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A graph-theoretic method to study the existence of periodic solutions for a coupled Rayleigh system via inequality techniques

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Abstract

In the paper, we are concerned with the existence of periodic solutions for a coupled Rayleigh system. By combining graph theory with coincidence degree theory as well as Lyapunov function method, two new sufficient conditions on the existence of periodic solutions for the coupled Rayleigh system are established. Our results on the existence of periodic solutions for the coupled Rayleigh system improve those obtained in the existing literature for coupled Rayleigh system. Hence, our results are new and complementary to the existing papers.

First part title: Introduction

Second part title: Preliminaries

Third part title: The existence of periodic solutions

Fourth part title: Numerical test

Five part title: Conclusion

Keywords: Periodic solutions; Coupled Rayleigh system; Graph theory; Continuation theorem of coincidence degree theory; Lyapunov function method

1 Introduction

An important class of Rayleigh systems is described by the following form:

$$x''(t) + f\big(t, x'(t)\big) + g\big(t, x(t)\big) = e(t), \tag{1.1}$$

where $f, g : R \times R \to R$ and $e : R \to R$ are continuous functions. The dynamic behaviors of system (1.1) have been an active research topic due to its extensive applications in physics, mechanics, engineering technique, and other areas (see [1–4] and the references therein). Such successful applications are greatly dependent on the existence of periodic solutions for system (1.1). Hence, the periodicity analysis of system (1.1) has been a subject of intense activities, and many results have been obtained, for example, see [5–8] and the references therein.

In [7], the authors investigated the following Rayleigh type equation:

$$x''(t) + f\big(x'(t)\big) + g\big(t, x(t)\big) = e(t), \tag{1.2}$$

where $f : R \to R$ is continuous, $g : R^2 \to R$ is continuous and T-periodic with respect to the first variable. Some criteria to guarantee the existence of periodic solutions of this equation were presented in [7] by using Mawhin's continuation theorem, Floquet theory, Lyapunov stability theory, and some analysis techniques. In [6], the authors studied the existence of periodic solutions of Rayleigh equations:

$$x''(t) + f\bigl(t, x'\bigr) + g(x) = e(t), \tag{1.3}$$

where $f : R^2 \to R$ is continuous and T-periodic with respect to the first variable, $g, e : R \to R$ are continuous, and e is T-periodic. They proved that the given equation possesses at least one T-periodic solution under some conditions. In [5], by employing the continuation theorem of coincidence degree theory, the authors studied a kind of Rayleigh equation with a deviating argument as follows:

$$x''(t) + f\bigl(x'(t)\bigr) + g\bigl(x\bigl(t - \tau(t)\bigr)\bigr) = p(t), \tag{1.4}$$

where $g, f : R \to R$ are two continuous functions, $\tau(t)$ and $p(t)$ are continuous and T-periodic functions, and established some new results on the existence of periodic solutions for system (1.4).

With the popularity of coupled systems, so far, the existence and global stability of periodic solutions of coupled systems on neural networks have gained increasing research [9–13], the existence of periodic solutions of coupled systems on the predator-prey systems [14, 15] has been widely studied, the existence of periodic solutions and stability of equilibrium point for coupled systems on networks have been widely investigated, for example, see [16–23] and the references therein.

In [21], the authors were concerned with the following coupled Rayleigh system:

$$x_k''(t) + f_k\bigl(t, x_k'(t)\bigr) + g_k\bigl(t, x_k(t)\bigr) = e_k(t), \tag{1.5}$$

where $k = 1, 2, \ldots, n$, n is a positive integer, $f_k, g_k : R \to R$ and $e_k : R \to R$ are continuous ω-periodic functions in the first argument with period $\omega > 0$, $f_k(t, x_k)$ is continuously differentiable in x_k.

In [21], by taking $y_k(t) = x_k'(t) + \eta x_k(t)$, $\eta > 0$, system (1.5) was rewritten as

$$\begin{cases} x_k'(t) = y_k(t) - \eta x_k(t), \\ y_k'(t) = -\eta^2 x_k(t) + \eta y_k(t) - f_k(t, y_k(t) - \eta x_k(t)) - g_k(t, x_k(t)) + e(t). \end{cases} \tag{1.6}$$

By adding $-\sum_{h=1}^{l} a_{kh}(y_k(t) - y_h(t))$ into the second equation of system (1.6), in [21], the authors established the following linear coupled Rayleigh system:

$$\begin{cases} x_k'(t) = y_k(t) - \eta x_k(t), \\ y_k'(t) = -\eta^2 x_k(t) + \eta y_k(t) - f_k(t, y_k(t) - \eta x_k(t)) - g_k(t, x_k(t)) + e(t) \\ \qquad - \sum_{h=1}^{l} a_{kh}(y_k(t) - y_h(t)), \quad k \in K, \end{cases} \tag{1.7}$$

where $a_{kh}(y_h - y_k)$ represents the influence of vertex h on vertex k, $a_{kh} > 0$, and $a_{kh} = 0$ if and only if there exists no arc from vertex h to vertex k in g, K, g are defined in Definition 2.1.

In [21, 24–26], by combining graph theory with coincidence degree theory as well as Lyapunov method, a sufficient criterion for the existence of periodic solutions for system (1.7) was provided under these conditions (A_1)–(A_5).

However, the conditions in the results obtained in [21] on the existence of periodic solutions for the coupled system (1.7) are too complicated and there are too many of them. This motivates us to obtain more concise and easily verified new sufficient conditions for system (1.7).

Up to now, the global existence of periodic solutions for differential systems has been investigated mainly by employing the following five methods: (1) Fixed point theorem methods [26]; (2) Combining continuation theorem of coincidence degree theory with the a priori estimate of periodic solutions [9, 11, 13–15, 27–30]; (3) Combining continuation theorem of coincidence degree theory with LMI [12]; (4) Combining continuation theorem of coincidence degree theory with Lyapunov function method [16, 18–21, 31]; (5) The method of upper and lower functions. But, in the above-mentioned methods, (3) and (4) are used in recent years to study the existence of periodic solutions for different systems. In this paper, we apply (4) to study the existence of periodic solution for system (1.7), but the concrete analysis techniques in our paper are different from those used in [16, 18–21]. In this paper, our purpose is, by combining graph theory with Mawhin's continuation theorem of coincidence degree theory as well as Lyapunov functional method, to improve the results on the existence of periodic solutions obtained in [21] for system (1.7) by removing conditions (A_4) and (A_5) in [21]. Consequently, the contribution of this paper lies in the following two aspects: (1) Novel inequality techniques are cited to study the existence of periodic solutions for different equations; (2) Novel sufficient conditions are gained for system (1.7) by improving the results obtained in the existing papers.

This paper is organized as follows. Some preliminaries and lemmas are given in Sect. 2. In Sect. 3, two sufficient conditions are derived for the existence of periodic solutions for system (1.7). In Sect. 4, two illustrative examples are given to show the effectiveness of the proposed theory. In Sect. 5, a conclusion is given.

2 Preliminaries

Let R and R^n be the set of real numbers and an n-dimensional Euclidean space, respectively. Let $|\cdot|$ and $\|\cdot\|$ respectively be norms of R and R^n.

We cite the notation as follows:

$$\bar{f} = \max_{t\in[0,\omega]}\{|f(t)|\},$$

where $f(t)$ is a continuous ω-periodic function.

We make the assumptions as follows:

(H_1) There exist constants $b > 0$, $d > 0$ such that, for $k \in K$,

$$|g_k(t,x_k)| \le b|x_k(t)| + d.$$

(H_2) There exist constants $\delta < 0$, $r > 0$, $e > 0$, and a with $A = -\eta - \delta\eta^2 + 0.5b^2 + 0.5bd + \eta^2 + 0.5\eta r + 0.5\eta|a| + 0.5\eta e < 0$ such that, for $k \in K$,

$$x_k f_k(t,x_k) \ge \delta x_k^2 + a x_k,\ |f_k(t,x_k)| \le r|x_k| + e.$$

(A_1) There exist constants δ and μ_1 satisfying $2\delta - \mu_1 \geq 1 - \frac{\eta^2}{2}$ such that, for $k \in K$,

$$x_k g_k(t,x_k) \geq \delta x_k^2, g_k^2(t,x_k) \leq \mu_1 x_k^2.$$

(A_2) Function $f_k(t,x_k)$ satisfies, for $k \in K$,

$$0 < \frac{2(\eta+1)}{2-\eta} \leq \frac{f_k(t,x_k)}{x_k} \leq 2, x_k \neq 0.$$

(A_3) The digraph (g,B) $(B = (a_{kh})_{n\times l})$ is strongly connected.

(A_4) There exists $\varepsilon > 0$ such that, for $k \in K$,

$$m_k(x_k) = \frac{\frac{1}{\omega}\int_0^\omega g_k(t,x_k)\,dt}{x_k} \geq \varepsilon, \quad x_k \neq 0,$$

where $m_k(x_k) \in C^1(R,R)$.

(A_5) For $k \in K$,

$$\int_0^\omega e_k(t)\,dt = 0.$$

For the sake of convenience, we introduce Gaines and Mawhin's continuation theorem about coincidence degree theory [24] and graph theory [25] as follows.

Lemma 2.1 ([24]) *Assume that X and Z are two Banach spaces, $L : D(L) \subset X \to Z$ is a Fredholm operator with index zero. Let $\Omega \in X$ be an open bounded set and $N : \overline{\Omega} \to Z$ be L-compact on $\overline{\Omega}$. Assume that*

(1) *for each $\lambda \in (0,1)$, $u \in \partial\Omega \cap \operatorname{Dom} L$, $Lu \neq \lambda Nu$;*

(2) *for each $u \in \partial\Omega \cap \operatorname{Ker} L$, $QNu \neq 0$;*

(3) $\deg\{JQNu, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$, *where* deg *denotes the Brouwer degree.*

Then the operator equation $Lu = Nu$ has at least one solution in $\overline{\Omega} \cap \operatorname{Dom} L$.

Definition 2.1 ([23]) A directed graph $g = (U,K)$ contains a set $U = \{1,2,\dots,n\}$ of vertices and a set K of arcs (i,j) leading from initial vertex i to terminal vertex j. A subgraph Γ of g is said to be spanning if Γ and g have the same vertex set. A directed graph g is weighed if each arc (j,i) is assigned a positive weight b_{ij}. The weight $W(\Gamma)$ of a subgraph H is the product of the weights on all its arc. A directed path δ in g is subgraph with distinct vertices $\{i_1,i_2,\dots,i_m\}$ such that its set of arcs is $\{(i_k,i_{k+1}) : k = 1,2,\dots,m-1\}$. For a weighted digraph g with l vertices, we define the weight matrix $B = (b_{ij})_{n\times n}$ whose entry $b_{ij} > 0$ is equal to the weight of arc (j,i) if it exists, and 0 otherwise. A digraph g is strongly connected if, for any pair of distinct vertices, there exists a directed path from one to the other. The Laplacian matrix of (g,B) is defined as $L = (p_{ij})_{l\times l}$, where $p_{ij} = -b_{ij}$ for $i \neq j$ and $p_{ij} = \sum_{k\neq i} b_{ik}$ for $i = j$.

Lemma 2.2 ([24]) *Suppose that $l \geq 2$ and c_k denotes the cofactor of the kth diagonal element of the Laplacian matrix of (g,B). Then $\sum_{k,h=1}^{l} c_k a_{kh} G_{kh}(x_k,x_h) = \sum_{Q\in\Omega} W(Q) \times \sum_{(k,h)\in K(C_Q)} G_{hk}(x_h,x_k)$, where $G_{kh}(x_k,x_h)$ is an arbitrary function, Q is the set of all spanning unicyclic graphs of (g,B), $W(Q)$ is the weight of Q, C_Q denotes the directed cycle of Q, and $K(C_Q)$ is the set of arcs in C_Q. In particular, if (g,B) is strongly connected, then $c_k > 0$ for $1 \leq k \leq l$.*

Lemma 2.3 *For any* $\lambda \in (0,1)$, *consider the following system*:

$$\begin{cases} x_k'(t) = \lambda[y_k(t) - \eta x_k(t)], \\ y_k'(t) = \lambda[-\eta^2 x_k(t) + \eta y_k(t) - f_k(t, y_k(t) - \eta x_k(t)) - g_k(t, x_k(t)) + e_k(t) \\ \qquad - \sum_{h=1}^{l} a_{kh}(y_k(t) - y_h(t))], \quad k \in K. \end{cases} \tag{2.1}$$

If the periodic solutions of system (2.1) *exist, then they are bounded and the boundary is independent of the choice of* λ *under assumptions* (H_1), (H_2), *and* (A_3). *Namely, there exists a positive constant H such that*

$$\left\| \left(x(t), y(t)\right)^T \right\| = \left\| \left(x_1(t), x_2(t), \ldots, x_l(t), y_1(t), y_2(t), \ldots, y_l(t)\right)^T \right\| \le H,$$

the norm $\|\cdot\|$ *is defined in the proof of Theorem* 3.1.

Proof Suppose that $(x(t), y(t))^T = (x_1(t), x_2(t), \ldots, x_l(t), y_1(t), y_2(t), \ldots, y_l(t))^T$ is a periodic solution of system (2.1) for some $\lambda \in (0,1)$. Letting $V(x,y) = 0.5\sum_{k=1}^{l} c_k(x_k^2 + y_k^2)$, where c_k denotes the cofactor of the kth diagonal element of Laplacian matrix of $(g, (b_{kh})_{l\times l})$. According to assumption (A_3) and Lemma 2.2, one has $c_k > 0$, $k \in K$. Making use of assumptions (H_1) and (H_2), we have

$$\begin{aligned} &\frac{dV(x,y)}{dt} \\ &\quad = \lambda \sum_{k=1}^{l} c_k \Biggl[-\eta x_k^2(t) - \eta^2 x_k(t) y_k(t) + \eta y_k^2(t) - y_k(t) f_k\bigl(t, y_k(t) - \eta x_k(t)\bigr) \\ &\qquad + y_k(t)\bigl(x_k(t) - g_k\bigl(t, x_k(t)\bigr)\bigr) + y_k(t) e_k(t) - \sum_{h=1}^{l} a_{kh} y_k(t)\bigl(y_k(t) - y_h(t)\bigr) \Biggr] \\ &\quad \le \lambda \sum_{k=1}^{l} c_k \Biggl\{ -\eta x_k^2(t) - \eta^2 x_k(t) y_k(t) + \eta y_k^2(t) - \delta\bigl[y_k(t) - \eta x_k(t)\bigr]^2 - a\bigl[y_k(t) - \eta x_k(t)\bigr] \\ &\qquad + y_k(t) \times e_k(t) + x_k(t) y_k(t) + 0.5 y_k^2(t) + 0.5 g_k^2(t, x_k) - \eta x_k(t) f_k\bigl(t, y_k(t) - \eta x_k(t)\bigr) \\ &\qquad - \sum_{h=1}^{l} a_{kh} y_k^2(t) + \frac{1}{2} \sum_{h=1}^{l} a_{kh}\bigl[y_k^2(t) + y_h^2(t)\bigr] \Biggr\} \\ &\quad \le \lambda \sum_{k=1}^{l} c_k \Biggl\{ -\eta x_k^2(t) - \eta^2 x_k(t) y_k(t) + \eta y_k^2(t) - \delta\bigl[y_k(t) - \eta x_k(t)\bigr]^2 - a\bigl[y_k(t) - \eta x_k(t)\bigr] \\ &\qquad + y_k(t) e_k + 0.5 y_k^2(t) + y_k(t) x_k(t) + 0.5\bigl[b^2 x_k^2(t) + d^2 + 2bd\bigl|x_k(t)\bigr|\bigr] \\ &\qquad + \eta\bigl|x_k(t)\bigr|\bigl[r\bigl|y_k(t)\bigr| + r\eta\bigl|x_k(t)\bigr| + e\bigr] + \frac{1}{2} \sum_{h=1}^{l} a_{kh}\bigl[y_h^2(t) - y_k^2(t)\bigr] \Biggr\} \\ &\quad \le \lambda \sum_{k=1}^{l} c_k \bigl\{ \bigl(-\eta - \delta\eta^2 + 0.5b^2 + 0.5bd + \eta^2 r + 0.5\eta l + 0.5\eta|a| + 0.5\eta e\bigr) x_k^2(t) \end{aligned}$$

$$
\begin{aligned}
&+ \big(\eta - \delta + 0.5\eta r + 0.5|a| + 1\big)y_k^2(t) + \big(-\eta^2 + 2\eta\delta +\big)x_k(t)y_k(t) \\
&+ 0.5\big((\overline{e_k})^2 + bd + d^2 + \eta e + |a|\eta + |a|\big)\Big\} + \frac{\lambda}{2}\sum_{h=1,h}^{l} c_k a_{kh} F_{hk}(y_k, y_h),
\end{aligned} \tag{2.2}
$$

where $F_{hk}(y_k, y_h) = y_h^2 - y_k^2$. By employing Lemma 2.2, we obtain

$$
\sum_{k,h=1}^{l} c_k a_{kh} F_{hk}(y_k, y_h) = 0,
$$

from which, together with (2.2), it follows that

$$
\begin{aligned}
&\frac{dV(x,y)}{dt} \\
&\quad\le \lambda \sum_{k=1}^{l} c_k \Big\{\big[-\eta + \big(-\delta\eta^2 + 0.5b^2 + 0.5bd + \eta^2 l + 0.5\eta l + 0.5\eta|a| + 0.5\eta e\big)\big]x_k^2(t) \\
&\qquad + \big(\eta - \delta + 0.5\eta l + 0.5|a| + 1\big)y_k^2(t) + \big(-\eta^2 + 2\eta\delta + 1\big)x_k(t)y_k(t) \\
&\qquad + 0.5\big((\overline{e_k})^2 + bd + d^2 + \eta e + |a|\eta + |a|\big)\Big\}.
\end{aligned} \tag{2.3}
$$

Since $A < 0$, $\delta < 0$, then $(-\eta^2 + 2\eta\delta + 1)^2 y_k^2(t) - 4A(\eta - \delta + 0.5\eta l + 0.5|a| + 1)y_k^2(t) > 0$, $\forall y_k(t) \neq 0$, $x_k(t) \neq 0$. So the equation in $x_k(t) : Ax_k^2(t) + (\eta - \delta + 0.5\eta l + 0.5|a| + 1)y_k^2(t) + (-\eta^2 + 2\eta\delta + 1)x_k(t)y_k(t) = 0$ has two real roots x_1, x_2 ($x_1 < x_2$) for fixed k and

$$
x_1 = \frac{-B - \sqrt{B^2 - 4AC}}{2A} y_k(t), \qquad x_2 = \frac{-B + \sqrt{B^2 - 4AC}}{2A} y_k(t).
$$

Hence, when $x_k > x_2$, or $x_k < x_1$, $Ax_k^2(t) + (\eta - \delta + 0.5\eta l + 0.5|a| + 1)y_k^2(t) + (-\eta^2 + 2\eta\delta + 1)x_k(t)y_k(t) < 0$. Namely, when $|x_k| > \max\{|x_1|, |x_2|\} = r^*|y_k|$, $Ax_k^2(t) + (\eta - \delta + 0.5\eta l + 0.5|a| + 1)y_k^2(t) + (-\eta^2 + 2\eta\delta + 1)x_k(t)y_k(t) < 0$. So when

$$
\begin{aligned}
\big\|(x,y)^T\big\| &= \big\|(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n)^T\big\| \\
&= \sum_{k=1}^{n} \Big[\max_{t\in[0,\omega]} \big(\big|x_k(t)\big| + \big|y_k(t)\big|\big)\Big] > n\big(r^* + 1\big)|y_k|,
\end{aligned}
$$

$$
Ax_k^2(t) + \big(\eta - \delta + 0.5\eta r + 0.5|a| + 1\big)y_k^2(t) + \big(-\eta^2 + 2\eta\delta + 1\big)x_k(t)y_k(t) < 0.
$$

So there exists a positive constant r_1 such that, when $\|(x,y)^T\| > r_1$,

$$
Ax_k^2(t) + \big(\eta - \delta + 0.5\eta r + 0.5|a| + 1\big)y_k^2(t) + \big(-\eta^2 + 2\eta\delta + 1\big)x_k(t)y_k(t) < 0.
$$

Since $Ax_k^2(t) + (\eta - \delta + 0.5\eta r + 0.5|a| + 1)y_k^2(t) + (-\eta^2 + 2\eta\delta + 1)x_k(t)y_k(t)$ is decreasing in x_k when $x_k > r|y_k|$, and is increasing in x_k when $x_k < -r^*|y_k|$, hence there exists a positive H such that, when $\|(x,y)^T\| > H$,

$$
\begin{aligned}
&Ax_k^2(t) + q_k\big(\eta - \delta + 0.5\eta r + 0.5|a| + 1\big)y_k^2(t) + \big(-\eta^2 + 2\eta\delta + 1\big)x_k(t)y_k(t) \\
&\quad + 0.5\big((\overline{e_k})^2 + bd + d^2 + \eta e + |a|\eta + |a|\big) < 0.
\end{aligned}
$$

From (2.3), it follows that there exists a positive constant H, which is independent of λ, such that when

$$\left\|(x,y)^T\right\| \geq H, \qquad \frac{dV(x,y)}{dt} \leq 0. \tag{2.4}$$

Recalling the fact that $(x(t),y(t))^T$ is an ω-periodic solution, we see that $V(x(t),y(t))$ is an ω-periodic solution. On the other hand, if $\|(x(t),y(t))^T\| \geq H$, then $\frac{dV(x(t),y(t))}{dt} \leq 0$, which is in contradiction to the fact $V(x(t),y(t))$ is a continuous ω-periodic solution. Thus $\|(x(t),y(t))^T\| \leq H$. □

Remark 1 From the proof of Lemma 2.3, there exists a positive H such that when $\|(x,y)^T\| > H$, then

$$\begin{aligned} &Ax_k^2(t) + q_k\big(\eta - \delta + 0.5\eta r + 0.5|a| + 1\big)y_k^2(t) + \big(-\eta^2 + 2\eta\delta + 1\big)x_k(t)y_k(t) \\ &\quad + 0.5\big((\overline{e_k})^2 + bd + d^2 + \eta e + |a|\eta + |a|\big) < 0. \end{aligned}$$

3 The existence of periodic solutions

In this section, we establish two sufficient conditions on the existence of periodic solutions for system (1.7) by combining graph theory with Mawhin's continuation theorem of coincidence degree theory.

Theorem 3.1 *Under assumptions* (H_1), (H_2), *and assumption* (A_3), *system* (1.7) *has at least an* ω*-periodic solution.*

Proof We will establish the existence of periodic solutions of system (1.7) by using Lemma 2.1. Let

$$\begin{aligned} X = Z = \big\{&z = \big(x(t),y(t)\big)^T = \big(x_1(t),x_2(t),\dots,x_l(t),y_1(t),y_2(t),\dots,y_l(t)\big)^T : \\ &\big(x(t),y(t)\big)^T \in C^1\big(R,R^{2l}\big), x_i(t+\omega) = x_i(t), y_i(t+\omega) = y_i(t) \\ &(i = 1,2,\dots,l), t \in R\big\}. \end{aligned}$$

Denote

$$\left\|\big(x(t),y(t)\big)^T\right\| = \sum_{k=1}^{l}\Big[\max_{t\in[0,\omega]}\big|x_k(t)\big| + \max_{t\in[0,\omega]}\big|y_k(t)\big|\Big].$$

Then X and Z are Banach spaces with the norm $\|\cdot\|$. Set

$$\begin{aligned} G_k &= y_k(t) - \eta x_k(t), \\ F_k(t) &= -\eta^2 x_k(t) + \eta y_k(t) - f_k\big(t, y_k(t) - \eta x_k(t)\big) - g_k\big(t, x_k(t)\big) + e_k(t) \\ &\quad - \sum_{h=1}^{l} a_{kh}\big(y_k(t) - y_h(t)\big), \quad k = 1,2,\dots,l. \\ Lz &= z' = \big(x'(t),y'(t)\big)^T = \big(x_1'(t),x_2'(t),\dots,x_l'(t),y_1'(t),\dots,y_l'(t)\big)^T, \end{aligned}$$

$$Nz = \big(G_1(t), G_2(t), \ldots, G_l(t), F_1(t), F_2(t), \ldots, F_l(t)\big),$$
$$Pz = \frac{1}{\omega}\int_0^{\omega} z(t)\,dt, \quad z \in X; \qquad Qz = \frac{1}{\omega}\int_0^{\omega} z(t)\,dt, \quad z \in Z.$$

It is easy to show that $\operatorname{Dim}\operatorname{Ker} L = \operatorname{Dim} R^{2l} = 2l = \operatorname{codim}\operatorname{Im} L$. Hence, L is a Fredholm mapping of index zero. We can prove that

$$\operatorname{Im} P = \operatorname{Ker} L, \qquad \operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I - Q).$$

Furthermore, the generalized inverse K_P of L is as follows: $K_P : \operatorname{Im} L \to \operatorname{Ker} P \cap \operatorname{Dom} L$ exists and

$$K_P(z) = \int_0^t z(s)\,ds - \frac{1}{\omega}\int_0^{\omega}\int_0^t z(s)\,ds\,dt.$$

Thus

$$\begin{aligned} &QNz \\ &\quad = \left(\frac{1}{\omega}\int_0^{\omega} G_1(t)\,dt, \frac{1}{\omega}\int_0^{\omega} G_2(t)\,dt, \ldots, \frac{1}{\omega}\int_0^{\omega} G_n(t)\,dt,\right. \\ &\qquad \left.\frac{1}{\omega}\int_0^{\omega} F_1(t)\,dt, \frac{1}{\omega}\int_0^{\omega} F_2(t)\,dt, \frac{1}{\omega}\int_0^{\omega} F_n(t)\,dt\right)^T \end{aligned}$$

and

$$\begin{aligned} &K_P(I - Q)Nz \\ &\quad = \begin{pmatrix} \int_0^t G_1(s)\,ds - \frac{1}{\omega}\int_0^{\omega}\int_0^t G_1(s)\,ds\,dt - (\frac{t}{\omega} - \frac{1}{2})\int_0^{\omega} G_1(t)\,dt \\ \int_0^t G_2(s)\,ds - \frac{1}{\omega}\int_0^{\omega}\int_0^t G_2(s)\,ds\,dt - (\frac{t}{\omega} - \frac{1}{2})\int_0^{\omega} G_2(t)\,dt \\ \cdots \\ \int_0^t G_n(s)\,ds - \frac{1}{\omega}\int_0^{\omega}\int_0^t G_n(s)\,ds\,dt - (\frac{t}{\omega} - \frac{1}{2})\int_0^{\omega} G_n(t)\,dt \\ \int_0^t F_1(s)\,ds - \frac{1}{\omega}\int_0^{\omega}\int_0^t F_1(s)\,ds\,dt - (\frac{t}{\omega} - \frac{1}{2})\int_0^{\omega} F_1(t)\,dt \\ \int_0^t F_2(s)\,ds - \frac{1}{\omega}\int_0^{\omega}\int_0^t F_2(s)\,ds\,dt - (\frac{t}{\omega} - \frac{1}{2})\int_0^{\omega} F_2(t)\,dt \\ \cdots \\ \int_0^t F_n(s)\,ds - \frac{1}{\omega}\int_0^{\omega}\int_0^t F_n(s)\,ds\,dt - (\frac{t}{\omega} - \frac{1}{2})\int_0^{\omega} F_n(t)\,dt \end{pmatrix}^T \end{aligned}$$

Clearly, QN and $K_P(I-Q)N$ are continuous and $QN(\overline{\Omega})$ is bounded, where Ω is an open set in X. Then by Arzela–Ascoli theorem, we can prove that $\overline{K_P(I - Q)N(\overline{\Omega})}$ is compact. Hence, N is L-compact on $\overline{\Omega}$.

Corresponding to the operator equation $Lx = \lambda Nx$, $\lambda \in (0, 1)$, we have system (2.1). By Lemma 2.3, for every periodic solution $(x(t), y(t))^T = (x_1(t), x_2(t), \ldots, x_l(t), y_1(t), \ldots, y_l(t))^T$ of $Lz = \lambda Nz$, there is $H > 0$, which is independent of the choice of λ, such that $\|(x(t), y(t))^T\| < H$.

We set $\Omega = \{(x(t), y(t))^T \in X : \|(x, y)^T\| < H + r\}$, where $r > 0$ is chosen so that the bound is larger. Hence, for any $\lambda \in (0, 1)$, $z \in \partial\Omega \cap \operatorname{Dom} L$, $Lz \neq \lambda Nz$. When $z \in \partial\Omega \cap \operatorname{Ker} P$, we will show $QNz \neq 0$. When $z \in \partial\Omega \cap \operatorname{Ker} L$, $z \in R^{2l}$ (namely z is a constant vector) with

$\|z\| = \|(x,y)^T\| = H + r$. If z is a constant vector with $\|z\| = H + r$, $QNz = 0$, then it follows that the constant vector z with $\|z\| = H + r$ satisfies, for $k = 1, 2, \ldots, l$,

$$\frac{1}{\omega}\int_0^\omega G_k(t)\,dt = 0, \qquad \frac{1}{\omega}\int_0^\omega F_k(t)\,dt = 0.$$

Hence, there exist t_k $(i = 1, 2)$, $\xi_i \in [0, \omega]$ $(k = 1, 2, \ldots, l)$ such that

$$G_k(t_k) = 0, \qquad F_k(\xi_k) = 0. \tag{3.1}$$

From (3.1), we have

$$0 = \sum_{k=1}^{l} c_k\bigl(x_k G_k(t_k) + y_k F_k(\xi_k)\bigr). \tag{3.2}$$

By using the same proof as those of (2.4) in Lemma 2.3, from (3.2), it follows that

$$\begin{aligned} 0 &= \sum_{k=1}^{l} c_k\bigl(x_k G_k(t_k) + y_k F_k(\xi_k)\bigr) \\ &\le \sum_{k=1}^{l} c_k\bigl\{\bigl[-\eta + \bigl(-\delta\eta^2 + 0.5b^2 + 0.5bd + \eta^2 r + 0.5\eta l + 0.5\eta|a| + 0.5\eta e\bigr)\bigr]x_k^2 \\ &\quad + \bigl(\eta - \delta + 0.5\eta r + 0.5|a| + 1\bigr)y_k^2 + \bigl(-\eta^2 + 2\eta\delta + 1\bigr)x_k y_k \\ &\quad + 0.5\bigl((\overline{e_k})^2 + bd + d^2 + \eta e + |a|\eta + |a|\bigr)\bigr\}. \end{aligned} \tag{3.3}$$

It follows from Remark 1 that, since $\|(x,y)^T\| > H$,

$$\begin{aligned} &\sum_{k=1}^{l} \bigl\{\bigl[-\eta + \bigl(-\delta\eta^2 + 0.5b^2 + 0.5bd + \eta^2 r + 0.5\eta l + 0.5\eta|a| + 0.5\eta e\bigr)\bigr]x_k^2 \\ &\quad + \bigl(\eta - \delta + 0.5\eta r + 0.5|a| + 1\bigr)y_k^2 + \bigl(-\eta^2 + 2\eta\delta + 1\bigr)x_k y_k \\ &\quad + 0.5\bigl((\overline{e_k})^2 + bd + d^2 + \eta + |a|\eta + |a|\bigr)\bigr\} < 0, \end{aligned} \tag{3.4}$$

which contradicts with (3.3). Hence, for each $z \in \partial \cap \operatorname{Ker} L$, $QNz \neq 0$.

Finally, we show that $\deg_B\{JQN, \Omega \cap \operatorname{Ker} L, (0, 0, \ldots, 0)\} \neq 0$. We only show that $\deg_B\{JQNz, \Omega \cap \operatorname{Ker} L, (0, 0, \ldots, 0)\} \neq 0$, when $z \in \partial\Omega \cap \operatorname{Ker} L$. To this end, we construct the following mapping for $k = 1, 2, \ldots, l$:

$$\begin{aligned} &L(x, y, \mu) \\ &\quad = (1-\mu)\Biggl(y_1 - \eta x_1, y_2 - \eta x_2, \ldots, y_n - \eta x_n, \\ &\qquad -\eta^2 x_1 + \eta y_1 - f_1(\xi_1, y_1 - \eta x_1) - g_1(\xi_1, x_1) + e_1(\xi_1) - \sum_{h=1}^{l} a_{1h}(y_1 - y_h), \\ &\qquad -\eta^2 x_2 + \eta y_2 - f_2(\xi_2, y_2 - \eta x_2) - g_2(\xi_2, x_2) + e_2(\xi_2) - \sum_{h=1}^{l} a_{2h} \times (y_2 - y_h), \end{aligned}$$

$$\begin{aligned}
&\cdots,\\
&-\eta^2x_l+\eta y_l-f_l(\xi_l,y_l-\eta x_l)-g_l(\xi_l,x_l)+e_l(\xi_l)-\sum_{h=1}^{l}a_{lh}(y_l-y_h)\Bigg)\\
&+\mu(m_1x_1+n_1y_1,m_2x_2+n_2y_2,\dots,m_lx_l+n_ly_l,u_1x_1+v_1y_1,u_2x_2+v_2y_2,\\
&\cdots,u_lx_l+v_ly_l),
\end{aligned}$$

where $\mu\in[0,1]$ is a parameter, m_k, n_k, u_k, v_k $(k=1,2,\dots,l)$ are chosen constants. We show that the mapping $L(x,y,\mu)$ is a homotopic mapping. Namely, we show when $(x,y)^T\in\partial\Omega\cap\operatorname{Ker}L$, $\mu\in[0,1]$, $L(x,y,\mu)\neq0$. If when $(x,y)^T\in\partial\Omega\cap\operatorname{Ker}L$, $\mu\in[0,1]$, $L(x,y,\mu)=0$, then for $k=1,2,\dots,l$,

$$(1-\mu)(y_k-\eta x_k)+\mu(m_kx_k+n_ky_k)=0,\tag{3.5}$$

$$\begin{aligned}
&(1-\mu)\Bigg[-\eta^2x_k+\eta y_k-f_k(\xi_k,y_k-\eta x_k)-g_k(\xi_k,x_k)+e_k(\xi_k)\\
&\quad-\sum_{h=1}^{l}a_{kh}(y_k-y_h)\Bigg]+\mu(u_kx_k+v_ky_k)=0.
\end{aligned}\tag{3.6}$$

From (3.5) and (3.6), it follows that

$$\begin{aligned}
0 &= \sum_{k=1}^{l}c_k\Bigg\{x_k\big[(1-\mu)(y_k-\eta x_k)+\mu(m_kx_k+n_ky_k)\big]\\
&\quad+y_k\Bigg[(1-\mu)\Bigg(-\eta^2x_k-f_k(\xi_k,y_k-\eta\times x_k)+g_k(\xi_k,x_k)+e_k(\xi_k)\\
&\quad-\sum_{h=1}^{l}a_{kh}(y_k-y_h)\Bigg)+\mu(u_kx_k+v_ky_k)\Bigg]\Bigg\}\\
&\le\sum_{k=1}^{l}c_k\Bigg\{(1-\mu)p_k\big(x_ky_k-\eta x_k^2\big)+\mu^*\big(m_kx_k^2+n_kx_ky_k\big)-(1-\mu)\eta^2x_ky_k\\
&\quad+(1-\mu)\eta y_k^2+\mu\times\big(v_ky_k^2+u_kx_ky_k\big)\\
&\quad+(1-\mu)[-\delta(y_k-\eta x_k)^2-a(y_k-\eta x_k)+\eta r|x_k||y_k|+\eta^2rx_k^2+\eta e|x_k|\\
&\quad+y_k^2+0.5\Bigg[bd+bd+(\overline{e_k})^2-\sum_{h=1}^{l}a_{kh}\big(y_k^2-y_h^2\big)+\eta e+\eta|a|+|a|\Bigg]\Bigg\}\\
&\le\sum_{k=1}^{l}c_k\Bigg\{\big[-\eta+\big(-\delta\eta^2+0.5b^2+0.5bd+\eta^2r-\eta l+0.5\eta|a|+0.5\eta e\big)\\
&\quad+\mu\big(m_k+\eta-\eta^2r+\delta\eta^2-0.5\eta e-0.5\eta r\big)\big]x_k^2\\
&\quad+\big[-\eta^2+2\eta\delta+1+\mu\big(l_k+u_k-1+\eta^2-2\delta\eta\big)\big]x_ky_k\\
&\quad+\big[\big(\eta-\delta+0.5\eta r+0.5|a|+1\big)+\mu\big(v_k-\eta+\delta-1-0.5\eta l\big)\big]y_k^2
\end{aligned}$$

$$\begin{aligned}&+0.5\bigl((\overline{e_k})^2+bd+d^2+\eta e+|a|\eta+|a|\bigr)\\&-0.5(1-\mu)\sum_{h=1,k=1}^{l}c_ka_{kh}\bigl(y_h^2-y_k^2\bigr)\Biggr\},\end{aligned}\tag{3.7}$$

from which it follows that, since $\sum_{h=1,k=1}^{l}c_ka_{kh}(y_h^2-y_k^2)=0$,

$$\begin{aligned}0\le\sum_{k=1}^{l}c_k\bigl\{&\bigl[-\eta-\delta\eta^2+0.5b^2+0.5bd+\eta^2r-\eta l+0.5\eta|a|+0.5\eta e\\&+\mu\bigl(m_k+\eta-\eta^2r+\delta\eta^2-0.5\eta e-0.5\eta r\bigr)\bigr]x_k^2\\&+\bigl[-\eta^2+2\eta\delta+1+\mu\bigl(n_k+u_k-1+\eta^2-2\delta\eta\bigr)\bigr]x_ky_k\\&+\bigl[\bigl(\eta-\delta+0.5\eta r+0.5|a|+1\bigr)+\mu(v_k-\eta+\delta-1-0.5\eta r)\bigr]y_k^2\\&+0.5\bigl((\overline{e_k})^2+bd+d^2+\eta e+|a|\eta+|a|\bigr)\bigr\}.\end{aligned}\tag{3.8}$$

Choose m_k, v_k, l_k, u_k such that

$$m_k+\eta-\eta^2r+\delta\eta^2-0.5\eta e-0.5\eta r=0,\tag{3.9}$$

$$n_k+u_k-1+\eta^2-2\delta\eta=0,\tag{3.10}$$

and

$$v_k-\eta+\delta-1-0.5\eta r=0.\tag{3.11}$$

Substituting (3.9)–(3.11) into (3.8) gives

$$\begin{aligned}0\le\sum_{k=1}^{l}\Bigl\{&\bigl[-\eta+\bigl(-\delta\eta^2+0.5b^2+0.5bd+\eta^2r+0.5\eta r+0.5\eta|a|+0.5\eta e\bigr)\bigr]x_k^2\\&+\bigl(-\eta^2+2\eta\delta+1\bigr)x_ky_k+\bigl(\eta-\delta+0.5\eta r+0.5|a|+1\bigr)y_k^2\\&+0.5\bigl((\overline{e_k})^2+bd+d^2+\eta e+|a|\eta+|a|\bigr)\Bigr\}.\end{aligned}\tag{3.12}$$

Since $\|(x,y)^T\|>H$, we have from Remark 1

$$\begin{aligned}\sum_{k=1}^{l}\Bigl\{&\bigl[-\eta+\bigl(-\delta\eta^2+0.5b^2+0.5bd+\eta^2r+0.5\eta l+0.5\eta|a|+0.5\eta e\bigr)\bigr]\\&+x_k^2\bigl(-\eta^2+2\eta\delta+1\bigr)x_ky_k+\bigl(\eta-\delta+0.5\eta r+0.5|a|+1\bigr)y_k^2\\&+0.5\bigl((\overline{e_k})^2+bd+d^2+\eta e+|a|\eta+|a|\bigr)\Bigr\}<0.\end{aligned}\tag{3.13}$$

Equation (3.13) contradicts with (3.12). Hence, $L(x,y,\mu)$ is a homotopic mapping, by topological degree theory, we have

$$\begin{aligned}&\deg_B\bigl(JQN,\partial\Omega\cap\operatorname{Ker}L,(0,0,\dots,0)\bigr)\\&\quad=\deg_B\bigl(L(x,y,0),\partial\Omega\cap\operatorname{Ker}L,(0,0,\dots,0)\bigr)\end{aligned}$$

$$= \deg_B\bigl(L(x,y,1), \partial\Omega \cap \operatorname{Ker} L, (0,0,\dots,0)\bigr)$$
$$= \deg_B(m_1x_1 + n_1y_1, m_2x_2 + n_2y_2, \dots, m_lx_l + n_ly_l, u_1x_1 + v_1y_1, \dots, u_lx_l + v_ly_l)$$
$$= \operatorname{sign}\begin{vmatrix} E & F \\ M & N \end{vmatrix}, \tag{3.14}$$

where

$$E = \operatorname{diag}(m_1, m_2, \dots, m_l), \qquad F = \operatorname{diag}(n_1, n_2, \dots, n_l),$$
$$M = \operatorname{diag}(u_1, u_2, \dots, u_l), \qquad N = \operatorname{diag}(v_1, v_2, \dots, v_l).$$

Since

$$\begin{vmatrix} E & F \\ M & N \end{vmatrix} = |EM - FN| = \prod_{k=1}^{l}(m_kv_k - l_ku_k). \tag{3.15}$$

Then substituting (3.15) into (3.14) gives

$$\deg_B\bigl(JQN, \partial\Omega \cap \operatorname{Ker} L, (0,0,\dots,0)\bigr)$$
$$= \operatorname{sign}\prod_{k=1}^{l}(m_kv_k - n_ku_k)$$
$$= \operatorname{sign}\prod_{k=1}^{l}\bigl((-\eta + \eta^2r - \delta\eta^2 + 0.5\eta e + 0.5\eta l)(\eta - \delta + 1 + 0.5\eta r) - n_ku_k\bigr).$$

Again choose n_k, u_k such that

$$n_ku_k \neq \bigl(-\eta + \eta^2r - \delta\eta^2 + 0.5\eta e + 0.5\eta r\bigr)(\eta - \delta + 1 + 0.5\eta r).$$

Then

$$\deg_B\bigl(JQN, \partial\Omega \cap \operatorname{Ker} L, (0,0,\dots,0)\bigr) \neq 0.$$

By Lemma 2.1, system (1.7) has at least an ω-periodic solution. This completes the proof of Theorem 3.1. □

Remark 2 In the proof of Theorem 3.1, m_k, n_k, u_k, v_k are chosen such that $m_k + \eta - \eta^2r + \delta\eta^2 - 0.5\eta e - 0.5\eta r = 0$, $n_k + u_k - 1 + \eta^2 - 2\delta\eta = 0$, $v_k - \eta + \delta - 1 - 0.5\eta r = 0$, $n_ku_k \neq (-\eta + \eta^2r - \delta\eta^2 + 0.5\eta e + 0.5\eta r)(\eta - \delta + 1 + 0.5\eta r)$. Such n_k, u_k indeed exist, for example, letting $\delta = -1$, $\eta = 0.5$, $b = d = 0.001$, $|a| = 0.003$, $r = 3$, $e = 0.003$, then $v_k = 3.25$, $m_k = 1.25075$, n_k, u_k satisfy $n_k + u_k = -0.25$, $n_ku_k \neq 3.25 \times 1.25075$. Thus by taking $n_k = -0.5$, $u_k = 0.25$, the task can be fulfilled.

Remark 3 In our Theorem 3.1, conditions (A_4) and (A_5) in Theorem 1 in [21] are removed and conditions (A_1) and (A_2) in Theorem 1 in [21] are replaced with conditions (H_1) and (H_2). Hence, our result on the existence of periodic solutions for a coupled Rayleigh system is different from that obtained in Theorem 1 in [21].

Theorem 3.2 *Under* (A_1)–(A_3), *system* (1.7) *has at least one* ω*-periodic solution.*

Proof Define the same X, Z, G_k, F_k, L, N, P, and Q as those in the proof of Theorem 3.1, where the norm of X is different from that of X in the proof of Theorem 3.1. Here, we define the norm of X by $\|x\| = (\sum_{k=1}^{l} \max_{t\in[0,\omega]}[|x_k(t)|^2 + |y_k(t)|^2])^{\frac{1}{2}}$.

Clearly, QN and $K_P(I-Q)N$ are continuous and $QN(\overline{\Omega})$ is bounded, where Ω is an open set in X. Then by Arzela–Ascoli theorem, we can prove that $\overline{K_P(I-Q)N(\overline{\Omega})}$ is compact. Hence, N is L-compact on $\overline{\Omega}$.

Corresponding to the operator equation $Lx = \lambda Nx$, $\lambda \in (0,1)$, we have system (2.1), a.e., system (4) in Lemma 3 of [21]. By Lemma 3 in [21], for every periodic solution $(x(t),y(t))^T = (x_1(t),x_2(t),\dots,x_l(t),y_1(t),\dots,y_l(t))^T$ of $Lz = \lambda Nz$, there is $H > 0$, which is independent of the choice of λ such that $\|(x(t),y(t))^T\| < H$. We set $\Omega = \{(x(t),y(t))^T \in X : \|(x,y)^T\| < H + r\}$, where $r > 0$ is a chosen positive constant such that the bound of Ω is larger. Hence, for any $\lambda \in (0,1)$, $z \in \partial\Omega \cap \operatorname{Dom} L$, $Lz \neq \lambda Nz$. When $z \in \partial\Omega \cap \operatorname{Ker} P$, we will show $QNz \neq 0$. When $z \in \partial\Omega \cap \operatorname{Ker} L$, $z \in R^{2l}$ (namely z is a constant vector) with $\|z\| = \|(x,y)^T\| = H + r$. If z is a constant vector with $\|z\| = H + r$, $QNz = 0$, then it follows that the constant vector z with $\|z\| = H + r$ satisfies, for $k = 1,2,\dots,l$,

$$\frac{1}{\omega}\int_0^{\omega} G_k(t)\,dt = 0, \qquad \frac{1}{\omega}\int_0^{\omega} F_k(t)\,dt = 0.$$

Hence, there exist t_k $(i = 1,2)$, $\xi_i \in [0,\omega]$ $(k = 1,2,\dots,l)$ such that

$$G_k(t_k) = 0, \qquad F_k(\xi_k) = 0. \tag{3.16}$$

From (3.16), we have

$$0 = \sum_{k=1}^{l} c_k\bigl(x_k G_k(t_k) + y_k F_k(\xi_k)\bigr). \tag{3.17}$$

From the proof of page 4 in Lemma 3 of [21], it follows from (3.17) that

$$\begin{aligned} 0 &= \sum_{k=1}^{l}\bigl(x_k G_k(t_k) + y_k F_k(\xi_k)\bigr) \\ &= \sum_{k=1}^{l}\Bigl\{-\eta x_k^2(t) - \eta^2 x_k(t)y_k(t) + \eta y_k^2(t) - y_k(t)f_k\bigl(t, y_k(t) - \eta x_k(t)\bigr) \\ &\quad + y_k(t)\bigl[x_k(t) - g_k\bigl(t,x_k(t)\bigr)\bigr] + y_k(t)e_k(t) - \sum_{h=1}^{l} a_{kh}y_k(t)\bigl[y_k(t) - y_h(t)\bigr]\Bigr\} \\ &< \sum_{k=1}^{l} c_k\biggl[-\frac{\eta^2}{4}x_k^2 - \frac{\eta^2}{2}y_k^2 + (\overline{e_k})^2\biggr] < 0. \end{aligned} \tag{3.18}$$

This a contradiction. Hence, for each $z \in \partial \cap \operatorname{Ker} L$, $QNz \neq 0$.

Finally, we show that $\deg_B\{JQN, \Omega \cap \operatorname{Ker} L, (0,0,\dots,0)\} \neq 0$. We only show that $\deg_B\{JQNz, \Omega \cap \operatorname{Ker} L, (0,0,\dots,0)\} \neq 0$ when $z \in \partial\Omega \cap \operatorname{Ker} L$. To this end, we construct the following mapping for $k = 1,2,\dots,l$:

$$\begin{aligned}
&L_1(x,y,\mu^*) \\
&\quad = (1-\mu^*)\Bigg[y_1 - \eta x_1, y_2 - \eta x_2, \dots, y_n - \eta x_n, \\
&\qquad -\eta^2 x_1 + \eta y_1 - f_1(\xi_1, y_1 - \eta x_1) - g_1(\xi_1, x_1) + e_1(\xi_1) - \sum_{h=1}^{l} a_{1h}(y_1 - y_h), \\
&\qquad -\eta^2 x_2 + \eta y_2 - f_2(\xi_2, y_2 - \eta x_2) - g_2(\xi_2, x_2) - \sum_{h=1}^{l} a_{2h}(y_2 - y_h) + e_2(\xi_2), \dots, \\
&\qquad -\eta^2 x_l + \eta y_l - f_l(\xi_l, y_l - \eta x_l) - g_l(\xi_l, x_l) + e_l(\xi_l) - \sum_{h=1}^{l} a_{lh}(y_l - y_h)\Bigg] \\
&\qquad + \mu^* \times \big(m_1^* x_1 + n_1^* y_1, m_2^* x_2 + n_2^* y_2, \dots, m_l^* x_l + n_l^* y_l, u_1^* x_1 + v_1^* y_1, \\
&\qquad u_2^* x_2 + v_2^* y_2, \dots, u_l^* x_l + v_l^* y_l\big),
\end{aligned}$$

where $\mu^* \in [0,1]$ is a parameter, m_k^*, r_k^*, u_k^*, v_k^* $(k = 1,2,\dots,l)$ are chosen constants. We show that the mapping $L_1(x,y,\mu^*)$ is a homotopic mapping. Namely, we show when $(x,y)^T \in \partial\Omega \cap \operatorname{Ker} L$, $\mu^* \in [0,1]$, $L_1(x,y,\mu^*) \neq 0$. If when $(x,y)^T \in \partial\Omega \cap \operatorname{Ker} L$, $\mu^* \in [0,1]$, $L_1(x,y,\mu^*) = 0$, then for $k = 1,2,\dots,l$,

$$(1-\mu^*)(y_k - \eta x_k) + \mu^*(m_k^* x_k + n_k^* y_k) = 0, \tag{3.19}$$

$$\begin{aligned}
&(1-\mu^*)\Bigg[-\eta^2 x_k + \eta y_k - f_k(\xi_k, y_k - \eta x_k) - g_k(\xi_k, x_k) + e_k(\xi_k) \\
&\quad - \sum_{h=1}^{l} a_{kh}(y_k - y_h)\Bigg] + \mu^*(u_k^* x_k + v_k^* y_k) = 0.
\end{aligned} \tag{3.20}$$

From (3.19) and (3.20), it follows that

$$\begin{aligned}
0 = \sum_{k=1}^{l} c_k \Bigg\{ & x_k\big[(1-\mu^*)(y_k - \eta x_k) + \mu^*(m_k^* x_k + l_k^* y_k)\big] \\
& + y_k\Bigg[(1-\mu^*)\Bigg(-\eta^2 x_k + \eta y_k - f_k(\xi_k, y_k - \eta x_k) - g_k(\xi_k, x_k) + e_k(\xi_k) \\
& - \sum_{h=1}^{l} a_{kh}(y_k - y_h)\Bigg) + \mu^*(u_k^* x_k + v_k^* y_k)\Bigg]\Bigg\}.
\end{aligned} \tag{3.21}$$

Let $f_k(\xi_k, y_k - \eta x_k) = (y_k - \eta x_k)\beta_k(\xi_k, y_k - \eta x_k)$, then from (A_2) we can easily obtain that $0 < \frac{2(\eta+1)}{2-\eta} \le \beta_k(\xi_k, y_k - \eta x_k) \le 2$. By using (A_1), (A_2), we have from (3.21)

$$\begin{aligned}
0 &= \sum_{k=1}^{l} c_k \Biggl\{ (1-\mu^*)(x_k y_k - \eta x_k^2) + \mu^*(m_k^* x_k^2 + n_k^* x_k y_k) \\
&\quad + (1-\mu^*)\Biggl(-\eta^2 x_k y_k + \eta y_k^2 - y_k^2 \beta_k(\xi_k, y_k - \eta x_k) \\
&\quad + \eta x_k y_k \beta_k(\xi_k, y_k - \eta x_k) + y_k\bigl[x_k(t) - g_k(\xi_k, x_k)\bigr] - x_k y_k + e_k(\xi_k) \times y_k \\
&\quad - \sum_{h=1}^{l} a_{kh}(y_k^2 - y_h y_k) \Biggr) + \mu^*(u_k^* x_k y_k + v_k^* y_k^2) \Biggr\} \\
&\le \sum_{k=1}^{l} c_k \Biggl\{ (1-\mu^*)(x_k y_k - \eta x_k^2) \\
&\quad + \mu^*(m_k^* x_k^2 + n_k^* x_k y_k) + (1-\mu^*)\Biggl(-\eta^2 x_k y_k + \eta y_k^2 - y_k^2 \times \beta_k(\xi_k, y_k - \eta x_k) \\
&\quad + \eta x_k y_k \beta_k(\xi_k, y_k - \eta x_k) + y_k^2 + 0.5\bigl[x_k(t) - g_k(\xi_k, x_k)\bigr]^2 - x_k y_k \\
&\quad + 0.5 \times e_k^2(\xi_k) + 0.5 \sum_{h=1}^{l} a_{kh}(y_h^2 - y_k^2) \Biggr) + \mu^*(u_k^* x_k y_k + v_k^* y_k^2) \Biggr\},
\end{aligned}$$

from which it follows that noting $\sum_{k=1}^{l} c_k a_{kh}(y_h^2 - y_k^2) = 0$,

$$\begin{aligned}
0 &\le \sum_{k=1}^{l} c_k \bigl\{ (1-\mu^*)(x_k y_k - \eta x_k^2) + \mu^*(m_k^* x_k^2 + n_k^* x_k y_k) \\
&\quad + (1-\mu^*)\bigl(-\eta^2 x_k y_k + \eta y_k^2 - y_k^2 \times \beta_k(\xi_k, y_k - \eta x_k) \\
&\quad + \eta x_k y_k \beta_k(\xi_k, y_k - \eta x_k) + y_k^2 + 0.5 x_k^2 - \delta x_k^2 + 0.5\mu_1 x_k^2 - x_k y_k + 0.5 \\
&\quad \times e_k^2(\xi_k)\bigr) + \mu^*(u_k^* x_k y_k + v_k^* y_k^2) \bigr\} \\
&= \sum_{k=1}^{l} c_k \bigl\{ \bigl(\eta\bigl(\beta_k(\xi_k, y_k - \eta x_k) - \eta\bigr) + \mu^*\bigl[n_k^* + \eta\bigl(\eta - \beta_k(\xi_k, y_k - \eta x_k)\bigr) + u_k^*\bigr]\bigr) x_k y_k \\
&\quad + \bigl[0.5 - \delta + 0.5\mu_1 - \eta + \mu^*(m_k^* + \eta - 0.5 + \delta - 0.5\mu_1)\bigr] x_k^2 \\
&\quad + \bigl(\eta - \beta_k(\xi_k, y_k - \eta x_k) + 1 + \mu^*(-\eta\beta_k(\xi_k, y_k - \eta x_k) - 1 + v_k^*)\bigr) y_k^2 \\
&\quad + 0.5(\overline{e_k})^2 \bigr\}. \qquad (3.22)
\end{aligned}$$

Noting that

$$\eta\bigl[\beta_k(\xi_k, y_k - \eta x_k) - \eta\bigr] \ge \eta\left(\frac{2(\eta+1)}{2-\eta} - \eta\right) = \eta\left(\frac{\eta^2+2}{2-\eta}\right) > 0,$$

$$\eta\bigl[\beta_k(\xi_k, y_k - \eta x_k) - \eta\bigr] x_k y_k \le 0.5\eta\bigl[\beta_k(\xi_k, y_k - \eta x_k) - \eta\bigr](x_k^2 + y_k^2)$$

and

$$\begin{aligned}&\mu^*\{n_k^*+u_k^*+\eta[\eta-\beta_k(\xi_k,y_k-\eta x_k)]\}x_ky_k\\&\quad\le 0.5\mu^*\{|n_k^*|+|u_k^*|+\eta[\beta_k(\xi_k,y_k-\eta x_k)-\eta]\}(x_k^2+y_k^2),\end{aligned}$$

it follows from (3.22) that

$$\begin{aligned}0&\le\sum_{k=1}^{l}c_k\{(0.5-\delta+0.5\mu_1-\eta+0.5\eta[\beta_k(\xi_k,y_k-\eta x_k)-\eta]\\&\quad+\mu^*[m_k^*+\eta-0.5+\delta-0.5\mu_1+0.5(|n_k^*|+|u_k^*|)+0.5\eta[\beta_k(\xi_k,y_k-\eta x_k)-\eta]])x_k^2\\&\quad+(0.5\eta[\beta_k(\xi_k,y_k-\eta x_k)-\eta]+\eta+1-\beta_k(\xi_k,y_k-\eta x_k)\\&\quad+\mu^*[-\eta\beta_k(\xi_k,y_k-\eta x_k)-1+v_k^*+0.5(|n_k^*|+|u_k^*|)+\eta[-\eta\beta_k(\xi_k,y_k-\eta x_k)]])y_k^2\\&\quad+0.5(\overline{e_k})^2\}\\&=\sum_{k=1}^{l}c_k\biggl\{\biggl[-\frac{1}{2}(2\delta-\mu_1-1+\eta^2+\eta[2-\beta_k(\xi_k,y_k-\eta x_k)])\\&\quad-\frac{1}{2}\mu^*(-2m_k^*-4\eta+1-2\delta+\mu_1+\eta^2-|n_k^*|-|u_k^*|+\eta[2-\beta_k(\xi_k,y_k-\eta x_k)])\biggr]x_k^2\\&\quad+\biggl[-\frac{1}{2}\Bigl(\eta^2+(2-\eta)\beta_k(\xi_k,y_k-\eta x_k)-2-2\eta\Bigr)+\mu^*\Bigl(-\eta\beta_k(\xi_k,y_k-\eta x_k)-1\\&\quad+v_k^*+0.5(|n_k^*+|u_k^*|)+\eta[\beta_k(\xi_k,y_k-\eta x_k)-\eta]\Bigr)\biggr]y_k^2+(\overline{e_k})^2\biggr\}\qquad(3.23)\\&\le\sum_{k=1}^{l}c_k\biggl\{-\frac{\eta^2}{4}x_k^2-\frac{\eta^2}{2}y_k^2+(\overline{e_k})^2-\frac{1}{2}\mu^*(-2m_k^*-4\eta+1-2\delta+\mu_1+\eta^2\\&\quad-|n_k^*|-|u_k^*|)|x_k^2-\frac{1}{2}\mu^*(2-2v_k^*-|n_k^*|-|u_k^*|+2\eta^2)y_k^2\biggr\}.\qquad(3.24)\end{aligned}$$

Choose v_k^*, m_k^*, u_k^*, n_k^* such that

$$2v_k^*=2-|n_k^*|-|u_k^*|+2\eta^2\tag{3.25}$$

and

$$2m_k^*=1-4\eta-2\delta+\mu_1+\eta^2-|n_k^*|-|u_k^*|.\tag{3.26}$$

Substituting (3.25) and (3.26) into (3.24) gives

$$0\le\sum_{k=1}^{l}c_k\biggl[-\frac{\eta^2}{4}x_k^2-\frac{\eta^2}{2}y_k^2+(\overline{e_k})^2\biggr].\tag{3.27}$$

From the proof of Lemma 3 in [21], we have

$$\sum_{k=1}^{l}c_k\biggl[-\frac{\eta^2}{4}x_k^2-\frac{\eta^2}{2}y_k^2+(\overline{e_k})^2\biggr]<0.\tag{3.28}$$

The rest of the proof is similar to that of the corresponding part in Theorem 3.1, and it is omitted. □

Remark 4 In our Theorem 3.2, conditions (A_4) and (A_5) in Theorem 1 in [21] are removed, the remaining conditions (A_1)–(A_3) are the same. Hence, our result improves Theorem 1 in [21].

Remark 5 By applying new inequality techniques, we establish new sufficient conditions for the existence of periodic solutions of a coupled Rayleigh system. Our method can be applied to studying the existence of periodic solutions for any second-order differential system.

4 Numerical test

Example 1 Consider the following Rayleigh system:

$$\begin{cases} x_k'(t) = y_k(t) - \eta x_k(t), \\ y_k'(t) = -\eta^2 x_k(t) + \eta y_k(t) - f_k(t, y_k(t) - \eta x_k(t)) - g_k(t, x_k(t)) \\ \qquad + e_k(t) - \sum_{h=1}^{l} a_{kh}[y_k(t) - y_h(t)], \end{cases} \tag{4.1}$$

where $k = 1,2,3,4$, $\eta = 0.4$ and $g_k(t, x_k(t)) = 0.05|x_k(t)| + 0.05\cos x_k(t) + 0.05\sin x_k$, $f_k(t, x_k(t)) = (0.5 + 0.6\sin x_k(t))x_k(t) + 0.003$, $e_k(t) = 1 + \cos t$.

We can check that $|g_k(t, x_k)| \le 0.05|x_k(t)| + 0.06$, and we take $b = 0.005$, $d = 0.06$. $x_k f_k(t, x_k) \ge -0.1x_k^2 + 0.03x_k$, and $\delta = -0.1$, $a = 0.03$. Taking $\eta = 0.4$, we get $A = -\eta - \delta\eta^2 + 0.5b^2 + 0.5bd + \eta^2 + 0.5\eta r + 0.5\eta|a| + 0.5\eta e < 0$, thus conditions (H_1), (H_2) are satisfied.

Since $g_k(t, x)$ is not differential in x_k, thus condition (A_4) in [21] cannot be satisfied; since $\int_0^1 (1 + \cos t)\,dt \neq 0$, hence condition (A_5) in [21] cannot be satisfied, hence the existence of periodic solutions of system (4.1) cannot be verified by these results in [21]. Assuming that

$$B = (a_{kh})_{4\times 4} = \begin{pmatrix} 0 & 2 & 0.6 & 1 \\ 0.3 & 0 & 3 & 0.4 \\ 3 & 0.5 & 0 & 2 \\ 2 & 0.6 & 2 & 0 \end{pmatrix},$$

Figure 1 The phrase plan $(x_1(t), y_1(t))$

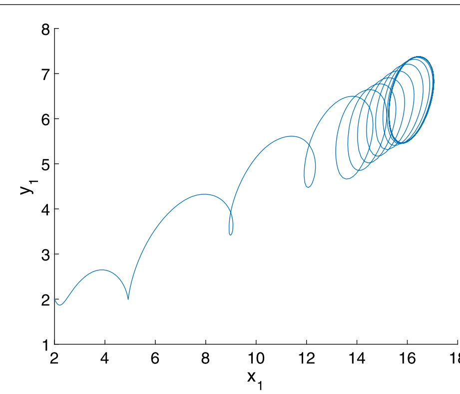


Figure 2 The phrase plan $(x_2(t), y_2(t))$

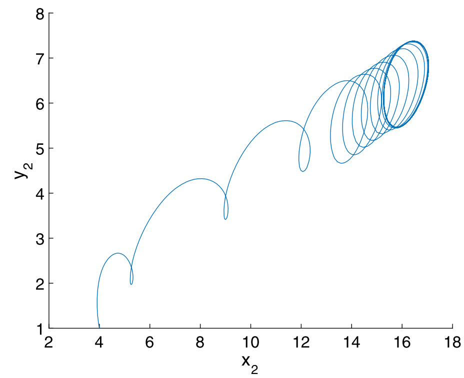


Figure 3 The phrase plan $(x_3(t), y_3(t))$

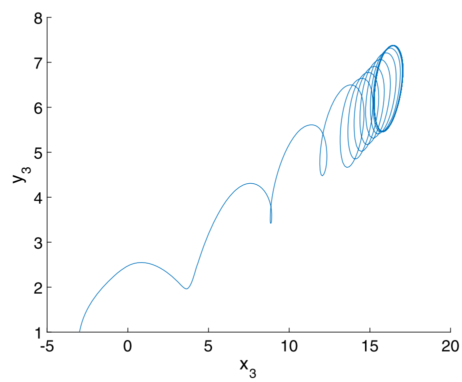
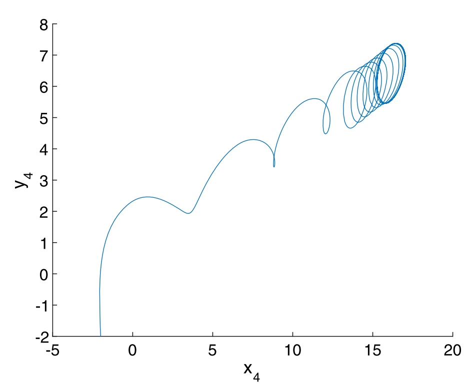


Figure 4 The phrase plan $(x_4(t), y_4(t))$



we can check that condition (A_3) holds. Now, all the conditions in Theorem 3.1 in our paper are satisfied. The solution of system (4.1) is shown in Figs. 1–4, from which we can clearly see that system (4.1) has at least one periodic solution.

Example 2 In system (4.1), we set $g_k(t,x_k(t)) = (1 + 0.001|\sin x_k(t)| + 0.001\sin t)x_k(t)$, $f_k(t,x_k(t)) = 0.2x_k(t)\sin 2t$, $e_k(t) = \sin t + 1$. It is easy to verify that (A_1), (A_2), and (A_3) are

Figure 5 The phrase plan $(x_1(t), y_1(t))$

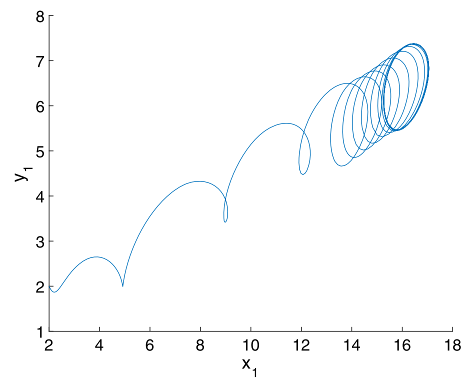
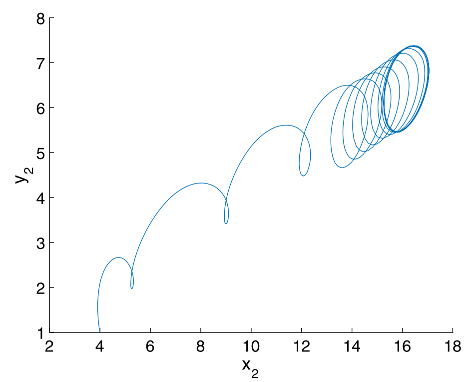


Figure 6 The phrase plan $(x_2(t), y_2(t))$



satisfied assuming that

$$B = (a_{kh})_{4\times 4} = \begin{pmatrix} 0 & 2 & 6 & 1 \\ 0.3 & 0 & 1 & 0.4 \\ 3 & 0.5 & 0 & 2 \\ 2 & 0.6 & 2 & 0 \end{pmatrix}.$$

But (A_4) is not satisfied since $m_k(x_k)$ contains $|\sin x_k(t)|$, which is not differential. Hence, the existence of periodic solutions of system (4.1) cannot be verified by the results in [21]. On the other hand, by our Theorem 3.1, system (4.1) has at least one ω-periodic solution.

The solution of system (4.1) is shown in Figs. 5–8, from which we can clearly see that system (4.1) has at least one periodic solution.

5 Conclusion

In the paper, we discuss the existence of periodic solutions for a class of coupled Rayleigh systems by combining graph theory with continuation theorem as well as Lyapunov functions. By the above study methods and by using novel inequality techniques, we obtain new sufficient conditions to ensure the existence of periodic solutions for system (1.7). Our results and method are completely new.

Figure 7 The phrase plan $(x_3(t), y_3(t))$

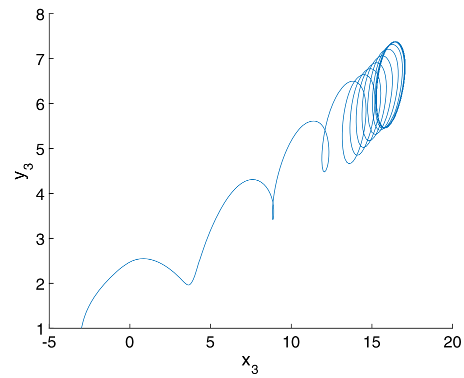
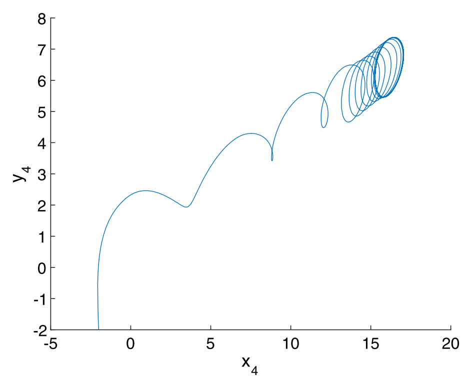


Figure 8 The phrase plan $(x_4(t), y_4(t))$



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