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Oscillation criteria for even-order neutral differential equations with distributed deviating arguments

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Abstract

In this work, new conditions are obtained for the oscillation of solutions of the even-order equation

$$(r(\zeta)z^{(n-1)}(\zeta))' + \int_a^b q(\zeta,s)f(x(g(\zeta,s)))\,\mathrm{d}s = 0, \quad \zeta \geq \zeta_0,$$

where $n \geq 2$ is an even integer and $z(\zeta) = x^{\alpha}(\zeta) + p(\zeta)x(\sigma(\zeta))$. By using the theory of comparison with first-order delay equations and the technique of Riccati transformation, we get two various conditions to ensure oscillation of solutions of this equation. Moreover, the importance of the obtained conditions is illustrated via some examples.

MSC: 34C10; 34K11

Keywords: Distributed deviating argument; Even order; Neutral differential equation; Oscillation

1 Introduction

In this work, we establish the oscillatory behavior of the nth-order neutral equation

$$\left(rz^{(n-1)}\right)'(\zeta) + \int_a^b q(\zeta,s)f\left(x\left(g(\zeta,s)\right)\right)\mathrm{d}s = 0, \quad \zeta \geq \zeta_0, \tag{1.1}$$

where α is a ratio of odd positive integers, n is an even integer, $n \geq 2$,

$$z(\zeta) = x^{\alpha}(\zeta) + p(\zeta)x\left(\sigma(\zeta)\right). \tag{1.2}$$

Throughout this work, we assume that:

(H_1) $p, r \in C([\zeta_0,\infty))$, $r(\zeta) > 0$, $r'(\zeta) \geq 0$, and $0 \leq p(\zeta) < 1$;

(H_2) $q \in C([\zeta_0,\infty) \times (a,b), \mathbb{R})$, $q(\zeta,s) \geq 0$, and

$$\int_{\zeta_0}^{\infty} \frac{1}{r(s)}\,\mathrm{d}s = \infty;$$

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(H_3) $f \in C(\mathbb{R}, \mathbb{R})$, $|f(x)| \geq k|x^{\alpha}|$ for $x \neq 0$, and k is a positive constant;

(H_4) $\sigma \in C([\zeta_0, \infty), (0, \infty))$, $\sigma(\zeta) \leq \zeta$, and $\lim_{\zeta \to \infty} \sigma(\zeta) = \infty$;

(H_5) $g \in C([\zeta_0, \infty) \times (a, b), \mathbb{R})$, $g(\zeta, s) \leq \zeta$, g has nonnegative partial derivatives, and $\lim_{\zeta \to \infty} g(\zeta, s) = \infty$.

By a solution of Eq. (1.1), we purpose a function $x(\zeta) \in C([\zeta_k, \infty), \mathbb{R})$ for some $\zeta_k \geq \zeta_0$ such that $z(\zeta) \in C^{(n)}([\zeta_k, \infty), \mathbb{R})$ and $(r(\zeta)z^{(n-1)}(\zeta)) \in C^1([\zeta_k, \infty), \mathbb{R})$ and satisfies Eq. (1.1) on $[\zeta_k, \infty)$. If x is neither positive nor negative eventually, then $x(\zeta)$ is called oscillatory, or it will be non-oscillatory.

The theory of oscillation of differential equation has been the subject of many papers [1–37]. During the recent decades, a great amount of work has been done on development the oscillation theory of the nth-order equations with delay and advanced argument, see [4–12, 23, 25, 27, 28, 31–37]. In the following, we present some related examples:

In [36], Zhang et al. established the conditions of oscillation of the equation

$$\left(r\left(x^{(n-1)}\right)^{\alpha}\right)'(\zeta) + q(\zeta)f\left(x\left(g(\zeta)\right)\right) = 0, \tag{1.3}$$

where $f(x) = x^{\beta}$, β is a ratio of odd positive integers, $\beta \leq \alpha$, and

$$\int_{\zeta_0}^{\infty} r^{-1/\alpha}(s)\, \mathrm{d}s < \infty. \tag{1.4}$$

Moreover, in [35], some oscillation results have been presented, which improves the results in [36]. As well, Baculikova et al. in [8] studied the properties of oscillation of the solutions of equation (1.3) under conditions (1.4) and

$$\int_{\zeta_0}^{\infty} r^{-1/\alpha}(s)\, \mathrm{d}s = \infty. \tag{1.5}$$

For more oscillation results about (1.3), see [3–5]. The asymptotic properties and oscillation of equation

$$\left(r\left(y^{(n-1)}\right)^{\alpha}\right)'(\zeta) + q(\zeta)f\left(x\left(g(\zeta)\right)\right) = 0,$$

where $y(\zeta) = x(\zeta) + p(\zeta)x(\sigma(\zeta))$, have been considered in [7, 23, 32, 37].

In [31], the oscillatory behavior of the neutral differential equation

$$\left(r\left(|x|^{\gamma-1}x + px(\sigma)\right)^{(n-1)}\right)'(\zeta) + q(\zeta)f\left(x\left(g(\zeta)\right)\right) = 0,$$

where $\gamma \geq 1$ is a real number, is established.

In this paper, by using the technique of comparison with first order delay equations and technique of Riccati transformation, we obtain a two different conditions ensure oscillation of solutions of this equation, which extend and improve results of [31]. Moreover, we establish some new criterion for oscillation of Eq. (1.1) by using an integral averages condition of Philos-type. We illustrate the importance of our results by presenting some examples.

During the following sections of our paper, we shall need the next definition and lemmas.

Definition 1 ([29]) Let

$$D_0 = \{(\zeta, s) : \zeta > s > \zeta_0\} \quad \text{and} \quad D = \{(\zeta, s) : \zeta \geq s \geq \zeta_0\}.$$

Let H be a continuous real functions on D. It is said that H belongs to the function class $\Im$, written by $H \in \Im$, if

(i) $H(\zeta, \zeta) = 0$ for $\zeta \geq \zeta_0$, $H(\zeta, s) > 0$ on D_0;

(ii) The partial derivative $\partial H/\partial s \in C(D_0, [0, \infty))$ such that the condition

$$\frac{\partial H(\zeta, s)}{\partial s} = -h(\zeta, s)\sqrt{H(\zeta, s)},$$

for all $(\zeta, s) \in D_0$ is satisfied for some $h \in C(D, \mathbb{R})$.

Lemma 1.1 ([3]) *Suppose that n be an even, $w \in C^n([\zeta_0, \infty))$, w of constant sign, $w^{(n)}(\zeta) \neq 0$ on $[\zeta_0, \infty)$ and $w(\zeta)w^{(n)}(\zeta) \leq 0$. Then,*

(I) *The derivatives $w^{(i)}(\zeta)$, $i = 1, 2, \ldots, n-1$, are of constant sign on $[\zeta_1, \infty)$ for some $\zeta_1 \geq \zeta_0$;*

(II) *There exists an odd integer $l \in [1, n)$, such that, for $\zeta \geq \zeta_1$,*

$$y(\zeta)y^{(i)}(\zeta) > 0$$

for all $i = 0, 1, \ldots, l$ and

$$(-1)^{n+i+1}y(\zeta)y^{(i)}(\zeta) > 0$$

for all $i = l + 1, \ldots, n$.

Lemma 1.2 ([3]) *Let w be as in Lemma 1.1 and $w^{(n-1)}(\zeta)w^{(n)}(\zeta) \leq 0$ for $\zeta \geq \zeta_0$. Then there exists a constant $M > 0$ such that*

$$\left|y(\lambda\zeta)\right| \geq M\zeta^{n-1}\left|y^{(n-1)}(\zeta)\right|$$

for all large ζ.

Lemma 1.3 ([3]) *Let w be as in Lemma 1.1 and $w^{(n-1)}(\zeta)w^{(n)}(\zeta) \leq 0$ for $\zeta \geq \zeta_0$. If $\lim_{\zeta\to\infty} w(\zeta) \neq 0$, then for every $\mu \in (0, 1)$ there exists a $\zeta_\mu \geq \zeta_0$ such that*

$$\left|y(\zeta)\right| \geq \frac{\mu}{(n-1)!}\zeta^{n-1}\left|y^{(n-1)}(\zeta)\right|$$

for all $\zeta \geq \zeta_\mu$.

2 Main results

Lemma 2.1 *Assume that $x(\zeta)$ is an eventually positive solution of equation (1.1). If*

$$\omega(\zeta) := \rho(\zeta)\frac{r(\zeta)z^{(n-1)}(\zeta)}{z(\lambda g(\zeta, a))},$$

where $\rho \in C'([\zeta_0,\infty),\mathbb{R}^+)$ *and* $\lambda \in (0,1)$, *then*

$$\omega'(\zeta) \le \frac{\rho'(\zeta)}{\rho(\zeta)}\omega(\zeta) - k\rho(\zeta)Q(\zeta) - \frac{\lambda}{\eta(\zeta)}\omega^2(\zeta), \tag{2.1}$$

where M *is a positive real constant and*

$$Q(\zeta) := \int_a^b q(\zeta,s)\bigl(1-p\bigl(g(\zeta,s)\bigr)\bigr)\,\mathrm{d}s$$

and

$$\eta(\zeta) := \frac{r(\zeta)\rho(\zeta)}{Mg^{n-2}(\zeta,s)g'(\zeta,a)}.$$

Proof Let $x(\zeta)$ be an eventually positive solution of equation (1.1). Then we can assume that $x(\zeta) > 0$, $x(\sigma(\zeta)) > 0$, and $x(g(\zeta,s)) > 0$ for $\zeta \ge \zeta_1$. Hence, we deduce $z(\zeta) > 0$ for $\zeta \ge \zeta_1$ and

$$\bigl(rz^{(n-1)}\bigr)'(\zeta) = -\int_a^b q(\zeta,s)f\bigl(x\bigl(g(\zeta,s)\bigr)\bigr)\,\mathrm{d}s \le 0. \tag{2.2}$$

Therefore, the function $r(\zeta)z^{(n-1)}(\zeta)$ is decreasing and $z^{(n-1)}(\zeta)$ is eventually of one sign. We claim that $z^{(n-1)}(\zeta) \ge 0$. Otherwise, if there exists $\zeta_2 \ge \zeta_1$ such that $z^{(n-1)}(\zeta) < 0$ for $\zeta \ge \zeta_2$, and

$$\bigl(rz^{(n-1)}\bigr)(\zeta) \le \bigl(rz^{(n-1)}\bigr)(\zeta_2) = -m,$$

where m is a positive constant. Integrating the above inequality from ζ_2 to ζ, we have

$$z^{(n-2)}(\zeta) \le z^{(n-2)}(\zeta_2) - m\int_{\zeta_2}^{\zeta} \frac{1}{r(s)}\,\mathrm{d}s.$$

Letting $\zeta \to \infty$, we get $\lim_{\zeta\to\infty} z^{(n-2)}(\zeta) = -\infty$, which implies $z(\zeta)$ is eventually negative by Lemma 1.1. This is a contradiction. Hence, we have that $z^{(n-1)}(\zeta) \ge 0$ for $\zeta \ge \zeta_1$. Furthermore, from Eq. (1.1) and (H_1), we get

$$\bigl(rz^{(n)}\bigr)(\zeta) = -\bigl(r'z^{(n-1)}\bigr)(\zeta) - \int_a^b q(\zeta,s)f\bigl(x\bigl(g(\zeta,s)\bigr)\bigr)\,\mathrm{d}s \le 0,$$

this implies that $z^{(n)}(\zeta) \le 0$, $\zeta \ge \zeta_1$. From Lemma 1.1, we obtain that

$$z(\zeta) > 0, \qquad z'(\zeta) > 0, \qquad z^{(n-1)}(\zeta) \ge 0, \quad \text{and} \quad z^{(n)}(\zeta) \le 0 \tag{2.3}$$

for $\zeta \ge \zeta_2$ are satisfied.

Next, from definition (1.2), we get

$$\begin{aligned} x^{\alpha}(\zeta) &= z(\zeta) - p(\zeta)x\bigl(\sigma(\zeta)\bigr) \ge z(\zeta) - p(\zeta)z\bigl(\sigma(\zeta)\bigr) \ge z(\zeta) - p(\zeta)z(\zeta) \\ &\ge \bigl(1-p(\zeta)\bigr)z(\zeta), \end{aligned}$$

and so

$$x^{\alpha}\big(g(\zeta,s)\big) \geq z\big(g(\zeta,s)\big)\big(1-p\big(g(\zeta,s)\big)\big). \tag{2.4}$$

By (H_3) and (2.4), we find

$$f\big(x\big(g(\zeta,s)\big)\big) \geq kz\big(g(\zeta,s)\big)\big(1-p\big(g(\zeta,s)\big)\big). \tag{2.5}$$

Combining (1.1) and (2.5), we have

$$\big(rz^{(n-1)}\big)'(\zeta) \leq -k\int_a^b q(\zeta,s)z\big(g(\zeta,s)\big)\big(1-p\big(g(\zeta,s)\big)\big)\,\mathrm{d}s.$$

Since $g(\zeta,s)$ is nondecreasing with respect to s, we get $g(\zeta,s) \geq g(\zeta,a)$ for $s \in (a,b)$, and so

$$\big(rz^{(n-1)}\big)'(\zeta) \leq -kz\big(g(\zeta,a)\big)Q(\zeta). \tag{2.6}$$

Using Lemma 1.2 with $u = z'$, there exists $M > 0$ such that

$$z'\big(\lambda g(\zeta,s)\big) \geq Mg^{n-2}(\zeta,s)z^{(n-1)}\big(g(\zeta,s)\big) \geq Mg^{n-2}(\zeta,s)z^{(n-1)}(\zeta). \tag{2.7}$$

From the definition of ω, we see that $\omega(\zeta) > 0$ and

$$\omega'(\zeta) = \frac{\rho'(\zeta)}{\rho(\zeta)}\omega(\zeta) + \rho(\zeta)\frac{(r(\zeta)z^{(n-1)}(\zeta))'}{z(\lambda g(\zeta,a))} - \lambda\rho(\zeta)\frac{r(\zeta)z^{(n-1)}(\zeta)z'(\lambda g(\zeta,a))g'(\zeta,a)}{(z(\lambda g(\zeta,a)))^2}.$$

From (2.6), we obtain

$$\omega'(\zeta) \leq \frac{\rho'(\zeta)}{\rho(\zeta)}\omega(\zeta) - k\rho(\zeta)Q(\zeta) - \lambda\frac{z'(\lambda g(\zeta,a))g'(\zeta,a)}{z(\lambda g(\zeta,a))}\omega(\zeta).$$

By using (2.7), we have

$$\begin{aligned}\omega'(\zeta) &\leq \frac{\rho'(\zeta)}{\rho(\zeta)}\omega(\zeta) - k\rho(\zeta)Q(\zeta) - \lambda\frac{Mg^{n-2}(\zeta,s)z^{(n-1)}(\zeta)g'(\zeta,a)}{z(\lambda g(\zeta,a))}\omega(\zeta)\\ &\leq \frac{\rho'(\zeta)}{\rho(\zeta)}\omega(\zeta) - k\rho(\zeta)Q(\zeta) - \frac{\lambda}{\eta(\upsilon)}\omega^2(\zeta).\end{aligned}$$

This completes the proof. □

Theorem 2.1 *If there exist a function $\rho \in C^1([\zeta_0,\infty),\mathbb{R}^+)$ and constants $\lambda \in (0,1)$, $M > 0$ such that*

$$\int_{\zeta_0}^{\infty}\left(k\rho(\upsilon)Q(\upsilon) - \frac{1}{4\lambda}\left(\frac{\rho'(\upsilon)}{\rho(\upsilon)}\right)^2\eta(\upsilon)\right)\mathrm{d}\upsilon = \infty, \tag{2.8}$$

then Eq. (1.1) *is oscillatory.*

Proof Suppose that Eq. (1.1) has a nonoscillatory solution in $[\zeta_0,\infty)$. Without loss of generality, we assume that $x(\zeta)$ is an eventually positive solution of equation (1.1). From Lemma 2.1, we get that (2.1) holds. Using the inequality

$$Uy-\upsilon y^{\frac{\gamma+1}{\gamma}}\leq\frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{U^{\gamma+1}}{\upsilon^{\gamma}},$$

with $U=\rho'/\rho$, $\upsilon=\lambda Mg^{n-2}(\zeta,s)g'(\zeta,a)/(r(\zeta)\rho(\zeta))$ and $y=\omega(\zeta)$, we find

$$\omega'(\zeta)\leq-k\rho(\zeta)Q(\zeta)+\frac{1}{4\lambda}\left(\frac{\rho'(\zeta)}{\rho(\zeta)}\right)^{2}\frac{r(\zeta)\rho(\zeta)}{Mg^{n-2}(\zeta,s)g'(\zeta,a)}.$$

Integrating this inequality from ζ_1 to ζ, we obtain

$$\begin{aligned}\int_{\zeta_1}^{\zeta}\left(k\rho(\upsilon)Q(\upsilon)-\frac{1}{4\lambda}\left(\frac{\rho'(\upsilon)}{\rho(\upsilon)}\right)^{2}\eta(\upsilon)\right)\mathrm{d}\upsilon&\leq\omega(\zeta_1)-\omega(\zeta)\\&\leq\omega(\zeta_1),\end{aligned}$$

which contradicts (2.8) and this completes the proof. □

Theorem 2.2 *If, for some constant* $\mu\in(0,1)$, *the differential equation*

$$u'(\zeta)+\widehat{Q}(\zeta)u\bigl(g(\zeta,a)\bigr)=0\tag{2.9}$$

is oscillatory, where

$$\widehat{Q}(\zeta):=\frac{k\mu g^{n-1}(\zeta,a)}{(n-1)!r(g(\zeta,a))}Q(\zeta),$$

then Eq. (1.1) *is oscillatory.*

Proof Suppose that Eq. (1.1) has a nonoscillatory solution in $[\zeta_0,\infty)$. Without loss of generality, we assume that $x(\zeta)$ is an eventually positive solution of equation (1.1). From Lemma 2.1, we get that (2.3)–(2.6) hold. By using Lemma 1.3, we find

$$z(\zeta)\geq\frac{\mu}{(n-1)!}\zeta^{n-1}z^{(n-1)}(\zeta)$$

for all $\zeta\geq\zeta_2\geq\max\{\zeta_1,\zeta_\mu\}$. Thus, from (2.6), we obtain

$$\bigl(r(\zeta)z^{(n-1)}(\zeta)\bigr)'+\frac{k\mu g^{n-1}(\zeta,a)Q(\zeta)}{(n-1)!r(g(\zeta,a))}\bigl(r\bigl(g(\zeta,a)\bigr)z^{(n-1)}\bigl(g(\zeta,a)\bigr)\bigr)\leq0.$$

Therefore, we see that $u(\zeta):=r(\zeta)z^{(n-1)}(\zeta)$ is a positive solution of the differential inequality

$$u'(\zeta)+\widehat{Q}(\zeta)u\bigl(g(\zeta,a)\bigr)\leq0.$$

From [29, Corollary 1], we have that Eq. (2.9) also has a positive solution, a contradiction. This completes the proof. □

By using Theorem 2.1.1 in [20], we get the following corollary.

Corollary 2.1 *If, for some constant $\mu \in (0,1)$,*

$$\liminf_{\zeta\to\infty} \int_{g(\zeta,a)}^{\zeta} \frac{g^{n-1}(s,a)}{r(g(s,a))} Q(s)\,\mathrm{d}s > \frac{(n-1)!}{k\mu \mathrm{e}},$$

then Eq. (1.1) *is oscillatory.*

Theorem 2.3 *If there exist $H \in \Im$, $\rho \in C^1([\zeta_0,\infty),\mathbb{R}^+)$ and constants $\lambda \in (0,1)$, $M > 0$ such that*

$$\limsup_{\zeta\to\infty} \frac{1}{H(\zeta,\zeta_0)} \int_{\zeta_0}^{\zeta} H(\zeta,\upsilon)\left(k\rho(\upsilon)Q(\upsilon) - \frac{1}{4\lambda}\eta(\upsilon)\Phi^2(\zeta,\upsilon)\right)\mathrm{d}\upsilon = \infty, \tag{2.10}$$

where

$$\Phi(\zeta,s) = \frac{\rho'(s)}{\rho(s)} - \frac{h(\zeta,s)}{\sqrt{H(\zeta,s)}},$$

then Eq. (1.1) *is oscillatory.*

Proof Suppose that Eq. (1.1) has a nonoscillatory solution in $[\zeta_0,\infty)$. Without loss of generality, we assume that $x(\zeta)$ is an eventually positive solution of equation (1.1). From Lemma 2.1, we get that (2.1) holds. Multiplying (2.1) by $H(\zeta,s)$ and integrating from ζ_2 to ζ, we get

$$\begin{aligned}
&\omega'(s) \le \frac{\rho'(s)}{\rho(s)}\omega(s) - k\rho(s)Q(s) - \frac{\lambda}{\eta(s)}\omega^2(s),\\
&k\int_{\zeta_2}^{\zeta} H(\zeta,\upsilon)\rho(\upsilon)Q(\upsilon)\,\mathrm{d}\upsilon \le -\int_{\zeta_2}^{\zeta} H(\zeta,\upsilon)\omega'(\upsilon)\,\mathrm{d}\upsilon - \int_{\zeta_2}^{\zeta} H(\zeta,\upsilon)\frac{\lambda}{\eta(\upsilon)}\omega^2(\upsilon)\,\mathrm{d}\upsilon\\
&\qquad + \int_{\zeta_2}^{\zeta} H(\zeta,\upsilon)\frac{\rho'(\upsilon)}{\rho(\upsilon)}\omega(\upsilon)\,\mathrm{d}\upsilon\\
&\qquad \le H(\zeta,\zeta_2)\omega(\zeta_2) - \int_{\zeta_2}^{\zeta} H(\zeta,\upsilon)\frac{\lambda}{\eta(\upsilon)}\omega^2(\upsilon)\,\mathrm{d}\upsilon\\
&\qquad + \int_{\zeta_2}^{\zeta} H(\zeta,\upsilon)\omega(\upsilon)\Phi(\zeta,\upsilon)\,\mathrm{d}\upsilon
\end{aligned}$$

and hence,

$$\begin{aligned}
k\int_{\zeta_2}^{\zeta} H(\zeta,\upsilon)\rho(\upsilon)Q(\upsilon)\,\mathrm{d}\upsilon &\le H(\zeta,\zeta_2)\omega(\zeta_2)\\
&\quad - \int_{\zeta_2}^{\zeta} H(\zeta,\upsilon)\frac{\lambda}{\eta(\upsilon)}\left(\omega^2(\upsilon) - \frac{\eta(\upsilon)}{\lambda}\Phi(\zeta,\upsilon)\omega(\upsilon)\right)\mathrm{d}\upsilon.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\frac{1}{H(\zeta,\zeta_2)}\int_{\zeta_2}^{\zeta} H(\zeta,\upsilon)\left(k\rho(\upsilon)Q(\upsilon) - \frac{1}{4\lambda}\eta(\upsilon)\Phi^2(\zeta,\upsilon)\right)\mathrm{d}\upsilon\\
&\quad \le \omega(\zeta_2) - \frac{1}{H(\zeta,\zeta_2)}\int_{\zeta_2}^{\zeta} H(\zeta,\upsilon)\frac{\lambda}{\eta(\upsilon)}\left(\omega(\upsilon) - \frac{1}{2\lambda}\eta(\upsilon)\Phi(\zeta,\upsilon)\right)^2 \mathrm{d}\upsilon,
\end{aligned}$$

which implies

$$\limsup_{\zeta\to\infty}\frac{1}{H(\zeta,\zeta_2)}\int_{\zeta_2}^{\zeta}H(\zeta,\upsilon)\left(k\rho(\upsilon)Q(\upsilon)-\frac{1}{4\lambda}\eta(\upsilon)\Phi^2(\zeta,\upsilon)\right)\mathrm{d}\upsilon\le\omega(\zeta_2).$$

From (2.10), we have a contradiction. This completes the proof. □

The following oscillation criteria treat the cases when it is not possible to verify easily conditions (2.10).

Theorem 2.4 *Assume that*

$$0<\inf_{s\ge\zeta}\left(\liminf_{\zeta\to\infty}\frac{H(\zeta,s)}{H(\zeta,\zeta_0)}\right)\le\infty$$

and

$$\limsup_{\zeta\to\infty}\frac{1}{H(\zeta,\zeta_0)}\int_{\zeta_0}^{\zeta}H(\zeta,\upsilon)\eta(\upsilon)\Phi^2(\zeta,\upsilon)\,\mathrm{d}\upsilon<\infty.$$

If there exists $\psi\in C([\zeta_0,\infty),\mathbb{R})$ *such that, for* $\zeta\ge\zeta_0$,

$$\limsup_{\zeta\to\infty}\int_{\zeta_0}^{\zeta}\frac{\psi_+^2(s)}{\eta(s)}\,\mathrm{d}s=\infty$$

and

$$\limsup_{\zeta\to\infty}\frac{1}{H(\zeta,\zeta_0)}\int_{\zeta_0}^{\zeta}H(\zeta,\upsilon)\left(k\rho(\upsilon)Q(\upsilon)-\frac{1}{4\lambda}\eta(\upsilon)\Phi^2(\zeta,\upsilon)\right)\mathrm{d}\upsilon\ge\sup_{\zeta\ge\zeta_0}\psi(\zeta),$$

where $\psi_+(\zeta)=\max\{\psi(\zeta),0\}$, *then every solution of Eq.* (1.1) *is oscillatory.*

The proof of Theorem 2.4 is similar to the proof of Theorem 2.5 in [18] and hence is omitted.

Example 2.1 Consider the following nth-order neutral differential equation:

$$\left(\left(x^3(\zeta)+\left(1-\frac{1}{\zeta}\right)x(\zeta-\sigma)\right)'\right)'+\int_{1/2}^{1}\zeta^2 s x^3(\zeta s)\,\mathrm{d}s=0,\tag{2.11}$$

where $n=2$, $\alpha=3$, $r(\zeta)=1$, $p(\zeta)=1-\frac{1}{\zeta}$, $\sigma(\zeta)=\zeta-\sigma$, $q(\zeta,s)=\zeta^2 s$, $f(x)=x^3$, $g(\zeta,s)=\zeta s$, and let $\rho(\zeta)=1$, then for any constants $\lambda\in(0,1)$ and $M>0$ we have

$$\int_{\zeta_0}^{\infty}\left(k\rho(\upsilon)Q(\upsilon)-\frac{1}{4\lambda}\left(\frac{\rho'(\upsilon)}{\rho(\upsilon)}\right)^2\eta(\upsilon)\right)\mathrm{d}\upsilon=\infty.$$

From Theorem 2.1, it follows that Eq. (2.11) is oscillatory.

Example 2.2 Consider the equation

$$\left(\zeta\left(x^\alpha(\zeta)+p_0x(\delta\sigma)\right)^{n-1}\right)'+\frac{q_0}{\zeta^{n-1}}x^\alpha(\beta\zeta)=0,\tag{2.12}$$

where $p_0 \in [0,1)$, $\delta, \beta \in (0,1)$, and $q_0 > 0$. We note that $a = 0$, $b = 1$, $r(\zeta) := \zeta$, $q(\zeta) := q_0/\zeta^{n-1}$, and $f(x) := x^{\alpha}$. Hence,

$$Q(\zeta) := q_0(1-p_0)\zeta^{1-n}.$$

Let $\rho(\zeta) := \zeta^n$. Then we have (2.8) holds if

$$q_0(1-p_0)\beta^{n-1} > \frac{n^2}{4\lambda M} \tag{2.13}$$

for every positive constant M. By using Theorem 2.1, Eq. (2.11) is oscillatory if (2.13) holds. Note that there is difficulty in applying Condition (2.13) due to a constant M. But, by using Corollary 2.1, we get that Eq. (2.11) is oscillatory if

$$\liminf_{\zeta\to\infty} \int_{g(\zeta,a)}^{\zeta} \beta^{n-2} q_0(1-p_0)\frac{1}{s}\,\mathrm{d}s > \frac{(n-1)!}{k\mu\mathrm{e}},$$

that is,

$$q_0(1-p_0)\beta^{n-2}\ln\frac{1}{\beta} > \frac{(n-1)!}{\mu\mathrm{e}}. \tag{2.14}$$

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