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A delay differential equation model of mealybugs and green lacewings

Kittipol Jankaew¹, Chontita Rattanukul^{1,2*}  and Warunee Sarika¹

*Correspondence:

chontita.rat@mahidol.ac.th

¹Department of Mathematics,
Faculty of Science, Mahidol

University, Bangkok, Thailand
²Centre of Excellence in
Mathematics, Commission on
Higher Education, Bangkok,
Thailand

Abstract

In this paper, we propose and analyze a mathematical model of mealybugs and green lacewings with time delay to investigate the population dynamics of mealybugs (a major insect pest of cassava) and green lacewings (a natural enemy of mealybugs) when the time delay in the development of green lacewings is taken in to account. Hopf bifurcation theorem and Routh–Hurwitz criteria are utilized so that the conditions on the model parameters which differentiate various dynamic behaviors of the model are obtained. Computer simulations are also carried out to illustrate our theoretical predictions. Chaotic behavior observed in the field data is also investigated numerically.

Keywords: Mathematical model; Mealybugs; Green lacewings; Time delay; Hopf-bifurcation; Periodic solutions

1 Introduction

Cassava (*Manihot esculenta* Crantz) is extensively cultivated as an annual crop in tropical and subtropical regions for its edible starchy storage roots. Apart from being the world's second largest producer of cassava, Thailand is the world largest exporter of cassava products as well [1].

On the other hand, the demand of energy increases continuously. In 2006, it has been reported in [2] that the use of energy worldwide is 11,266.7 million tons of oil equivalent and increases to 13,276.3 million tons of oil equivalent, which is an increase of 17.84%, and tends to be increasing continuously. However, the resource of energy is limited and it might not be sufficient to reach the demand in the future. Therefore it is necessary to seek for alternative energy. Department of Alternative Energy Development and Efficiency found that dry crops is one of effective choices that can be transmuted to the alternative energy. The most suitable dry crops in Thailand are sugar cane (*Saccharum officinarum* L.) and cassava (*Manihot esculenta* L. crantz) that can be transmuted to ethanol. According to [3], 1 ton of cassava roots can be transmuted to 140 liters of ethanol, 1 ton of tapioca chips can be transmuted to 333 liters of ethanol, 1 ton of tapioca starch can be transmuted to 500 liters of ethanol, 1 ton of sugarcane can be transmuted to 70 liters of ethanol, and 1 ton of molasses can be transmuted to 238 liters of ethanol.

In 2008, the outbreak of cassava mealybugs (a major insect pest of cassava) caused severe damage in cassava fields in Thailand and rapidly spread to other areas [4]. According

to Thailand’s office of agricultural economics, cassava yields reduced to 23 million tons in the year 2009–2010, which is approximately 10 million tons lower than the expected cassava yields [5]. Mealybug is a major insect pest of cassava. It can destroy up to 54% of roots and 100% of leaves in locations of infestation [6]. It damages cassava plants directly by sucking the sap and contaminating the plant with its toxic saliva and indirectly by favoring the development of sooty molds [4]. Without control strategy, mealybugs can reduce cassava yields to less than 80% [7]. To control the spread of mealybugs, there are many strategies suggested such as insecticide, biological control, or mixed insecticide and biological control. With biological control, the use of mealybug’s natural enemies has proved experimentally to be successful [8].

Green lacewing is one of natural enemies mostly used to control the spread of mealybugs. The green lacewing larvae can destroy 100 aphids of mealybugs or more in a week. They damage the prey by using the hollow suckle-shaped jaws that allow them to withdraw fluids from many soft-bodied insects. In this work, the population dynamics of mealybugs and green lacewings will be investigated.

In this paper, we propose and analyze a mathematical model of mealybugs and green lacewings with time delay to investigate the population dynamics of mealybugs (a major insect pest of cassava) and green lacewings (a natural enemy of mealybug) when the time delay in the development of green lacewings is taken in to account.

2 A mathematical model

In this paper, we assume that the following nonlinear system of the delay differential equations can be used to investigate the population dynamics of mealybugs and green lacewings:

$$\frac{dM}{dt} = a_1M\left(1 - \frac{M}{k_1}\right) - \frac{a_2MG}{k_2 + M} - b_1M, \tag{2.1}$$

$$\frac{dG}{dt} = \frac{\alpha_1 a_2 M(t - \tau)G(t - \tau)}{k_2 + M(t - \tau)} - b_2G, \tag{2.2}$$

where $M(t)$ and $G(t)$ account for the population densities of mealybugs and green lacewing larvae at time t , respectively; $M(t - \tau)$ and $G(t - \tau)$ account for the population densities of mealybugs and green lacewings at time $t - \tau$, respectively. Here $a_1, a_2, k_1, b_1, b_2, \alpha_1$ are assumed to be positive.

In Eq. (2.1), the term $a_1M(1 - \frac{M}{k_1})$ represents the the growth rate of mealybugs, $\frac{a_2MG}{k_2+M}$ represents the death rate of mealybugs due to the predation by green lacewing larvae, and b_1 represents the natural death rate of mealybugs.

In Eq. (2.2), since only the green lacewing larvae can prey on mealybugs and it takes approximately 39 days for green lacewing larvae to develop into pupa, an adult that can lay eggs which will develop into green lacewing larvae [9, 10], we then assume that the term $\frac{\alpha_1 a_2 M(t - \tau)G(t - \tau)}{(k_2 + M(t - \tau))}$ represents the growth rate of green lacewings due to the consumption on mealybugs which incorporates the time delay in the increase of green lacewings; b_2 represents the natural death rate of green lacewings.

3 Modeling analysis

The points (0, 0) and

$$\left(\frac{b_2 k_2}{\alpha_1 a_2 - b_2}, \frac{\alpha_1 k_2 (a_1 a_2 \alpha_1 k_1 - a_1 b_2 k_1 - a_1 b_2 k_2 - a_2 b_1 \alpha_1 k_1 + b_1 b_2 k_1)}{k_1 (\alpha_1 a_2 - b_2)^2} \right)$$

are two steady states of the system of Eqs. (2.1)–(2.2).

Firstly, let us consider the local stability of the washout steady state (0, 0). The Jacobian matrix of system (2.1)–(2.2) evaluated at the washout steady state (0, 0) is

$$J|_{(0,0)} = \begin{pmatrix} a_1 - b_1 & 0 \\ 0 & -b_2 \end{pmatrix}. \tag{3.1}$$

We can see that the eigenvalues of $J|_{(0,0)}$ are $\lambda_1 = a_1 - b_1$ and $\lambda_2 = -b_2$, and hence, we obtain the following theorem.

Theorem 1 *The washout steady state (0, 0) of system (2.1)–(2.2) is locally asymptotically stable if and only if*

$$a_1 < b_1 \quad \text{and} \quad b_2 > 0. \tag{3.2}$$

Proof Since all eigenvalues of $J|_{(0,0)}$ will have negative real parts if and only if inequalities in (3.2) hold, the washout steady state (0, 0) of system (2.1)–(2.2) is locally asymptotically stable if and only if (3.2) is satisfied. \square

Next, let us consider the non-washout steady state

$$\left(\frac{b_2 k_2}{\alpha_1 a_2 - b_2}, \frac{\alpha_1 k_2 (a_1 a_2 \alpha_1 k_1 - a_1 b_2 k_1 - a_1 b_2 k_2 - a_2 b_1 \alpha_1 k_1 + b_1 b_2 k_1)}{k_1 (\alpha_1 a_2 - b_2)^2} \right) \equiv (M_s, G_s).$$

Note that $M_s > 0$ and $G_s > 0$ if and only if

$$\alpha_1 a_2 > b_2 \tag{3.3}$$

and

$$(a_1 a_2 \alpha_1 - a_1 b_2 - a_2 b_1 \alpha_1 + b_1 b_2) > \frac{a_1 b_2 k_2}{k_1}. \tag{3.4}$$

Letting $m = M - M_s$ and $g = G - G_s$, we will be led to following linearized system of (2.1)–(2.2):

$$\begin{pmatrix} \dot{m}(t) \\ \dot{g}(t) \end{pmatrix} = J_s \begin{pmatrix} m(t) \\ g(t) \end{pmatrix}, \tag{3.5}$$

where J_s is the corresponding Jacobian matrix evaluated at (M_s, G_s) , namely

$$J_s = \begin{pmatrix} \frac{(a_1 - b_1)b_2}{\alpha_1 a_2} + \frac{a_1 b_2 k_2}{\alpha_1 a_2 k_1} - \frac{2a_1 b_2 k_2}{k_1 (\alpha_1 a_2 - b_2)} & -\frac{b_2}{\alpha_1} \\ \frac{(k_1 (a_1 - b_1) (\alpha_1 a_2 - b_2) - a_1 b_2 k_2) e^{-\lambda_1 \tau}}{a_2 k_1} & -b_2 (1 - e^{-\lambda_1 \tau}) \end{pmatrix}. \tag{3.6}$$

For simplicity, we introduce new parameters by letting

$$\begin{aligned}
 a &= b_2 - \left(\frac{(a_1 - b_1)b_2}{\alpha_1 a_2} + \frac{a_1 b_2 k_2}{\alpha_1 a_2 k_1} - \frac{2a_1 b_2 k_2}{k_1(\alpha_1 a_2 - b_2)} \right), \\
 b &= -b_2 \left(\frac{(a_1 - b_1)b_2}{\alpha_1 a_2} + \frac{a_1 b_2 k_2}{\alpha_1 a_2 k_1} - \frac{2a_1 b_2 k_2}{k_1(\alpha_1 a_2 - b_2)} \right), \\
 c &= b_2 \left[\left(\frac{(a_1 - b_1)b_2}{\alpha_1 a_2} + \frac{a_1 b_2 k_2}{\alpha_1 a_2 k_1} - \frac{2a_1 b_2 k_2}{k_1(\alpha_1 a_2 - b_2)} \right) \right. \\
 &\quad \left. + \frac{(k_1(a_1 - b_1)(\alpha_1 a_2 - b_2) - a_1 b_2 k_2)}{\alpha_1 a_2 k_1} \right], \\
 d &= -b_2.
 \end{aligned}$$

Then, the characteristic equation of J_s can be written as

$$F(\lambda) = \lambda^2 + a\lambda + b + (c + d\lambda)e^{-\lambda\tau} = 0. \tag{3.7}$$

According to the Hopf bifurcation theory, for a periodic solution to exist, it is necessary that Eq. (3.7) has a pair of purely imaginary complex roots $\lambda = \pm i\omega$ for some value of τ . In order that such a pair can be found, one must have $F(i\omega) = 0$, that is,

$$(i\omega)^2 + a(i\omega) + b + (c + d(i\omega))e^{(-i\omega)\tau} = 0. \tag{3.8}$$

Equating real and imaginary parts on the left of Eq. (3.8) to zero, we obtain the following equations:

$$-\omega^2 + b + c \cos(\omega\tau) - d\omega \sin(\omega\tau) = 0, \tag{3.9}$$

$$a\omega + c \sin(\omega\tau) + d\omega \cos(\omega\tau) = 0. \tag{3.10}$$

By squaring (3.9) and (3.10), and then adding them, it follows that

$$\omega^4 + (a^2 - 2b - d^2)\omega^2 + (b^2 - c^2) = 0. \tag{3.11}$$

Let $\eta = \omega^2$, then (3.11) can be written as

$$\delta(\eta) \equiv \eta^2 + (a^2 - 2b - d^2)\eta + (b^2 - c^2) = 0. \tag{3.12}$$

Hence, Eq. (3.7) will have a pair of complex solutions $\lambda = \pm i\omega$ provided that (3.12) has a positive real solution $\eta = \omega^2 > 0$.

Lemma 1 *If*

$$b^2 - c^2 < 0 \tag{3.13}$$

then Eq. (3.12) has at least one positive real root.

Lemma 2 *If*

$$b^2 - c^2 \geq 0 \tag{3.14}$$

then the necessary conditions that Eq. (3.12) has at least one positive real root are

$$a^2 - 2b - d^2 < 0 \quad \text{and} \quad (a^2 - 2b - d^2)^2 \geq 4(b^2 - c^2). \tag{3.15}$$

Without loss of generality, we assume that (3.10) has two positive real roots denoted by η_1 and η_2 . Then, (3.9) has two positive roots $\omega_k = \sqrt{\eta_k}$, $k = 1, 2$.

Now, let $\tau_0 > 0$ be the smallest of such τ for which $\lambda = \pm i\omega$. We substitute $\omega = \omega_k$ into Eqs. (3.9)–(3.10) and have

$$-\omega_k^2 + b + c \cos(\omega_k \tau) - d\omega_k \sin(\omega_k \tau) = 0, \tag{3.16}$$

$$a\omega_k + c \sin(\omega_k \tau) + d\omega_k \cos(\omega_k \tau) = 0. \tag{3.17}$$

Rewrite Eq. (3.17), we obtain

$$\sin(\omega_k \tau) = \frac{-a\omega_k - d\omega_k \cos(\omega_k \tau)}{c}. \tag{3.18}$$

Substituting (3.18) into (3.16) and rearranging, we then have

$$\cos(\omega_k \tau) = \frac{(c - ad)\omega_k^2 - bc}{c^2 + d^2\omega_k^2}. \tag{3.19}$$

Solving for τ , we obtain

$$\tau_k^{(j)} = \frac{1}{\omega_k} \arccos \left[\frac{(c - ad)\omega_k^2 - bc}{c^2 + d^2\omega_k^2} \right] + \frac{(j - 1)\pi}{\omega_k}, \tag{3.20}$$

where $k = 1, 2, j = 1, 2, \dots$

Then $\pm i\omega_k$ is a pair of purely imaginary roots of Eq. (3.7) with $\tau = \tau_k^{(j)}$, $j = 1, 2, \dots$. Thus, we can define

$$\tau_0 = \min_{1 \leq k \leq 2, j \geq 1} (\tau_k^{(j)}), \quad \tau_k^{(j)} > 0, \tag{3.21}$$

with $\tau_k^{(j)}$ defined in Eq. (3.20).

Theorem 2

(a) If (3.14) holds and

$$a^2 - 2b - d^2 \geq 0 \quad \text{or} \quad (a^2 - 2b - d^2)^2 < 4(b^2 - c^2), \tag{3.22}$$

then all roots of (3.7) have nonzero real parts for all $\tau \geq 0$.

(b) If

$$a + d > 0 \quad \text{and} \quad b + c > 0, \tag{3.23}$$

and either (3.13) or ((3.14) and (3.15)) holds, then all roots of (3.7) have negative real parts for $\tau \in [0, \tau_0)$ where τ_0 is as defined in (3.21).

Proof (a) Arguing by contradiction, suppose that (3.7) has a root with zero real part for some $\tau \geq 0$. This implies that (3.12) has a positive real root. According to Lemma 2, the necessary conditions that (3.12) has a positive real root are $a^2 - 2b - d^2 < 0$ and $(a^2 - 2b - d^2)^2 \geq 4(b^2 - c^2)$, which is a contradiction. Hence, all roots of (3.7) have nonzero real parts for all $\tau \geq 0$.

(b) For $\tau = 0$, (3.7) is reduced to

$$F(\lambda) = \lambda^2 + (a + d)\lambda + (b + c) = 0.$$

Since all inequalities in (3.23) hold, the Routh–Hurwitz criterion then implies that all roots of (3.7) have negative real parts at $\tau = 0$. The continuity of $\lambda(\tau)$ implies that all roots of (3.7) will have negative real parts for values of τ in some open interval containing $\tau = 0$. Hence, there is a $\tau_c > 0$ such that all roots of (3.7) have negative real parts for $\tau \in [0, \tau_c)$.

Since τ_0 defined in (3.21) is the minimum of $\tau_k^{(j)}$ defined in (3.20) for which the real part of at least one root of (3.7) vanishes provided that (3.22) holds, $\tau_c = \tau_0$ and the proof is complete. \square

Theorem 2 implies that if (3.23) and either (3.13) or ((3.14) and (3.15)) hold, the steady state (M_s, G_s) of our system of Eqs. (2.1)–(2.2) is stable for some values of $\tau \in [0, \tau_0)$. At $\tau = \tau_0$, $\text{Re}(\lambda(\tau)) = 0$ by the definition of τ_0 , and hence the stability of steady state (M_s, G_s) is lost at $\tau = \tau_0$.

In order for a Hopf bifurcation to occur, and hence a periodic solution of our system of Eqs. (2.1)–(2.2) to be expected, we still need to show that

$$\left. \frac{d(\text{Re}(\lambda(\tau)))}{d\tau} \right|_{\tau=\tau_0} \neq 0.$$

Theorem 3 *Suppose that (3.13) or ((3.14) and (3.15)) holds, then $\lambda = \pm i\omega_0$ is a pair of purely imaginary roots of (3.7). Moreover,*

$$\left. \frac{d(\text{Re}(\lambda(\tau)))}{d\tau} \right|_{\tau=\tau_0} \neq 0, \tag{3.24}$$

where $\omega_0 = \omega_k|_{\tau=\tau_0}$, provided that

$$\delta'(\beta_0) \neq 0, \tag{3.25}$$

where $\beta_0 = \omega_0^2$ and $\omega_0 = \omega_k|_{\tau=\tau_0}$.

Proof In order to prove that $\left. \frac{d(\text{Re}(\lambda(\tau)))}{d\tau} \right|_{\tau=\tau_0} \neq 0$, let us consider Eq. (3.7), namely

$$F(\lambda) = \lambda^2 + a\lambda + b + (c + d\lambda)e^{-\lambda\tau} = 0.$$

Then,

$$\frac{dF(\lambda)}{d\tau} = (2\lambda + a - (c + d\lambda)\tau e^{-\lambda\tau} + de^{-\lambda\tau}) \frac{d\lambda}{d\tau} - (c + d\lambda)\lambda e^{-\lambda\tau} = 0,$$

and hence,

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{2\lambda + a}{(c + d\lambda)\lambda e^{-\lambda\tau}} - \frac{\tau}{\lambda} + \frac{d}{(c + d\lambda)\lambda}.$$

Since $(c + d\lambda)e^{-\lambda\tau} = -(\lambda^2 + a\lambda + b)$, we get

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{2\lambda + a}{-\lambda^3 - a\lambda^2 - b\lambda} - \frac{\tau}{\lambda} + \frac{d}{(c + d\lambda)\lambda}$$

at $\tau = \tau_0, \lambda = i\omega_0$, and thus

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} \Big|_{\tau=\tau_0, \lambda=i\omega_0} = \frac{2i\omega_0 + a}{a\omega_0^2 + i(\omega_0^3 - b\omega_0)} + i\left(\frac{\tau}{\omega_0}\right) + \frac{d}{-d\omega_0^2 + ic\omega_0}.$$

Therefore,

$$\operatorname{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1} \Big|_{\tau=\tau_0, \lambda=i\omega_0} = \frac{2\omega_0^2 + a^2 - 2b}{\omega_0^4 + (a^2 - 2b)\omega_0^2 + b^2} - \frac{d^2}{d^2\omega_0^2 + c^2}.$$

Equation (3.11) implies that

$$\omega_0^4 + (a^2 - 2b)\omega_0^2 + b^2 = d^2\omega_0^2 + c^2,$$

and then

$$\operatorname{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1} \Big|_{\tau=\tau_0, \lambda=i\omega_0} = \frac{2\omega_0^2 + a^2 - 2b - d^2}{d^2\omega_0^2 + c^2}.$$

Equation (3.12) implies that

$$\operatorname{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1} \Big|_{\tau=\tau_0, \lambda=i\omega_0} = \frac{\delta'(\omega_0^2)}{d^2\omega_0^2 + c^2} \neq 0.$$

Hence, $\frac{d(\operatorname{Re}(\lambda(\tau)))}{d\tau} \Big|_{\tau=\tau_0} \neq 0$ and the proof is complete. □

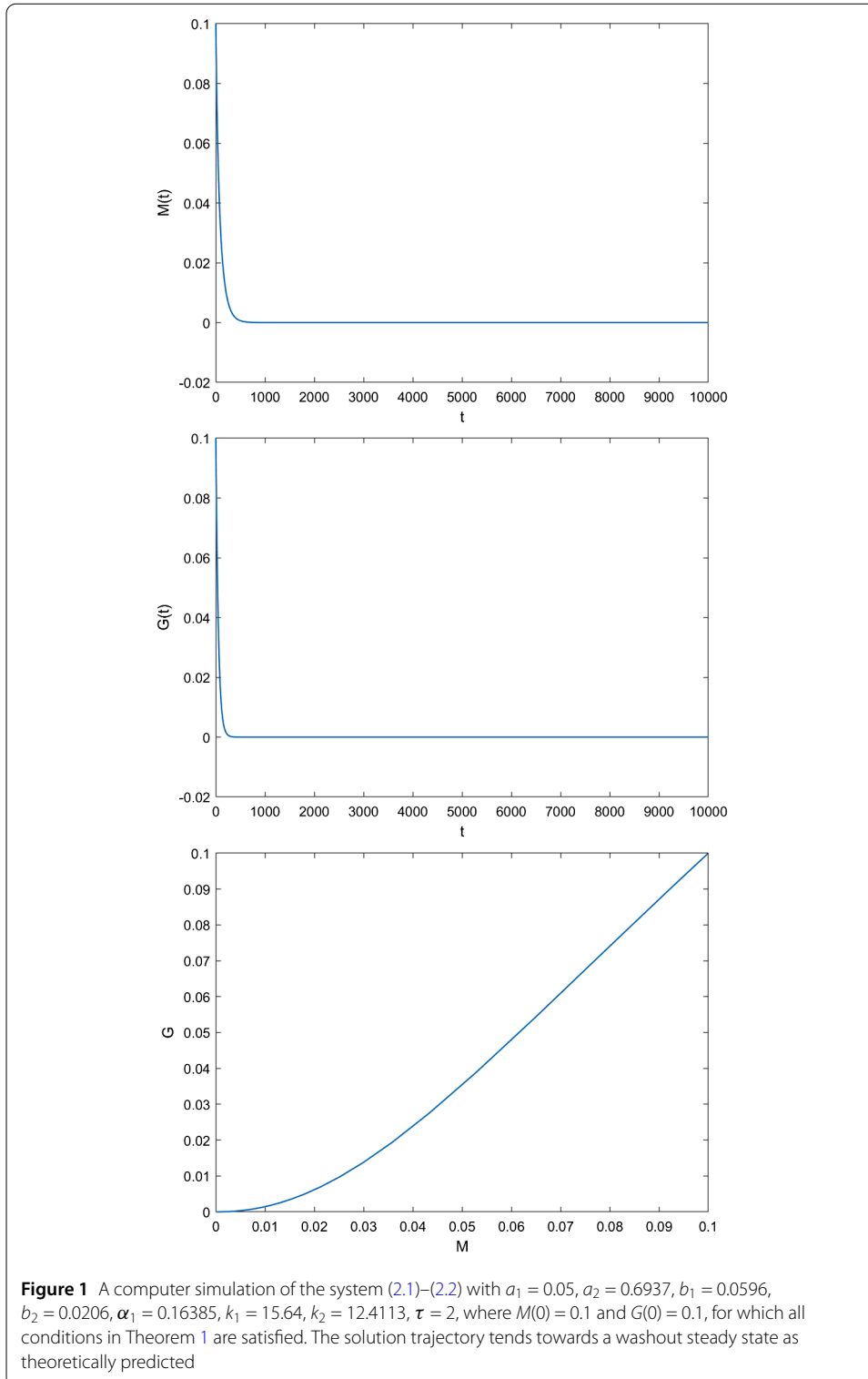
Theorem 4 *If either (3.13) or ((3.14) and (3.15)) holds, then a Hopf bifurcation occurs in our model Eqs. (2.1)–(2.2) for a positive time delay $\tau = \tau_0$ given by (3.20) provided that (3.3), (3.4), (3.23), and (3.25) are satisfied.*

Proof Since all necessary conditions of the Hopf bifurcation theorem are satisfied, then Eqs. (2.1)–(2.2) undergo a Hopf bifurcation when $\tau = \tau_0$, therefore the proof is complete. □

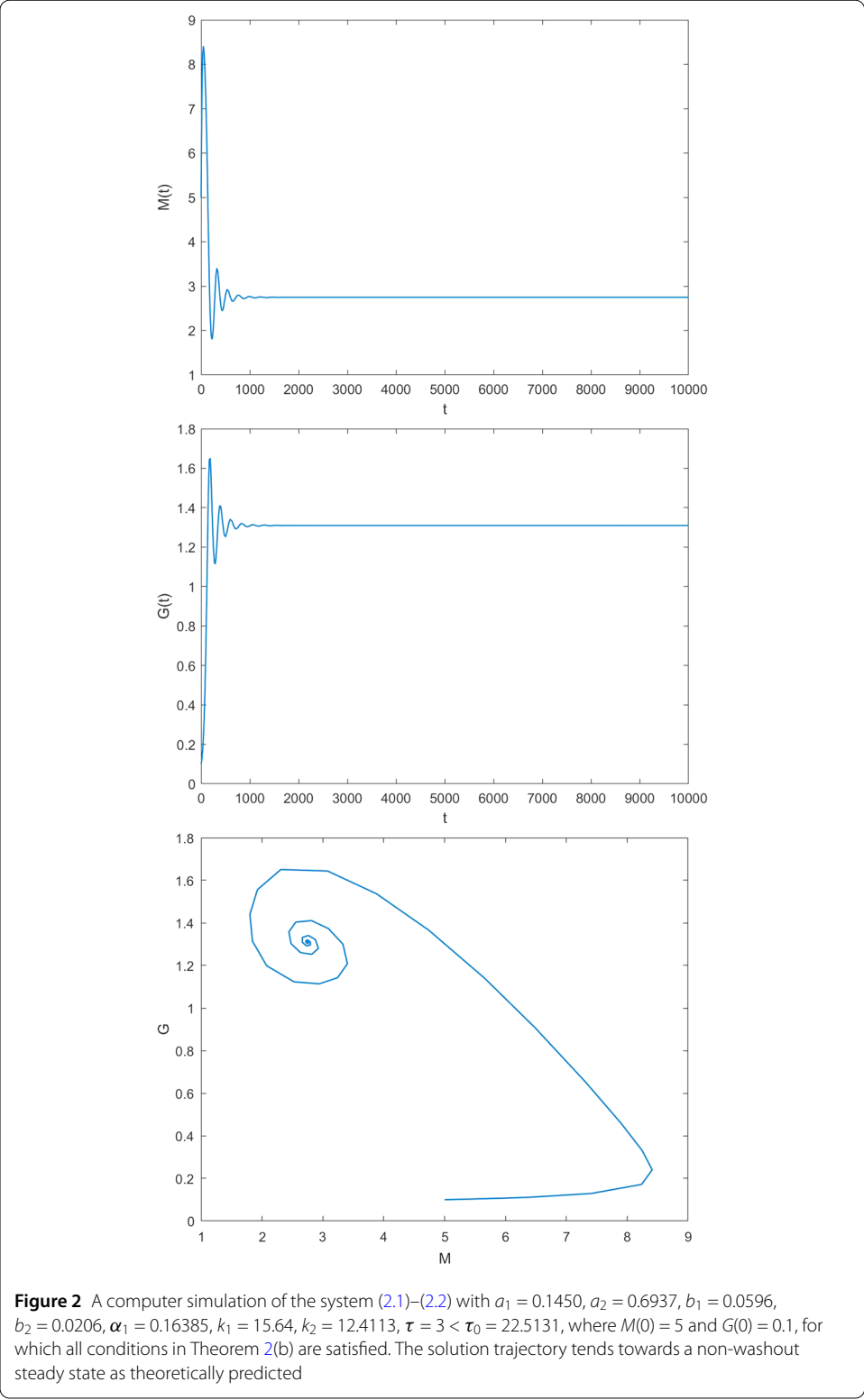
4 Numerical simulations

In this section, computer simulations of our model Eqs. (2.1)–(2.2) are presented to illustrate our theoretical predictions. Some parametric values are obtained from [10–12].

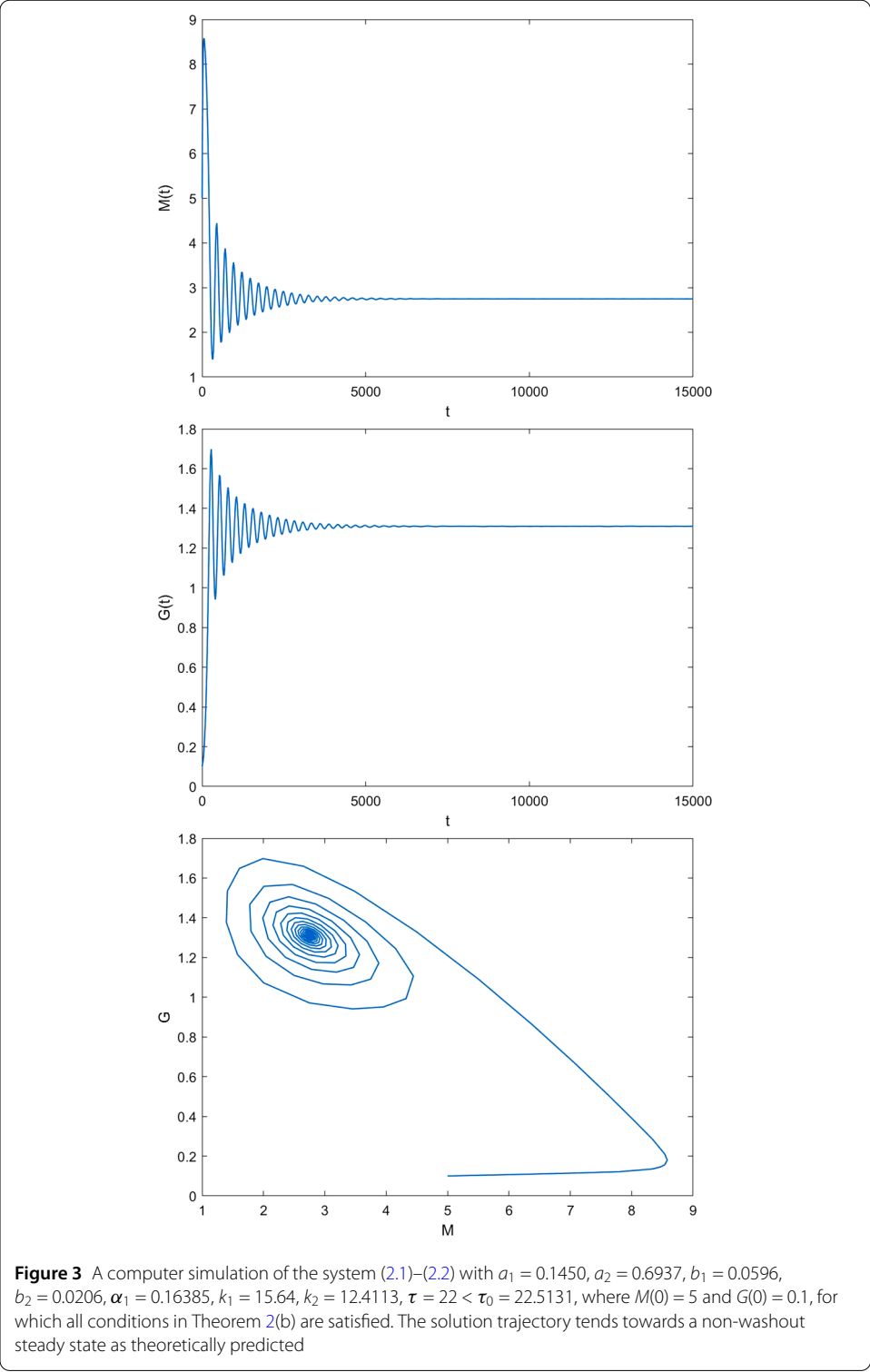
In Fig. 1, the parameters b_1 and b_2 are taken from [10–12]. The other parameters are chosen to satisfy conditions in Theorem 1. A computer simulation is run for the system



(2.1)–(2.2) with $a_1 = 0.05$, $a_2 = 0.6937$, $b_1 = 0.0596$, $b_2 = 0.0206$, $\alpha_1 = 0.16385$, $k_1 = 15.64$, $k_2 = 12.4113$, $\tau = 2$, where $M(0) = 0.1$ and $G(0) = 0.1$, for which all conditions in Theorem 1 are satisfied, showing that the solution trajectory tends towards a washout steady state as theoretically predicted.



In Figs. 2–8, the parameters a_1, b_1 and b_2 are taken from [10–12]. The other parameters are chosen to satisfy conditions in Theorems 2 and 4.



A computer simulation for the system (2.1)–(2.2) with $a_1 = 0.1450$, $a_2 = 0.6937$, $b_1 = 0.0596$, $b_2 = 0.0206$, $\alpha_1 = 0.16385$, $k_1 = 15.64$, $k_2 = 12.4113$, $\tau = 3 < \tau_0 = 22.5131$, where $M(0) = 5$ and $G(0) = 0.1$, for which all conditions in Theorem 2(b) are satisfied, is shown

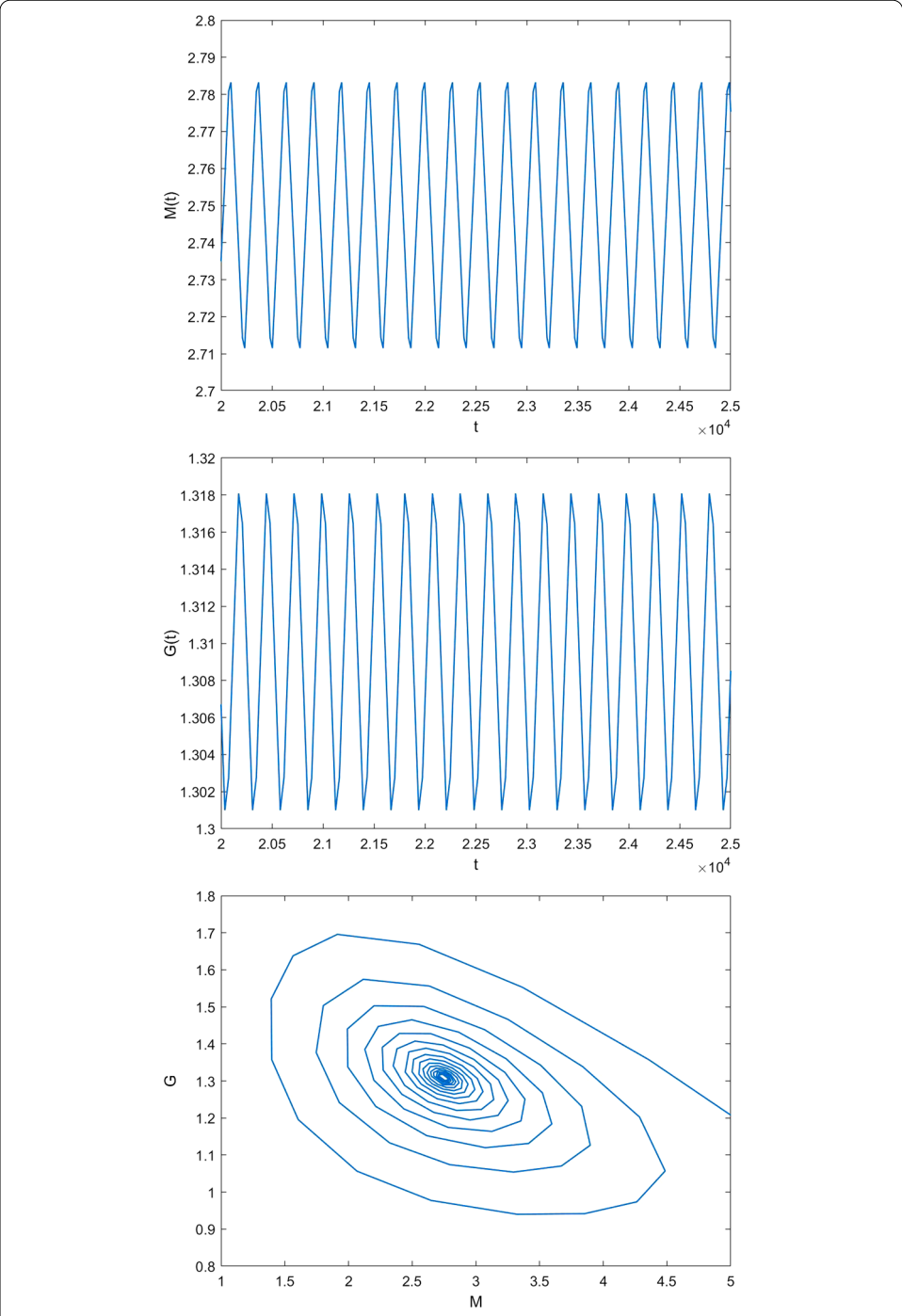
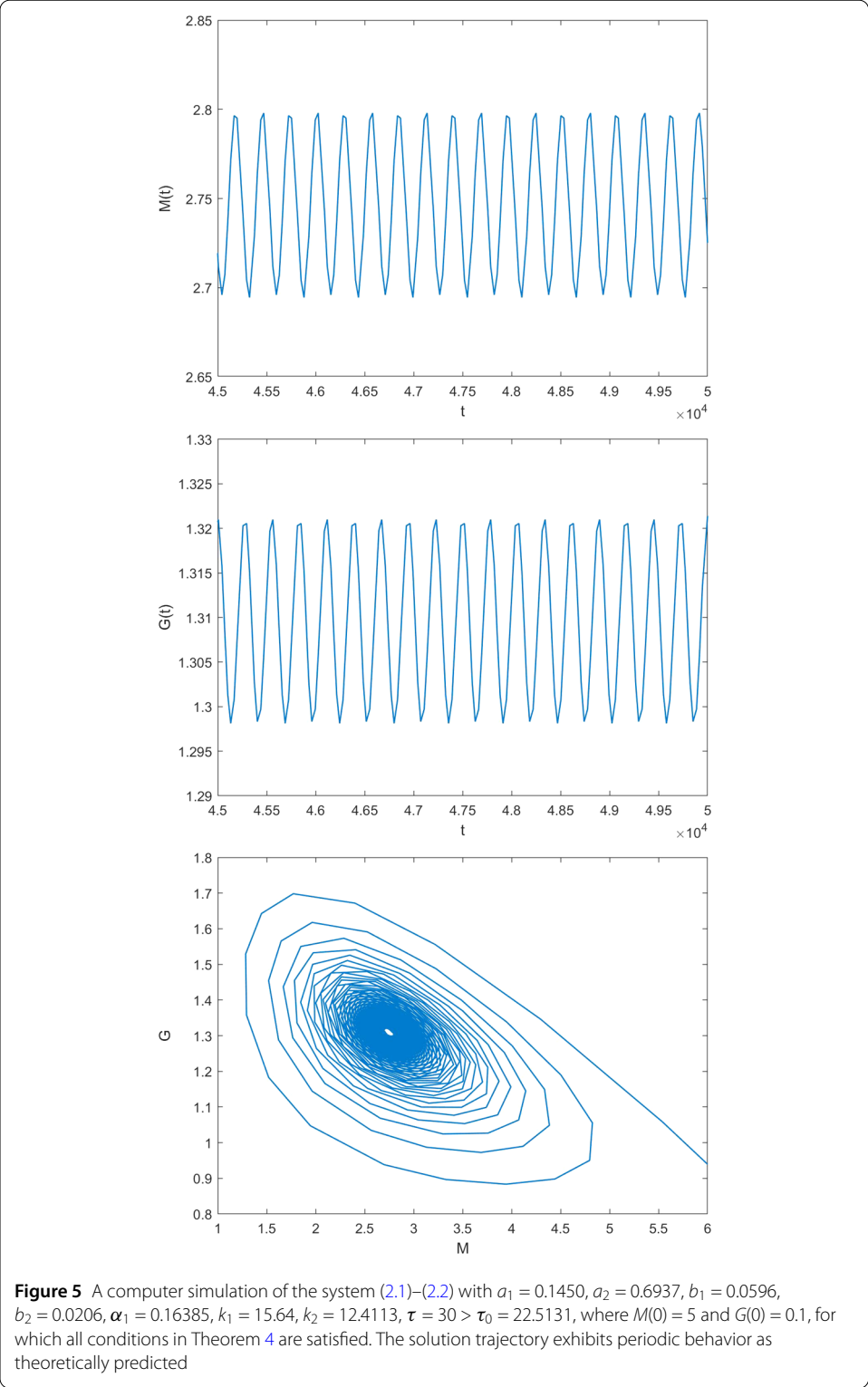


Figure 4 A computer simulation of the system (2.1)–(2.2) with $a_1 = 0.1450, a_2 = 0.6937, b_1 = 0.0596, b_2 = 0.0206, \alpha_1 = 0.16385, k_1 = 15.64, k_2 = 12.4113, \tau = \tau_0 = 22.5131$, where $M(0) = 5$ and $G(0) = 0.1$, for which all conditions in Theorem 4 are satisfied. The solution trajectory exhibits periodic behavior as theoretically predicted

in Fig. 2. The solution trajectory tends towards a non-washout steady state as theoretically predicted.

A computer simulation of the system (2.1)–(2.2) with $a_1 = 0.1450, a_2 = 0.6937, b_1 = 0.0596, b_2 = 0.0206, \alpha_1 = 0.16385, k_1 = 15.64, k_2 = 12.4113, \tau = 22 < \tau_0 = 22.5131$, where



$M(0) = 5$ and $G(0) = 0.1$, for which all conditions in Theorem 2(b) are satisfied, is shown in Fig. 3. The solution trajectory tends towards a non-washout steady state as theoretically predicted.

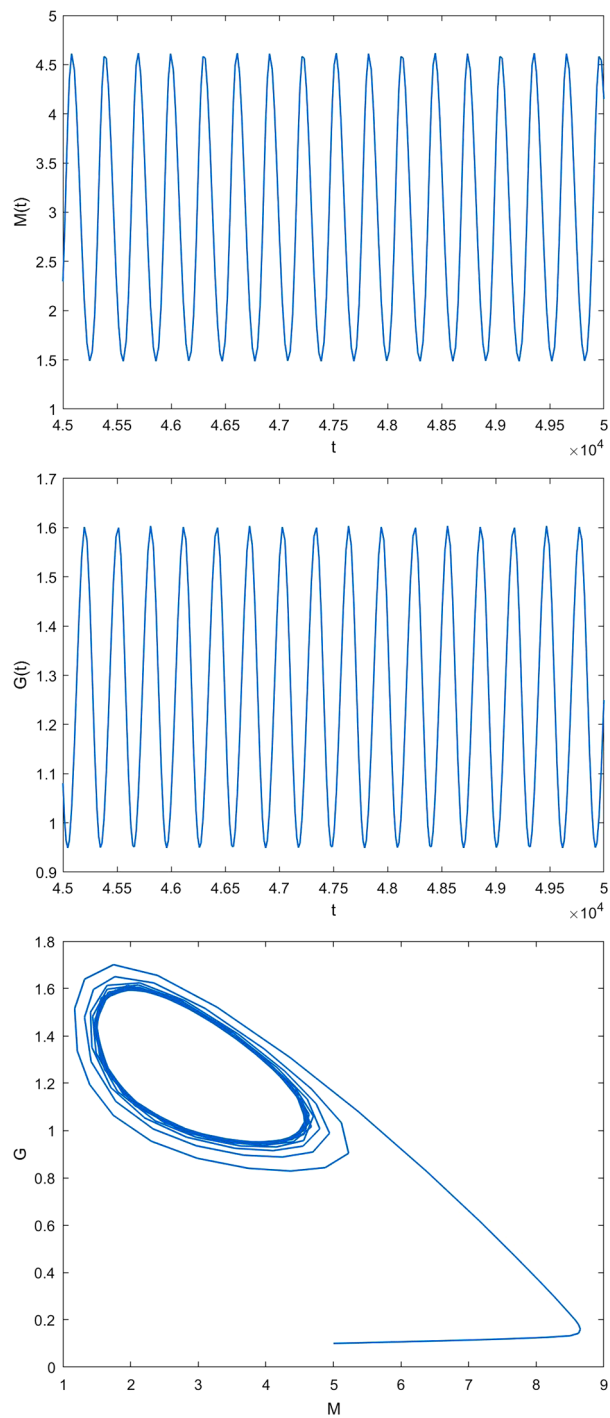
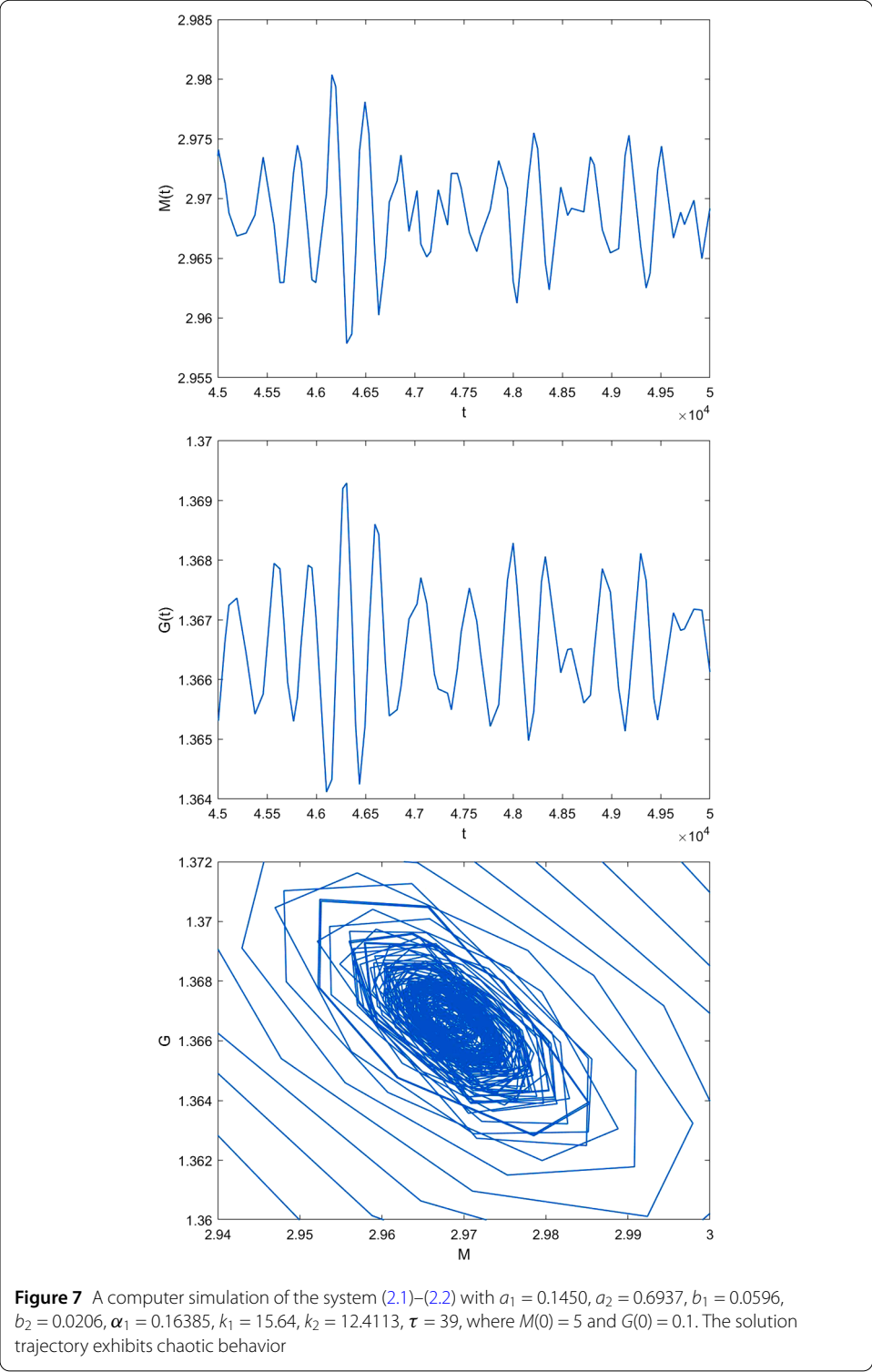


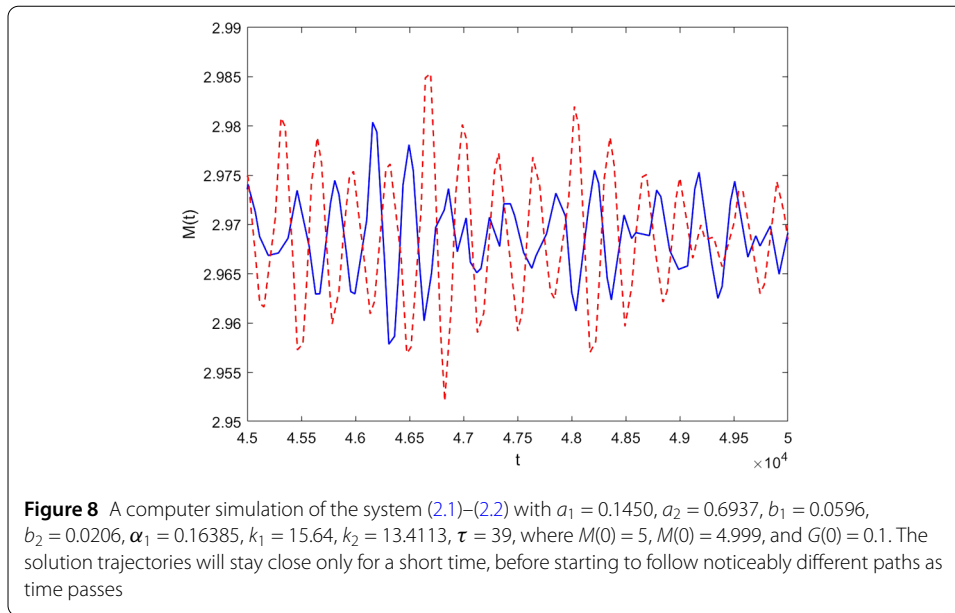
Figure 6 A computer simulation of the system (2.1)–(2.2) with $a_1 = 0.1450$, $a_2 = 0.6937$, $b_1 = 0.0596$, $b_2 = 0.0206$, $\alpha_1 = 0.16385$, $k_1 = 15.64$, $k_2 = 12.4113$, $\tau = 35 > \tau_0 = 22.5131$, where $M(0) = 5$ and $G(0) = 0.1$, for which all conditions in Theorem 4 are satisfied. The solution trajectory exhibits periodic behavior as theoretically predicted

A computer simulation of the system (2.1)–(2.2) with $a_1 = 0.1450$, $a_2 = 0.6937$, $b_1 = 0.0596$, $b_2 = 0.0206$, $\alpha_1 = 0.16385$, $k_1 = 15.64$, $k_2 = 12.4113$, $\tau = \tau_0 = 22.5131$, where $M(0) =$



5 and $G(0) = 0.1$, for which all conditions in Theorem 4 are satisfied is shown in Fig. 4. The solution trajectory exhibits periodic behavior as theoretically predicted.

A computer simulation of the system (2.1)–(2.2) with $a_1 = 0.1450, a_2 = 0.6937, b_1 = 0.0596, b_2 = 0.0206, \alpha_1 = 0.16385, k_1 = 15.64, k_2 = 12.4113, \tau = 30 > \tau_0 = 22.5131$, where



$M(0) = 5$ and $G(0) = 0.1$, for which all conditions in Theorem 4 are satisfied, is shown in Fig. 5. The solution trajectory exhibits periodic behavior as theoretically predicted.

A computer simulation of the system (2.1)–(2.2) with $a_1 = 0.1450, a_2 = 0.6937, b_1 = 0.0596, b_2 = 0.0206, \alpha_1 = 0.16385, k_1 = 15.64, k_2 = 12.4113, \tau = 35 > \tau_0 = 22.5131$, where $M(0) = 5$ and $G(0) = 0.1$, for which all conditions in Theorem 4 are satisfied, is shown in Fig. 6. The solution trajectory exhibits periodic behavior as theoretically predicted.

A computer simulation of the system (2.1)–(2.2) with $a_1 = 0.1450, a_2 = 0.6937, b_1 = 0.0596, b_2 = 0.0206, \alpha_1 = 0.16385, k_1 = 15.64, k_2 = 12.4113, \tau = 39$, where $M(0) = 5$ and $G(0) = 0.1$, is shown in Fig. 7. The solution trajectory exhibits chaotic behavior.

A computer simulation of the system (2.1)–(2.2) with $a_1 = 0.1450, a_2 = 0.6937, b_1 = 0.0596, b_2 = 0.0206, \alpha_1 = 0.16385, k_1 = 15.64, k_2 = 12.4113, \tau = 39$, where $M(0) = 5, M(0) = 4.999$, and $G(0) = 0.1$, is as shown in Fig. 8. The solution trajectories will stay close only for a short time, before starting to follow noticeably different paths as time passes.

5 Conclusion and discussion

In this paper, a system of nonlinear delay differential equations is utilized to investigate the dynamic behaviors of mealybugs and green lacewings when the time delay on the reproduction of green lacewing larvae is taken into account. The system is then analyzed by using Hopf bifurcation theorem. The conditions on the system parameters that ensure the existence of a periodic solution are obtained. We found that the non-washout steady state of our system is asymptotically stable when the time delay is less than a critical value, i.e., $\tau < \tau_0$. A Hopf bifurcation occurs at the critical value of τ , i.e., $\tau = \tau_0$. The time delay on the reproduction of green lacewing larvae plays an important role in controlling the population of mealybugs. We can see that when the time delay is less than the critical value, the reproduction of green lacewing larvae is fast enough to control the population of mealybugs to some certain levels. When the time delay is more than the critical value, the population of mealybugs could be controlled to lie within some certain ranges and not at any specific level as we could expect to occur. In addition, our model can also exhibit a chaotic behavior which has been observed in the field data [13]. Hence, our model might

be modified further to investigate the effects of biological control, insecticide or pathogen in controlling the outbreak of mealybugs in cassava fields.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The first and second author developed and analyzed the model both theoretically and numerically. The third author investigated the literature reviews. All authors read and approved the manuscript.

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