_i)} \int_0^{u(t_i)} I_i(t)\,dt - \lambda \int_0^T e^{G(t)} F\big(t,u(t)\big)\,dt \\
&= \frac{1}{2}\|u\|_H^2 + \sum_{i=1}^{p} e^{G(t_i)} \int_0^{u(t_i)} I_i(t)\,dt \\
&\quad - \lambda\biggl(\int_{T_n} e^{G(t)} F\big(t,u(t)\big)\,dt + \int_{[0,T]\setminus T_n} e^{G(t)} F\big(t,u(t)\big)\,dt\biggr) \\
&\geq \frac{1}{2}\|u\|_H^2 - \lambda\beta\|b\|_{L^1}\bigl(a + M^l \|u\|_H{}^l\bigr) - \lambda\beta C_1.
\end{aligned}$$

Taking account of $l < 2$, one has $\lim_{\|u\|_H \to +\infty}(\varphi(u) - \lambda\phi(u)) = +\infty$ for any $\lambda \in \Lambda_r$, so condition (ii) in Theorem 2.1 is also satisfied. Hence, combining with Lemmas 2.5 and 2.6, we get the conclusion. □

Particularly, we consider $f(t,u) = v(t)h(u)$, where $v \in L^1[0,T], h \in C(\mathbb{R},\mathbb{R})$. Letting $H(u) = \int_0^u h(s)\,ds$, we get the following result.

Theorem 3.2 *Suppose that (H_1) and (H_3) hold, in addition, assume that*

$(H_2)'$ *there exist positive constants C_0, a, l with $l < 2$, and a function $b \in L^1[0,T]$ such that for all $|u| > C_0$ and $t \in [0,T]$*

$$\big|v(t)H(u)\big| \leq b(t)\big(a + |u|^l\big),$$

$(H_4)'$ *there exist $c, d > 0$ such that $\frac{M^2 \max_{|s|\leq c} H(s)\|v(t)\|_{L_1}}{c^2} < \frac{H(d)\|e^{G(t)}v(t)\|_{L^1}}{d^2\|e^{G(t)}\|_{L^1} + 2\sum_{i=1}^p e^{G(t_i)}\int_0^d I_i(t)\,dt}$ hold, then for any $\lambda \in (\frac{d^2\|e^{G(t)}\|_{L^1} + 2\sum_{i=1}^p e^{G(t_i)}\int_0^d I_i(t)\,dt}{2H(d)\|e^{G(t)}v(t)\|_{L^1}}, \frac{c^2}{2M^2\max_{|s|\leq c} H(s)\|v(t)\|_{L_1}})$, problem (1.1) has at least three classical solutions.*

Example 3.3 Taking $T = 1$, $t_1 = \frac{1}{2}$, $g(t) = -1$, $f(t,u) = e^{-t}u^{\frac{1}{4}}$, $I_1(u) = u$. Consider the problem

$$\begin{cases} -u''(t) + u(t) - u'(t) = \lambda e^{-t}u^{\frac{1}{4}}, & t \in [0,1]\setminus\{\frac{1}{2}\}, \\ \Delta u'(\frac{1}{2}) = u(\frac{1}{2}), \\ u(0) = u(1) = 0. \end{cases} \tag{3.3}$$

We choose $d = 1$ and two positive constants c and M such that $\frac{c^2}{M^2} < 3.366$, by direct calculations and applying Theorem 3.1 or Theorem 3.2, for any $\lambda \in (2.103, \frac{5ec^{\frac{3}{4}}}{8M^2(e-1)})$ problem (3.3) has at least three classical solutions.

4 Conclusion

We get the existence of at least three classical solutions for nonlinear impulsive problems with a parameter. Our approach relies on variational methods, more specifically on a three critical points theorem by Bonanno and Marano. We show the results by neither adding superlinear or local sublinear assumptions on nonlinear terms at zero nor adding superlinear assumptions on the nonlinearity at infinity.

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