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Positive solutions for a class of fractional difference systems with coupled boundary conditions

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Abstract

In this paper we use the fixed point index and nonnegative matrices to study the existence of positive solutions for a class of fractional difference systems with coupled boundary conditions.

Keywords: Fractional difference systems; Positive solutions; Fixed point index; Nonnegative matrices

1 Introduction

For $a, b \in \mathbf{R}$, let $\mathbf{N}_a = \{a, a+1, a+2, \ldots\}$ and $[a,b]_{\mathbf{N}_a} = \{a, a+1, a+2, \ldots, b\}$ with $b-a \in \mathbf{N}_1$. In this paper we study the existence of positive solutions for the following fractional difference system with coupled boundary conditions:

$$\begin{cases} -\Delta_{\nu-3}^{\nu} x(t) = f_1(t+\nu-1, x(t+\nu-1), y(t+\nu-1)), & t \in [0, T-1]_{\mathbf{N}_0}, \\ -\Delta_{\nu-3}^{\nu} y(t) = f_2(t+\nu-1, x(t+\nu-1), y(t+\nu-1)), & t \in [0, T-1]_{\mathbf{N}_0}, \\ x(\nu-3) = [\Delta_{\nu-3}^{\alpha} x(t)]|_{t=\nu-\alpha-2} = 0, \qquad y(\nu-3) = [\Delta_{\nu-3}^{\alpha} y(t)]|_{t=\nu-\alpha-2} = 0, \\ x(T+\nu-1) = ay(\xi+\nu), \qquad y(T+\nu-1) = bx(\eta+\nu), \end{cases} \tag{1.1}$$

where $\nu \in (2,3]$, $\alpha \in (0,1)$ are two real numbers, $\Delta_{\nu-3}^{\nu}$, $\Delta_{\nu-3}^{\alpha}$ are discrete fractional operators, $\nu-\alpha-2 > 0$, $\xi, \eta \in [0, T-2]_{\mathbf{N}_0}$, $a, b > 0$ with $ab < \frac{(\xi+1)!(\eta+1)!}{\Gamma(\xi+\nu+1)\Gamma(\eta+\nu+1)}[\frac{\Gamma(T+\nu)}{T!}]^2$, and the nonlinearities $f_i(t,x,y) : [\nu-1, T+\nu-2]_{\mathbf{N}_{\nu-1}} \times \mathbf{R}^+ \times \mathbf{R}^+ \to \mathbf{R}^+$ are continuous functions ($i = 1, 2, \mathbf{R}^+ = [0, +\infty)$).

In recent years, the fractional calculus and fractional differential equations have been of great interest in the literature, and they have been widely applied in numerous diverse fields including electrical engineering, chemistry, mathematical biology, control theory, and the calculus of variations. For example, papers [1, 2] have introduced a fractional order model for infection of CD4$^+$T cells in HIV, which can be depicted by the system

$$\begin{cases} D^{\alpha_1}(T) = s - KVT - dT + bI, \\ D^{\alpha_2}(I) = KVT - (b+\delta)I, \\ D^{\alpha_3}(V) = N\delta I - cV, \end{cases}$$

where D^{α_i} are fractional derivatives, $i = 1, 2, 3$. Till now, we have noted that by using the techniques of nonlinear analysis, a large number of results concerning the existence and multiplicity of solutions (or positive solutions) of nonlinear fractional differential equations can be found in the literature, we refer the reader to [3–27] and the references cited therein. In [3], the authors studied the singular fractional p-Laplacian boundary value system

$$\begin{cases} D_{0+}^{\alpha}(\varphi_p(D_{0+}^{\gamma}u))(t) + \lambda^{1/(q-1)}f(t, u(t), D_{0+}^{\mu_1}u(t), D_{0+}^{\mu_2}u(t), \ldots, D_{0+}^{\mu_{n-1}}u(t), v(t)) = 0, \\ \quad 0 < t < 1, \\ D_{0+}^{\beta}(\varphi_p(D_{0+}^{\delta}v))(t) + \mu^{1/(q-1)}g(t, u(t), D_{0+}^{\eta_1}u(t), D_{0+}^{\eta_2}u(t), \ldots, D_{0+}^{\eta_{m-1}}u(t)) = 0, \\ \quad 0 < t < 1, \\ u(0) = D_{0+}^{\mu_i}u(0) = 0, \qquad D_{0+}^{\gamma}u(0) = D_{0+}^{\gamma+\mu_i}u(0) = 0, \quad i = 1, 2, \ldots, n-2, \\ D_{0+}^{\mu_{n-1}}u(1) = \chi \int_0^{\eta} h(t) D_{0+}^{\mu_{n-1}}u(t)\, dA(t), \\ v(0) = D_{0+}^{\eta_i}v(0) = 0, \qquad D_{0+}^{\delta}v(0) = D_{0+}^{\delta+\eta_i}v(0) = 0, \quad i = 1, 2, \ldots, m-2, \\ D_{0+}^{\eta_{m-1}}v(1) = \iota \int_0^{\vartheta} a(t) D_{0+}^{\eta_{m-1}}v(t)\, dB(t). \end{cases} \tag{1.2}$$

Here, they used the mixed monotone methods to obtain the uniqueness of positive solutions for (1.2) and established an iterative sequence, which can converge uniformly to the unique solution.

In [4], the authors studied the system of nonlinear fractional differential equations with coupled integral boundary conditions

$$\begin{cases} D_{0+}^{\alpha}u(t) + \lambda f(t, u(t), v(t)) = 0, \quad 0 < t < 1, \\ D_{0+}^{\beta}v(t) + \mu g(t, u(t), v(t)) = 0, \quad 0 < t < 1, \\ u(0) = u^{(i)}(0) = 0, \qquad u'(1) = \int_0^1 v(s)\, dH(s), \quad i = 1, 2, \ldots, n-2, \\ v(0) = v^{(i)}(0) = 0, \qquad v'(1) = \int_0^1 u(s)\, dK(s), \quad i = 1, 2, \ldots, m-2, \end{cases} \tag{1.3}$$

where the nonlinear terms f, g are sign-changing nonsingular or singular functions. They used the Guo–Krasnosel'skii fixed point theorem to obtain the existence of positive solutions for (1.3), and they also presented intervals for parameters λ and μ for the positive solutions.

However, as is mentioned by Christopher S. Goodrich in [28], there has been little work done in fractional difference equations, we only refer to [29–43]. For example, in [29] the authors studied discrete fractional calculus and offered some important properties of the fractional sum and the fractional difference operators. Also, they studied the uniqueness of solutions for the nonlinear fractional difference equation

$$\begin{cases} \Delta^{\nu}y(t) = f(t + \nu - 1, y(t + \nu - 1)), \quad t = 0, 1, 2, \ldots, \\ \Delta^{\nu-1}y(t)|_{t=0} = a_0. \end{cases} \tag{1.4}$$

Christopher S. Goodrich has made a great contribution to the development of the theory for discrete fractional calculus and associated difference equations (see [31, 32, 35–37, 39, 44]), presented and summarized many excellent results in his monograph with A. Peterson [43] in this direction. For example, in [35, 36] the authors studied the following two fractional difference equations boundary value problems:

$$\begin{cases} -\Delta^{\nu_1} y_1(t) = \lambda_1 f_1(t+\nu_1-1, y_1(t+\nu_1-1), y_2(t+\nu_2-1)), & t \in [1, b+1], \\ -\Delta^{\nu_2} y_2(t) = \lambda_2 f_2(t+\nu_2-1, y_1(t+\nu_1-1), y_2(t+\nu_2-1)), & t \in [1, b+1], \\ y_1(\nu_1-2) = y_1(\nu_1+b+1) = y_2(\nu_2-2) = y_2(\nu_2+b+1) = 0, \end{cases} \tag{1.5}$$

and

$$\begin{cases} -\Delta^{\nu_1} y_1(t) = \lambda_1 a_1(t+\nu_1-1) f_1(y_1(t+\nu_1-1), y_2(t+\nu_2-1)), & t \in [0, b], \\ -\Delta^{\nu_2} y_2(t) = \lambda_2 a_2(t+\nu_2-1) f_2(y_1(t+\nu_1-1), y_2(t+\nu_2-1)), & t \in [0, b], \\ y_1(\nu_1-2) = \psi_1(y_1), \qquad y_1(\nu_1+b) = \phi_1(y_1), \\ y_2(\nu_2-2) = \psi_2(y_2), \qquad y_2(\nu_2+b) = \phi_2(y_2), \end{cases} \tag{1.6}$$

where $\nu_1, \nu_2 \in (1, 2]$. They used the Guo–Krasnosel'skii fixed point theorem to obtain the existence of positive solutions for the above two problems, where the nonlinearities in (1.5) can be sign-changing.

Motivated by works aforementioned and some results from integer-order equations (including differential and difference equations, see [45–55]), we study the existence of positive solutions for the fractional difference systems (1.1). We use the fixed point index theory to establish our main results based on a priori estimates achieved by utilizing nonnegative matrices (see [10, 54, 55]) that involve some useful inequalities associated with the Green's functions for (1.1). Moreover, our nonlinearities f_i $(i = 1, 2)$ are allowed to grow superlinearly and sublinearly about the linear combinations of unknown functions x, y, see conditions (H1)–(H4) in Sect. 3.

2 Preliminaries

In this section, we first offer some necessary definitions from discrete fractional calculus. These materials can be found in some recent papers.

Definition 2.1 (see [43]) We define $t^{\underline{\nu}} := \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)}$ for any $t, \nu \in \mathbf{R}$ for which the right-hand side is well-defined. We use the convention that if $t+1-\nu$ is a pole of the gamma function and $t+1$ is not a pole, then $t^{\underline{\nu}} = 0$.

Definition 2.2 (see [43]) For $\nu > 0$, the νth fractional sum of a function f is

$$\Delta_a^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{\underline{\nu-1}} f(s) \quad \text{for } t \in \mathbf{N}_{a+\nu}.$$

We also define the νth fractional difference for $\nu > 0$ by

$$\Delta_a^{\nu} f(t) = \Delta_a^N \Delta_a^{\nu-N} f(t) \quad \text{for } t \in \mathbf{N}_{a+N-\nu},$$

where $N \in \mathbf{N}$ with $0 \le N-1 < \nu \le N$.

Lemma 2.3 (see [43]) *Let $N \in \mathbf{N}$ with $0 \le N-1 < \nu \le N$. Then*

$$\Delta_0^{-\nu} \Delta_{\nu-N}^{\nu} f(t) = f(t) + c_1 t^{\underline{\nu-1}} + c_2 t^{\underline{\nu-2}} + \cdots + C_N t^{\underline{\nu-N}} \quad \textit{for } c_i \in \mathbf{R}, 1 \le i \le N.$$

Lemma 2.4 (see [44, Lemma 4.1]) *For all* $\nu \in \mathbf{R}$, *we have* $\Delta_a^\alpha t^{\underline{\nu}} = \frac{\Gamma(\nu+1)t^{\underline{\nu-\alpha}}}{\Gamma(\nu+1-\alpha)}$ *with* $\alpha > 0$, *if* $t^{\underline{\nu}}$, $t^{\underline{\nu-\alpha}}$ *are well-defined.*

Next, we use Lemmas 2.3 and 2.4 to calculate the Green's functions associated with (1.1). For convenience, we let $L = [\frac{\Gamma(T+\nu)}{T!}]^2 - ab\frac{\Gamma(\xi+\nu+1)\Gamma(\eta+\nu+1)}{(\xi+1)!(\eta+1)!}$, and

$$G(t,s) = \frac{1}{\Gamma(\nu)} \begin{cases} \frac{t^{\underline{\nu-1}}(T+\nu-s-2)^{\underline{\nu-1}}}{(T+\nu-1)^{\underline{\nu-1}}} - (t-s-1)^{\underline{\nu-1}}, & 0 \le s < t-\nu+1 \le T-1; \\ \frac{t^{\underline{\nu-1}}(T+\nu-s-2)^{\underline{\nu-1}}}{(T+\nu-1)^{\underline{\nu-1}}}, & 0 \le t-\nu+1 \le s \le T-1. \end{cases} \tag{2.1}$$

The following lemma is as in [40] (for completeness, we present its proof).

Lemma 2.5 *Let* $\nu \in (2,3]$, $\alpha \in (0,1)$, *and* $h_i(t) : [\nu-1, T+\nu-2]_{\mathbf{N}_{\nu-1}} \to \mathbf{R}$ $(i = 1,2)$. *Then the fractional difference system*

$$\begin{cases} -\Delta_{\nu-3}^{\nu} x(t) = h_1(t+\nu-1), & t \in [0, T-1]_{\mathbf{N_0}}, \\ -\Delta_{\nu-3}^{\nu} y(t) = h_2(t+\nu-1), & t \in [0, T-1]_{\mathbf{N_0}}, \\ x(\nu-3) = [\Delta_{\nu-3}^{\alpha} x(t)]|_{t=\nu-\alpha-2} = 0, & y(\nu-3) = [\Delta_{\nu-3}^{\alpha} y(t)]|_{t=\nu-\alpha-2} = 0, \\ x(T+\nu-1) = ay(\xi+\nu), & y(T+\nu-1) = bx(\eta+\nu), \end{cases} \tag{2.2}$$

has the unique solution, which takes the form

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \sum_{s=0}^{T-1} H_1(t,s)h_1(s+\nu-1) + \sum_{s=0}^{T-1} K_1(t,s)h_2(s+\nu-1) \\ \sum_{s=0}^{T-1} H_2(t,s)h_2(s+\nu-1) + \sum_{s=0}^{T-1} K_2(t,s)h_1(s+\nu-1) \end{pmatrix}, \tag{2.3}$$

where

$$\begin{aligned} H_1(t,s) &= G(t,s) + \frac{ab(\xi+\nu)^{\underline{\nu-1}}t^{\underline{\nu-1}}}{L} G(\eta+\nu,s), \\ K_1(t,s) &= \frac{a(T+\nu-1)^{\underline{\nu-1}}t^{\underline{\nu-1}}}{L} G(\xi+\nu,s), \end{aligned} \tag{2.4}$$

$$\begin{aligned} H_2(t,s) &= G(t,s) + \frac{ab(\eta+\nu)^{\underline{\nu-1}}t^{\underline{\nu-1}}}{L} G(\xi+\nu,s), \\ K_2(t,s) &= \frac{b(T+\nu-1)^{\underline{\nu-1}}t^{\underline{\nu-1}}}{L} G(\eta+\nu,s). \end{aligned} \tag{2.5}$$

Proof From Lemma 2.3 we have

$$\begin{aligned} x(t) &= -\frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} (t-s-1)^{\underline{\nu-1}} h_1(s+\nu-1) + C_1 t^{\underline{\nu-1}} + C_2 t^{\underline{\nu-2}} + C_3 t^{\underline{\nu-3}}, \\ &\quad (C_i \in \mathbf{R}, i = 1,2,3) \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} y(t) &= -\frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} (t-s-1)^{\underline{\nu-1}} h_2(s+\nu-1) + \overline{C}_1 t^{\underline{\nu-1}} + \overline{C}_2 t^{\underline{\nu-2}} + \overline{C}_3 t^{\underline{\nu-3}}, \\ &\quad (\overline{C}_i \in \mathbf{R}, i = 1,2,3). \end{aligned} \tag{2.7}$$

Substituting $x(\nu-3)=y(\nu-3)=0$ into (2.6), (2.7), we obtain $C_3=\overline{C}_3=0$. Because of

$$\begin{aligned}\triangle_{\nu-3}^{\alpha}x(t) &= C_1\,\triangle_{\nu-3}^{\alpha}\,t^{\underline{\nu-1}}+C_2\,\triangle_{\nu-3}^{\alpha}\,t^{\underline{\nu-2}}-\triangle_{\nu-3}^{-(\nu-\alpha)}h_1(t+\nu-1)\\ &= C_1\frac{\Gamma(\nu)t^{\underline{\nu-\alpha-1}}}{\Gamma(\nu-\alpha)}+C_2\frac{\Gamma(\nu-1)t^{\underline{\nu-\alpha-2}}}{\Gamma(\nu-\alpha-1)}\\ &\quad-\frac{1}{\Gamma(\nu-\alpha)}\sum_{s=0}^{t-\nu+\alpha}(t-s-1)^{\underline{\nu-\alpha-1}}h_1(s+\nu-1),\end{aligned}$$

and using the boundary condition $[\triangle_{\nu-3}^{\alpha}x(t)]|_{t=\nu-\alpha-2}=0$ to obtain $C_2=0$. Similarly, we have $\overline{C}_2=0$. By virtue of the conditions $x(T+\nu-1)=ay(\xi+\nu)$, $y(T+\nu-1)=bx(\eta+\nu)$, we respectively obtain

$$\begin{aligned}&-\frac{1}{\Gamma(\nu)}\sum_{s=0}^{T-1}(T+\nu-s-2)^{\underline{\nu-1}}h_1(s+\nu-1)+C_1(T+\nu-1)^{\underline{\nu-1}}\\ &\quad=-\frac{a}{\Gamma(\nu)}\sum_{s=0}^{\xi}(\xi+\nu-s-1)^{\underline{\nu-1}}h_2(s+\nu-1)+a\overline{C}_1(\xi+\nu)^{\underline{\nu-1}},\end{aligned}$$

and

$$\begin{aligned}&-\frac{1}{\Gamma(\nu)}\sum_{s=0}^{T-1}(T+\nu-s-2)^{\underline{\nu-1}}h_2(s+\nu-1)+\overline{C}_1(T+\nu-1)^{\underline{\nu-1}}\\ &\quad=-\frac{b}{\Gamma(\nu)}\sum_{s=0}^{\eta}(\eta+\nu-s-1)^{\underline{\nu-1}}h_1(s+\nu-1)+bC_1(\eta+\nu)^{\underline{\nu-1}}.\end{aligned}$$

Note that

$$\begin{aligned}&\begin{vmatrix}(T+\nu-1)^{\underline{\nu-1}} & -a(\xi+\nu)^{\underline{\nu-1}}\\ -b(\eta+\nu)^{\underline{\nu-1}} & (T+\nu-1)^{\underline{\nu-1}}\end{vmatrix}\\ &\quad=\bigl((T+\nu-1)^{\underline{\nu-1}}\bigr)^2-ab(\xi+\nu)^{\underline{\nu-1}}(\eta+\nu)^{\underline{\nu-1}}\\ &\quad=\left(\frac{\Gamma(T+\nu)}{T!}\right)^2-\frac{ab\Gamma(\xi+\nu+1)\Gamma(\eta+\nu+1)}{(\xi+1)!(\eta+1)!}\\ &\quad=L>0,\end{aligned}$$

so we have

$$\begin{aligned}C_1={}&\frac{1}{L\Gamma(\nu)}\Biggl[(T+\nu-1)^{\underline{\nu-1}}\Biggl[\sum_{s=0}^{T-1}(T+\nu-s-2)^{\underline{\nu-1}}h_1(s+\nu-1)\\ &-a\sum_{s=0}^{\xi}(\xi+\nu-s-1)^{\underline{\nu-1}}h_2(s+\nu-1)\Biggr]\\ &+a(\xi+\nu)^{\underline{\nu-1}}\Biggl[\sum_{s=0}^{T-1}(T+\nu-s-2)^{\underline{\nu-1}}h_2(s+\nu-1)\\ &-b\sum_{s=0}^{\eta}(\eta+\nu-s-1)^{\underline{\nu-1}}h_1(s+\nu-1)\Biggr]\Biggr],\end{aligned}$$

$$
\begin{aligned}
\overline{C}_1 = \frac{1}{L\Gamma(\nu)}\Biggl[&(T+\nu-1)^{\underline{\nu-1}}\Biggl[-b\sum_{s=0}^{\eta}(\eta+\nu-s-1)^{\underline{\nu-1}}h_1(s+\nu-1)\\
&+\sum_{s=0}^{T-1}(T+\nu-s-2)^{\underline{\nu-1}}h_2(s+\nu-1)\Biggr]\\
&+b(\eta+\nu)^{\underline{\nu-1}}\Biggl[-a\sum_{s=0}^{\xi}(\xi+\nu-s-1)^{\underline{\nu-1}}h_2(s+\nu-1)\\
&+\sum_{s=0}^{T-1}(T+\nu-s-2)^{\underline{\nu-1}}h_1(s+\nu-1)\Biggr]\Biggr].
\end{aligned}
$$

As a result, we have

$$
\begin{aligned}
x(t) = &-\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-s-1)^{\underline{\nu-1}}h_1(s+\nu-1)\\
&+\frac{t^{\underline{\nu-1}}}{L\Gamma(\nu)}\Biggl[(T+\nu-1)^{\underline{\nu-1}}\Biggl[\sum_{s=0}^{T-1}(T+\nu-s-2)^{\underline{\nu-1}}h_1(s+\nu-1)\\
&-a\sum_{s=0}^{\xi}(\xi+\nu-s-1)^{\underline{\nu-1}}h_2(s+\nu-1)\Biggr]\\
&+a(\xi+\nu)^{\underline{\nu-1}}\Biggl[\sum_{s=0}^{T-1}(T+\nu-s-2)^{\underline{\nu-1}}h_2(s+\nu-1)\\
&-b\sum_{s=0}^{\eta}(\eta+\nu-s-1)^{\underline{\nu-1}}h_1(s+\nu-1)\Biggr]\Biggr]\\
= &-\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-s-1)^{\underline{\nu-1}}h_1(s+\nu-1)\\
&+\frac{t^{\underline{\nu-1}}(T+\upsilon-1)^{\underline{\nu-1}}}{L\Gamma(\nu)}\sum_{s=0}^{T-1}(T+\nu-s-2)^{\underline{\nu-1}}h_1(s+\nu-1)\\
&-\frac{t^{\underline{\nu-1}}}{\Gamma(\nu)(T+\nu-1)^{\underline{\nu-1}}}\sum_{s=0}^{T-1}(T+\nu-s-2)^{\underline{\nu-1}}h_1(s+\nu-1)\\
&+\frac{t^{\underline{\nu-1}}}{\Gamma(\nu)(T+\nu-1)^{\underline{\nu-1}}}\sum_{s=0}^{T-1}(T+\nu-s-2)^{\underline{\nu-1}}h_1(s+\nu-1)\\
&-\frac{ab(\xi+\nu)^{\underline{\nu-1}}t^{\underline{\nu-1}}}{L\Gamma(\nu)}\sum_{s=0}^{\eta}(\eta+\nu-s-1)^{\underline{\nu-1}}h_1(s+\nu-1)\\
&-\frac{a(T+\nu-1)^{\underline{\nu-1}}t^{\underline{\nu-1}}}{L\Gamma(\nu)}\sum_{s=0}^{\xi}(\xi+\nu-s-1)^{\underline{\nu-1}}h_2(s+\nu-1)\\
&+\frac{a(\xi+\nu)^{\underline{\nu-1}}t^{\underline{\nu-1}}}{L\Gamma(\nu)}\sum_{s=0}^{T-1}(T+\nu-s-2)^{\underline{\nu-1}}h_2(s+\nu-1)\\
= &\sum_{s=0}^{T-1}G(t,s)h_1(s+\nu-1)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{abt^{\underline{\nu-1}}(\xi+\nu)^{\underline{\nu-1}}(\eta+\nu)^{\underline{\nu-1}}}{L\Gamma(\nu)(T+\nu-1)^{\underline{\nu-1}}}\sum_{s=0}^{T-1}(T+\nu-s-2)^{\underline{\nu-1}}h_1(s+\nu-1)\\
&-\frac{ab(\xi+\nu)^{\underline{\nu-1}}t^{\underline{\nu-1}}}{L\Gamma(\nu)}\sum_{s=0}^{\eta}(\eta+\nu-s-1)^{\underline{\nu-1}}h_1(s+\nu-1)\\
&-\frac{a(T+\nu-1)^{\underline{\nu-1}}t^{\underline{\nu-1}}}{L\Gamma(\nu)}\sum_{s=0}^{\xi}(\xi+\nu-s-1)^{\underline{\nu-1}}h_2(s+\nu-1)\\
&+\frac{a(\xi+\nu)^{\underline{\nu-1}}t^{\underline{\nu-1}}}{L\Gamma(\nu)}\sum_{s=0}^{T-1}(T+\nu-s-2)^{\underline{\nu-1}}h_2(s+\nu-1)\\
=&\sum_{s=0}^{T-1}G(t,s)h_1(s+\nu-1)\\
&+\frac{abt^{\underline{\nu-1}}(\xi+\nu)^{\underline{\nu-1}}}{L}\sum_{s=0}^{T-1}G(\eta+\nu,s)h_1(s+\nu-1)\\
&+\frac{a(T+\nu-1)^{\underline{\nu-1}}t^{\underline{\nu-1}}}{L}\sum_{s=0}^{T-1}G(\xi+\nu,s)h_2(s+\nu-1)\\
=&\sum_{s=0}^{T-1}\left[G(t,s)+\frac{abt^{\underline{\nu-1}}(\xi+\nu)^{\underline{\nu-1}}}{L}G(\eta+\nu,s)\right]h_1(s+\nu-1)\\
&+\sum_{s=0}^{T-1}\frac{a(T+\nu-1)^{\underline{\nu-1}}t^{\underline{\nu-1}}}{L}G(\xi+\nu,s)]h_2(s+\nu-1).
\end{aligned}
$$

Similarly, we can obtain

$$
\begin{aligned}
y(t)=&-\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-s-1)^{\underline{\nu-1}}h_2(s+\nu-1)\\
&+\frac{t^{\underline{\nu-1}}}{L\Gamma(\nu)}\left[(T+\nu-1)^{\underline{\nu-1}}\left[-b\sum_{s=0}^{\eta}(\eta+\nu-s-1)^{\underline{\nu-1}}h_1(s+\nu-1)\right.\right.\\
&\left.+\sum_{s=0}^{T-1}(T+\nu-s-2)^{\underline{\nu-1}}h_2(s+\nu-1)\right]\\
&+b(\eta+\nu)^{\underline{\nu-1}}\left[-a\sum_{s=0}^{\xi}(\xi+\nu-s-1)^{\underline{\nu-1}}h_2(s+\nu-1)\right.\\
&\left.\left.+\sum_{s=0}^{T-1}(T+\nu-s-2)^{\underline{\nu-1}}h_1(s+\nu-1)\right]\right]\\
=&-\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-s-1)^{\underline{\nu-1}}h_2(s+\nu-1)\\
&+\frac{t^{\underline{\nu-1}}(T+\nu-1)^{\underline{\nu-1}}}{L\Gamma(\nu)}\sum_{s=0}^{T-1}(T+\nu-s-2)^{\underline{\nu-1}}h_2(s+\nu-1)
\end{aligned}
$$

$$
\begin{aligned}
& - \frac{t^{\underline{\nu-1}}}{\Gamma(\nu)(T+\nu-1)^{\underline{\nu-1}}} \sum_{s=0}^{T-1} (T+\nu-s-2)^{\underline{\nu-1}} h_2(s+\nu-1) \\
& + \frac{t^{\underline{\nu-1}}}{\Gamma(\nu)(T+\nu-1)^{\underline{\nu-1}}} \sum_{s=0}^{T-1} (T+\nu-s-2)^{\underline{\nu-1}} h_2(s+\nu-1) \\
& - \frac{ab(\eta+\nu)^{\underline{\nu-1}} t^{\underline{\nu-1}}}{L\Gamma(\nu)} \sum_{s=0}^{\xi} (\xi+\nu-s-1)^{\underline{\nu-1}} h_2(s+\nu-1) \\
& - \frac{b(T+\nu-1)^{\underline{\nu-1}} t^{\underline{\nu-1}}}{L\Gamma(\nu)} \sum_{s=0}^{\eta} (\eta+\nu-s-1)^{\underline{\nu-1}} h_1(s+\nu-1) \\
& + \frac{b(\eta+\nu)^{\underline{\nu-1}} t^{\underline{\nu-1}}}{L\Gamma(\nu)} \sum_{s=0}^{T-1} (T+\nu-s-2)^{\underline{\nu-1}} h_1(s+\nu-1) \\
= & \sum_{s=0}^{T-1} G(t,s) h_2(s+\nu-1) \\
& + \frac{abt^{\underline{\nu-1}} (\xi+\nu)^{\underline{\nu-1}} (\eta+\nu)^{\underline{\nu-1}}}{L\Gamma(\nu)(T+\nu-1)^{\underline{\nu-1}}} \sum_{s=0}^{T-1} (T+\nu-s-2)^{\underline{\nu-1}} h_2(s+\nu-1) \\
& - \frac{ab(\eta+\nu)^{\underline{\nu-1}} t^{\underline{\nu-1}}}{L\Gamma(\nu)} \sum_{s=0}^{\xi} (\xi+\nu-s-1)^{\underline{\nu-1}} h_2(s+\nu-1) \\
& - \frac{b(T+\nu-1)^{\underline{\nu-1}} t^{\underline{\nu-1}}}{L\Gamma(\nu)} \sum_{s=0}^{\eta} (\eta+\nu-s-1)^{\underline{\nu-1}} h_1(s+\nu-1) \\
& + \frac{b(\eta+\nu)^{\underline{\nu-1}} t^{\underline{\nu-1}} (T+\nu-1)^{\underline{\nu-1}}}{L\Gamma(\nu)} \sum_{s=0}^{T-1} \frac{(T+\nu-s-2)^{\underline{\nu-1}}}{(T+\nu-1)} h_1(s+\nu-1) \\
= & \sum_{s=0}^{T-1} G(t,s) h_2(s+\nu-1) \\
& + \frac{abt^{\underline{\nu-1}} (\eta+\nu)^{\underline{\nu-1}}}{L} \sum_{s=0}^{T-1} G(\xi+\nu,s) h_2(s+\nu-1) \\
& + \frac{b(T+\nu-1)^{\underline{\nu-1}} t^{\underline{\nu-1}}}{L} \sum_{s=0}^{T-1} G(\eta+\nu,s) h_1(s+\nu-1) \\
= & \sum_{s=0}^{T-1} \left[G(t,s) + \frac{abt^{\underline{\nu-1}} (\eta+\nu)^{\underline{\nu-1}}}{L} G(\xi+\nu,s) \right] h_2(s+\nu-1) \\
& + \sum_{s=0}^{T-1} \frac{b(T+\nu-1)^{\underline{\nu-1}} t^{\underline{\nu-1}}}{L} G(\eta+\nu,s)] h_1(s+\nu-1).
\end{aligned}
$$

This completes the proof. □

Lemma 2.6 (see [40, Theorems 2.2, 2.3 and Remark 2.4]) *Let* $L_1 = \frac{\nu-1}{T(T+\nu-1)^{\underline{\nu-1}}(T+\nu-2)}$ *and* $\rho(s) = (T+\nu-s-2)^{\underline{\nu-1}}$ *for* $s \in [0, T-1]_{\mathbf{N}_0}$. *Then we have*

(i) $G(t,s) > 0$, *for* $(t,s) \in [\nu-1, T+\nu-2]_{\mathbf{N}_{\nu-1}} \times [0, T-1]_{\mathbf{N}_0}$;

(ii) *for all* $(t,s) \in [\nu-1, T+\nu-2]_{\mathbf{N}_{\nu-1}} \times [0, T-1]_{\mathbf{N}_0}$, *there holds*

$$\begin{aligned}
&\frac{abL_1(\xi+\nu)^{\underline{\nu-1}}(\eta+\nu)^{\underline{\nu-1}}t^{\underline{\nu-1}}\rho(s)}{L\Gamma(\nu)} \\
&\quad \le H_1(t,s) \le \frac{[L+ab(\xi+\nu)^{\underline{\nu-1}}(T+\nu-2)^{\underline{\nu-1}}]\rho(s)}{L\Gamma(\nu)}, \\
&\frac{aL_1(\xi+\nu)^{\underline{\nu-1}}(\eta+\nu)^{\underline{\nu-1}}t^{\underline{\nu-1}}\rho(s)}{L\Gamma(\nu)} \\
&\quad \le K_1(t,s) \le \frac{a(T+\nu-1)^{\underline{\nu-1}}(T+\nu-2)^{\underline{\nu-1}}\rho(s)}{L\Gamma(\nu)};
\end{aligned}$$

(iii) *for all* $(t,s) \in [\nu-1, T+\nu-2]_{\mathbf{N}_{\nu-1}} \times [0, T-1]_{\mathbf{N}_0}$, *there holds*

$$\begin{aligned}
&\frac{abL_1(\xi+\nu)^{\underline{\nu-1}}(\eta+\nu)^{\underline{\nu-1}}t^{\underline{\nu-1}}\rho(s)}{L\Gamma(\nu)} \\
&\quad \le H_2(t,s) \le \frac{[L+ab(\eta+\nu)^{\underline{\nu-1}}(T+\nu-2)^{\underline{\nu-1}}]\rho(s)}{L\Gamma(\nu)}, \\
&\frac{bL_1(\xi+\nu)^{\underline{\nu-1}}(\eta+\nu)^{\underline{\nu-1}}t^{\underline{\nu-1}}\rho(s)}{L\Gamma(\nu)} \\
&\quad \le K_2(t,s) \le \frac{b(T+\nu-1)^{\underline{\nu-1}}(T+\nu-2)^{\underline{\nu-1}}\rho(s)}{L\Gamma(\nu)}.
\end{aligned}$$

Lemma 2.7 *Let* $\rho^*(t) = (T+2\nu-t-3)^{\underline{\nu-1}}$ *for* $t \in [\nu-1, T+\nu-2]_{\mathbf{N}_{\nu-1}}$. *Then, for all* $s \in [0, T-1]_{\mathbf{N}_0}$, *we have the following inequalities*:

$$h_{\mu_1}\rho(s) \le \sum_{t=\nu-1}^{T+\nu-2} H_1(t,s)\rho^*(t) \le h_{\mu_2}\rho(s), \tag{2.8}$$

and

$$k_{\mu_1}\rho(s) \le \sum_{t=\nu-1}^{T+\nu-2} K_1(t,s)\rho^*(t) \le k_{\mu_2}\rho(s), \tag{2.9}$$

and

$$h_{\mu_3}\rho(s) \le \sum_{t=\nu-1}^{T+\nu-2} H_2(t,s)\rho^*(t) \le h_{\mu_4}\rho(s), \tag{2.10}$$

and

$$k_{\mu_3}\rho(s) \le \sum_{t=\nu-1}^{T+\nu-2} K_2(t,s)\rho^*(t) \le k_{\mu_4}\rho(s), \tag{2.11}$$

where

$$\begin{pmatrix} h_{\mu_1} & h_{\mu_2} \\ k_{\mu_1} & k_{\mu_2} \\ h_{\mu_3} & h_{\mu_4} \\ k_{\mu_3} & k_{\mu_4} \end{pmatrix} = \begin{pmatrix} \sum_{t=0}^{T-1} \frac{abL_1(\xi+\nu)^{\underline{\nu-1}}(\eta+\nu)^{\underline{\nu-1}}(t+\nu-1)^{\underline{\nu-1}}\rho(t)}{L\Gamma(\nu)} & \sum_{t=0}^{T-1} \frac{[L+ab(\xi+\nu)^{\underline{\nu-1}}(T+\nu-2)^{\underline{\nu-1}}]\rho(t)}{L\Gamma(\nu)} \\ \sum_{t=0}^{T-1} \frac{aL_1(\xi+\nu)^{\underline{\nu-1}}(\eta+\nu)^{\underline{\nu-1}}(t+\nu-1)^{\underline{\nu-1}}\rho(t)}{L\Gamma(\nu)} & \sum_{t=0}^{T-1} \frac{a(T+\nu-1)^{\underline{\nu-1}}(T+\nu-2)^{\underline{\nu-1}}\rho(t)}{L\Gamma(\nu)} \\ \sum_{t=0}^{T-1} \frac{abL_1(\xi+\nu)^{\underline{\nu-1}}(\eta+\nu)^{\underline{\nu-1}}(t+\nu-1)^{\underline{\nu-1}}\rho(t)}{L\Gamma(\nu)} & \sum_{t=0}^{T-1} \frac{[L+ab(\eta+\nu)^{\underline{\nu-1}}(T+\nu-2)^{\underline{\nu-1}}]\rho(t)}{L\Gamma(\nu)} \\ \sum_{t=0}^{T-1} \frac{bL_1(\xi+\nu)^{\underline{\nu-1}}(\eta+\nu)^{\underline{\nu-1}}(t+\nu-1)^{\underline{\nu-1}}\rho(t)}{L\Gamma(\nu)} & \sum_{t=0}^{T-1} \frac{b(T+\nu-1)^{\underline{\nu-1}}(T+\nu-2)^{\underline{\nu-1}}\rho(t)}{L\Gamma(\nu)} \end{pmatrix}.$$

Proof We only prove (2.8). Indeed, for all $s \in [0, T-1]_{\mathbf{N}_0}$, from Lemma 2.6(ii) we have

$$\begin{aligned} \sum_{t=\nu-1}^{T+\nu-2} H_1(t,s)\rho^*(t) &= \sum_{t=0}^{T-1} H_1(t-\nu+1,s)\rho^*(t+\nu-1) \\ &\leq \sum_{t=0}^{T-1} \frac{[L+ab(\xi+\nu)^{\underline{\nu-1}}(T+\nu-2)^{\underline{\nu-1}}]\rho(s)}{L\Gamma(\nu)}\rho^*(t+\nu-1) \\ &\leq \sum_{t=0}^{T-1} \frac{[L+ab(\xi+\nu)^{\underline{\nu-1}}(T+\nu-2)^{\underline{\nu-1}}]\rho(s)}{L\Gamma(\nu)}\rho(t) = h_{\mu_2}\rho(s). \end{aligned}$$

On the other hand, we obtain

$$\begin{aligned} \sum_{t=\nu-1}^{T+\nu-2} H_1(t,s)\rho^*(t) &= \sum_{t=0}^{T-1} H_1(t-\nu+1,s)\rho^*(t+\nu-1) \\ &\geq \sum_{t=0}^{T-1} \frac{abL_1(\xi+\nu)^{\underline{\nu-1}}(\eta+\nu)^{\underline{\nu-1}}(t+\nu-1)^{\underline{\nu-1}}\rho(s)}{L\Gamma(\nu)}\rho^*(t+\nu-1) \\ &\geq \sum_{t=0}^{T-1} \frac{abL_1(\xi+\nu)^{\underline{\nu-1}}(\eta+\nu)^{\underline{\nu-1}}(t+\nu-1)^{\underline{\nu-1}}\rho(s)}{L\Gamma(\nu)}\rho(t) \\ &= h_{\mu_1}\rho(s). \end{aligned}$$

This completes the proof. □

Let E be the collection of all maps from $[\nu-3, T+\nu-2]_{\mathbf{N}_{\nu-3}}$ to $\mathbf{R}$ with the norm $\|z\| = \max_{t\in[\nu-3,T+\nu-2]_{\mathbf{N}_{\nu-3}}} |z(t)|$. Then $(E, \|\cdot\|)$ is a Banach space, and $P = \{z \in E : z(t) \geq 0, t \in [\nu-3, T+\nu-2]_{\mathbf{N}_{\nu-3}}\}$ is a cone on E. From Lemma 2.5, we know that the fractional difference system (1.1) can be expressed in the following form:

$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= \begin{pmatrix} \sum_{s=0}^{T-1} H_1(t,s)f_1(s+\nu-1, x(s+\nu-1), y(s+\nu-1)) + \sum_{s=0}^{T-1} K_1(t,s)f_2(s+\nu-1, x(s+\nu-1), y(s+\nu-1)) \\ \sum_{s=0}^{T-1} H_2(t,s)f_2(s+\nu-1, x(s+\nu-1), y(s+\nu-1)) + \sum_{s=0}^{T-1} K_2(t,s)f_1(s+\nu-1, x(s+\nu-1), y(s+\nu-1)) \end{pmatrix} \\ &:= \begin{pmatrix} A_1(x,y)(t) \\ A_2(x,y)(t) \end{pmatrix}, \quad \forall t \in [\nu-3, T+\nu-2]_{\mathbf{N}_{\nu-3}}. \end{aligned} \tag{2.12}$$

Then we define an operator $A : P \times P \to P \times P$ as follows:

$$A(x,y)(t) = (A_1, A_2)(x,y)(t), \quad \forall t \in [\nu - 3, T + \nu - 2]_{\mathbf{N}_{\nu-3}}.$$

Then positive solutions for the fractional difference system (1.1) exist if and only if positive fixed points for A exist, i.e., if there exists $(\overline{x}, \overline{y}) \in P$ such that $A(\overline{x}, \overline{y}) = (\overline{x}, \overline{y})$, and $A_1(\overline{x}, \overline{y})(t) = \overline{x}(t)$, $A_2(\overline{x}, \overline{y})(t) = \overline{y}(t)$, from (2.12) we have $(\overline{x}, \overline{y})(t)$ is a positive solution for (1.1), for $t \in [\nu - 3, T + \nu - 2]_{\mathbf{N}_{\nu-3}}$. Now, we turn to study the existence of fixed points for the operator A. In what follows, we provide two lemmas involving the fixed point index; for more details, we refer to the book [56].

Lemma 2.8 *Let E be a real Banach space and P be a cone on E. Suppose that $\Omega \subset E$ is a bounded open set and that $A : \overline{\Omega} \cap P \to P$ is a continuous compact operator. If there exists $\omega_0 \in P \setminus \{0\}$ such that*

$$\omega - A\omega \neq \lambda \omega_0, \quad \forall \lambda \geq 0, \omega \in \partial \Omega \cap P,$$

then $i(A, \Omega \cap P, P) = 0$, where i denotes the fixed point index on P.

Lemma 2.9 *Let E be a real Banach space and P be a cone on E. Suppose that $\Omega \subset E$ is a bounded open set with $0 \in \Omega$ and that $A : \overline{\Omega} \cap P \to P$ is a continuous compact operator. If*

$$\omega - \lambda A\omega \neq 0, \quad \forall \lambda \in [0,1], \omega \in \partial \Omega \cap P,$$

then $i(A, \Omega \cap P, P) = 1$.

3 Main results

In this section, we first provide some assumptions for our nonlinearities f_i, $i = 1, 2$. Here, we make an explanation: in $P \times P$, if $\binom{x_1}{x_2} \geq (\text{or } \leq) \binom{y_1}{y_2}$, we mean that $x_1(t) \geq (\text{or } \leq) y(t)$, $x_2(t) \geq (\text{or } \leq) y_2(t)$ for $t \in [\nu - 1, T + \nu - 2]_{\mathbf{N}_{\nu-1}}$.

(H1) There exist $a_1, b_1, c_1, d_1 \geq 0$ and $l_1, l_2 > 0$ such that

$$\begin{pmatrix} f_1(t,x,y) \\ f_2(t,x,y) \end{pmatrix} \geq \begin{pmatrix} a_1 x + b_1 y - l_1 \\ c_1 x + d_1 y - l_2 \end{pmatrix}, \quad \forall (t,x,y) \in [\nu - 1, T + \nu - 2]_{\mathbf{N}_{\nu-1}} \times \mathbf{R}^+ \times \mathbf{R}^+,$$

and

$$h_{\mu_1} a_1 + k_{\mu_1} c_1 < 1, \qquad h_{\mu_3} d_1 + k_{\mu_3} b_1 < 1,$$
$$\det \begin{pmatrix} h_{\mu_1} b_1 + k_{\mu_1} d_1 & h_{\mu_1} a_1 + k_{\mu_1} c_1 - 1 \\ h_{\mu_3} d_1 + k_{\mu_3} b_1 - 1 & h_{\mu_3} c_1 + k_{\mu_3} a_1 \end{pmatrix} := \kappa_1 > 0.$$

(H2) There exist $a_2, b_2, c_2, d_2 \geq 0$ and $r_1 > 0$ such that

$$\begin{pmatrix} f_1(t,x,y) \\ f_2(t,x,y) \end{pmatrix} \leq \begin{pmatrix} a_2 x + b_2 y \\ c_2 x + d_2 y \end{pmatrix}, \quad \forall (t,x,y) \in [\nu - 1, T + \nu - 2]_{\mathbf{N}_{\nu-1}} \times [0, r_1] \times [0, r_1],$$

and

$$h_{\mu_2}a_2 + k_{\mu_2}c_2 < 1, \qquad h_{\mu_4}d_2 + k_{\mu_4}b_2 < 1,$$
$$\det\begin{pmatrix} 1 - h_{\mu_2}a_2 - k_{\mu_2}c_2 & -h_{\mu_2}b_2 - k_{\mu_2}d_2 \\ -h_{\mu_4}c_2 - k_{\mu_4}a_2 & 1 - h_{\mu_4}d_2 - k_{\mu_4}b_2 \end{pmatrix} := \kappa_2 > 0.$$

(H3) There exist $a_3, b_3, c_3, d_3 \geq 0$ and $r_2 > 0$ such that

$$\begin{pmatrix} f_1(t,x,y) \\ f_2(t,x,y) \end{pmatrix} \geq \begin{pmatrix} a_3x + b_3y \\ c_3x + d_3y \end{pmatrix},$$
$$\forall (t,x,y) \in [\nu - 1, T + \nu - 2]_{\mathbf{N}_{\nu-1}} \times [0, r_2] \times [0, r_2],$$

and

$$h_{\mu_1}a_3 + k_{\mu_1}c_3 < 1, \qquad h_{\mu_3}d_3 + k_{\mu_3}b_3 < 1,$$
$$\det\begin{pmatrix} h_{\mu_1}b_3 + k_{\mu_1}d_3 & h_{\mu_1}a_3 + k_{\mu_1}c_3 - 1 \\ h_{\mu_3}d_3 + k_{\mu_3}b_3 - 1 & h_{\mu_3}c_3 + k_{\mu_3}a_3 \end{pmatrix} := \kappa_3 > 0.$$

(H4) There exist $a_4, b_4, c_4, d_4 \geq 0$ and $l_3, l_4 > 0$ such that

$$\begin{pmatrix} f_1(t,x,y) \\ f_2(t,x,y) \end{pmatrix} \leq \begin{pmatrix} a_4x + b_4y + l_3 \\ c_4x + d_4y + l_4 \end{pmatrix},$$
$$\forall (t,x,y) \in [\nu - 1, T + \nu - 2]_{\mathbf{N}_{\nu-1}} \times \mathbf{R}^+ \times \mathbf{R}^+,$$

and

$$h_{\mu_2}a_4 + k_{\mu_2}c_4 < 1, \qquad h_{\mu_4}d_4 + k_{\mu_4}b_4 < 1,$$
$$\det\begin{pmatrix} 1 - h_{\mu_2}a_4 - k_{\mu_2}c_4 & -h_{\mu_2}b_4 - k_{\mu_2}d_4 \\ -h_{\mu_4}c_4 - k_{\mu_4}a_4 & 1 - h_{\mu_4}d_4 - k_{\mu_4}b_4 \end{pmatrix} := \kappa_4 > 0.$$

Theorem 3.1 *Suppose that* (H1)–(H2) *hold. Then the fractional difference system* (1.1) *has at least one positive solution.*

Proof Define a set

$$S_1 = \{(x,y) \in P \times P : (x,y) = A(x,y) + \lambda(\varphi_0, \varphi_0) \text{ for some } \lambda \geq 0\},$$

where $\varphi_0 \in P$ is a fixed element. Then we will claim that S_1 is a bounded set in $P \times P$. In fact, if $(x,y) \in S_1$, we have $x(t) = A_1(x,y)(t) + \lambda\varphi_0(t)$, $y(t) = A_2(x,y)(t) + \lambda\varphi_0(t)$ for $t \in [\nu - 1, T + \nu - 2]_{\mathbf{N}_{\nu-1}}$. Together with (H1), we obtain

$$\begin{aligned}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &\geq \begin{pmatrix} A_1(x,y)(t) \\ A_2(x,y)(t) \end{pmatrix} \\ &\geq \begin{pmatrix} \sum_{s=0}^{T-1} H_1(t,s)(a_1x(s+\nu-1) + b_1y(s+\nu-1) - l_1) + \sum_{s=0}^{T-1} K_1(t,s)(c_1x(s+\nu-1) + d_1y(s+\nu-1) - l_2) \\ \sum_{s=0}^{T-1} H_2(t,s)(c_1x(s+\nu-1) + d_1y(s+\nu-1) - l_2) + \sum_{s=0}^{T-1} K_2(t,s)(a_1x(s+\nu-1) + b_1y(s+\nu-1) - l_1) \end{pmatrix}\end{aligned}$$

for $t \in [\nu - 1, T + \nu - 2]_{\mathbf{N}_{\nu-1}}$. Multiplying both sides of the above inequality by $\rho^*(t)$ and summing from $\nu - 1$ to $T + \nu - 2$, together with (2.8)–(2.11), we obtain

$$\begin{aligned}
&\begin{pmatrix} \sum_{t=\nu-1}^{T+\nu-2} x(t)\rho^*(t) \\ \sum_{t=\nu-1}^{T+\nu-2} y(t)\rho^*(t) \end{pmatrix} \\
&\geq \begin{pmatrix} \sum_{t=\nu-1}^{T+\nu-2} \rho^*(t)[\sum_{s=0}^{T-1} H_1(t,s)(a_1x(s+\nu-1) + b_1y(s+\nu-1) - l_1) + \sum_{s=0}^{T-1} K_1(t,s)(c_1x(s+\nu-1) + d_1y(s+\nu-1) - l_2)] \\ \sum_{t=\nu-1}^{T+\nu-2} \rho^*(t)[\sum_{s=0}^{T-1} H_2(t,s)(c_1x(s+\nu-1) + d_1y(s+\nu-1) - l_2) + \sum_{s=0}^{T-1} K_2(t,s)(a_1x(s+\nu-1) + b_1y(s+\nu-1) - l_1)] \end{pmatrix} \\
&\geq \begin{pmatrix} h_{\mu_1} \sum_{s=0}^{T-1} \rho(s)(a_1x(s+\nu-1) + b_1y(s+\nu-1)) + k_{\mu_1} \sum_{s=0}^{T-1} \rho(s)(c_1x(s+\nu-1) + d_1y(s+\nu-1)) - (h_{\mu_2}l_1 + k_{\mu_2}l_2)\sum_{s=0}^{T-1} \rho(s) \\ h_{\mu_3} \sum_{s=0}^{T-1} \rho(s)(c_1x(s+\nu-1) + d_1y(s+\nu-1)) + k_{\mu_3} \sum_{s=0}^{T-1} \rho(s)(a_1x(s+\nu-1) + b_1y(s+\nu-1)) - (k_{\mu_4}l_1 + h_{\mu_4}l_2)\sum_{s=0}^{T-1} \rho(s) \end{pmatrix} \\
&= \begin{pmatrix} \sum_{s=0}^{T-1} \rho^*(s+\nu-1)[h_{\mu_1}(a_1x(s+\nu-1) + b_1y(s+\nu-1)) + k_{\mu_1}(c_1x(s+\nu-1) + d_1y(s+\nu-1))] - (h_{\mu_2}l_1 + k_{\mu_2}l_2)\sum_{s=0}^{T-1} \rho(s) \\ \sum_{s=0}^{T-1} \rho^*(s+\nu-1)[h_{\mu_3}(c_1x(s+\nu-1) + d_1y(s+\nu-1)) + k_{\mu_3}(a_1x(s+\nu-1) + b_1y(s+\nu-1))] - (k_{\mu_4}l_1 + h_{\mu_4}l_2)\sum_{s=0}^{T-1} \rho(s) \end{pmatrix} \\
&= \begin{pmatrix} h_{\mu_1} \sum_{t=\nu-1}^{T+\nu-2} \rho^*(t)(a_1x(t) + b_1y(t)) + k_{\mu_1} \sum_{t=\nu-1}^{T+\nu-2} \rho^*(t)(c_1x(t) + d_1y(t)) - (h_{\mu_2}l_1 + k_{\mu_2}l_2)\sum_{s=0}^{T-1} \rho(s) \\ h_{\mu_3} \sum_{t=\nu-1}^{T+\nu-2} \rho^*(t)(c_1x(t) + d_1y(t)) + k_{\mu_3} \sum_{t=\nu-1}^{T+\nu-2} \rho^*(t)(a_1x(t) + b_1y(t)) - (k_{\mu_4}l_1 + h_{\mu_4}l_2)\sum_{s=0}^{T-1} \rho(s) \end{pmatrix}.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\begin{pmatrix} h_{\mu_1}b_1 + k_{\mu_1}d_1 & h_{\mu_1}a_1 + k_{\mu_1}c_1 - 1 \\ h_{\mu_3}d_1 + k_{\mu_3}b_1 - 1 & h_{\mu_3}c_1 + k_{\mu_3}a_1 \end{pmatrix} \begin{pmatrix} \sum_{t=\nu-1}^{T+\nu-2} y(t)\rho^*(t) \\ \sum_{t=\nu-1}^{T+\nu-2} x(t)\rho^*(t) \end{pmatrix} \\
&\quad \leq \begin{pmatrix} (h_{\mu_2}l_1 + k_{\mu_2}l_2)\sum_{s=0}^{T-1} \rho(s) \\ (k_{\mu_4}l_1 + h_{\mu_4}l_2)\sum_{s=0}^{T-1} \rho(s) \end{pmatrix}.
\end{aligned}$$

Solving this matrix inequality, we have

$$\begin{aligned}
&\begin{pmatrix} \sum_{t=\nu-1}^{T+\nu-2} y(t)\rho^*(t) \\ \sum_{t=\nu-1}^{T+\nu-2} x(t)\rho^*(t) \end{pmatrix} \\
&\quad \leq \kappa_1^{-1} \begin{pmatrix} h_{\mu_3}c_1 + k_{\mu_3}a_1 & 1 - h_{\mu_1}a_1 - k_{\mu_1}c_1 \\ 1 - h_{\mu_3}d_1 - k_{\mu_3}b_1 & h_{\mu_1}b_1 + k_{\mu_1}d_1 \end{pmatrix} \begin{pmatrix} (h_{\mu_2}l_1 + k_{\mu_2}l_2)\sum_{s=0}^{T-1} \rho(s) \\ (k_{\mu_4}l_1 + h_{\mu_4}l_2)\sum_{s=0}^{T-1} \rho(s) \end{pmatrix}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\sum_{t=\nu-1}^{T+\nu-2} y(t)\rho^*(t) \\
&\quad \leq \kappa_1^{-1}\big[(h_{\mu_3}c_1 + k_{\mu_3}a_1)(h_{\mu_2}l_1 + k_{\mu_2}l_2) + (1 - h_{\mu_1}a_1 - k_{\mu_1}c_1)(k_{\mu_4}l_1 + h_{\mu_4}l_2)\big]\sum_{s=0}^{T-1} \rho(s), \\
&\sum_{t=\nu-1}^{T+\nu-2} x(t)\rho^*(t) \\
&\quad \leq \kappa_1^{-1}\big[(1 - h_{\mu_3}d_1 - k_{\mu_3}b_1)(h_{\mu_2}l_1 + k_{\mu_2}l_2) + (h_{\mu_1}b_1 + k_{\mu_1}d_1)(k_{\mu_4}l_1 + h_{\mu_4}l_2)\big]\sum_{s=0}^{T-1} \rho(s).
\end{aligned}$$

On the other hand, there exist $t_1, t_2 \in [\nu - 1, T + \nu - 2]_{\mathbf{N}_{\nu-1}}$ such that

$$\begin{aligned}
&x(t_1)\rho^*(t_1) = \|x\|\rho^*(t_1) \leq \sum_{t=\nu-1}^{T+\nu-2} x(t)\rho^*(t), \\
&y(t_2)\rho^*(t_2) = \|y\|\rho^*(t_2) \leq \sum_{t=\nu-1}^{T+\nu-2} y(t)\rho^*(t).
\end{aligned} \tag{3.1}$$

Consequently, we have

$$\begin{aligned}\|x\| &\le \frac{1}{\kappa_1\rho^*(t_1)}\big[(1-h_{\mu_3}d_1-k_{\mu_3}b_1)(h_{\mu_2}l_1+k_{\mu_2}l_2)\\ &\quad+(h_{\mu_1}b_1+k_{\mu_1}d_1)(k_{\mu_4}l_1+h_{\mu_4}l_2)\big]\sum_{s=0}^{T-1}\rho(s),\\ \|y\| &\le \frac{1}{\kappa_1\rho^*(t_2)}\big[(h_{\mu_3}c_1+k_{\mu_3}a_1)(h_{\mu_2}l_1+k_{\mu_2}l_2)\\ &\quad+(1-h_{\mu_1}a_1-k_{\mu_1}c_1)(k_{\mu_4}l_1+h_{\mu_4}l_2)\big]\sum_{s=0}^{T-1}\rho(s).\end{aligned}$$

This proves that S_1 is bounded in $P\times P$. Then we can choose a positive number $R_1>r_1$, $R_1>\frac{1}{\kappa_1\rho^*(t_1)}[(1-h_{\mu_3}d_1-k_{\mu_3}b_1)(h_{\mu_2}l_1+k_{\mu_2}l_2)+(h_{\mu_1}b_1+k_{\mu_1}d_1)(k_{\mu_4}l_1+h_{\mu_4}l_2)]\sum_{s=0}^{T-1}\rho(s)$, and $R_1>\frac{1}{\kappa_1\rho^*(t_2)}[(h_{\mu_3}c_1+k_{\mu_3}a_1)(h_{\mu_2}l_1+k_{\mu_2}l_2)+(1-h_{\mu_1}a_1-k_{\mu_1}c_1)(k_{\mu_4}l_1+h_{\mu_4}l_2)]\sum_{s=0}^{T-1}\rho(s)$ such that

$$(x,y)\neq A(x,y)+\lambda(\varphi_0,\varphi_0),\quad \text{for } (x,y)\in\partial B_{R_1}\cap(P\times P),\lambda\ge 0. \tag{3.2}$$

As a result, Lemma 2.8 implies

$$i\big(A,B_{R_1}\cap(P\times P),P\times P\big)=0. \tag{3.3}$$

In what follows, we prove that

$$(x,y)\neq\lambda A(x,y),\quad \text{for } (x,y)\in\partial B_{r_1}\cap(P\times P),\lambda\in[0,1], \tag{3.4}$$

where r_1 is defined by (H2). Argument by contrary, there exist $(x,y)\in\partial B_{r_1}\cap(P\times P)$, $\lambda_0\in[0,1]$ such that $(x,y)=\lambda_0 A(x,y)$, and thus from (H2) we obtain

$$\begin{aligned}\begin{pmatrix}x(t)\\y(t)\end{pmatrix}&\le\begin{pmatrix}A_1(x,y)(t)\\A_2(x,y)(t)\end{pmatrix}\\ &\le\begin{pmatrix}\sum_{s=0}^{T-1}H_1(t,s)(a_2x(s+\nu-1)+b_2y(s+\nu-1))+\sum_{s=0}^{T-1}K_1(t,s)(c_2x(s+\nu-1)+d_2y(s+\nu-1))\\ \sum_{s=0}^{T-1}H_2(t,s)(c_2x(s+\nu-1)+d_2y(s+\nu-1))+\sum_{s=0}^{T-1}K_2(t,s)(a_2x(s+\nu-1)+b_2y(s+\nu-1))\end{pmatrix}.\end{aligned}$$

Multiplying both sides of the above inequality by $\rho^*(t)$ and summing from $\nu-1$ to $T+\nu-2$, together with (2.8)–(2.11), we obtain

$$\begin{aligned}&\begin{pmatrix}\sum_{t=\nu-1}^{T+\nu-2}x(t)\rho^*(t)\\ \sum_{t=\nu-1}^{T+\nu-2}y(t)\rho^*(t)\end{pmatrix}\\ &\le\begin{pmatrix}\sum_{t=\nu-1}^{T+\nu-2}\rho^*(t)\sum_{s=0}^{T-1}H_1(t,s)(a_2x(s+\nu-1)+b_2y(s+\nu-1))+\sum_{t=\nu-1}^{T+\nu-2}\rho^*(t)\sum_{s=0}^{T-1}K_1(t,s)(c_2x(s+\nu-1)+d_2y(s+\nu-1))\\ \sum_{t=\nu-1}^{T+\nu-2}\rho^*(t)\sum_{s=0}^{T-1}H_2(t,s)(c_2x(s+\nu-1)+d_2y(s+\nu-1))+\sum_{t=\nu-1}^{T+\nu-2}\rho^*(t)\sum_{s=0}^{T-1}K_2(t,s)(a_2x(s+\nu-1)+b_2y(s+\nu-1))\end{pmatrix}\\ &\le\begin{pmatrix}h_{\mu_2}\sum_{s=0}^{T-1}\rho(s)(a_2x(s+\nu-1)+b_2y(s+\nu-1))+k_{\mu_2}\sum_{s=0}^{T-1}\rho(s)(c_2x(s+\nu-1)+d_2y(s+\nu-1))\\ h_{\mu_4}\sum_{s=0}^{T-1}\rho(s)(c_2x(s+\nu-1)+d_2y(s+\nu-1))+k_{\mu_4}\sum_{s=0}^{T-1}\rho(s)(a_2x(s+\nu-1)+b_2y(s+\nu-1))\end{pmatrix}\\ &=\begin{pmatrix}h_{\mu_2}\sum_{s=0}^{T-1}\rho^*(s+\nu-1)(a_2x(s+\nu-1)+b_2y(s+\nu-1))+k_{\mu_2}\sum_{s=0}^{T-1}\rho^*(s+\nu-1)(c_2x(s+\nu-1)+d_2y(s+\nu-1))\\ h_{\mu_4}\sum_{s=0}^{T-1}\rho^*(s+\nu-1)(c_2x(s+\nu-1)+d_2y(s+\nu-1))+k_{\mu_4}\sum_{s=0}^{T-1}\rho^*(s+\nu-1)(a_2x(s+\nu-1)+b_2y(s+\nu-1))\end{pmatrix}\\ &=\begin{pmatrix}h_{\mu_2}\sum_{t=\nu-1}^{T+\nu-2}\rho^*(t)(a_2x(t)+b_2y(t))+k_{\mu_2}\sum_{t=\nu-1}^{T+\nu-2}\rho^*(t)(c_2x(t)+d_2y(t))\\ h_{\mu_4}\sum_{t=\nu-1}^{T+\nu-2}\rho^*(t)(c_2x(t)+d_2y(t))+k_{\mu_4}\sum_{t=\nu-1}^{T+\nu-2}\rho^*(t)(a_2x(t)+b_2y(t))\end{pmatrix}.\end{aligned}$$

Solving this matrix inequality, we have

$$\begin{pmatrix} 1-h_{\mu_2}a_2-k_{\mu_2}c_2 & -h_{\mu_2}b_2-k_{\mu_2}d_2 \\ -h_{\mu_4}c_2-k_{\mu_4}a_2 & 1-h_{\mu_4}d_2-k_{\mu_4}b_2 \end{pmatrix} \begin{pmatrix} \sum_{t=\nu-1}^{T+\nu-2} x(t)\rho^*(t) \\ \sum_{t=\nu-1}^{T+\nu-2} y(t)\rho^*(t) \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Consequently, we have

$$\begin{pmatrix} \sum_{t=\nu-1}^{T+\nu-2} x(t)\rho^*(t) \\ \sum_{t=\nu-1}^{T+\nu-2} y(t)\rho^*(t) \end{pmatrix} \leq \kappa_2^{-1} \begin{pmatrix} 1-h_{\mu_4}d_2-k_{\mu_4}b_2 & h_{\mu_2}b_2+k_{\mu_2}d_2 \\ h_{\mu_4}c_2+k_{\mu_4}a_2 & 1-h_{\mu_2}a_2-k_{\mu_2}c_2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Note that $\rho^*(t) \not\equiv 0$ for $t \in [\nu-1, T+\nu-2]_{\mathbf{N}_{\nu-1}}$, whence $x(t) = y(t) \equiv 0$ for $t \in [\nu-1, T+\nu-2]_{\mathbf{N}_{\nu-1}}$, and this contradicts $(x,y) \in \partial B_{r_1} \cap (P \times P)$ with $r_1 > 0$. As a result, (3.4) holds, and from Lemma 2.9 we have

$$i\bigl(A, B_{r_1} \cap (P \times P), P \times P\bigr) = 1. \tag{3.5}$$

Up to now, (3.3) and (3.5) enabled us to obtain $i(A, (B_{R_1} \backslash \overline{B}_{r_1}) \cap (P \times P), P \times P) = -1 \neq 0$. Hence the operator A has at least one fixed point on $(B_{R_1} \backslash \overline{B}_{r_1}) \cap (P \times P)$, and therefore (1.1) has at least one positive solution. This completes the proof. □

Theorem 3.2 *Suppose that* (H3)–(H4) *hold. Then the fractional difference system* (1.1) *has at least one positive solution.*

Proof We first prove that

$$(x,y) \neq A(x,y) + \lambda(\varphi_1, \varphi_1), \quad \text{for } (x,y) \in \partial B_{r_2} \cap (P \times P), \lambda \geq 0, \tag{3.6}$$

where $\varphi_1 \in P$ is a given element, and r_2 is defined by (H3). Suppose the contrary. Then there exist $(x,y) \in \partial B_{r_2} \cap (P \times P)$ and $\lambda_0 \geq 0$ such that

$$(x,y) = A(x,y) + \lambda_0(\varphi_1, \varphi_1).$$

Associated with condition (H3), this means that

$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &\geq \begin{pmatrix} A_1(x,y)(t) \\ A_2(x,y)(t) \end{pmatrix} \\ &\geq \begin{pmatrix} \sum_{s=0}^{T-1} H_1(t,s)(a_3x(s+\nu-1)+b_3y(s+\nu-1)) + \sum_{s=0}^{T-1} K_1(t,s)(c_3x(s+\nu-1)+d_3y(s+\nu-1)) \\ \sum_{s=0}^{T-1} H_2(t,s)(c_3x(s+\nu-1)+d_3y(s+\nu-1)) + \sum_{s=0}^{T-1} K_2(t,s)(a_3x(s+\nu-1)+b_3y(s+\nu-1)) \end{pmatrix}. \end{aligned}$$

Multiplying both sides of the above inequality by $\rho^*(t)$ and summing from $\nu-1$ to $T+\nu-2$, together with (2.8)–(2.11), we obtain

$$\begin{aligned}
&\begin{pmatrix} \sum_{t=\nu-1}^{T+\nu-2} x(t)\rho^*(t) \\ \sum_{t=\nu-1}^{T+\nu-2} y(t)\rho^*(t) \end{pmatrix} \\
&\geq \begin{pmatrix} \sum_{t=\nu-1}^{T+\nu-2} \rho^*(t)[\sum_{s=0}^{T-1} H_1(t,s)(a_3x(s+\nu-1)+b_3y(s+\nu-1))+\sum_{s=0}^{T-1} K_1(t,s)(c_3x(s+\nu-1)+d_3y(s+\nu-1))] \\ \sum_{t=\nu-1}^{T+\nu-2} \rho^*(t)[\sum_{s=0}^{T-1} H_2(t,s)(c_3x(s+\nu-1)+d_3y(s+\nu-1))+\sum_{s=0}^{T-1} K_2(t,s)(a_3x(s+\nu-1)+b_3y(s+\nu-1))] \end{pmatrix} \\
&\geq \begin{pmatrix} h_{\mu_1}\sum_{s=0}^{T-1} \rho(s)(a_3x(s+\nu-1)+b_3y(s+\nu-1))+k_{\mu_1}\sum_{s=0}^{T-1} \rho(s)(c_3x(s+\nu-1)+d_3y(s+\nu-1)) \\ h_{\mu_3}\sum_{s=0}^{T-1} \rho(s)(c_3x(s+\nu-1)+d_3y(s+\nu-1))+k_{\mu_3}\sum_{s=0}^{T-1} \rho(s)(a_3x(s+\nu-1)+b_3y(s+\nu-1)) \end{pmatrix} \\
&= \begin{pmatrix} h_{\mu_1}\sum_{s=0}^{T-1} \rho^*(s+\nu-1)(a_3x(s+\nu-1)+b_3y(s+\nu-1))+k_{\mu_1}\sum_{s=0}^{T-1} \rho^*(s+\nu-1)(c_3x(s+\nu-1)+d_3y(s+\nu-1)) \\ h_{\mu_3}\sum_{s=0}^{T-1} \rho^*(s+\nu-1)(c_3x(s+\nu-1)+d_3y(s+\nu-1))+k_{\mu_3}\sum_{s=0}^{T-1} \rho^*(s+\nu-1)(a_3x(s+\nu-1)+b_3y(s+\nu-1)) \end{pmatrix} \\
&= \begin{pmatrix} h_{\mu_1}\sum_{t=\nu-1}^{T+\nu-2} \rho^*(t)(a_3x(t)+b_3y(t))+k_{\mu_1}\sum_{t=\nu-1}^{T+\nu-2} \rho^*(t)(c_3x(t)+d_3y(t)) \\ h_{\mu_3}\sum_{t=\nu-1}^{T+\nu-2} \rho^*(t)(c_3x(t)+d_3y(t))+k_{\mu_3}\sum_{t=\nu-1}^{T+\nu-2} \rho^*(t)(a_3x(t)+b_3y(t)) \end{pmatrix}.
\end{aligned}$$

This leads us to obtain

$$\begin{pmatrix} h_{\mu_1}b_3+k_{\mu_1}d_3 & h_{\mu_1}a_3+k_{\mu_1}c_3-1 \\ h_{\mu_3}d_3+k_{\mu_3}b_3-1 & h_{\mu_3}c_3+k_{\mu_3}a_3 \end{pmatrix} \begin{pmatrix} \sum_{t=\nu-1}^{T+\nu-2} y(t)\rho^*(t) \\ \sum_{t=\nu-1}^{T+\nu-2} x(t)\rho^*(t) \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Solving this matrix inequality, we have

$$\begin{pmatrix} \sum_{t=\nu-1}^{T+\nu-2} y(t)\rho^*(t) \\ \sum_{t=\nu-1}^{T+\nu-2} x(t)\rho^*(t) \end{pmatrix} \leq \kappa_3^{-1} \begin{pmatrix} h_{\mu_3}c_3+k_{\mu_3}a_3 & 1-h_{\mu_1}a_3-k_{\mu_1}c_3 \\ 1-h_{\mu_3}d_3-k_{\mu_3}b_3 & h_{\mu_1}b_3+k_{\mu_1}d_3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence, we find

$$\begin{pmatrix} \sum_{t=\nu-1}^{T+\nu-2} y(t)\rho^*(t) \\ \sum_{t=\nu-1}^{T+\nu-2} x(t)\rho^*(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Note that $\rho^*(t) \not\equiv 0$ for $t \in [\nu-1, T+\nu-2]_{\mathbf{N}_{\nu-1}}$, whence $x(t) = y(t) \equiv 0$ for $t \in [\nu-1, T+\nu-2]_{\mathbf{N}_{\nu-1}}$, and this contradicts $(x,y) \in \partial B_{r_2} \cap (P\times P)$ with $r_2 > 0$. Consequently, (3.6) is satisfied, and Lemma 2.8 implies that

$$i\big(A, B_{r_2} \cap (P\times P), P\times P\big) = 0. \tag{3.7}$$

On the other hand, we claim that the set

$$S_2 = \big\{(x,y) \in P\times P : (x,y) = \lambda A(x,y) \text{ for some } \lambda \in [0,1]\big\}$$

is bounded in $P\times P$. If there exists $(x,y) \in S_2$, then from (H4) we have

$$\begin{aligned}
&\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \\
&\leq \begin{pmatrix} A_1(x,y)(t) \\ A_2(x,y)(t) \end{pmatrix} \\
&\leq \begin{pmatrix} \sum_{s=0}^{T-1} H_1(t,s)(a_4x(s+\nu-1)+b_4y(s+\nu-1)+l_3)+\sum_{s=0}^{T-1} K_1(t,s)(c_4x(s+\nu-1)+d_4y(s+\nu-1)+l_4) \\ \sum_{s=0}^{T-1} H_2(t,s)(c_4x(s+\nu-1)+d_4y(s+\nu-1)+l_4)+\sum_{s=0}^{T-1} K_2(t,s)(a_4x(s+\nu-1)+b_4y(s+\nu-1)+l_3) \end{pmatrix}
\end{aligned}$$

for $t \in [\nu-1, T+\nu-2]_{\mathbb{N}_{\nu-1}}$. Multiplying both sides of the above inequality by $\rho^*(t)$ and summing from $\nu-1$ to $T+\nu-2$, together with (2.8)–(2.11), we obtain

$$\begin{aligned}
&\begin{pmatrix} \sum_{t=\nu-1}^{T+\nu-2} x(t)\rho^*(t) \\ \sum_{t=\nu-1}^{T+\nu-2} y(t)\rho^*(t) \end{pmatrix} \\
&\leq \begin{pmatrix} \sum_{t=\nu-1}^{T+\nu-2} \rho^*(t)[\sum_{s=0}^{T-1} H_1(t,s)(a_4x(s+\nu-1)+b_4y(s+\nu-1)+l_3)+\sum_{s=0}^{T-1} K_1(t,s)(c_4x(s+\nu-1)+d_4y(s+\nu-1)+l_4)] \\ \sum_{t=\nu-1}^{T+\nu-2} \rho^*(t)[\sum_{s=0}^{T-1} H_2(t,s)(c_4x(s+\nu-1)+d_4y(s+\nu-1)+l_4)+\sum_{s=0}^{T-1} K_2(t,s)(a_4x(s+\nu-1)+b_4y(s+\nu-1)+l_3)] \end{pmatrix} \\
&\leq \begin{pmatrix} h_{\mu_2} \sum_{s=0}^{T-1} \rho(s)(a_4x(s+\nu-1)+b_4y(s+\nu-1))+k_{\mu_2} \sum_{s=0}^{T-1} \rho(s)(c_4x(s+\nu-1)+d_4y(s+\nu-1))+(h_{\mu_2}l_3+k_{\mu_2}l_4)\sum_{s=0}^{T-1} \rho(s) \\ h_{\mu_4} \sum_{s=0}^{T-1} \rho(s)(c_4x(s+\nu-1)+d_4y(s+\nu-1))+k_{\mu_4} \sum_{s=0}^{T-1} \rho(s)(a_4x(s+\nu-1)+b_4y(s+\nu-1))+(k_{\mu_4}l_3+h_{\mu_4}l_4)\sum_{s=0}^{T-1} \rho(s) \end{pmatrix} \\
&= \begin{pmatrix} \sum_{s=0}^{T-1} \rho^*(s+\nu-1)[h_{\mu_2}(a_4x(s+\nu-1)+b_4y(s+\nu-1))+k_{\mu_2}(c_4x(s+\nu-1)+d_4y(s+\nu-1))]+(h_{\mu_2}l_3+k_{\mu_2}l_4)\sum_{s=0}^{T-1} \rho(s) \\ \sum_{s=0}^{T-1} \rho^*(s+\nu-1)[h_{\mu_4}(c_4x(s+\nu-1)+d_4y(s+\nu-1))+k_{\mu_4}(a_4x(s+\nu-1)+b_4y(s+\nu-1))]+(k_{\mu_4}l_3+h_{\mu_4}l_4)\sum_{s=0}^{T-1} \rho(s) \end{pmatrix} \\
&= \begin{pmatrix} h_{\mu_2} \sum_{t=\nu-1}^{T+\nu-2} \rho^*(t)(a_4x(t)+b_4y(t))+k_{\mu_2} \sum_{t=\nu-1}^{T+\nu-2} \rho^*(t)(c_4x(t)+d_4y(t))+(h_{\mu_2}l_3+k_{\mu_2}l_4)\sum_{s=0}^{T-1} \rho(s) \\ h_{\mu_4} \sum_{t=\nu-1}^{T+\nu-2} \rho^*(t)(c_4x(t)+d_4y(t))+k_{\mu_4} \sum_{t=\nu-1}^{T+\nu-2} \rho^*(t)(a_4x(t)+b_4y(t))+(k_{\mu_4}l_3+h_{\mu_4}l_4)\sum_{s=0}^{T-1} \rho(s) \end{pmatrix}.
\end{aligned}$$

Solving this matrix inequality, we have

$$\begin{aligned}
&\begin{pmatrix} 1-h_{\mu_2}a_4-k_{\mu_2}c_4 & -h_{\mu_2}b_4-k_{\mu_2}d_4 \\ -h_{\mu_4}c_4-k_{\mu_4}a_4 & 1-h_{\mu_4}d_4-k_{\mu_4}b_4 \end{pmatrix} \begin{pmatrix} \sum_{t=\nu-1}^{T+\nu-2} x(t)\rho^*(t) \\ \sum_{t=\nu-1}^{T+\nu-2} y(t)\rho^*(t) \end{pmatrix} \\
&\quad \leq \begin{pmatrix} (h_{\mu_2}l_3+k_{\mu_2}l_4)\sum_{s=0}^{T-1} \rho(s) \\ (k_{\mu_4}l_3+h_{\mu_4}l_4)\sum_{s=0}^{T-1} \rho(s) \end{pmatrix}.
\end{aligned}$$

This indicates that

$$\begin{aligned}
&\begin{pmatrix} \sum_{t=\nu-1}^{T+\nu-2} x(t)\rho^*(t) \\ \sum_{t=\nu-1}^{T+\nu-2} y(t)\rho^*(t) \end{pmatrix} \\
&\quad \leq \kappa_4^{-1} \begin{pmatrix} 1-h_{\mu_4}d_4-k_{\mu_4}b_4 & h_{\mu_2}b_4+k_{\mu_2}d_4 \\ h_{\mu_4}c_4+k_{\mu_4}a_4 & 1-h_{\mu_2}a_4-k_{\mu_2}c_4 \end{pmatrix} \begin{pmatrix} (h_{\mu_2}l_3+k_{\mu_2}l_4)\sum_{s=0}^{T-1} \rho(s) \\ (k_{\mu_4}l_3+h_{\mu_4}l_4)\sum_{s=0}^{T-1} \rho(s) \end{pmatrix}.
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
\sum_{t=\nu-1}^{T+\nu-2} x(t)\rho^*(t) &\leq \kappa_4^{-1}\bigl[(1-h_{\mu_4}d_4-k_{\mu_4}b_4)(h_{\mu_2}l_3+k_{\mu_2}l_4) \\
&\qquad + (h_{\mu_2}b_4+k_{\mu_2}d_4)(k_{\mu_4}l_3+h_{\mu_4}l_4)\bigr]\sum_{s=0}^{T-1} \rho(s), \\
\sum_{t=\nu-1}^{T+\nu-2} y(t)\rho^*(t) &\leq \kappa_4^{-1}\bigl[(h_{\mu_4}c_4+k_{\mu_4}a_4)(h_{\mu_2}l_3+k_{\mu_2}l_4) \\
&\qquad + (1-h_{\mu_2}a_4-k_{\mu_2}c_4)(k_{\mu_4}l_3+h_{\mu_4}l_4)\bigr]\sum_{s=0}^{T-1} \rho(s).
\end{aligned}$$

Similarly, using (3.1) we have

$$\begin{aligned}\|x\| &\le \frac{1}{\kappa_4\rho^*(t_1)}\bigl[(1-h_{\mu_4}d_4-k_{\mu_4}b_4)(h_{\mu_2}l_3+k_{\mu_2}l_4)\\ &\quad+(h_{\mu_2}b_4+k_{\mu_2}d_4)(k_{\mu_4}l_3+h_{\mu_4}l_4)\bigr]\sum_{s=0}^{T-1}\rho(s),\\ \|y\| &\le \frac{1}{\kappa_4\rho^*(t_2)}\bigl[(h_{\mu_4}c_4+k_{\mu_4}a_4)(h_{\mu_2}l_3+k_{\mu_2}l_4)\\ &\quad+(1-h_{\mu_2}a_4-k_{\mu_2}c_4)(k_{\mu_4}l_3+h_{\mu_4}l_4)\bigr]\sum_{s=0}^{T-1}\rho(s).\end{aligned}$$

Then we can choose a positive number $R_2 > r_2$, $R_2 > \frac{1}{\kappa_4\rho^*(t_1)}[(1-h_{\mu_4}d_4-k_{\mu_4}b_4)(h_{\mu_2}l_3+k_{\mu_2}l_4)+(h_{\mu_2}b_4+k_{\mu_2}d_4)(k_{\mu_4}l_3+h_{\mu_4}l_4)]\sum_{s=0}^{T-1}\rho(s)$, and $R_2 > \frac{1}{\kappa_4\rho^*(t_2)}[(h_{\mu_4}c_4+k_{\mu_4}a_4)(h_{\mu_2}l_3+k_{\mu_2}l_4)+(1-h_{\mu_2}a_4-k_{\mu_2}c_4)(k_{\mu_4}l_3+h_{\mu_4}l_4)]\sum_{s=0}^{T-1}\rho(s)$ such that

$$(x,y)\neq\lambda A(x,y),\quad \text{for } (x,y)\in\partial B_{R_2}\cap(P\times P),\lambda\in[0,1]. \tag{3.8}$$

As a result, Lemma 2.9 implies

$$i\bigl(A,B_{R_2}\cap(P\times P),P\times P\bigr)=1. \tag{3.9}$$

Now, (3.7) and (3.9) enable us to obtain $i(A,(B_{R_2}\backslash\overline{B}_{r_2})\cap(P\times P),P\times P)=1\neq 0$. Hence the operator A has at least one fixed point on $(B_{R_2}\backslash\overline{B}_{r_2})\cap(P\times P)$, and therefore (1.1) has at least one positive solution. This completes the proof. □

Example 3.3 Consider equation (1.1) with $\nu=\frac{5}{2}$, $T=4$, $\alpha=\frac{1}{3}$, $\xi=1$, $\eta=2$, $a=\frac{2}{3}$, $b=\frac{4}{3}$. Then we need to calculate the following values: $L=(\frac{\Gamma(T+\nu)}{T!})^2-\frac{ab\Gamma(\xi+\nu+1)\Gamma(\eta+\nu+1)}{(\xi+1)!(\eta+1)!}=(\frac{\Gamma(\frac{13}{2})}{24})^2-\frac{\frac{8}{9}\Gamma(\frac{9}{2})\Gamma(\frac{11}{2})}{12}\approx 98.9>0$, $L_1=\frac{\nu-1}{T(T+\nu-1)^{\underline{\nu-1}}(T+\nu-2)}\approx 0.007$, $\Gamma(\nu)\approx 1.33$, $(\xi+\nu)^{\underline{\nu-1}}\approx 5.82$, $(\eta+\nu)^{\underline{\nu-1}}=(T+\nu-2)^{\underline{\nu-1}}\approx 8.72$, $(T+\nu-1)^{\underline{\nu-1}}\approx 12$, $\sum_{t=0}^{3}\rho(t)=\sum_{t=0}^{3}(T+\nu-t-2)^{\underline{\nu-1}}\approx 19.19$, $\sum_{t=0}^{3}(t+\nu-1)^{\underline{\nu-1}}\rho(t)=\sum_{t=0}^{3}(t+\nu-1)^{\underline{\nu-1}}(T+\nu-t-2)^{\underline{\nu-1}}\approx 61.84$. Then we have

$$\begin{aligned}&\begin{pmatrix} h_{\mu_1} & h_{\mu_2}\\ k_{\mu_1} & k_{\mu_2}\\ h_{\mu_3} & h_{\mu_4}\\ k_{\mu_3} & k_{\mu_4}\end{pmatrix}\\ &=\begin{pmatrix} \sum_{t=0}^{T-1}\frac{abL_1(\xi+\nu)^{\underline{\nu-1}}(\eta+\nu)^{\underline{\nu-1}}(t+\nu-1)^{\underline{\nu-1}}\rho(t)}{L\Gamma(\nu)} & \sum_{t=0}^{T-1}\frac{[L+ab(\xi+\nu)^{\underline{\nu-1}}(T+\nu-2)^{\underline{\nu-1}}]\rho(t)}{L\Gamma(\nu)}\\ \sum_{t=0}^{T-1}\frac{aL_1(\xi+\nu)^{\underline{\nu-1}}(\eta+\nu)^{\underline{\nu-1}}(t+\nu-1)^{\underline{\nu-1}}\rho(t)}{L\Gamma(\nu)} & \sum_{t=0}^{T-1}\frac{a(T+\nu-1)^{\underline{\nu-1}}(T+\nu-2)^{\underline{\nu-1}}\rho(t)}{L\Gamma(\nu)}\\ \sum_{t=0}^{T-1}\frac{abL_1(\xi+\nu)^{\underline{\nu-1}}(\eta+\nu)^{\underline{\nu-1}}(t+\nu-1)^{\underline{\nu-1}}\rho(t)}{L\Gamma(\nu)} & \sum_{t=0}^{T-1}\frac{[L+ab(\eta+\nu)^{\underline{\nu-1}}(T+\nu-2)^{\underline{\nu-1}}]\rho(t)}{L\Gamma(\nu)}\\ \sum_{t=0}^{T-1}\frac{bL_1(\xi+\nu)^{\underline{\nu-1}}(\eta+\nu)^{\underline{\nu-1}}(t+\nu-1)^{\underline{\nu-1}}\rho(t)}{L\Gamma(\nu)} & \sum_{t=0}^{T-1}\frac{b(T+\nu-1)^{\underline{\nu-1}}(T+\nu-2)^{\underline{\nu-1}}\rho(t)}{L\Gamma(\nu)}\end{pmatrix}\\ &\approx\begin{pmatrix}0.15 & 21\\ 0.11 & 10.35\\ 0.15 & 24.23\\ 0.23 & 20.7\end{pmatrix}.\end{aligned}$$

Let $a_1 = a_3 = 3$, $b_1 = b_3 = 2$, $c_1 = c_3 = 4.5$, $d_1 = d_3 = 3$, $a_2 = a_4 = \frac{1}{500}$, $b_2 = b_4 = \frac{1}{420}$, $c_2 = c_4 = \frac{1}{210}$, $d_2 = d_4 = \frac{1}{500}$ and $f_1(t,x,y) = (3x+2y)^{\gamma_1}$, $f_2(t,x,y) = (4.5x+3y)^{\gamma_2}$, for $(t,x,y) \in [\nu-1, T+\nu-2]_{\mathbf{N}_{\nu-1}} \times \mathbf{R}^+ \times \mathbf{R}^+$. Then we can calculate:

$$h_{\mu_1}a_1 + k_{\mu_1}c_1 = h_{\mu_1}a_3 + k_{\mu_1}c_3 = 0.15 \times 3 + 0.11 \times 4.5 < 1,$$
$$h_{\mu_3}d_1 + k_{\mu_3}b_1 = h_{\mu_3}d_3 + k_{\mu_3}b_3 = 0.15 \times 3 + 0.23 \times 2 < 1,$$

and

$$\kappa_1 = \kappa_3 = \begin{vmatrix} h_{\mu_1}b_1 + k_{\mu_1}d_1 & h_{\mu_1}a_1 + k_{\mu_1}c_1 - 1 \\ h_{\mu_3}d_1 + k_{\mu_3}b_1 - 1 & h_{\mu_3}c_1 + k_{\mu_3}a_1 \end{vmatrix} = \begin{vmatrix} 0.63 & -0.055 \\ -0.09 & 1.365 \end{vmatrix} = 0.86 > 0.$$

Moreover,

$$h_{\mu_2}a_2 + k_{\mu_2}c_2 = h_{\mu_2}a_4 + k_{\mu_2}c_4 = 21 \times \frac{1}{500} + 10.35 \times \frac{1}{210} < 1,$$
$$h_{\mu_4}d_2 + k_{\mu_4}b_2 = h_{\mu_4}d_4 + k_{\mu_4}b_4 = 24.23 \times \frac{1}{500} + 20.7 \times \frac{1}{420} < 1,$$

and

$$\kappa_2 = \kappa_4 = \begin{vmatrix} 1 - h_{\mu_2}a_2 - k_{\mu_2}c_2 & -h_{\mu_2}b_2 - k_{\mu_2}d_2 \\ -h_{\mu_4}c_2 - k_{\mu_4}a_2 & 1 - h_{\mu_4}d_2 - k_{\mu_4}b_2 \end{vmatrix} = \begin{vmatrix} 0.91 & -0.071 \\ -0.16 & 0.90 \end{vmatrix} = 0.81 > 0.$$

Case 1. When $\gamma_i > 1$, $i = 1,2$. Then we have

$$\liminf_{a_1x+b_1y\to+\infty} \frac{f_1(t,x,y)}{a_1x+b_1y} = \liminf_{a_1x+b_1y\to+\infty} \frac{(3x+2y)^{\gamma_1}}{3x+2y} = +\infty,$$
uniformly on $t \in [\nu-1, T+\nu-2]_{\mathbf{N}_{\nu-1}}$,

and

$$\liminf_{c_1x+d_1y\to+\infty} \frac{f_2(t,x,y)}{c_1x+d_1y} = \liminf_{c_1x+d_1y\to+\infty} \frac{(4.5x+3y)^{\gamma_2}}{4.5x+3y} = +\infty,$$
uniformly on $t \in [\nu-1, T+\nu-2]_{\mathbf{N}_{\nu-1}}$.

On the other hand, we also have

$$\limsup_{a_2x+b_2y\to0^+} \frac{f_1(t,x,y)}{a_2x+b_2y} = \limsup_{a_2x+b_2y\to0^+} \frac{(3x+2y)^{\gamma_1}}{\frac{x}{500}+\frac{y}{420}} = 0,$$
uniformly on $t \in [\nu-1, T+\nu-2]_{\mathbf{N}_{\nu-1}}$,

and

$$\limsup_{c_2x+d_2y\to0^+} \frac{f_2(t,x,y)}{c_2x+d_2y} = \limsup_{c_2x+d_2y\to0^+} \frac{(4.5x+3y)^{\gamma_2}}{\frac{x}{210}+\frac{y}{500}} = 0,$$
uniformly on $t \in [\nu-1, T+\nu-2]_{\mathbf{N}_{\nu-1}}$.

As a result, (H1)–(H2) hold.

Case 2. When $\gamma_i \in (0,1)$, $i = 1,2$. Then we have

$$\liminf_{a_3x+b_3y\to 0^+} \frac{f_1(t,x,y)}{a_3x+b_3y} = \liminf_{a_3x+b_3y\to 0^+} \frac{(3x+2y)^{\gamma_1}}{3x+2y} = +\infty,$$
$$\text{uniformly on } t \in [\nu-1, T+\nu-2]_{\mathbf{N}_{\nu-1}},$$

and

$$\liminf_{c_3x+d_3y\to 0^+} \frac{f_2(t,x,y)}{c_3x+d_3y} = \liminf_{c_3x+d_3y\to 0^+} \frac{(4.5x+3y)^{\gamma_2}}{4.5x+3y} = +\infty,$$
$$\text{uniformly on } t \in [\nu-1, T+\nu-2]_{\mathbf{N}_{\nu-1}}.$$

On the other hand, we also have

$$\limsup_{a_4x+b_4y\to +\infty} \frac{f_1(t,x,y)}{a_4x+b_4y} = \limsup_{a_4x+b_4y\to +\infty} \frac{(3x+2y)^{\gamma_1}}{\frac{x}{500}+\frac{y}{420}} = 0,$$
$$\text{uniformly on } t \in [\nu-1, T+\nu-2]_{\mathbf{N}_{\nu-1}},$$

and

$$\limsup_{c_4x+d_4y\to +\infty} \frac{f_2(t,x,y)}{c_4x+d_4y} = \limsup_{c_4x+d_4y\to +\infty} \frac{(4.5x+3y)^{\gamma_2}}{\frac{x}{210}+\frac{y}{500}} = 0,$$
$$\text{uniformly on } t \in [\nu-1, T+\nu-2]_{\mathbf{N}_{\nu-1}}.$$

As a result, (H3)–(H4) hold.

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