Nwaeze *Advances in Difference Equations* (2019) 2019:262
https://doi.org/10.1186/s13662-019-2204-3

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Generalized fractional integral inequalities by means of quasiconvexity

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Abstract

Using the newly introduced fractional integral operators in (Fasc. Math. 20(4):5-27, 2016) and (East Asian Math. J. 21(2):191-203, 2005), we establish some novel inequalities of the Hermite–Hadamard type for functions whose second derivatives in absolute value are η-quasiconvex. Results obtained herein give a broader generalization to some existing results in the literature by choosing appropriate values of the parameters under consideration. We apply our results to some special means such as the arithmetic, geometric, harmonic, logarithmic, generalized logarithmic, and identric means to obtain more results in this direction.

MSC: 26A51; 26D15; 26E60; 41A55

Keywords: Hermite–Hadamard inequality; convex functions; quasiconvex functions; special means; Riemann–Liouville fractional integral operators

1 Introduction

In many areas of mathematics, convex functions play an important role. In the study of optimization problems, they are particularly important where they are distinguished by a number of convenient properties. For example, there is no more than one minimum for a (strictly) convex function on an open set. Even in infinite-dimensional spaces, convex functions remain in place under suitable additional hypotheses. Fractional integral inequalities established by means of (quasi)convexity have been a subject of immense investigation in recent times. In this paper, it is our objective to contribute to this subject area.

Let $\mathfrak{D} \subset \mathbb{R}$ be an interval. We start by collecting the following preliminaries:

Definition 1 A function $\mathcal{K} : \mathfrak{D} \to \mathbb{R}$ is called convex on $\mathfrak{D}$ if

$$\mathcal{K}\big(\tau x + (1-\tau)y\big) \le \tau \mathcal{K}(x) + (1-\tau)\mathcal{K}(y)$$

for all $x, y \in \mathfrak{D}$ and $\tau \in [0,1]$.

Thanks to the following fractional integral operators introduced by Raina [10] and Agarwal et al. [1]: if $\mathcal{K} \in L([\alpha,\beta])$, $\rho, \lambda > 0$, and $\sigma : \mathbb{N} \cup \{0\} \to (0,\infty)$ is a bounded sequence, then we define

$$\left(\mathfrak{J}^{\sigma}_{\rho,\lambda,\alpha^{+};\omega}\mathcal{K}\right)(x) := \int_{\alpha}^{x} (x-\tau)^{\lambda-1}\mathfrak{F}^{\sigma}_{\rho,\lambda}\left[\omega(x-\tau)^{\rho}\right]\mathcal{K}(\tau)\,d\tau \quad (x > \alpha > 0) \tag{1}$$

and

$$\left(\mathfrak{J}^{\sigma}_{\rho,\lambda,\beta^{-};\omega}\mathcal{K}\right)(x) := \int_{x}^{\beta} (\tau-x)^{\lambda-1}\mathfrak{F}^{\sigma}_{\rho,\lambda}\left[\omega(\tau-x)^{\rho}\right]\mathcal{K}(\tau)\,d\tau \quad (0 < x < \beta), \tag{2}$$

where

$$\mathfrak{F}^{\sigma}_{\rho,\lambda}(x) = \mathfrak{F}^{\sigma(0),\sigma(1),\ldots}_{\rho,\lambda}(x) = \sum_{j=0}^{\infty} \frac{\sigma(j)}{\Gamma(j\rho+\lambda)}x^{j}. \tag{3}$$

Set et al. [11] obtained, among other things, the following three consequences of their main results for the class of convex functions.

Theorem 2 ([11]) *Let $\alpha,\beta,\omega \in \mathbb{R}$ with $\alpha < \beta$, and $\rho,\lambda > 0$. Suppose that $\mathcal{K} : [\alpha,\beta] \to \mathbb{R}$ is a twice differentiable function on (α,β) and $\mathcal{K}'' \in L([\alpha,\beta])$. If, in addition, $|\mathcal{K}''|$ is convex on $[\alpha,\beta]$, then the following inequality for generalized fractional integrals holds*:

$$\begin{aligned} &\left|\mathfrak{F}^{\sigma}_{\rho,\lambda+1}\left[\omega(\beta-\alpha)^{\rho}\right]\frac{\mathcal{K}(\alpha)+\mathcal{K}(\beta)}{2} - \frac{1}{2(\beta-\alpha)^{\lambda}}\left[\left(\mathfrak{J}^{\sigma}_{\rho,\lambda,\beta^{-};\omega}\mathcal{K}\right)(\alpha) + \left(\mathfrak{J}^{\sigma}_{\rho,\lambda,\alpha^{+};\omega}\mathcal{K}\right)(\beta)\right]\right| \\ &\quad \le \frac{(\beta-\alpha)^{2}}{4}\mathfrak{F}^{\sigma_{1,1}}_{\rho,\lambda+2}\left[|\omega|(\beta-\alpha)^{\rho}\right]\left(\left|\mathcal{K}''(\alpha)\right| + \left|\mathcal{K}(\beta)\right|\right), \end{aligned} \tag{4}$$

where

$$\sigma_{1,1}(j) = \sigma(j)\left[\frac{\lambda+j\rho}{2(\lambda+j\rho+2)}\right], \quad j \in \mathbb{N}\cup\{0\}.$$

Theorem 3 ([11]) *Let $\alpha,\beta,\omega \in \mathbb{R}$ with $\alpha < \beta$, and $\rho,\lambda > 0$. Suppose that $\mathcal{K} : [\alpha,\beta] \to \mathbb{R}$ is a twice differentiable function on (α,β), $\mathcal{K}'' \in L([\alpha,\beta])$, and $p > 1$ with $1/p + 1/q = 1$. If, in addition, $|\mathcal{K}''|^{q}$ is convex on $[\alpha,\beta]$, then the following inequality for generalized fractional integrals holds*:

$$\begin{aligned} &\left|\mathfrak{F}^{\sigma}_{\rho,\lambda+1}\left[\omega(\beta-\alpha)^{\rho}\right]\frac{\mathcal{K}(\alpha)+\mathcal{K}(\beta)}{2} - \frac{1}{2(\beta-\alpha)^{\lambda}}\left[\left(\mathfrak{J}^{\sigma}_{\rho,\lambda,\beta^{-};\omega}\mathcal{K}\right)(\alpha) + \left(\mathfrak{J}^{\sigma}_{\rho,\lambda,\alpha^{+};\omega}\mathcal{K}\right)(\beta)\right]\right| \\ &\quad \le \frac{(\beta-\alpha)^{2}}{2}\mathfrak{F}^{\sigma_{2}}_{\rho,\lambda+2}\left[|\omega|(\beta-\alpha)^{\rho}\right]\left[\frac{|\mathcal{K}''(\alpha)|^{q} + |\mathcal{K}(\beta)|^{q}}{2}\right]^{\frac{1}{q}}, \end{aligned} \tag{5}$$

where

$$\sigma_{2}(j) = 2\sigma(j)\left[\frac{1}{\lambda+j\rho}\mathcal{B}\left(\frac{p+1}{\lambda+j\rho}, p+1\right)\right]^{\frac{1}{p}}, \quad j \in \mathbb{N}\cup\{0\}$$

and $\mathcal{B}(\cdot,\cdot)$ is the Euler beta function defined as follows:

$$\mathcal{B}(x,y) = \int_{0}^{1} \tau^{x-1}(1-\tau)^{y-1}\,d\tau, \quad x,y > 0.$$

Theorem 4 ([11]) *Let $\alpha, \beta, \omega \in \mathbb{R}$ with $\alpha < \beta$, and $\rho, \lambda > 0$. Suppose that $\mathcal{K} : [\alpha, \beta] \to \mathbb{R}$ is a twice differentiable function on (α, β) and $\mathcal{K}'' \in L([\alpha, \beta])$. If, in addition, $|\mathcal{K}''|^q$ is a convex function on $[\alpha, \beta]$ for $q \geq 1$, then the following inequality for generalized fractional integrals holds*:

$$\begin{aligned} &\left|\mathfrak{F}^{\sigma}_{\rho,\lambda+1}\bigl[\omega(\beta-\alpha)^{\rho}\bigr]\frac{\mathcal{K}(\alpha)+\mathcal{K}(\beta)}{2} - \frac{1}{2(\beta-\alpha)^{\lambda}}\bigl[\bigl(\mathfrak{J}^{\sigma}_{\rho,\lambda,\beta^-;\omega}\mathcal{K}\bigr)(\alpha) + \bigl(\mathfrak{J}^{\sigma}_{\rho,\lambda,\alpha^+;\omega}\mathcal{K}\bigr)(\beta)\bigr]\right| \\ &\quad \leq \frac{(\beta-\alpha)^2}{2}\mathfrak{F}^{\sigma_{3,1}}_{\rho,\lambda+2}\bigl[|\omega|(\beta-\alpha)^{\rho}\bigr], \end{aligned} \tag{6}$$

where, for each $j \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned} \sigma_{3,1}(j) &= \sigma(j)\left[\frac{\lambda+j\rho}{2(\lambda+j\rho+2)}\right]^{1-\frac{1}{q}} \\ &\quad \times \left[\left(\frac{\lambda+j\rho}{3(\lambda+j\rho+3)}\bigl|\mathcal{K}''(\alpha)\bigr|^q + \frac{(\lambda+j\rho)(\lambda+j\rho+5)}{6(\lambda+j\rho+2)(\lambda+j\rho+3)}\bigl|\mathcal{K}''(\beta)\bigr|^q\right)^{\frac{1}{q}} \right. \\ &\quad \left. + \left(\frac{(\lambda+j\rho)(\lambda+j\rho+5)}{6(\lambda+j\rho+2)(\lambda+j\rho+3)}\bigl|\mathcal{K}''(\alpha)\bigr|^q + \frac{\lambda+j\rho}{3(\lambda+j\rho+3)}\bigl|\mathcal{K}''(\beta)\bigr|^q\right)^{\frac{1}{q}}\right]. \end{aligned}$$

Remark 5 It is easy to verify that $\mathfrak{J}^{\sigma}_{\rho,\lambda,\alpha^+;\omega}\mathcal{K}$ and $\mathfrak{J}^{\sigma}_{\rho,\lambda,\beta^-;\omega}\mathcal{K}$ are bounded integral operators. In addition, if $\omega = 0$ and $\sigma(0) = 1$, then (1) and (2) reduce to the left- and right- Riemann–Liouville fractional integral operators, respectively. That is,

$$\bigl(\mathfrak{J}^{\sigma}_{\rho,\lambda,\alpha^+;0}\mathcal{K}\bigr)(x) = \bigl(\mathfrak{J}^{1,\sigma(1),\sigma(2),\ldots}_{\rho,\lambda,\alpha^+;0}\mathcal{K}\bigr)(x) = \frac{1}{\Gamma(\lambda)}\int_{\alpha}^{x}(x-\tau)^{\lambda-1}\mathcal{K}(\tau)\,d\tau =: \mathcal{J}^{\lambda}_{\alpha^+}\mathcal{K}(x)$$

and

$$\bigl(\mathfrak{J}^{\sigma}_{\rho,\lambda,\beta^-;0}\mathcal{K}\bigr)(x) = \bigl(\mathfrak{J}^{1,\sigma(1),\sigma(2),\ldots}_{\rho,\lambda,\beta^-;0}\mathcal{K}\bigr)(x) = \frac{1}{\Gamma(\lambda)}\int_{x}^{\beta}(\tau-x)^{\lambda-1}\mathcal{K}(\tau)\,d\tau =: \mathcal{J}^{\lambda}_{\beta^-}\mathcal{K}(x),$$

where the gamma function $\Gamma(\lambda) = \int_0^{\infty} e^{-v}v^{\lambda-1}\,dv$.

The class of convex functions has been generalized as follows.

Definition 6 A function $\mathcal{K} : \mathfrak{D} \to \mathbb{R}$ is called quasiconvex on $\mathfrak{D}$ if

$$\mathcal{K}\bigl(\tau x + (1-\tau)y\bigr) \leq \max\bigl\{\mathcal{K}(x), \mathcal{K}(y)\bigr\}$$

for all $x, y \in \mathfrak{D}$ and $\tau \in [0,1]$.

Recently, Gordji et al. further generalized the class of quasiconvex functions in the following manner.

Definition 7 ([4]) A function $\mathcal{K} : \mathfrak{D} \to \mathbb{R}$ is called η-quasiconvex on $\mathfrak{D}$ with respect to $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ if

$$\mathcal{K}\bigl(\tau x + (1-\tau)y\bigr) \leq \max\bigl\{\mathcal{K}(y), \mathcal{K}(y) + \eta\bigl(\mathcal{K}(x), \mathcal{K}(y)\bigr)\bigr\}$$

for all $x, y \in \mathfrak{D}$ and $\tau \in [0,1]$.

Some inequalities via η-(quasi)convex functions can be found in [3–5, 9].

Remark 8 We summarize Definitions 1, 6, and 7 as follows:

$$\underset{[\text{with } \eta(x,y)=x-y]}{\eta\text{-quasiconvexity}} \implies \text{quasiconvexity} \underset{\not\Rightarrow}{\Leftarrow} \text{convexity}.$$

To see that quasiconvexity does not always imply convexity, consider the following example: let $\mathcal{K}:[-2,2]\to\mathbb{R}$ be defined by

$$\mathcal{K}(v) := \begin{cases} 1, & v\in[-2,-1], \\ v^2, & v\in(-1,2]. \end{cases}$$

The function $\mathcal{K}$ is quasiconvex on $[-2,2]$ but not convex on $[-2,2]$.

Theorems 2, 3, and 4 cannot be applied to functions whose second derivatives in absolute value, raise to some powers, are not convex. For this reason, it is our primary objective in this paper to extend Theorems 2, 3, and 4 to a more general class of functions—the class of η-quasiconvex functions. More precisely, we obtain inequalities akin to (4), (5), and (6) in the case when $\mathcal{K}:[\alpha,\beta]\to\mathbb{R}$ is twice differentiable on (α,β) and $|\mathcal{K}''|$ or $|\mathcal{K}''|^q, q>1$ is η-quasiconvex on $[\alpha,\beta]$. By taking the bifunction $\eta(x,y)=x-y$, our results boil down to inequalities of the Hermite–Hadamard type for functions whose second derivatives in absolute value are quasiconvex (see Remarks 12, 15, and 18 of Sect. 2).

This article is organized in the following manner: Sect. 2 contains the main results and the proofs. In Sect. 3, we apply our results to some special means.

2 Main results

The following lemma will be useful in the proofs of our main results:

Lemma 9 ([11]) *Let $\alpha,\beta,\omega\in\mathbb{R}$ with $\alpha<\beta$, and $\rho,\lambda>0$. If $\mathcal{K}:[\alpha,\beta]\to\mathbb{R}$ is a twice differentiable function on (α,β) and $\mathcal{K}''\in L([\alpha,\beta])$, then the following equality for generalized fractional integrals holds*:

$$\begin{aligned} &\mathfrak{F}^{\sigma}_{\rho,\lambda+1}\bigl[\omega(\beta-\alpha)^{\rho}\bigr]\frac{\mathcal{K}(\alpha)+\mathcal{K}(\beta)}{2} - \frac{1}{2(\beta-\alpha)^{\lambda}}\bigl[\bigl(\mathfrak{J}^{\sigma}_{\rho,\lambda,\beta^-;\omega}\mathcal{K}\bigr)(\alpha)+\bigl(\mathfrak{J}^{\sigma}_{\rho,\lambda,\alpha^+;\omega}\mathcal{K}\bigr)(\beta)\bigr] \\ &\quad = \frac{(\beta-\alpha)^2}{2}\int_0^1 \bigl\{\bigl(\tau\mathfrak{F}^{\sigma}_{\rho,\lambda+2}\bigl[\omega(\beta-\alpha)^{\rho}\bigr]-\tau^{\lambda+1}\mathfrak{F}^{\sigma}_{\rho,\lambda+2}\bigl[\omega(\beta-\alpha)^{\rho}\tau^{\rho}\bigr]\bigr) \\ &\qquad \times\bigl(\mathcal{K}''\bigl(\tau\alpha+(1-\tau)\beta\bigr)+\mathcal{K}''\bigl(\tau\beta+(1-\tau)\alpha\bigr)\bigr)\bigr\}\,d\tau. \end{aligned} \tag{7}$$

For the sake of convenience, we will make use of the following notations: for $q\geq 1$, denote

$$\mathbf{M}^{\beta}_{\alpha}\bigl(|\mathcal{K}''|^q,\eta\bigr) := \max\bigl\{\bigl|\mathcal{K}''(\beta)\bigr|^q, \bigl|\mathcal{K}''(\beta)\bigr|^q+\eta\bigl(\bigl|\mathcal{K}''(\alpha)\bigr|^q,\bigl|\mathcal{K}''(\beta)\bigr|^q\bigr)\bigr\}$$

and

$$\mathbf{N}^{\beta}_{\alpha}\bigl(|\mathcal{K}''|^q,\eta\bigr) := \max\bigl\{\bigl|\mathcal{K}''(\alpha)\bigr|^q, \bigl|\mathcal{K}''(\alpha)\bigr|^q+\eta\bigl(\bigl|\mathcal{K}''(\beta)\bigr|^q,\bigl|\mathcal{K}''(\alpha)\bigr|^q\bigr)\bigr\}.$$

We now state and justify our first result.

Theorem 10 *Let $\alpha,\beta,\omega\in\mathbb{R}$ with $\alpha<\beta$, and $\rho,\lambda>0$. Suppose that $\mathcal{K}:[\alpha,\beta]\to\mathbb{R}$ is a twice differentiable function on (α,β) and $\mathcal{K}''\in L([\alpha,\beta])$. If, in addition, $|\mathcal{K}''|$ is η-quasiconvex on $[\alpha,\beta]$, then the following inequality for generalized fractional integrals holds*:

$$\begin{aligned}&\left|\mathfrak{F}^{\sigma}_{\rho,\lambda+1}\big[\omega(\beta-\alpha)^{\rho}\big]\frac{\mathcal{K}(\alpha)+\mathcal{K}(\beta)}{2}-\frac{1}{2(\beta-\alpha)^{\lambda}}\big[\big(\mathfrak{J}^{\sigma}_{\rho,\lambda,\beta^-;\omega}\mathcal{K}\big)(\alpha)+\big(\mathfrak{J}^{\sigma}_{\rho,\lambda,\alpha^+;\omega}\mathcal{K}\big)(\beta)\big]\right|\\&\quad\le\frac{(\beta-\alpha)^2}{4}\sum_{j=0}^{\infty}\frac{\sigma(j)(\lambda+j\rho)}{\Gamma(j\rho+\lambda+3)}|\omega|^j(\beta-\alpha)^{j\rho}\big[\mathbf{M}^{\beta}_{\alpha}\big(\big|\mathcal{K}''\big|,\eta\big)+\mathbf{N}^{\beta}_{\alpha}\big(\big|\mathcal{K}''\big|,\eta\big)\big].\end{aligned}$$

Proof Using the η-quasiconvexity of $|\mathcal{K}''|$ on $[\alpha,\beta]$, we obtain

$$\big|\mathcal{K}''\big(\tau\alpha+(1-\tau)\beta\big)\big|\le\mathbf{M}^{\beta}_{\alpha}\big(\big|\mathcal{K}''\big|,\eta\big)\tag{8}$$

and

$$\big|\mathcal{K}''\big(\tau\beta+(1-\tau)\alpha\big)\big|\le\mathbf{N}^{\beta}_{\alpha}\big(\big|\mathcal{K}''\big|,\eta\big)\tag{9}$$

for $\tau\in[0,1]$.

Now applying Lemma 9, inequalities (8)–(9), and relation (3), one obtains

$$\begin{aligned}&\left|\mathfrak{F}^{\sigma}_{\rho,\lambda+1}\big[\omega(\beta-\alpha)^{\rho}\big]\frac{\mathcal{K}(\alpha)+\mathcal{K}(\beta)}{2}-\frac{1}{2(\beta-\alpha)^{\lambda}}\big[\big(\mathfrak{J}^{\sigma}_{\rho,\lambda,\beta^-;\omega}\mathcal{K}\big)(\alpha)+\big(\mathfrak{J}^{\sigma}_{\rho,\lambda,\alpha^+;\omega}\mathcal{K}\big)(\beta)\big]\right|\\
&\quad\le\frac{(\beta-\alpha)^2}{2}\int_0^1\big\{\big|\tau\mathfrak{F}^{\sigma}_{\rho,\lambda+2}\big[\omega(\beta-\alpha)^{\rho}\big]-\tau^{\lambda+1}\mathfrak{F}^{\sigma}_{\rho,\lambda+2}\big[\omega(\beta-\alpha)^{\rho}\tau^{\rho}\big]\big|\\
&\qquad\times\big|\mathcal{K}''\big(\tau\alpha+(1-\tau)\beta\big)+\mathcal{K}''\big(\tau\beta+(1-\tau)\alpha\big)\big|\big\}\,d\tau\\
&\quad\le\frac{(\beta-\alpha)^2}{2}\\
&\qquad\times\int_0^1\left\{\left|\sum_{j=0}^{\infty}\frac{\sigma(j)}{\Gamma(j\rho+\lambda+2)}\omega^j(\beta-\alpha)^{j\rho}\tau-\sum_{j=0}^{\infty}\frac{\sigma(j)}{\Gamma(j\rho+\lambda+2)}\omega^j(\beta-\alpha)^{j\rho}\tau^{j\rho+\lambda+1}\right|\right.\\
&\qquad\left.\times\big|\mathcal{K}''\big(\tau\alpha+(1-\tau)\beta\big)+\mathcal{K}''\big(\tau\beta+(1-\tau)\alpha\big)\big|\right\}d\tau\\
&\quad\le\frac{(\beta-\alpha)^2}{2}\int_0^1\left\{\left|\sum_{j=0}^{\infty}\frac{\sigma(j)}{\Gamma(j\rho+\lambda+2)}\omega^j(\beta-\alpha)^{j\rho}\big[\tau-\tau^{j\rho+\lambda+1}\big]\right|\right.\\
&\qquad\left.\times\big|\mathcal{K}''\big(\tau\alpha+(1-\tau)\beta\big)+\mathcal{K}''\big(\tau\beta+(1-\tau)\alpha\big)\big|\right\}d\tau\\
&\quad\le\frac{(\beta-\alpha)^2}{2}\sum_{j=0}^{\infty}\frac{\sigma(j)}{\Gamma(j\rho+\lambda+2)}|\omega|^j(\beta-\alpha)^{j\rho}\\
&\qquad\times\int_0^1\big|\tau-\tau^{j\rho+\lambda+1}\big|\big[\big|\mathcal{K}''\big(\tau\alpha+(1-\tau)\beta\big)\big|+\big|\mathcal{K}''\big(\tau\beta+(1-\tau)\alpha\big)\big|\big]\,d\tau\\
&\quad\le\frac{(\beta-\alpha)^2}{2}\sum_{j=0}^{\infty}\frac{\sigma(j)}{\Gamma(j\rho+\lambda+2)}|\omega|^j(\beta-\alpha)^{j\rho}\end{aligned}$$

$$
\begin{aligned}
&\times \int_0^1 \left|\tau - \tau^{j\rho+\lambda+1}\right| \left[\mathbf{M}_\alpha^\beta\left(\left|\mathcal{K}''\right|, \eta\right) + \mathbf{N}_\alpha^\beta\left(\left|\mathcal{K}''\right|, \eta\right)\right] d\tau \\
&= \frac{(\beta-\alpha)^2}{2} \sum_{j=0}^{\infty} \frac{\sigma(j)}{\Gamma(j\rho+\lambda+2)} |\omega|^j (\beta-\alpha)^{j\rho} \\
&\quad \times \left[\mathbf{M}_\alpha^\beta\left(\left|\mathcal{K}''\right|, \eta\right) + \mathbf{N}_\alpha^\beta\left(\left|\mathcal{K}''\right|, \eta\right)\right] \int_0^1 \left(\tau - \tau^{j\rho+\lambda+1}\right) d\tau \\
&= \frac{(\beta-\alpha)^2}{4} \sum_{j=0}^{\infty} \frac{\sigma(j)(\lambda+j\rho)}{\Gamma(j\rho+\lambda+3)} |\omega|^j (\beta-\alpha)^{j\rho} \left[\mathbf{M}_\alpha^\beta\left(\left|\mathcal{K}''\right|, \eta\right) + \mathbf{N}_\alpha^\beta\left(\left|\mathcal{K}''\right|, \eta\right)\right],
\end{aligned}
$$

since

$$
\int_0^1 \left(\tau - \tau^{j\rho+\lambda+1}\right) d\tau = \frac{\lambda + j\rho}{2(\lambda + j\rho + 2)}.
$$

Hence, the intended inequality is obtained. □

Corollary 11 *Let $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$, and $\lambda > 0$. Suppose that $\mathcal{K} : [\alpha, \beta] \to \mathbb{R}$ is a twice differentiable function on (α, β) and $\mathcal{K}'' \in L([\alpha, \beta])$. If, in addition, $|\mathcal{K}''|$ is η-quasiconvex on $[\alpha, \beta]$, then the following inequality for fractional integrals holds*:

$$
\begin{aligned}
&\left|\frac{\mathcal{K}(\alpha) + \mathcal{K}(\beta)}{2} - \frac{\Gamma(\lambda+1)}{2(\beta-\alpha)^\lambda}\left[\mathcal{J}_{\beta^-}^{\lambda}\mathcal{K}(\alpha) + \mathcal{J}_{\beta^-}^{\lambda}\mathcal{K}(\alpha)\right]\right| \\
&\quad \leq \frac{(\beta-\alpha)^2 \lambda}{4(\lambda+1)(\lambda+2)}\left[\mathbf{M}_\alpha^\beta\left(\left|\mathcal{K}''\right|, \eta\right) + \mathbf{N}_\alpha^\beta\left(\left|\mathcal{K}''\right|, \eta\right)\right].
\end{aligned} \tag{10}
$$

Proof By taking $\sigma(0) = 1$ and $w = 0$, one observes that

$$
\begin{aligned}
&\mathfrak{F}_{\rho,\lambda+1}^{\sigma}\left[\omega(\beta-\alpha)^\rho\right] = \frac{1}{\Gamma(\lambda+1)}, \\
&\left(\mathfrak{I}_{\rho,\lambda,\beta^-;\omega}^{\sigma}\mathcal{K}\right)(\alpha) = \mathcal{J}_{\beta^-}^{\lambda}\mathcal{K}(\alpha),
\end{aligned}
$$

and

$$
\left(\mathfrak{I}_{\rho,\lambda,\alpha^+;\omega}^{\sigma}\mathcal{K}\right)(\beta) = \mathcal{J}_{\alpha^+}^{\lambda}\mathcal{K}(\beta).
$$

Now, using Theorem 10, we arrive at the intended inequality. □

Remark 12 By setting the bifunction $\eta(x, y) = x - y$, the inequality in (10) boils down to

$$
\begin{aligned}
&\left|\frac{\mathcal{K}(\alpha) + \mathcal{K}(\beta)}{2} - \frac{\Gamma(\lambda+1)}{2(\beta-\alpha)^\lambda}\left[\mathcal{J}_{\beta^-}^{\lambda}\mathcal{K}(\alpha) + \mathcal{J}_{\alpha^+}^{\lambda}\mathcal{K}(\beta)\right]\right| \\
&\quad \leq \frac{(\beta-\alpha)^2 \lambda}{2(\lambda+1)(\lambda+2)} \max\left\{\left|\mathcal{K}''(\alpha)\right|, \left|\mathcal{K}''(\beta)\right|\right\}.
\end{aligned} \tag{11}
$$

Theorem 13 *Let $\alpha, \beta, \omega \in \mathbb{R}$ with $\alpha < \beta$, and $\rho, \lambda > 0$. Suppose that $\mathcal{K} : [\alpha, \beta] \to \mathbb{R}$ is a twice differentiable function on (α, β), $\mathcal{K}'' \in L([\alpha, \beta])$, and $p > 1$ with $1/p + 1/q = 1$. If, in* *addition, $|\mathcal{K}''|^q$ is η-quasiconvex on $[\alpha,\beta]$, then the following inequality for generalized fractional integrals holds*:

$$\begin{aligned}&\left|\mathfrak{F}^{\sigma}_{\rho,\lambda+1}\big[\omega(\beta-\alpha)^{\rho}\big]\frac{\mathcal{K}(\alpha)+\mathcal{K}(\beta)}{2}-\frac{1}{2(\beta-\alpha)^{\lambda}}\big[\big(\mathfrak{J}^{\sigma}_{\rho,\lambda,\beta^-;\omega}\mathcal{K}\big)(\alpha)+\big(\mathfrak{J}^{\sigma}_{\rho,\lambda,\alpha^+;\omega}\mathcal{K}\big)(\beta)\big]\right|\\&\quad\le\frac{(\beta-\alpha)^2}{2}\sum_{j=0}^{\infty}\frac{\sigma(j)}{\Gamma(j\rho+\lambda+2)}|\omega|^j(\beta-\alpha)^{j\rho}\left[\frac{1}{\lambda+j\rho}\mathcal{B}\left(\frac{p+1}{\lambda+j\rho},p+1\right)\right]^{\frac{1}{p}}\\&\qquad\times\big[\big(\mathbf{M}^{\beta}_{\alpha}\big(|\mathcal{K}''|^q,\eta\big)\big)^{\frac{1}{q}}+\big(\mathbf{N}^{\beta}_{\alpha}\big(|\mathcal{K}''|^q,\eta\big)\big)^{\frac{1}{q}}\big],\end{aligned}$$

where $\mathcal{B}(\cdot,\cdot)$ is the Euler beta function.

Proof We start by first observing that if one lets $u=\tau^{\lambda+j\rho}$, one gets

$$\begin{aligned}\int_0^1\tau^p\big(1-\tau^{\lambda+j\rho}\big)^p\,d\tau&=\frac{1}{\lambda+j\rho}\int_0^1 u^{\frac{p+1}{\lambda+j\rho}-1}(1-u)^{(p+1)-1}\,du\\&=\frac{1}{\lambda+j\rho}\mathcal{B}\left(\frac{p+1}{\lambda+j\rho},p+1\right).\end{aligned}\tag{12}$$

Employing Lemma 9 and mimicking the idea from the proof of Theorem 10, we get

$$\begin{aligned}&\left|\mathfrak{F}^{\sigma}_{\rho,\lambda+1}\big[\omega(\beta-\alpha)^{\rho}\big]\frac{\mathcal{K}(\alpha)+\mathcal{K}(\beta)}{2}-\frac{1}{2(\beta-\alpha)^{\lambda}}\big[\big(\mathfrak{J}^{\sigma}_{\rho,\lambda,\beta^-;\omega}\mathcal{K}\big)(\alpha)+\big(\mathfrak{J}^{\sigma}_{\rho,\lambda,\alpha^+;\omega}\mathcal{K}\big)(\beta)\big]\right|\\&\quad\le\frac{(\beta-\alpha)^2}{2}\sum_{j=0}^{\infty}\frac{\sigma(j)}{\Gamma(j\rho+\lambda+2)}|\omega|^j(\beta-\alpha)^{j\rho}\\&\qquad\times\int_0^1\big|\tau-\tau^{j\rho+\lambda+1}\big|\big[\big|\mathcal{K}''\big(\tau\alpha+(1-\tau)\beta\big)\big|+\big|\mathcal{K}''\big(\tau\beta+(1-\tau)\alpha\big)\big|\big]\,d\tau.\end{aligned}\tag{13}$$

Using Hölder's inequality, the η-quasiconvexity of $|\mathcal{K}''|$ on $[\alpha,\beta]$, and (12), one has

$$\begin{aligned}&\int_0^1\big|\tau-\tau^{j\rho+\lambda+1}\big|\big[\big|\mathcal{K}''\big(\tau\alpha+(1-\tau)\beta\big)\big|+\big|\mathcal{K}''\big(\tau\beta+(1-\tau)\alpha\big)\big|\big]\,d\tau\\&\quad\le\left[\int_0^1\big(\tau-\tau^{j\rho+\lambda+1}\big)^p\,d\tau\right]^{\frac{1}{p}}\left[\left(\int_0^1\big|\mathcal{K}''\big(\tau\alpha+(1-\tau)\beta\big)\big|^q\,d\tau\right)^{\frac{1}{q}}\right.\\&\qquad\left.+\left(\int_0^1\big|\mathcal{K}''\big(\tau\beta+(1-\tau)\alpha\big)\big|^q\,d\tau\right)^{\frac{1}{q}}\right]\\&\quad\le\left[\int_0^1\tau^p\big(1-\tau^{j\rho+\lambda}\big)^p\,d\tau\right]^{\frac{1}{p}}\big[\big(\mathbf{M}^{\beta}_{\alpha}\big(|\mathcal{K}''|^q,\eta\big)\big)^{\frac{1}{q}}+\big(\mathbf{N}^{\beta}_{\alpha}\big(|\mathcal{K}''|^q,\eta\big)\big)^{\frac{1}{q}}\big]\\&\quad=\left[\frac{1}{\lambda+j\rho}\mathcal{B}\left(\frac{p+1}{\lambda+j\rho},p+1\right)\right]^{\frac{1}{p}}\big[\big(\mathbf{M}^{\beta}_{\alpha}\big(|\mathcal{K}''|^q,\eta\big)\big)^{\frac{1}{q}}+\big(\mathbf{N}^{\beta}_{\alpha}\big(|\mathcal{K}''|^q,\eta\big)\big)^{\frac{1}{q}}\big].\end{aligned}\tag{14}$$

Combining (13) and (14) gives the desired result. □

Substituting $\sigma(0)=1$ and $w=0$ in Theorem 13, we derive the following corollary.

Corollary 14 *Let $\alpha,\beta \in \mathbb{R}$ with $\alpha < \beta$, and $\lambda > 0$. Suppose that $\mathcal{K} : [\alpha,\beta] \to \mathbb{R}$ is a twice differentiable function on (α,β), $\mathcal{K}'' \in L([\alpha,\beta])$, and $p > 1$ with $1/p + 1/q = 1$. If, in addition, $|\mathcal{K}''|^q$ is η-quasiconvex on $[\alpha,\beta]$, then the following inequality for fractional integrals holds*:

$$\begin{aligned}
&\left|\frac{\mathcal{K}(\alpha)+\mathcal{K}(\beta)}{2}-\frac{\Gamma(\lambda+1)}{2(\beta-\alpha)^{\lambda}}\left[\mathcal{J}^{\lambda}_{\beta^-}\mathcal{K}(\alpha)+\mathcal{J}^{\lambda}_{\alpha^+}\mathcal{K}(\beta)\right]\right| \\
&\quad\le \frac{(\beta-\alpha)^2}{2(\lambda+1)}\left[\frac{1}{\lambda}\mathcal{B}\left(\frac{p+1}{\lambda},p+1\right)\right]^{\frac{1}{p}}\left[\left(\mathbf{M}^{\beta}_{\alpha}\left(\left|\mathcal{K}''\right|^q,\eta\right)\right)^{\frac{1}{q}}+\left(\mathbf{N}^{\beta}_{\alpha}\left(\left|\mathcal{K}''\right|^q,\eta\right)\right)^{\frac{1}{q}}\right],
\end{aligned}\tag{15}$$

where $\mathcal{B}(\cdot,\cdot)$ is the Euler beta function.

Remark 15 If the bifunction $\eta(x,y) = x - y$, then (15) becomes

$$\begin{aligned}
&\left|\frac{\mathcal{K}(\alpha)+\mathcal{K}(\beta)}{2}-\frac{\Gamma(\lambda+1)}{2(\beta-\alpha)^{\lambda}}\left[\mathcal{J}^{\lambda}_{\beta^-}\mathcal{K}(\alpha)+\mathcal{J}^{\lambda}_{\alpha^+}\mathcal{K}(\beta)\right]\right| \\
&\quad\le \frac{(\beta-\alpha)^2}{\lambda+1}\left[\frac{1}{\lambda}\mathcal{B}\left(\frac{p+1}{\lambda},p+1\right)\right]^{\frac{1}{p}}\left(\max\left\{\left|\mathcal{K}''(\alpha)\right|^q,\left|\mathcal{K}''(\beta)\right|^q\right\}\right)^{\frac{1}{q}}.
\end{aligned}\tag{16}$$

Theorem 16 *Let $\alpha,\beta,\omega \in \mathbb{R}$ with $\alpha < \beta$, and $\rho,\lambda > 0$. Suppose that $\mathcal{K} : [\alpha,\beta] \to \mathbb{R}$ is a twice differentiable function on (α,β) and $\mathcal{K}'' \in L([\alpha,\beta])$. If, in addition, $|\mathcal{K}''|^q$ is an η-quasiconvex function on $[\alpha,\beta]$ for $q \ge 1$, then the following inequality for generalized fractional integrals holds*:

$$\begin{aligned}
&\left|\mathfrak{F}^{\sigma}_{\rho,\lambda+1}\left[\omega(\beta-\alpha)^{\rho}\right]\frac{\mathcal{K}(\alpha)+\mathcal{K}(\beta)}{2}-\frac{1}{2(\beta-\alpha)^{\lambda}}\left[\left(\mathfrak{J}^{\sigma}_{\rho,\lambda,\beta^-;\omega}\mathcal{K}\right)(\alpha)+\left(\mathfrak{J}^{\sigma}_{\rho,\lambda,\alpha^+;\omega}\mathcal{K}\right)(\beta)\right]\right| \\
&\quad\le \frac{(\beta-\alpha)^2}{4}\sum_{j=0}^{\infty}\frac{\sigma(j)(\lambda+j\rho)}{\Gamma(j\rho+\lambda+3)}|\omega|^j(\beta-\alpha)^{j\rho}\left[\left(\mathbf{M}^{\beta}_{\alpha}\left(\left|\mathcal{K}''\right|^q,\eta\right)\right)^{\frac{1}{q}}+\left(\mathbf{N}^{\beta}_{\alpha}\left(\left|\mathcal{K}''\right|^q,\eta\right)\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Proof Utilizing Lemma 9, the η-quasiconvexity of $|\mathcal{K}''|^q$ on $[\alpha,\beta]$, and the power mean inequality, the following inequalities are established:

$$\begin{aligned}
&\left|\mathfrak{F}^{\sigma}_{\rho,\lambda+1}\left[\omega(\beta-\alpha)^{\rho}\right]\frac{\mathcal{K}(\alpha)+\mathcal{K}(\beta)}{2}-\frac{1}{2(\beta-\alpha)^{\lambda}}\left[\left(\mathfrak{J}^{\sigma}_{\rho,\lambda,\beta^-;\omega}\mathcal{K}\right)(\alpha)+\left(\mathfrak{J}^{\sigma}_{\rho,\lambda,\alpha^+;\omega}\mathcal{K}\right)(\beta)\right]\right| \\
&\quad\le \frac{(\beta-\alpha)^2}{2}\sum_{j=0}^{\infty}\frac{\sigma(j)}{\Gamma(j\rho+\lambda+2)}|\omega|^j(\beta-\alpha)^{j\rho} \\
&\qquad\times\int_0^1\left|\tau-\tau^{j\rho+\lambda+1}\right|\left[\left|\mathcal{K}''\left(\tau\alpha+(1-\tau)\beta\right)\right|+\left|\mathcal{K}''\left(\tau\beta+(1-\tau)\alpha\right)\right|\right]d\tau \\
&\quad= \frac{(\beta-\alpha)^2}{2}\sum_{j=0}^{\infty}\frac{\sigma(j)}{\Gamma(j\rho+\lambda+2)}|\omega|^j(\beta-\alpha)^{j\rho} \\
&\qquad\times\int_0^1\left(\tau-\tau^{j\rho+\lambda+1}\right)\left[\left|\mathcal{K}''\left(\tau\alpha+(1-\tau)\beta\right)\right|+\left|\mathcal{K}''\left(\tau\beta+(1-\tau)\alpha\right)\right|\right]d\tau \\
&\quad\le \frac{(\beta-\alpha)^2}{2}\sum_{j=0}^{\infty}\frac{\sigma(j)}{\Gamma(j\rho+\lambda+2)}|\omega|^j(\beta-\alpha)^{j\rho} \\
&\qquad\times\left[\int_0^1\left(\tau-\tau^{j\rho+\lambda+1}\right)d\tau\right]^{1-\frac{1}{q}}\left[\left(\int_0^1\left(\tau-\tau^{j\rho+\lambda+1}\right)\left|\mathcal{K}''\left(\tau\alpha+(1-\tau)\beta\right)\right|^q d\tau\right)^{\frac{1}{q}}\right.
\end{aligned}$$

$$
\begin{aligned}
&+\left(\int_0^1 \left(\tau-\tau^{j\rho+\lambda+1}\right)\left|\mathcal{K}''\left(\tau\beta+(1-\tau)\alpha\right)\right|^q\,d\tau\right)^{\frac{1}{q}}\Bigg] \\
&\leq \frac{(\beta-\alpha)^2}{2}\sum_{j=0}^{\infty}\frac{\sigma(j)}{\Gamma(j\rho+\lambda+2)}|\omega|^j(\beta-\alpha)^{j\rho}\left[\frac{\lambda+j\rho}{2(\lambda+j\rho+2)}\right]^{1-\frac{1}{q}} \\
&\times\Bigg[\left(\int_0^1 \left(\tau-\tau^{j\rho+\lambda+1}\right)\mathbf{M}_\alpha^\beta\left(\left|\mathcal{K}''\right|^q,\eta\right)d\tau\right)^{\frac{1}{q}} \\
&+\left(\int_0^1 \left(\tau-\tau^{j\rho+\lambda+1}\right)\mathbf{N}_\alpha^\beta\left(\left|\mathcal{K}''\right|^q,\eta\right)d\tau\right)^{\frac{1}{q}}\Bigg] \\
&=\frac{(\beta-\alpha)^2}{4}\sum_{j=0}^{\infty}\frac{\sigma(j)(\lambda+j\rho)}{\Gamma(j\rho+\lambda+3)}|\omega|^j(\beta-\alpha)^{j\rho} \\
&\times\left[\left(\mathbf{M}_\alpha^\beta\left(\left|\mathcal{K}''\right|^q,\eta\right)\right)^{\frac{1}{q}}+\left(\mathbf{N}_\alpha^\beta\left(\left|\mathcal{K}''\right|^q,\eta\right)\right)^{\frac{1}{q}}\right]. \qquad \square
\end{aligned}
$$

Proceeding in a similar fashion, we get from Theorem 16 the succeeding drop out.

Corollary 17 *Let $\alpha,\beta\in\mathbb{R}$ with $\alpha<\beta$, and $\lambda>0$. Suppose that $\mathcal{K}:[\alpha,\beta]\to\mathbb{R}$ is a twice differentiable function on (α,β) and $\mathcal{K}''\in L([\alpha,\beta])$. If, in addition, $|\mathcal{K}''|^q$ is an η-quasiconvex function on $[\alpha,\beta]$ for $q\geq 1$, then the following inequality for fractional integrals holds*:

$$
\begin{aligned}
&\left|\frac{\mathcal{K}(\alpha)+\mathcal{K}(\beta)}{2}-\frac{\Gamma(\lambda+1)}{2(\beta-\alpha)^\lambda}\left[\mathcal{J}_{\beta^-}^\lambda\mathcal{K}(\alpha)+\mathcal{J}_{\alpha^+}^\lambda\mathcal{K}(\beta)\right]\right| \\
&\leq\frac{(\beta-\alpha)^2\lambda}{4(\lambda+1)(\lambda+2)}\left[\left(\mathbf{M}_\alpha^\beta\left(\left|\mathcal{K}''\right|^q,\eta\right)\right)^{\frac{1}{q}}+\left(\mathbf{N}_\alpha^\beta\left(\left|\mathcal{K}''\right|^q,\eta\right)\right)^{\frac{1}{q}}\right].
\end{aligned}
\tag{17}
$$

Remark 18 From Corollary 17, one gets

$$
\begin{aligned}
&\left|\frac{\mathcal{K}(\alpha)+\mathcal{K}(\beta)}{2}-\frac{\Gamma(\lambda+1)}{2(\beta-\alpha)^\lambda}\left[\mathcal{J}_{\beta^-}^\lambda\mathcal{K}(\alpha)+\mathcal{J}_{\alpha^+}^\lambda\mathcal{K}(\beta)\right]\right| \\
&\leq\frac{(\beta-\alpha)^2\lambda}{2(\lambda+1)(\lambda+2)}\left(\max\left\{\left|\mathcal{K}''(\alpha)\right|^q,\left|\mathcal{K}''(\beta)\right|^q\right\}\right)^{\frac{1}{q}}.
\end{aligned}
\tag{18}
$$

3 Applications to special means

We now apply inequalities (11), (16), and (18) to the following special means of distinct real numbers:

1. Arithmetic mean:

$$\mathcal{A}(u,v)=\frac{u+v}{2}.$$

2. Geometric mean:

$$\mathcal{G}(u,v)=\sqrt{uv},\quad u,v>0.$$

3. Harmonic mean:

$$\mathcal{H}(u,v)=\frac{2uv}{u+v}.$$

4. Logarithmic mean:

$$\mathcal{L}(u,v) = \frac{u-v}{\ln|u| - \ln|v|}, \quad |u| \neq |v|, \text{ and } u, v \neq 0.$$

5. Generalized logarithmic mean:

$$\mathcal{L}_m(u,v) = \left[\frac{v^{m+1} - u^{m+1}}{(m+1)(v-u)}\right]^{\frac{1}{m}}, \quad m \in \mathbb{N}.$$

6. Identric mean:

$$\mathcal{I}(u,v) = \frac{1}{e}\left(\frac{v^v}{u^u}\right)^{\frac{1}{v-u}}.$$

Proposition 19 *Suppose $u, v \in \mathbb{R}$ with $u < v$ and $m \geq 2$. Then the following inequality holds*:

$$\left|\mathcal{A}\left(u^m, v^m\right) - \mathcal{L}_m^m(u,v)\right| \leq \frac{(v-u)^2}{12}) \max\left\{|u|^{m-2}, |v|^{m-2}\right\}.$$

Proof Let $\mathcal{K}(x) = x^m$. In this case, $|\mathcal{K}''(x)| = m(m-1)|x|^{m-2}$ which is quasiconvex on $[u,v]$. Now applying inequality (11), with $\lambda = 1$, to the function $\mathcal{K}$, the desired result is established. □

Proposition 20 *Suppose $u, v \in \mathbb{R}$ with $0 < u < v$. Then the following inequality holds*:

$$\left|\mathcal{A}(u,v) - \mathcal{L}(u,v)\right| \leq \frac{(\ln v - \ln u)^2}{12} \max\{u, v\}.$$

Proof The result follows by applying (11) to the function $\mathcal{K}(x) = e^x$ with $\lambda = 1$. Since the function $|\mathcal{K}''(x)| = e^x$ is convex for all $x \in \mathbb{R}$, it is also quasiconvex. □

Proposition 21 *Suppose $u, v \in \mathbb{R}$, $0 < u < v$, and $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then the following inequality holds*:

$$\left|\ln \mathcal{G}(u,v) - \ln \mathcal{I}(u,v)\right| \leq \frac{(v-u)^2}{2}\left[\mathcal{B}(p+1, p+1)\right]^{\frac{1}{p}}\left(\max\left\{\frac{1}{u^{2q}}, \frac{1}{v^{2q}}\right\}\right)^{\frac{1}{q}}.$$

Proof Set $\lambda = 1$ in inequality (16) and take $\mathcal{K}(x) = \ln x$. □

Proposition 22 *Suppose $u, v \in \mathbb{R}$ with $0 < u < v$ and $q \geq 1$. Then the following inequality holds*:

$$\left|\mathcal{H}^{-1}(u,v) - \mathcal{L}^{-1}(u,v)\right| \leq \frac{(v-u)^2}{12}\left[\max\left\{\left(\frac{2}{u^3}\right)^q, \left(\frac{2}{v^3}\right)^q\right\}\right]^{\frac{1}{q}}.$$

Proof The intended inequality is obtained by taking $\lambda = 1$ in (18) and using the function $\mathcal{K}(x) = \frac{1}{x}$. □

4 Conclusion

The main contribution of this paper is to establish new inequalities of the Hermite–Hadamard kind for functions with second derivatives, involving generalized Riemann–Liouville fractional integrals introduced by Raina [10] and Agarwal et al. [1], via η-quasiconvexity. Applications to some special means are also provided. By taking $\omega = 0$ and $\sigma(0) = 1$, we extend some already known theorems to a larger class. To the best of our knowledge, the results obtained herein are novel, and we hope that they would trigger further interest in this direction. For more recent results around η-(quasi)convexity, we refer the interested reader to [2, 6–8].

Acknowledgements
Many thanks to the referees and editors for their recommendations.

Funding
There is no funding to report at this point in time.

Availability of data and materials
Not applicable in this work.

Competing interests
The author declares that there are no competing interests.

Author's contributions
The sole author contributed 100% to this article. Author read and approved the final manuscript.

Publisher's Note

Received: 14 May 2019 Accepted: 17 June 2019 Published online: 02 July 2019

References

1. Agarwal, R.P., Luo, M.-J., Raina, R.K.: On Ostrowski type inequalities. Fasc. Math. **20**(4), 5–27 (2016)
2. Awan, M.U., Noorb, M.A., Noorb, K.I., Safdarb, F.: On strongly generalized convex functions. Filomat **31**(18), 5783–5790 (2017)
3. Delavar, M.R., Dragomir, S.S.: On η-convexity. Math. Inequal. Appl. **20**(1), 203–216 (2017)
4. Gordji, M.E., Delavar, M.R., Sen, M.D.L.: On φ-convex functions. J. Math. Inequal. **10**(1), 173–183 (2016)
5. Gordji, M.E., Dragomir, S.S., Delavar, M.R.: An inequality related to η-convex functions (II). Int. J. Nonlinear Anal. Appl. **6**(2), 26–32 (2015)
6. Kermausuor, S., Nwaeze, E.R.: Some new inequalities involving the Katugampola fractional integrals for strongly η-convex functions. Tbil. Math. J. **12**(1), 117–130 (2019)
7. Kermausuor, S., Nwaeze, E.R., Tameru, A.M.: New integral inequalities via the Katugampola fractional integrals for functions whose second derivatives are strongly η-convex. Mathematics **7**(2), Article ID 183 (2019)
8. Khan, M.A., Khurshid, Y., Ali, T.: Hermite–Hadamard inequality for fractional integrals via η-convex functions. Acta Math. Univ. Comen. **LXXXVI**(1), 153–164 (2017)
9. Nwaeze, E.R., Kermausuor, S., Tameru, A.M.: Some new k-Riemann–Liouville fractional integral inequalities associated with the strongly η-quasiconvex functions with modulus $\mu \geq 0$. J. Inequal. Appl. **2018**, 139 (2018)
10. Raina, R.K.: On generalized Wright's hypergeometric functions and fractional calculus operators. East Asian Math. J. **21**(2), 191–203 (2005)
11. Set, E., Dragomir, S.S., Gözpinar, A.: Some generalized Hermite–Hadamard type inequalities involving fractional integral operator for functions whose second derivatives in absolute value are s-convex. Acta Math. Univ. Comen. **LXXXVIII**(1), 87–100 (2019)