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A class of new nonlinear dynamic integral inequalities containing integration on infinite interval on time scales

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Abstract

In this paper, we investigate some new nonlinear dynamic integral inequalities containing integration on infinite interval on time scales, which provide explicit bounds on unknown functions. Our results not only generalize some dynamic inequalities in related literature, but also are new even for the continuous and discrete time cases. Two examples are given to illustrate the present results.

Keywords: Time scales; Nonlinear; Integral inequality; Dynamic equation

1 Introduction

The theory of time scales, as a unification of the continuous and discrete analysis, was initiated by Stefan Hilger [1] in 1988. Since then, more and more authors have been interested in this area. Along with the in-depth research into the theory, researchers find it not only can be used in pure mathematics but also is an important tool in many branches of science and engineering such as 3D tracking of shape [2], DNA dynamics [3], and so on.

As one of the most fundamental topics, the analysis of dynamic equations on time scales has been extensively investigated in recent years, see [4–31]. Because dynamic inequalities play an important role in qualitative analysis of dynamic equations on time scales, there have been plenty of results focused on them, we refer the readers to [32–53]. Among these inequalities, the well-known Gronwall type inequalities have been intensively investigated due to their wide applications. However, to the best of our knowledge, Gronwall type inequalities containing integration on infinite interval have received less attention. Recent results in this direction include the works of [32, 33]. For instance, Meng et al. [33] investigated some integral inequalities on time scales containing integration on infinite interval

$$u^p(t) \leq a(t) + \int_t^\infty [f(s)u^p(s) + g(s)u(s) + h(s)] \Delta s, \quad t \in \mathbb{T}^\kappa, \quad (1)$$

where p is a real constant, $u, a, f, g, h: \mathbb{T}^\kappa \rightarrow \mathbb{R}_+$ are rd-continuous functions.

In this paper, we establish some new nonlinear dynamic inequalities containing integration on infinite interval on time scales. Our results not only generalize some dynamic inequalities that have been studied in [33], but also are new even for the continuous and discrete time cases.

2 Preliminaries

In what follows, we always assume that \mathbb{R} denotes the set of real numbers, $\mathbb{R}_+ = [0, \infty)$, \mathbb{T} is an arbitrary time scale (nonempty closed subset of \mathbb{R}), \mathbb{T}^κ is defined as follows: If \mathbb{T} has a maximum m and m is left-scattered, then $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$, otherwise $\mathbb{T}^\kappa = \mathbb{T}$. \mathcal{R} denotes the set of all regressive and rd-continuous functions, $\mathcal{R}^+ = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0 \text{ for all } t \in \mathbb{T}\}$. The ‘‘circle minus’’ addition \ominus defined by $(p \ominus q)(t) := \frac{p(t)-q(t)}{1+\mu(t)q(t)}$ for all $t \in \mathbb{T}^\kappa$.

The following lemmas are useful in the proof of the main results of this paper.

Lemma 2.1 *Let $m > 0, n > 0, p > 0, \alpha > 0$, and $\beta > 0$ be given, then for each $x \geq 0$,*

$$mx^\alpha - nx^\beta \leq \frac{m(\beta - \alpha)}{\beta - p} \left(\frac{(\beta - p)n}{(\alpha - p)m} \right)^{(\alpha-p)/(\alpha-\beta)} x^p \tag{2}$$

holds for the cases when $0 < p < \alpha < \beta$ or $0 < \beta < \alpha < p$.

Proof If $x = 0$, then it is easy to see that inequality (2) holds. So we only prove that inequality (2) holds when $x > 0$. For the case $0 < p < \alpha < \beta$, set $F(x) = mx^{\alpha-p} - nx^{\beta-p}$, $x > 0$, where $m > 0$ and $n > 0$. Let $F'(x) = 0$, we get $x_0 = (\frac{m(\alpha-p)}{n(\beta-p)})^{1/(\beta-\alpha)}$. Since $\forall x \in (0, x_0), F'(x) > 0$; $\forall x \in (x_0, +\infty), F'(x) < 0$, F attains its maximum at $x_0 = (\frac{m(\alpha-p)}{n(\beta-p)})^{1/(\beta-\alpha)}$ and $F_{\max} = F(x_0) = \frac{m(\beta-\alpha)}{\beta-p} (\frac{(\beta-p)n}{(\alpha-p)m})^{(\alpha-p)/(\alpha-\beta)}$. Thus, (2) holds. For the case when $0 < \beta < \alpha < p$, by a similar argument with the case $p < \alpha < \beta$, we can get (2) holds. The proof is complete. \square

Lemma 2.2 ([54]) *Assume that $x \geq 0, p \geq q \geq 0$, and $p \neq 0$, then for any $K > 0$,*

$$x^{q/p} \leq \frac{q}{p} K^{(q-p)/p} x + \frac{p-q}{p} K^{q/p}.$$

Lemma 2.3 ([4, Theorem 6.1]) *Suppose that $\sup_{t \in \mathbb{T}^\kappa} t = \infty, x, q \in C_{rd}(\mathbb{T}^\kappa, \mathbb{R}_+), p \in \mathcal{R}^+$, and x is delta differential at $t \in \mathbb{T}^\kappa$, then*

$$x^\Delta(t) \geq p(t)x(t) - q(t), \quad t \in \mathbb{T}^\kappa,$$

implies

$$x(t) \leq x(\infty)e_{\ominus p}(\infty, t) + \int_t^\infty q(s)e_p(t, \sigma(s))\Delta s, \quad t \in \mathbb{T}.$$

3 Main results

In this section, we deal with some nonlinear inequalities on time scales. For convenience, we always assume that $t \geq t_0, t \in \mathbb{T}^\kappa$.

Theorem 3.1 *Assume that $x, f, g, h, a, b, c, m, n \in C_{rd}(\mathbb{T}^\kappa, \mathbb{R}_+), k, l \in C_{rd}(\mathbb{T}^\kappa, (0, \infty)), \mu(t)F(t) < 1, e_{G \ominus (-F)}(\infty, t) < \infty$ for $t \in \mathbb{T}^\kappa, p, q, r, \alpha$, and β are constants satisfying*

- (i) $0 \leq q \leq p, 0 \leq r \leq p, p < \alpha < \beta$; or
- (ii) $0 \leq q \leq p, 0 \leq r \leq p, 0 < \beta < \alpha < p$.

Suppose that x satisfies

$$x^p(t) \leq f(t) + g(t) \int_t^\infty \left\{ a(s)x^p(s) + b(s)x^q(s) + c(s) + m(s) \int_s^\infty n(\xi)x^r(\xi)\Delta\xi + h(s)[l(s)x^\alpha(\sigma(s)) - k(s)x^\beta(\sigma(s))] \right\} \Delta s, \quad t \in \mathbb{T}^\kappa, \tag{3}$$

then, for any $K_1 > 0$ and $K_2 > 0$,

$$x(t) \leq \left(f(t) + g(t) \int_t^\infty C(s)e_{(-F)\ominus G}(t, \sigma(s))\Delta s \right)^{1/p}, \quad t \in \mathbb{T}^\kappa, \tag{4}$$

where

$$F(t) := a(t)g(t) + \frac{q}{p}K_1^{(q-p)/p}b(t)g(t) + \frac{r}{p}K_2^{(r-p)/p}m(t) \int_t^\infty n(\xi)g(\xi)\Delta\xi, \tag{5}$$

$$G(t) := h(t)B(t)g(\sigma(t)), \tag{6}$$

$$A(t) := a(t)f(t) + \frac{q}{p}K_1^{(q-p)/p}b(t)f(t) + \frac{p-q}{p}K_1^{q/p}b(t) + c(t) + m(t) \int_t^\infty n(\xi) \left(\frac{r}{p}K_2^{(r-p)/p}f(\xi) + \frac{p-r}{p}K_2^{r/p} \right) \Delta\xi + h(t)B(t)f(\sigma(t)), \tag{7}$$

$$B(t) := \frac{l(t)(\beta - \alpha)}{\beta - p} \left(\frac{(\beta - p)k(t)}{(\alpha - p)l(t)} \right)^{(\alpha-p)/(\alpha-\beta)}, \tag{8}$$

$$C(t) := \frac{A(t)}{1 + \mu(t)G(t)}. \tag{9}$$

Proof By Lemma 2.1 and (3), we have

$$x^p(t) \leq f(t) + g(t) \int_t^\infty \left[a(s)x^p(s) + b(s)x^q(s) + c(s) + m(s) \int_s^\infty n(\xi)x^r(\xi)\Delta\xi + h(s)B(s)x^p(\sigma(s)) \right] \Delta s, \quad t \in \mathbb{T}^\kappa, \tag{10}$$

where $B(t)$ is defined as in (8). Denote

$$z(t) = \int_t^\infty \left[a(s)x^p(s) + b(s)x^q(s) + c(s) + m(s) \int_s^\infty n(\xi)x^r(\xi)\Delta\xi + h(s)B(s)x^p(\sigma(s)) \right] \Delta s, \quad t \in \mathbb{T}^\kappa. \tag{11}$$

From the assumptions on x, a, b, c, m, n, h, B , (10), and (11), we obtain z is nonincreasing and

$$x(t) \leq (f(t) + g(t)z(t))^{1/p}, \quad t \in \mathbb{T}^\kappa. \tag{12}$$

In view of (11) and (12), we have

$$\begin{aligned}
 z^\Delta(t) &= - \left[a(t)x^p(t) + b(t)x^q(t) + c(t) + m(t) \int_t^\infty n(\xi)x^r(\xi)\Delta\xi + h(t)B(t)x^p(\sigma(t)) \right] \\
 &\geq - \left[a(t)(f(t) + g(t)z(t)) + b(t)(f(t) + g(t)z(t))^{q/p} + c(t) \right. \\
 &\quad \left. + m(t) \int_t^\infty n(\xi)(f(\xi) + g(\xi)z(\xi))^{r/p} \Delta\xi \right. \\
 &\quad \left. + h(t)B(t)(f(\sigma(t)) + g(\sigma(t))z(\sigma(t))) \right], \quad t \in \mathbb{T}^\kappa. \tag{13}
 \end{aligned}$$

Using Lemma 2.2 on the right-hand side of (13), for any $K_1 > 0$ and $K_2 > 0$, we obtain

$$\begin{aligned}
 z^\Delta(t) &\geq - \left[a(t)(f(t) + g(t)z(t)) + b(t) \left[\frac{q}{p} K_1^{(q-p)/p} (f(t) + g(t)z(t)) + \frac{p-q}{p} K_1^{q/p} \right] \right. \\
 &\quad \left. + c(t) + m(t) \int_t^\infty n(\xi) \left[\frac{r}{p} K_2^{(r-p)/p} (f(\xi) + g(\xi)z(\xi)) + \frac{p-r}{p} K_2^{r/p} \right] \Delta\xi \right. \\
 &\quad \left. + h(t)B(t)(f(\sigma(t)) + g(\sigma(t))z(\sigma(t))) \right] \\
 &\geq - \left[a(t)f(t) + a(t)g(t)z(t) + \frac{q}{p} K_1^{(q-p)/p} b(t)(f(t) + g(t)z(t)) + \frac{p-q}{p} K_1^{q/p} b(t) \right. \\
 &\quad \left. + c(t) + \frac{r}{p} K_2^{(r-p)/p} m(t)z(t) \int_t^\infty n(\xi)g(\xi)\Delta\xi \right. \\
 &\quad \left. + m(t) \int_t^\infty n(\xi) \left(\frac{r}{p} K_2^{(r-p)/p} f(\xi) + \frac{p-r}{p} K_2^{r/p} \right) \Delta\xi \right. \\
 &\quad \left. + h(t)B(t)g(\sigma(t))z(\sigma(t)) + h(t)B(t)f(\sigma(t)) \right] \\
 &= - [F(t)z(t) + G(t)z(\sigma(t)) + A(t)], \quad t \in \mathbb{T}^\kappa, \tag{14}
 \end{aligned}$$

where $F(t)$, $G(t)$, and $A(t)$ are defined as in (5), (6), and (7). From (14) we have

$$\begin{aligned}
 z^\Delta(t) &\geq - [F(t)z(t) + G(t)(z(t) + \mu(t)z^\Delta(t)) + A(t)] \\
 &= - [F(t) + G(t)]z(t) - G(t)\mu(t)z^\Delta(t) - A(t), \quad t \in \mathbb{T}^\kappa,
 \end{aligned}$$

which yields

$$[1 + \mu(t)G(t)]z^\Delta(t) \geq - [F(t) + G(t)]z(t) - A(t), \quad t \in \mathbb{T}^\kappa,$$

that is,

$$\begin{aligned}
 z^\Delta(t) &\geq - \frac{F(t) + G(t)}{1 + \mu(t)G(t)} z(t) - \frac{A(t)}{1 + \mu(t)G(t)} \\
 &= ((-F) \ominus G)(t)z(t) - \frac{A(t)}{1 + \mu(t)G(t)} \\
 &= ((-F) \ominus G)(t)z(t) - C(t), \quad t \in \mathbb{T}^\kappa, \tag{15}
 \end{aligned}$$

where $C(t)$ is defined as in (9). Note that z is rd-continuous, and from the assumption $\mu(t)F(t) < 1, t \in \mathbb{T}^\kappa$, we get $(-F) \ominus G \in \mathcal{R}^+$. Then, by Lemma 2.3 and (15), we obtain

$$z(t) \leq z(\infty)e_{\ominus((-F) \ominus G)}(\infty, t) + \int_t^\infty C(s)e_{(-F) \ominus G}(t, \sigma(s)) \Delta s, \quad t \in \mathbb{T}^\kappa. \tag{16}$$

From $e_{G \ominus (-F)}(\infty, t) < \infty$ and $\ominus((-F) \ominus G) = (G \ominus (-F))$, we have $e_{\ominus((-F) \ominus G)}(\infty, t) < \infty$. According to $z(\infty) = 0$ and (16), we obtain

$$z(t) \leq \int_t^\infty C(s)e_{(-F) \ominus G}(t, \sigma(s)) \Delta s, \quad t \in \mathbb{T}^\kappa. \tag{17}$$

Combining (12), we get the desired inequality (4). This completes the proof. □

If we let $h(t) \equiv 0$ in Theorem 3.1, then we obtain the following corollary.

Corollary 3.1 *Assume that $x, f, g, a, b, c, m, n, p, q, r$, and F are defined the same as in Theorem 3.1, $e_{\ominus(-F)}(\infty, t) < \infty$ for $t \in \mathbb{T}^\kappa$,*

$$\begin{aligned} x^p(t) \leq & f(t) + g(t) \int_t^\infty \left[a(s)x^p(s) + b(s)x^q(s) + c(s) \right. \\ & \left. + m(s) \int_s^\infty n(\xi)x^r(\xi) \Delta \xi \right] \Delta s, \quad t \in \mathbb{T}^\kappa, \end{aligned}$$

then, for any $K_1 > 0$ and $K_2 > 0$,

$$x(t) \leq \left(f(t) + g(t) \int_t^\infty A(s)e_{(-F)}(t, \sigma(s)) \Delta s \right)^{1/p}, \quad t \in \mathbb{T}^\kappa,$$

where

$$\begin{aligned} A(t) := & a(t)f(t) + \frac{q}{p}K_1^{(q-p)/p}b(t)f(t) + \frac{p-q}{p}K_1^{q/p}b(t) \\ & + c(t) + m(t) \int_t^\infty n(\xi) \left(\frac{r}{p}K_2^{(r-p)/p}f(\xi) + \frac{p-r}{p}K_2^{r/p} \right) \Delta \xi. \end{aligned}$$

Remark 3.1 If $g(t) \equiv 1, m(t) \equiv 0$, and $q = 1$, then Corollary 3.1 reduces to Theorem 3.3 in [33].

Theorem 3.2 *Assume that $x, f, g, h, a, b, c, k, l, p, q, \alpha, \beta, B$, and G are defined the same as in Theorem 3.1, $\mu(t)\tilde{F}(t) < 1$ and $e_{G \ominus (-\tilde{F})}(\infty, t) < \infty$ for $t \in \mathbb{T}^\kappa$. Suppose that x satisfies*

$$\begin{aligned} x^p(t) \leq & f(t) + g(t) \int_t^\infty \left[a(s)x^p(s) + b(s)x^q(s) + c(s) \right] \Delta s \\ & + g(t) \int_{t_0}^t h(s) \left[k(s)x^\beta(\sigma(s)) - l(s)x^\alpha(\sigma(s)) \right] \Delta s, \quad t \in \mathbb{T}^\kappa, \end{aligned} \tag{18}$$

then, for any $K_1 > 0$,

$$x(t) \leq \left(f(t) + g(t) \int_t^\infty \tilde{C}(s)e_{(-\tilde{F}) \ominus G}(t, \sigma(s)) \Delta s \right)^{1/p}, \quad t \in \mathbb{T}^\kappa, \tag{19}$$

where

$$\tilde{F}(t) := a(t)g(t) + \frac{q}{p}K_1^{(q-p)/p}b(t)g(t), \tag{20}$$

$$\tilde{A}(t) := a(t)f(t) + \frac{q}{p}K_1^{(q-p)/p}b(t)f(t) + \frac{p-q}{p}K_1^{q/p}b(t) + c(t) + h(t)B(t)f(\sigma(t)), \tag{21}$$

$$\tilde{C}(t) := \frac{\tilde{A}(t)}{1 + \mu(t)G(t)}. \tag{22}$$

Proof Denote

$$\begin{aligned} z(t) &= \int_t^\infty [a(s)x^p(s) + b(s)x^q(s) + c(s)]\Delta s \\ &\quad + \int_{t_0}^t h(s)[k(s)x^\beta(\sigma(s)) - l(s)x^\alpha(\sigma(s))]\Delta s, \quad t \in \mathbb{T}^\kappa. \end{aligned} \tag{23}$$

From (18) and (23), we have

$$x(t) \leq (f(t) + g(t)z(t))^{1/p}, \quad t \in \mathbb{T}^\kappa. \tag{24}$$

In view of Lemma 2.1, (23), and (24), we have

$$\begin{aligned} z^\Delta(t) &= -[a(t)x^p(t) + b(t)x^q(t) + c(t)] + h(t)[k(t)x^\beta(\sigma(t)) - l(t)x^\alpha(\sigma(t))] \\ &\geq -[a(t)x^p(t) + b(t)x^q(t) + c(t)] - h(t)B(t)x^p(\sigma(t)) \\ &\geq -[a(t)(f(t) + g(t)z(t)) + b(t)(f(t) + g(t)z(t))^{q/p} + c(t) \\ &\quad + h(t)B(t)(f(\sigma(t)) + g(\sigma(t))z(\sigma(t)))], \quad t \in \mathbb{T}^\kappa. \end{aligned} \tag{25}$$

Using Lemma 2.2 on the right-hand side of (25), for any $K_1 > 0$, we obtain

$$\begin{aligned} z^\Delta(t) &\geq -\left[a(t)(f(t) + g(t)z(t)) + b(t)\left[\frac{q}{p}K_1^{(q-p)/p}(f(t) + g(t)z(t)) + \frac{p-q}{p}K_1^{q/p} \right] \right. \\ &\quad \left. + c(t) + h(t)B(t)(f(\sigma(t)) + g(\sigma(t))z(\sigma(t))) \right] \\ &= -[\tilde{F}(t)z(t) + G(t)z(\sigma(t)) + \tilde{A}(t)], \end{aligned} \tag{26}$$

where $\tilde{F}(t)$, $G(t)$, and $\tilde{A}(t)$ are defined as in (20), (6), and (21). By a similar argument with Theorem 3.1 in the rest of the proof, one can prove that (19). This completes the proof. \square

Theorem 3.3 *Assume that $x, f, g, h, a, b, c, p, q$, and r are defined the same as in Theorem 3.1, $k, l \in C_{rd}(\mathbb{T}^\kappa, \mathbb{R}_+)$, and $\mu(t)P(t) < 1, e_{Q\ominus(-p)}(\infty, t) < \infty$ for $t \in \mathbb{T}^\kappa$. Suppose that x satisfies*

$$\begin{aligned} x^p(t) &\leq f(t) + g(t) \int_t^\infty [a(s)x^p(s) + b(s)x^q(s) + c(s)]\Delta s \\ &\quad - g(t) \int_{t_0}^t [h(s)x^p(\sigma(s)) + k(s)x^r(\sigma(s)) + l(s)]\Delta s, \quad t \in \mathbb{T}^\kappa, \end{aligned} \tag{27}$$

then, for any $K_1 > 0$ and $K_2 > 0$,

$$x(t) \leq \left(f(t) + g(t) \int_t^\infty M(s)e_{(-p)\ominus Q}(t, \sigma(s)) \Delta s \right)^{1/p}, \quad t \in \mathbb{T}^\kappa, \tag{28}$$

where

$$P(t) := a(t)g(t) + \frac{q}{p}K_1^{(q-p)/p}b(t)g(t), \tag{29}$$

$$Q(t) := h(t)g(\sigma(t)) + \frac{r}{p}K_2^{(r-p)/p}k(t)g(\sigma(t)), \tag{30}$$

$$R(t) := a(t)f(t) + b(t) \left[\frac{q}{p}K_1^{(q-p)/p}f(t) + \frac{p-q}{p}K_1^{q/p} \right] + c(t) + h(t)f(\sigma(t)) \\ + k(t) \left[\frac{r}{p}K_2^{(r-p)/p}f(\sigma(t)) + \frac{p-r}{p}K_2^{r/p} \right] + l(t), \tag{31}$$

$$M(t) := \frac{R(t)}{1 + \mu(t)Q(t)}. \tag{32}$$

Proof Denote

$$z(t) = \int_t^\infty [a(s)x^p(s) + b(s)x^q(s) + c(s)] \Delta s \\ - \int_{t_0}^t [h(s)x^p(\sigma(s)) + k(s)x^r(\sigma(s)) + l(s)] \Delta s, \quad t \in \mathbb{T}^\kappa. \tag{33}$$

From (27) and (33), we get

$$x(t) \leq (f(t) + g(t)z(t))^{1/p}, \quad t \in \mathbb{T}^\kappa. \tag{34}$$

In view of (33) and (34), we have

$$z^\Delta(t) = -[a(t)x^p(t) + b(t)x^q(t) + c(t) + h(t)x^p(\sigma(t)) + k(t)x^r(\sigma(t)) + l(t)] \\ \geq -[a(t)(f(t) + g(t)z(t)) + b(t)(f(t) + g(t)z(t))^{q/p} + c(t) \\ + h(t)(f(\sigma(t)) + g(\sigma(t))z(\sigma(t))) \\ + k(t)(f(\sigma(t)) + g(\sigma(t))z(\sigma(t)))^{r/p} + l(t)], \quad t \in \mathbb{T}^\kappa. \tag{35}$$

Using Lemma 2.2 on the right-hand side of (35), we obtain

$$z^\Delta(t) \geq - \left\{ a(t)(f(t) + g(t)z(t)) + b(t) \left[\frac{q}{p}K_1^{(q-p)/p}(f(t) + g(t)z(t)) + \frac{p-q}{p}K_1^{q/p} \right] \right. \\ \left. + c(t) + h(t)(f(\sigma(t)) + g(\sigma(t))z(\sigma(t))) \right. \\ \left. + k(t) \left[\frac{r}{p}K_1^{(r-p)/p}(f(\sigma(t)) + g(\sigma(t))z(\sigma(t))) + \frac{p-r}{p}K_1^{r/p} \right] + l(t) \right\} \\ = -[P(t)z(t) + Q(t)z(\sigma(t)) + R(t)], \tag{36}$$

where $P(t)$, $Q(t)$, and $R(t)$ are defined as in (29), (30), and (31). By a similar argument with Theorem 3.1 in the rest of the proof, one can prove that (28). This completes the proof. \square

4 Applications

In this section, we apply our results to study the boundedness of the solutions of two dynamic equations on time scales.

Example 4.1 Consider the following dynamic equation on time scales:

$$(x^p(t))^\Delta = -W\left(t, x(t), x(\sigma(t)), \int_t^\infty V(s, x(s)) \Delta s\right), \quad t \in \mathbb{T}^\kappa, \tag{37}$$

$$x(\infty) = x_0, \tag{38}$$

where $p > 0$ and x_0 are constants, $W \in C(\mathbb{T}^\kappa \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, and $V \in C(\mathbb{T}^\kappa \times \mathbb{R}, \mathbb{R})$.

Theorem 4.1 *Suppose that the functions W and V in (37) satisfy the conditions*

$$\begin{aligned} |W(t, u, v, w)| \leq & c(t) + a(t)|u|^p + b(t)|u|^q \\ & + l(t)|v|^\alpha - k(t)|v|^\beta + m(t)|w|, \quad t \in \mathbb{T}^\kappa, u, v, w \in \mathbb{R}, \end{aligned} \tag{39}$$

$$|V(t, u)| \leq n(t)|u|^r, \quad t \in \mathbb{T}^\kappa, u \in \mathbb{R}, \tag{40}$$

where $a, b, c, m, n \in C_{rd}(\mathbb{T}^\kappa, \mathbb{R}_+)$, and $k, l \in C_{rd}(\mathbb{T}^\kappa, (0, \infty))$, q, r, α , and β are constants satisfying

- (i) $0 \leq q \leq p, 0 \leq r \leq p, p < \alpha < \beta$; or
- (ii) $0 \leq q \leq p, 0 \leq r \leq p, 0 < \beta < \alpha < p$.

If x is a solution of Eq. (37) satisfying (38), $\mu(t)A(t) < 1$ and $e_{B \ominus (-F)}(\infty, t) < \infty$ for $t \in \mathbb{T}^\kappa$, then, for any $K_1 > 0$ and $K_2 > 0$,

$$|x(t)| \leq \left(|x_0^p| + \int_t^\infty C(s)e_{(-F) \ominus B}(t, \sigma(s)) \Delta s \right)^{1/p}, \quad t \in \mathbb{T}^\kappa, \tag{41}$$

where

$$\begin{aligned} F(t) &:= a(t) + \frac{q}{p}K_1^{(q-p)/p}b(t) + \frac{r}{p}K_2^{(r-p)/p}m(t) \int_t^\infty n(\xi) \Delta \xi, \\ A(t) &:= |x_0^p|a(t) + \frac{q}{p}K_1^{(q-p)/p}|x_0^p|b(t) + \frac{p-q}{p}K_1^{q/p}b(t) \\ &\quad + c(t) + \left(\frac{r}{p}K_2^{(r-p)/p}|x_0^p| + \frac{p-r}{p}K_2^{r/p} \right) m(t) \int_t^\infty n(\xi) \Delta \xi + |x_0^p|B(t), \\ B(t) &:= \frac{l(t)(\beta - \alpha)}{\beta - p} \left(\frac{(\beta - p)k(t)}{(\alpha - p)l(t)} \right)^{(\alpha-p)/(\alpha-\beta)}, \\ C(t) &:= \frac{A(t)}{1 + \mu(t)B(t)}. \end{aligned}$$

Proof Considering (38), then the equivalent integral equation of Eq. (37) is denoted by

$$x^p(t) = x_0^p + \int_t^\infty W\left(s, x(s), x(\sigma(s)), \int_t^\infty V(\xi, x(\xi)) \Delta \xi\right) \Delta s, \quad t \in \mathbb{T}^\kappa. \tag{42}$$

Using assumptions (37)–(40) and (42), we have

$$\begin{aligned}
 |x(t)|^p \leq & |x_0^p| + \int_t^\infty \left[a(s)|x(s)|^p + b(s)|x(s)|^q + c(s) + m(s) \int_s^\infty n(\xi)|x(\xi)|^r \Delta\xi \right. \\
 & \left. + l(s)|x(\sigma(s))|^\alpha - k(s)|x(\sigma(s))|^\beta \right] \Delta s, \quad t \in \mathbb{T}^\kappa.
 \end{aligned}
 \tag{43}$$

Then a suitable application of Theorem 3.1 to (43) yields (41). □

Example 4.2 Consider the following dynamic integral equation on time scales:

$$\begin{aligned}
 x^2(t) \leq & f(t) + \int_t^\infty [a(s)x^2(s) + b(s)x(s) + c(s)] \Delta s \\
 & + \int_{t_0}^t h(s)[k(s)x^5(\sigma(s)) - l(s)x^4(\sigma(s))] \Delta s, \quad t \in \mathbb{T}^\kappa,
 \end{aligned}
 \tag{44}$$

where $x, a, b, c, h, k,$ and l are defined the same as in Theorem 3.1, $f \in C_{rd}(\mathbb{T}^\kappa, \mathbb{R}), \mu(t) \times \tilde{F}(t) < 1,$ and $e_{G \ominus (-\tilde{F})}(\infty, t) < \infty$ for $t \in \mathbb{T}^\kappa,$ then, for any $K_1 > 0,$

$$|x(t)| \leq \left(|f(t)| + \int_t^\infty \tilde{C}(s)e_{(-\tilde{F}) \ominus G}(t, \sigma(s)) \Delta s \right)^{1/p}, \quad t \in \mathbb{T}^\kappa,
 \tag{45}$$

where

$$\begin{aligned}
 \tilde{F}(t) &:= a(t) + \frac{1}{2}K_1^{(-1)/p} b(t), \\
 \tilde{A}(t) &:= a(t)|f(t)| + \frac{1}{2}K_1^{(-1)/2} b(t)|f(t)| + \frac{1}{2}K_1^{1/2} b(t) + c(t) + h(t)B(t)|f(\sigma(t))|, \\
 G(t) &:= h(t)B(t), \quad B(t) := \frac{4l^3(t)}{27k^2(t)} \quad \text{and} \quad \tilde{C}(t) := \frac{\tilde{A}(t)}{1 + \mu(t)G(t)}.
 \end{aligned}$$

In fact, from (44), we have

$$\begin{aligned}
 |x(t)|^2 \leq & |f(t)| + \int_t^\infty [a(s)|x(s)|^2 + b(s)|x(s)| + c(s)] \Delta s \\
 & + \int_{t_0}^t h(s)[k(s)|x(\sigma(s))|^5 - l(s)|x(\sigma(s))|^4] \Delta s, \quad t \in \mathbb{T}^\kappa.
 \end{aligned}
 \tag{46}$$

Then a suitable application of Theorem 3.2 to (46) yields (45).

5 Conclusions

In this paper, we have established some new nonlinear dynamic integral inequalities containing integration on infinite interval on time scales which can be used as tools in the qualitative theory of certain classes of dynamic equations on time scales. Our results complement the results established in the literature.

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Authors' contributions

HDL carried out the main part of this article. All authors read and approved the final manuscript.

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