Advances in Difference Equations
a SpringerOpen Journal

Dynamical modeling of the control of brown planthoppers by *Beauveria bassiana* and *Cyrtorhinus lividipennis*

Chontita Rattanakul[1,2]*

*Correspondence:
chontita.rat@mahidol.ac.th
[1] Center of Excellence in
Mathematics, Commission on
Higher Education, Bangkok,
Thailand
[2] Department of Mathematics,
Faculty of Science, Mahidol
University, Bangkok, Thailand

Abstract

In this work, the control of brown planthopper, a major pest of rice, when a biological control agent (*Cyrtorhinus lividipennis*) and a pathogen (*Beauveria bassiana*) are utilized is investigated mathematically. An impulsive mathematical model accounted for the population densities of susceptible brown planthoppers (brown planthoppers that are susceptible to *Beauveria bassiana*), infected brown planthoppers (brown planthoppers that are infected by *Beauveria bassiana*), and *Cyrtorhinus lividipennis* (a natural enemy of brown planthopper) is developed. We analyze the model in terms of its stability and permanence so that we obtain the conditions that differentiate dynamic behaviors exhibited by the model. To illustrate our theoretical results, computer simulations are also presented.

Keywords: Mathematical model; Brown planthopper; *Cyrtorhinus lividipennis*; *Beauveria bassiana*

1 Introduction

Rice is recognized as an important food crop for the population of the world [1]. To meet with the increased demand of rice due to the increased world's population, the production of rice should be increased. Rice production has an important effect on the development of several countries' economics. Any crises that reduce the production of this commodity can adversely affect these countries [2]. Approximately 25% of rice crop losses occur due to its pests [3]. One of the major pests of rice is brown planthopper (BPH). The infestation of BPH in a rice field can cause the damage known as hopperburn [4]. In Thailand, during the dry season of the year 2010, the outbreak of BPH caused the loss of approximately $52 million as reported in [5].

To avoid the side effects of insecticide such as biodiversity reduction and the decrease in population of natural enemies of BPH, biological control and pathogen are considered as the alternative methods for controlling the outbreak of BPH in the paddy field [6, 7].

There are many natural enemies that have been used as the biological control agent for controlling the outbreak of brown planthoppers such as *Cyrtorhinus lividipennis* and *Lycosa pseudoannulata* [8–11]. In this study, we focus on *Cyrtorhinus lividipennis*. The predatory activity of *Cyrtorhinus lividipennis* against BPH has been investigated widely and the results indicated that the *Cyrtorhinus lividipennis*'s preying on BPH's eggs was an

important cause of the decrease in BPH population [8, 12]. When the outbreak of BPH is severe and the use of *Cyrtorhinus lividipennis* alone might not be effective because the reproduction of *Cyrtorhinus lividipennis* is not rapid enough to control the outbreak, additional use of the pathogen *Beauveria bassiana* might be the appropriate way for controlling the outbreak of BPH.

Hence, we then investigate the effects of impulsive applications/releases of pathogen *Beauveria bassiana* and *Cyrtorhinus lividipennis* on the population dynamics of brown planthoppers in this paper. In the next section, we state a mathematical model that will be used to investigate the dynamic behaviors of the model when the pathogen *Beauveria bassiana* and *Cyrtorhinus lividipennis* are utilized.

2 Model development

Let $x(t)$, $y(t)$, and $z(t)$ denote the population densities of susceptible brown planthoppers (brown planthoppers that are susceptible to *Beauveria bassiana* but not yet infected) at time t, infected brown planthoppers (brown planthoppers that have been infected by *Beauveria bassiana*) at time t, and *Cyrtorhinus lividipennis* (a natural enemy of brown planthopper) at time t, respectively. The following impulsive system is proposed to investigate the population dynamics of brown planthoppers when *Cyrtorhinus lividipennis* and the pathogen *Beauveria bassiana* are utilized:

$$\frac{dx}{dt} = a_1 x \left(1 - \frac{x}{k_1} \right) - a_2 xy - \frac{a_3 xz}{1 + k_2 x} - b_1 x \equiv F_1, \tag{1a}$$

$$\frac{dy}{dt} = a_2 xy - b_2 y \equiv F_2, \qquad t \neq mT, \tag{1b}$$

$$\frac{dz}{dt} = \frac{\alpha a_3 xz}{1 + k_2 x} - b_3 z \equiv F_3, \tag{1c}$$

with

$$\Delta x(t) = -\delta x(t), \tag{1d}$$

$$\Delta y(t) = \delta x(t), \qquad t = mT, \tag{1e}$$

$$\Delta z(t) = \gamma, \tag{1f}$$

where $0 \leq \alpha \leq 1$. All parametric values of (1a)–(1f) are assumed to be positive. Once the rice plants in the field are not infested by BPH yet, the susceptible brown planthoppers have many rice plants to feed on. When brown planthoppers spread in the field as hopperburn, the number of rice plants available for the susceptible brown planthoppers to feed on is then decreased, and hence the logistic growth is assumed for the reproduction of susceptible BPH. The infection rate of susceptible BPH by Beauveria bassiana is assumed to vary directly to the encounters of the susceptible BPH and the infected BPH and hence the term $a_2 xy$ is utilized. Since the consumption capability of a *Cyrtorhinus lividipennis* is limited, we then assume that the predation rate of susceptible BPH by *Cyrtorhinus lividipennis* is represented by the term $\frac{a_3 xz}{1 + k_2 x}$.

The impulsive period of application/release of the pathogen *Beauveria bassiana* and *Cyrtorhinus lividipennis* in the field is denoted by T, $m \in Z_+$, $Z_+ = \{1, 2, 3, \ldots\}$. $\Delta x(t) =$

$x(t^+) - x(t)$, $\Delta y(t) = y(t^+) - y(t)$, and $\Delta z(t) = z(t^+) - z(t)$. Once the pathogen *Beauveria bassiana* is applied, some of the population of the susceptible brown planthoppers will be infected. δ represents the fraction of the susceptible brown planthoppers that are infected by the pathogen *Beauveria bassiana* and becomes the infected brown planthoppers at time $t = mT$, $0 < \delta < 1$. However, there is no effect of the pathogen *Beauveria bassiana* on the population of *Cyrtorhinus lividipennis*. In addition, *Cyrtorhinus lividipennis* preys only on susceptible brown planthoppers, not on those infected by the pathogen *Beauveria bassiana*. γ represents the increase in the population of *Cyrtorhinus lividipennis* due to the release of *Cyrtorhinus lividipennis* at time $t = mT$.

3 Model analysis

In what follows, we let R_+ denote $[0, \infty)$. R_+^3 denotes $\{Q \in R^3 : Q = (x, y, z), \ x, y, z \in R_+\}$ and F denotes (F_1, F_2, F_3). We also assume that the solution of (1a)–(1f) is piecewise continuous. To prove the main results, we first provide some lemmas which need the following definitions [13].

Definition 1 Let $W : R_+ \times R_+^3 \to R_+$ satisfy the locally Lipschitzian condition in Q provided that W is continuous for $(t, Q) \in (mT, (m + 1)T] \times R_+^3$ and $\lim_{(t,S) \to (mT^+, Q)} W(t, S) = W(mT^+, Q)$ exists for each $Q \in R_+^3$, $m \in Z_+$. The upper right derivative of $W(t, Q)$ with respect to the impulsive equations (1a)–(1f) is then defined as

$$D^+ W(t, Q) = \limsup_{h \to 0^+} \left\{ \frac{W(t + h, Q + hF(t, Q)) - W(t, Q)}{h} \right\}$$

for $(t, Q) \in (mT, (m + 1)T] \times R_+^3$.

Note that $\frac{dx}{dt} = 0$, $\frac{dy}{dt} = 0$, and $\frac{dz}{dt} = 0$ whenever $x(t) = 0$, $y(t) = 0$, and $z(t) = 0$, respectively, for $t \neq mT$, $x(mT^+) = (1 - \delta)x(mT)$ and $y(mT^+) = y(mT) + \delta x(mT)$, $0 < \delta < 1$, $z(mT^+) = z(mT) + \gamma$. In addition, the smoothness properties of F also imply that the solution of the impulsive differential equations (1a)–(1f) exists and is unique [14].

Lemma 1 *The solution of the impulsive differential equations* (1a)–(1f), $Q(t) = (x(t), y(t), z(t))$, *is nonnegative for all* $t \geq 0$ *if* $Q(0^+) \geq 0$. *In addition,* $Q(t)$ *is positive for all* $t \geq 0$ *if* $Q(0^+) > 0$.

Proof We will prove by contradiction. Suppose that there exists $\hat{t} \in (0, T]$ such that $x(t) \geq 0$, $y(t) \geq 0$, $z(t) \geq 0$, $x(\hat{t}) = 0$, $\dot{x}(\hat{t}) < 0$, $y(\hat{t}) \geq 0$, and $z(\hat{t}) \geq 0$ for all $t \in (0, \hat{t})$. From (1a), $\dot{x}(\hat{t}) = 0$ which is a contradiction. Moreover, we also obtain

$$x(t) = x(0^+) \exp\left(\int_0^t \left[a_1 \left(1 - \frac{x(\tau)}{k_1} \right) - a_2 y(\tau) - \frac{a_3 z(\tau)}{1 + k_2 x(\tau)} - b_1 \right] d\tau \right), \quad t \in (0, T].$$

Therefore, $x(t) \geq 0$ if $x(0^+) \geq 0$ and $x(t) > 0$ if $x(0^+) > 0$ for $t \in (0, T]$. The proof for $y(t)$ and $z(t)$ can be conducted in the same manner. □

Lemma 2 *For sufficiently large* t, *the solution of the impulsive differential equations* (1a)–(1f), $Q(t) = (x(t), y(t), z(t))$, *is bounded above.*

Proof We let $M_1 = \frac{a_1 k_1}{4}$, $W(t, Q) = x + y + z$, and $c = \min\{b_1, b_2, b_3\}$. It is obvious that

$$\left\| W(t, Q_1) - W(t, Q_2) \right\| = \left\| (x_1 + y_1 + z_1) - (x_2 + y_2 + z_2) \right\| \leq L \|Q_1 - Q_2\|,$$

where $Q_1 = (x_1, y_1, z_1)$, $Q_2 = (x_2, y_2, z_2)$, and $L = 1$. Hence, $W(t, Q)$ is locally Lipschitz in Q.
For $t \neq mT$,

$$\begin{aligned}
D^+ W + cW &= \frac{dx}{dt} + \frac{dy}{dt} + \frac{dz}{dt} + cx + cy + cz \\
&= a_1 x \left(1 - \frac{x}{k_1} \right) - a_2 xy - \frac{a_3 xz}{1 + k_2 x} - b_1 x + a_2 xy - b_2 y + \frac{\alpha a_3 xz}{1 + k_2 x} - b_3 z \\
&\quad + cx + cy + cz \\
&\leq \frac{a_1 k_1}{4} + (c - b_1)x + (c - b_2)y + (c - b_3)z \\
&\leq M_1.
\end{aligned}$$

Hence $D^+ W \leq -cW + M_1$.
For $t = mT$,

$$\begin{aligned}
W\left(mT^+\right) &= x\left(mT^+\right) + y\left(mT^+\right) + z\left(mT^+\right) \\
&= (1 - \delta)x(mT) + y(mT) + \delta x(mT) + z(mT) + \gamma \\
&= x(mT) + y(mT) + z(mT) + \gamma \\
&= W(mT) + \gamma.
\end{aligned}$$

Hence, for $t \in (mT, (m + 1)T]$, Lemma 2.2 of [13] implies that

$$\begin{aligned}
W(t) &\leq W(0)e^{-ct} + \int_0^t M_1 e^{-c(t-s)} \, ds + \gamma \sum_{0 < t_m < t} e^{-\int_{t_m}^t c \, d\tau} \\
&\leq W(0)e^{-ct} + M_1 \left(\frac{1}{c} - \frac{e^{-ct}}{c} \right) + \gamma \left[\frac{e^{-c(t-T)} - e^{-c(t-t_{m+1})}}{1 - e^{-cT}} \right] \\
&\leq \frac{M_1}{c} \equiv M \quad \text{as } t \to \infty.
\end{aligned}$$

That is, $W(t)$ is uniformly ultimately bounded. Hence, when t is large enough, $x(t)$, $y(t)$, and $z(t)$ are bounded above, which implies that the solution $Q(t) = (x(t), y(t), z(t))$ of the impulsive differential equations (1a)–(1f) is bounded above as well. □

Next, let us consider the system of (1a)–(1f) when there is no brown planthopper ($x = 0$ and $y = 0$):

$$\frac{dz}{dt} = -b_3 z, \quad t \neq mT, \tag{2}$$

$$z\left(mT^+\right) = z(mT) + \gamma, \quad t = mT, \tag{3}$$

$$z\left(0^+\right) = z_0. \tag{4}$$

We can see that the function

$$\tilde{z}(t) = \frac{\gamma e^{-b_3(t-mT)}}{1 - e^{-b_3 T}}$$

is a positive solution of equations (2)–(4) for $t \in (mT, (m+1)T)$, $m \in Z_+$, such that

$$\tilde{z}(0^+) = \frac{\gamma}{1 - e^{-b_3 T}}.$$

Therefore,

$$z(t) = \left(z_0 - \frac{\gamma}{1 - e^{-b_3 T}}\right) e^{-b_3 t} + \tilde{z}(t)$$

is the solution of equations (2)–(4) for $t \in (mT, (m+1)T)$, $m \in Z_+$.

Lemma 3 *Equations* (2)–(4) *have* $\tilde{z}(t)$ *as a positive periodic solution. In addition, as* $t \to \infty$, $z(t) \to \tilde{z}(t)$ *for every solution* $z(t)$ *of equations* (2)–(4).

Therefore, at the vanishing of brown planthoppers, system (1a)–(1f) has a periodic solution

$$\left(0, 0, \tilde{z}(t)\right) = \left(0, 0, \frac{\gamma e^{-b_3(t-mT)}}{1 - e^{-b_3 T}}\right)$$

for $t \in (mT, (m+1)T]$ and $\tilde{z}(mT^+) = \tilde{z}(0^+) = \frac{\gamma}{1-e^{-b_3 T}}$, $m \in Z_+$.

Theorem 1 *Suppose that*

$$a_1 > b_1 \tag{5}$$

and

$$T < \frac{1}{(a_1 - b_1)} \left[\ln\left(\frac{1}{1-\delta}\right) - \frac{\gamma}{b_3}\right] \equiv T_{\max}, \tag{6}$$

then $(0, 0, \tilde{z}(t))$, *the solution of the impulsive differential equations* (1a)–(1f), *is locally asymptotically stable.*

Proof Consider a small perturbation of $(0, 0, \tilde{z}(t))$:

$$x_1(t) = x(t),$$
$$y_1(t) = y(t),$$
$$z_1(t) = z(t) - \tilde{z}(t).$$

Therefore, we have

$$\begin{pmatrix} x_1(t) \\ y_1(t) \\ z_1(t) \end{pmatrix} = \Phi(t) \begin{pmatrix} x_1(0) \\ y_1(0) \\ z_1(0) \end{pmatrix}, \quad 0 < t < T,$$

where $\Phi(t)$, which is the fundamental solution matrix, satisfies

$$\frac{d\Phi(t)}{dt} = \begin{pmatrix} a_1 - b_1 - a_3\tilde{z}(t) & 0 & 0 \\ 0 & -b_2 & 0 \\ * & 0 & -b_3 \end{pmatrix} \Phi(t)$$

with $\Phi(0) = I$, the identity matrix. Hence,

$$\Phi(t) = \begin{pmatrix} \exp\int_0^t (a_1 - b_1 - a_3\tilde{z}(t))\,ds & 0 & 0 \\ 0 & \exp\int_0^t (-b_2)\,ds & 0 \\ ** & 0 & \exp\int_0^t (-b_3)\,ds \end{pmatrix}.$$

We can see that the terms (*) and (**) will not be used in further calculation, then there is no need to obtain the exact expression for (*) and (**).

Linearization of (1d)–(1f) yields

$$\begin{pmatrix} x_1(mT^+) \\ y_1(mT^+) \\ z_1(mT^+) \end{pmatrix} = \begin{pmatrix} 1-\delta & 0 & 0 \\ \delta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1(mT) \\ y_1(mT) \\ z_1(mT) \end{pmatrix}.$$

The solution $(0, 0, \tilde{z}(t))$ of the impulsive differential equations (1a)–(1f) is locally asymptotically stable, according to Floquet theory, if $|\lambda_1|, |\lambda_2|, |\lambda_3| < 1$ where $\lambda_1, \lambda_2, \lambda_3$ are eigenvalues of

$$P = \begin{pmatrix} 1-\delta & 0 & 0 \\ \delta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Phi(T).$$

Here, the eigenvalues of P are

$$\lambda_1 = (1-\delta)\exp\int_0^t \left(a_1 - b_1 - a_3\tilde{z}(t)\right)ds = (1-\delta)\exp\left((a_1 - b_1)T + \frac{\gamma}{b_3}\right),$$

$$\lambda_2 = \exp\int_0^t (-b_2)\,ds = \exp(-b_2 t),$$

$$\lambda_3 = \exp\int_0^t (-b_3)\,ds = \exp(-b_3 t).$$

Since $0 < \delta < 1$ and (5)–(6) hold, then all eigenvalues are positive and less than 1. Hence, the solution $(0, 0, \tilde{z}(t))$ of the impulsive differential equations (1a)–(1f) is locally asymptotically stable. □

4 System permanence
Definition 2 If there are constants $\bar{m}, \bar{M} > 0$ and $t_0 > 0$ for which every solution with positive initial condition $x(0^+)$, $y(0^+)$, and $z(0^+)$,

$$\bar{m} \le x(t) \le \bar{M},$$

$$\bar{m} \le y(t) \le \bar{M},$$

$$\bar{m} \leq z(t) \leq \bar{M},$$

for all $t > t_0$, system (1a)–(1f) is said to be permanent.

Theorem 2 *Suppose that*

$$A > B \tag{7}$$

and

$$T > \left(\frac{1}{A - B}\right) \ln\left(\frac{1}{1 - \delta}\right) \equiv T^*, \tag{8}$$

where $A \equiv a_1 - b_1$ *and* $B \equiv (\frac{a_1}{k_1} + a_2 + a_3)(\frac{a_1 k_1}{4c} + \frac{\gamma e^{cT}}{e^{cT} - 1})$, $c = \min\{b_1, b_2, b_3\}$. *The impulsive differential equations* (1a)–(1f) *are permanent if* (7) *and* (8) *hold.*

Proof Let $Q(t)$ be a solution of the impulsive differential equations (1a)–(1f), $Q(t) = (x(t), y(t), z(t))$, with $x(0^+), y(0^+), z(0^+) > 0$. Then Lemma 2 guarantees that $Q(t)$ is bounded above when t is large enough. Hence, a constant $\bar{M} > 0$ exists for which, when t is sufficiently large, we have $x(t), y(t), z(t) \leq \bar{M}$.

Consider (1c) and (1f), we know that

$$\frac{dz}{dt} \geq -b_3 z, \quad t \neq mT, \tag{9}$$

$$z\left(mT^+\right) = z(mT) + \gamma, \quad t = mT, \tag{10}$$

$$z\left(0^+\right) = z_0. \tag{11}$$

Consider the comparison system

$$\frac{dr}{dt} = -b_3 r, \quad t \neq mT, \tag{12}$$

$$r\left(mT^+\right) = r(mT) + \gamma, \quad t = mT, \tag{13}$$

$$r\left(0^+\right) = z_0. \tag{14}$$

We can see that, for $t \in (mT, (m + 1)T)$, $m \in Z_+$,

$$\tilde{r}(t) = \frac{\gamma e^{-b_3(t - mT)}}{1 - e^{-b_3 T}}$$

is a positive solution of the comparison system (12)–(14) such that

$$\tilde{r}\left(0^+\right) = \frac{\gamma}{1 - e^{-b_3 T}}.$$

Therefore, the solution of this comparison system (12)–(14) is

$$r(t) = \left(z_0 - \frac{\gamma}{1 - e^{-b_3 T}}\right) e^{-b_3 t} + \tilde{r}(t), \quad t \in \left(mT, (m + 1)T\right)$$

and $r(t) \to \tilde{r}(t)$ as $t \to \infty$.

According to [14], we obtain $z(t) \geq r(t)$ by the comparison theorem and hence, when t is large enough,

$$z(t) \geq \frac{\gamma e^{-b_3 T}}{1 - e^{-b_3 T}} + \varepsilon_1 \equiv m_1 > 0. \tag{15}$$

Next, we will show that there exists a constant $m_2 > 0$ for which $x(t) \geq m_2$ when t is large enough.

Step I. By contradiction, we will show for $t \geq t_1$ that there is $t_1 > 0$ for which $x(t) \geq m_3$. For all $t \geq 0$, we suppose that $x(t) < m_3$.

Consider (1a) and (1d). Since Lemma 2 implies that there is $t_1 > 0$ in which we can select $\bar{M} = \frac{a_1 k_1}{4c} + \frac{\gamma e^{cT}}{e^{cT}-1}$ where $c = \min(b_1, b_2, b_3)$ for which, for $t \geq t_1$, we have $x(t), y(t), z(t) \leq \bar{M}$, then

$$\frac{dx}{dt} = a_1 x \left(1 - \frac{x}{k_1} \right) - a_2 xy - \frac{a_3 xz}{1 + k_2 x} - b_1 x, \quad t \neq mT \tag{16}$$

$$\geq \left(a_1 \left(1 - \frac{x}{k_1} \right) - a_2 y - a_3 z - b_1 \right) x \tag{17}$$

$$\geq \left[a_1 - b_1 - \left(\frac{a_1}{k_1} + a_2 + a_3 \right) \bar{M} \right] x \tag{18}$$

for $t \geq t_1$, with $x(mT^+) = (1 - \delta) x(mT)$, $t = mT$.

Let $N \in Z_+$ and $NT \geq T_1$. We integrate (18) over $(mT, (m+1)T]$, $m \geq N$ and obtain

$$x\big((m+1)T\big) \geq x(mT)(1-\delta) \exp\left(\int_{mT}^{(m+1)T} \left[a_1 - b_1 - \left(\frac{a_1}{k_1} + a_2 + a_3 \right) \bar{M} \right] dt \right)$$

$$= x(mT)(1-\delta) \exp\left[a_1 - b_1 - \left(\frac{a_1}{k_1} + a_2 + a_3 \right) \bar{M} \right] T$$

$$= x(mT)\eta,$$

where $\eta \equiv (1 - \delta) \exp[a_1 - b_1 - (\frac{a_1}{k_1} + a_2 + a_3)\bar{M}]T$.

Consider

$$\ln \eta = \ln(1 - \delta) + \left[a_1 - b_1 - \left(\frac{a_1}{k_1} + a_2 + a_3 \right) \bar{M} \right] T \tag{19}$$

$$= \ln(1 - \delta) + \left[a_1 - b_1 - \left(\frac{a_1}{k_1} + a_2 + a_3 \right) \left(\frac{a_1 k_1}{4c} + \frac{\gamma e^{cT}}{e^{cT} - 1} \right) \right] T. \tag{20}$$

Since (7) and (8) hold, then $\ln \eta > 0$, which implies that $\eta > 1$. Therefore, $x((m+k)T) \geq x(mT)\eta^k \to \infty$ as $k \to \infty$, and hence $x(t)$ is not bounded above when t is large enough, and this is a contradiction. This means that there is $t_1 > 0$ for which $x(t_1) \geq m_3$.

Step II. For all $t > t_1$, if $x(t) \geq m_3$, the proof is then complete, or else there must be $t' > t_1$ for which $x(t') < m_3$. Next, we consider the following possible cases when we let $t^* = \inf_{t > t_1} \{t : x(t) < m_3\}$.

Case 1. $t^* = k_1 T$ for some $k_1 \in Z_+$. This means that for $t \in [t_1, t^*]$, $x(t) \geq m_3$ and

$$m_3 > x\big(t^{*+}\big) = (1 - \delta) x\big(t^*\big) \geq m_3(1 - \delta).$$

Next, $k_2, k_3 \in Z_+$ are chosen so that

$$k_2 T > T_1,$$

$$(1 - \delta)^{k_2} \exp(k_2 \eta_1 T)\eta^{k_3} > (1 - \delta)^{k_2} \exp\big((k_2 + 1)\eta_1 T\big)\eta^{k_3} > 1,$$

where $\eta_1 = a_1 - b_1 - (\frac{a_1}{k_1} + a_2 + a_3)\bar{M} < 0$.

Now, we let $T' = k_2 T + k_3 T$ and claim that $t_2 \in (t^*, t^* + T']$ must exist for which

$$x(t_2) > m_3,$$

or else, for $t^* + k_2 T \leq t \leq t^* + T'$, (18) holds and

$$x\big(t^* + T'\big) \geq x\big(t^* + k_2 T\big)\eta^{n_3}. \tag{21}$$

For $t \in [t^*, t^* + k_2 T]$,

$$\frac{dx}{dt} \geq \left[a_1 - b_1 - \left(\frac{a_1}{k_1} + a_2 + a_3\right)\bar{M}\right]x, \quad t \neq mT, \tag{22}$$

$$x\big(t^+\big) = (1 - \delta)x(t), \quad t \neq mT. \tag{23}$$

By integrating (22) over $[t^*, t^* + k_2 T]$, we obtain

$$x\big(t^* + k_2 T\big) \geq m_3 (1 - \delta)^{k_2} \exp(k_2 \eta_1 T).$$

Therefore,

$$x\big(t^* + T'\big) \geq m_3 (1 - \delta)^{k_2} \exp(k_2 \eta_1 T)\eta^{k_3} > m_3$$

and a contradiction occurs. Thus, $t_2 \in (t^*, t^* + T']$ must exist for which $x(t_2) > m_3$.

Letting $\tilde{t} = \inf_{t > t^*}\{t : x(t) > m_3\}$. Since $x(t)$ is left continuous and $x(t^+) = (1 - \delta)x(t) \leq x(t)$ when $t = mT$, we can conclude that $x(t) \leq m_3$ whenever $t \in (t^*, \tilde{t})$ and $x(\tilde{t}) = m_3$.

Next, we assume for $t \in (t^*, \tilde{t})$ that $t \in (t^*, (l-1)T, t^* + lT]$ where $l \in Z_+$ and $l \leq k_2 + k_3$. Consider (21), we get

$$\begin{aligned}
x(t) &\geq x\big(t^{*+}\big)(1 - \delta)^{l-1} \exp\big((l-1)\eta_1 T\big) \exp\big(\eta_1\big(t - \big(t^* + (l-1)T\big)\big)\big) \\
&\geq m_3 (1 - \delta)^l \exp(l\eta_1 T) \\
&\geq m_3 (1 - \delta)^{k_2 + k_3} \exp\big((k_2 + k_3)\eta_1 T\big) \equiv m_1'.
\end{aligned}$$

Therefore, we can conclude that $x(t) \geq m_1'$ whenever $t \in (t^*, \tilde{t})$ and $x(\tilde{t}) \geq m_3$. With similar argument for $t > \tilde{t}$, when t is sufficiently large, we obtain $x(t) \geq m_2 > 0$.

Case 2. $t^* \neq mT$ for all $m \in Z_+$. For $t \in (t_1, t^*)$, we obtain $x(t) \geq m_3$ and $x(t^*) = m_3$. Suppose $t^* \in (k_1' T, (k_1' + 1)T)$ for some $k_1' \in Z_+$.

Case 2.1. For all $t \in (t^*, (k_1' + 1)T), x(t) \leq m_3$. We claim that $t_2' \in [(n_1' + 1)T, (n_1' + 1)T + T']$ exists for which $x(t_2') > m_3$. Otherwise, similar to Case 1, we get

$$x\big((k_1' + 1 + k_2 + k_3)T\big) \geq x\big((k_1' + 1 + k_2)T\big)\eta^{n_3}.$$

For $t \in (t^*, (k_1' + 1)T)$, (22) holds on $[t^*, (k_1' + 1 + k_2 + k_3)T]$ and $x(t) \leq m_3$, so that we obtain

$$x\big((k_1' + 1 + k_2)T\big) \geq m_3(1 - \delta)^{k_2} \exp\big((k_2 + 1)\eta_1 T\big).$$

Therefore,

$$x\big((k_1' + 1 + k_2 + k_3)T\big) \geq m_3(1 - \delta)^{k_2} \exp\big((k_2 + 1)\eta_1 T\big)\eta^{n_3} > m_3,$$

and a contradiction occurs.

Next, letting $\bar{t} = \inf_{t > t^*}\{t : x(t) > m_3\}$. Therefore, for $t \in (t^*, \bar{t})$,

$$x(t) \leq m_3$$

and

$$x(\bar{t}) = m_3.$$

Now we assume for $t \in (t^*, \bar{t})$ that $t \in (k_1'T + (l' - 1)T, k_1'T + l'T]$ where $l' \in Z_+$ and $l' \leq 1 + k_2 + k_3$. Thus, we obtain

$$\begin{aligned} x(t) &\geq m_3(1 - \delta)^{l' - 1} \exp\big(l'\eta_1 T\big) \\ &\geq m_3(1 - \delta)^{k_2 + k_3} \exp\big((k_2 + k_3 + 1)\eta_1 T\big) \equiv m_1. \end{aligned}$$

So, for $t \in (t^*, \bar{t})$, $x(t) \geq m_1$. For $t > \bar{t}$, since $x(\bar{t}) \geq m_3$, the similar arguments can be applied. We thus get $x(t) \geq m_2 > 0$ when t is sufficiently large.

Case 2.2. There is $t \in (t^*, (k_1 + 1)T)$ for which $x(t) > m_3$. Therefore, we let $\hat{t} = \inf_{t > t^*}\{t : x(t) > m_3\}$. Hence, for $t \in (t^*, \hat{t})$, we get $x(t) \leq m_3$ and $x(\hat{t}) = m_3$.

Next, we integrate (22) on $(t^*, \hat{t})$ where we note that (22) satisfies for $t \in (t^*, \hat{t})$ and obtain

$$x(t) \geq x\big(t^*\big) \exp\big(\eta_1\big(t - t^*\big)\big) \geq m_3 \exp(\eta_1 T) > m_1.$$

With the fact that $x(\hat{t}) \geq m_3$, the above argument can be applied again for $t > \hat{t}$. Thus, we obtain $x(t) \geq m_2 > 0$ for all $t \geq t_1$.

Finally, we can also show that there exists a constant $m_4 > 0$ for which $y(t) \geq m_4$ for sufficiently large t in the same manner as we showed above that $x(t) \geq m_2$ for sufficiently large t by using the fact that $x(t) \geq m_2$ for sufficiently large t, and hence the proof is omitted here.

Therefore, we can choose $\bar{m} = \min\{m_1, m_2, m_4\}$ so that $x(t), y(t), z(t) \geq \bar{m} > 0$, and hence the system is permanent provided that (7) and (8) hold and the proof is complete.　　□

5 Numerical simulations

Computer simulations are presented in this section to illustrate our theoretical results in Sect. 3.

Here, two sets of parametric values of the model are chosen as examples to illustrate the theoretical results in which the parameters a_3 and k_2 are obtained from the literature [15].
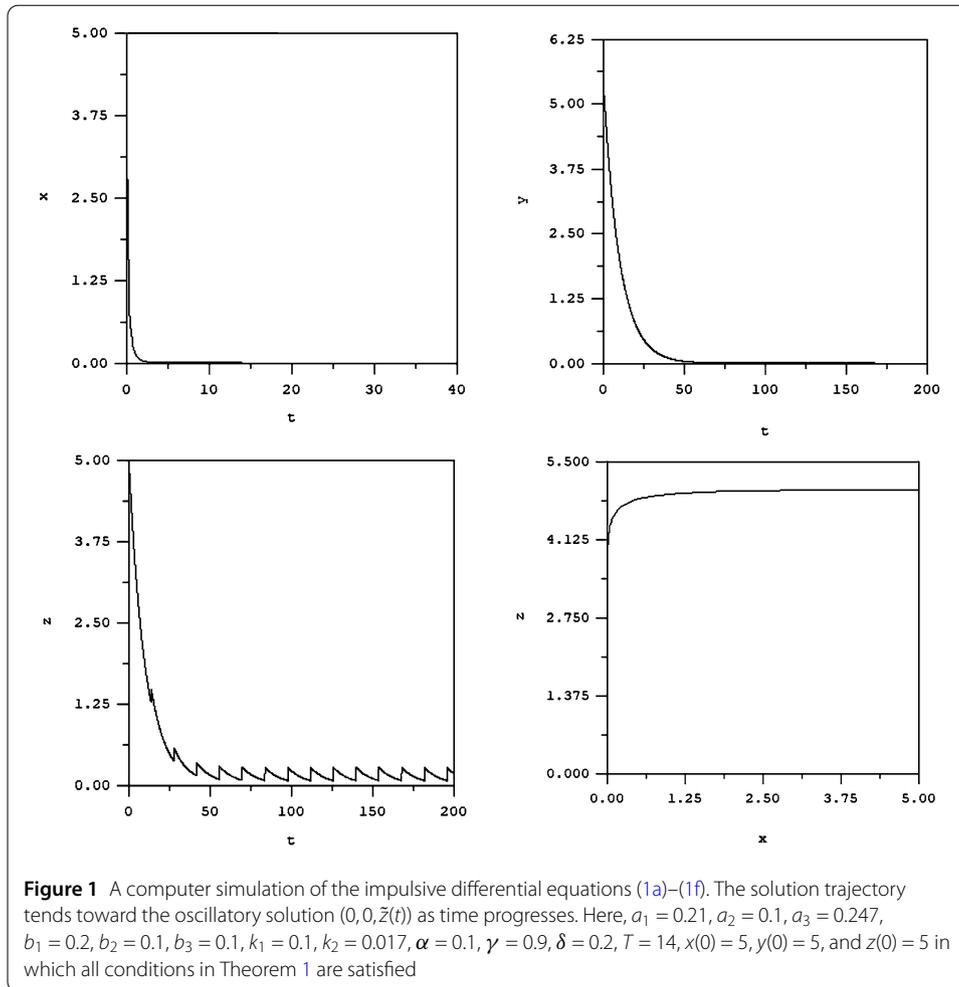
Figure 1 A computer simulation of the impulsive differential equations (1a)–(1f). The solution trajectory tends toward the oscillatory solution $(0, 0, \tilde{z}(t))$ as time progresses. Here, $a_1 = 0.21$, $a_2 = 0.1$, $a_3 = 0.247$, $b_1 = 0.2$, $b_2 = 0.1$, $b_3 = 0.1$, $k_1 = 0.1$, $k_2 = 0.017$, $\alpha = 0.1$, $\gamma = 0.9$, $\delta = 0.2$, $T = 14$, $x(0) = 5$, $y(0) = 5$, and $z(0) = 5$ in which all conditions in Theorem 1 are satisfied

A simulation result of the impulsive system of equations (1a)–(1f) with the parametric values $a_1 = 0.21$, $a_2 = 0.1$, $a_3 = 0.247$, $b_1 = 0.2$, $b_2 = 0.1$, $b_3 = 0.1$, $k_1 = 0.1$, $k_2 = 0.017$, $\alpha = 0.1$, $\gamma = 0.9$, $\delta = 0.2$, $T = 14$, $x(0) = 5$, $y(0) = 5$, and $z(0) = 5$, in which all the conditions in Theorem 1 are satisfied, are presented in Fig. 1. In Fig. 1, the time courses of the population densities of the susceptible BPH, infected BPH, and *Cyrtorhinus lividipennis* as well as the projection of the solution trajectory onto the (x, z)-plane are presented. The solution trajectory in this case tends to the oscillating solution $(0, 0, \tilde{z})$ for which both susceptible and infected BPH vanish as predicted in Theorem 1.

On the other hand, Fig. 2 shows a simulation result of the impulsive system of equations (1a)–(1f) with the parametric values $a_1 = 0.38$, $a_2 = 0.1$, $a_3 = 0.247$, $b_1 = 0.25$, $b_2 = 0.1$, $b_3 = 0.1$, $k_1 = 0.3$, $k_2 = 0.017$, $\alpha = 0.5$, $\gamma = 0.9$, $\delta = 0.2$, $T = 60$, $x(0) = 5$, $y(0) = 5$, and $z(0) = 5$ in which all the conditions in Theorem 2 are satisfied. The time courses of the population densities of the susceptible BPH, infected BPH, and *Cyrtorhinus lividipennis* as well as the projection of the solution trajectory onto the (x, z)-plane are as shown in Fig. 2. The solution of the system in this case is permanent as predicted in Theorem 2. We can see that the population densities of both susceptible and infected BPH can be controlled within a certain level, while its natural enemy *Cyrtorhinus lividipennis* also survives in the paddy field.

Figure 2 A computer simulation of the impulsive differential equations (1a)–(1f). The solution of the system is permanent. Here, $a_1 = 0.38$, $a_2 = 0.1$, $a_3 = 0.247$, $b_1 = 0.25$, $b_2 = 0.1$, $b_3 = 0.1$, $k_1 = 0.3$, $k_2 = 0.017$, $\alpha = 0.5$, $\gamma = 0.9$, $\delta = 0.2$, $T = 60$, $x(0) = 5$, $y(0) = 5$, and $z(0) = 5$ in which all conditions in Theorem 2 are satisfied
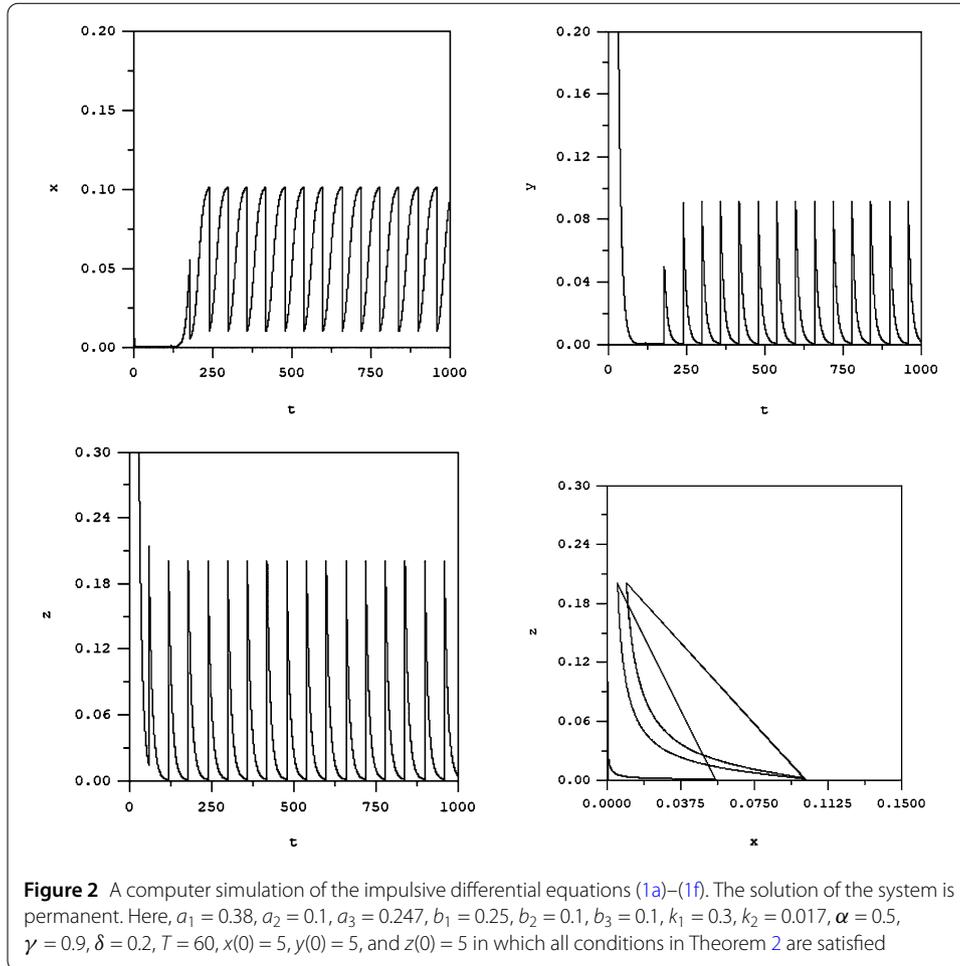
6 Conclusion

We investigate the dynamic behaviors of the populations of susceptible and infected BPH when *Cyrtorhinus lividipennis* and the pathogen *Beauveria bassiana* are utilized to control the population of BPH in the paddy field mathematically. Here, the pathogen *Beauveria bassiana* does not have an effect on *Cyrtorhinus lividipennis*.

Brown plant hoppers (BPH) are rice's insect pests. Therefore, the aim of this work is to obtain the conditions on the system parameters for which the populations of both susceptible and infected BPH tend to zero level as time passes. However, the cost for controlling BPH to zero level might be too high and the control of BPH levels $x(t)$ and $y(t)$ to lie within a certain range might be a better option in an economic point of view. Hence, we also provide the conditions for which the system is permanent and the levels of BPH can be controlled to lie within a certain range as well.

The examples of two sets of parameters are selected as examples to illustrate the theoretical results in which some parametric values are obtained from the literature [15]. Numerical simulations are as shown in Figs. 1 and 2. We can see that in Fig. 2, the population of BPH could be controlled below certain ranges, and we can also observe that the population of *Cyrtorhinus lividipennis* is approximately twice of the population of BPH. This result corresponds to what has been observed in the rice field in Thailand [16] that the population of BPH could be controlled to be lower than a certain level if the population

of *Cyrtorhinus lividipennis* in the field is approximately twice of the population of BPH in the rice field.

In addition, the conditions in Theorem 1 and 2 depend on the duration T between the two consecutive applications of *Beauveria bassiana* and *Cyrtorhinus lividipennis.* Hence, the appropriate duration T could play an important role in controlling the population of BPH in the rice field. The current practices in Thailand that natural enemies of BPH or pathogens will be applied in the field whenever the spread of BPH in the rice field is detected would take some time before the population of BPH can be controlled. Hence, the applications of *Beauveria bassiana* and *Cyrtorhinus lividipennis* in the rice field with the appropriate duration T could lead to the more efficient control of BPH in the rice field.

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References

1. Kole, C.: Cereals and Millets, Genome mapping and molecular breeding in plants, vol. 1. Springer, Berlin (2006)
2. Lichtfouse, E., Goyal, A.: Sustainable Agriculture Reviews, Sustainable Agriculture Reviews, vol. 16. Springer, Switzerland (2015)
3. Oerke, E.C.: Crop losses to pests. J. Agric. Sci. **144**, 13–43 (2006)
4. Stout, M.J.: Chapter I: Host-plant resistance in pest management. In: Abrol, D.P. (ed.) Integrated Pest Management. Elsevier, Amsterdam (2014)
5. Escalada, M.M., Luecha, M., Heong, K.L.: Chapter 10: Social impacts of planthopper outbreaks: case study from Thailand. In: Heong, K.L., Cheng, J.A., Escalada, M.M. (eds.) Rice Planthoppers: Ecology, Management, Socio Economics and Policy, pp. 209–226. Zhejiang University Press, Hangzhou; Springer, Dordrecht (2015). https://doi.org/10.1007/978-94-017-9535-7
6. Huang, J., Qiao, F., Zhang, L., Rozelle, S.: Farm Pesticide, Rice Production and Human Health, Centre for Chinese Agricultural Policy. Chinese Academy of Sciences, Beijing (2000)
7. Rola, A.C., Pingali, L.P.: Pesticides, Rice Productivity, and Farmers' Health: An Economic Assessment. International Rice Research Institute, Baños Los, Philippines; World Resource Institute, Washington D.C. (1993)
8. Katti, G., Pasalu, I.C., Padmakumari, A.P., Padmavathi, C., Jhansilakshmi, V., Krishnaiah, N.V., Bentur, J.S., Prasad, J.S., Rao, Y.K.: Biological control of insect pests of rice. Technical Bulletin, Directorate of Rice Research, Rajendranagar, Hyderabad, AP, India **22** 22 (2007)
9. Sigsgaard, L.: Early season natural control of the brown planthopper Nilaparvata lugens: the contribution and interaction of two spider species and a predatory bug. Bull. Entomol. Res. **97**, 533–544 (2007)
10. Khan, A.A., Misra, D.S.: Biology of woft spider Lycosa pseudoannulata Boesenberg and Strand (Araneae: Lycosidae). Environ. Ecol. **26**, 796–799 (2008)
11. Samiayyan, K.: Chapter 15: Spider-the generalist super predators in agro-ecosystem. In: Abrol, D.P. (ed.) Integrated Pest Management, 1st edn., pp. 283–310. Academic Press, Tamil nadu Agricultural University, Coimbatore (2014)
12. Preetha, G., Stanley, J., Suresh, S., Samiyappan, R.: Risk assessment of insecticides used in rice on miridbug, Cyrtorhinus lividipennis Reuter, the important predator of brown planthopper, Nilaparvata lugens (Stal.). Chemosphere **80**, 498–503 (2010)
13. Liu, B., Zhi, Y., Chen, L.: The dynamics of a predator-prey model with Ivlev's functional response concerning integrated pest management. Acta Math. Appl. Sin. **20**(1), 133–146 (2004)
14. Lakshmikantham, V., Bainov, D.D., Simeonov, P.S.: Theory of Impulsive Differential Equations. World Scientific, Singapore (1989)
15. Jiang, X., Huang, Q., Ling, Y., Chen, Y., Xiao, G., Huang, S., Wu, B., Huang, F., Cai, J., Long, L.: Functional and numerical responses of Cyrtorhinus lividipennis to eggs of Nilaparvata ligens are not affected by genetically modified herbicide-tolerant rice. J. Integr. Agr. **14**(10), 2019–2026 (2015)
16. Rice Knowledge Bank, http://www.ricethailand.go.th/rkb/disease