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Estimation for incomplete information stochastic systems from discrete observations

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Abstract

This paper is concerned with the estimation problem for incomplete information stochastic systems from discrete observations. The suboptimal estimation of the state is obtained by constructing the extended Kalman filtering equation. The approximate likelihood function is given by using a Riemann sum and an Itô sum to approximate the integrals in the continuous-time likelihood function. The consistency of the maximum likelihood estimator and the asymptotic normality of the error of estimation are proved by applying the martingale moment inequality, Hölder's inequality, the Chebyshev inequality, the Burkholder–Davis–Gundy inequality and the uniform ergodic theorem. An example is provided to verify the effectiveness of the estimation methods.

Keywords: Incomplete information stochastic system; Suboptimal estimation; Parameter estimation; Consistency; Asymptotic normality; Discrete observations

1 Introduction

Stochastic differential equations have been widely used in many application areas such as engineering, information and medical science [4, 5]. Recently, stochastic differential equations have been applied to a description of the dynamics of a financial asset, asset portfolio and term structure of interest rates, such as the Black–Scholes option pricing model [6], Vasicek and Cox–Ingersoll–Ross pricing formulas [8, 9, 27], the Chan–Karloyi–Longstaff–Sanders model [10], the Constantinides model [11], and the Ait–Sahalia model [1]. Some parameters in pricing formulas describe the related assets dynamic, however, these parameters are always unknown. In the past few decades, some authors studied the parameter estimation problem for economic models. For example, Yu and Phillips [33] used a Gaussian approach to the study of the parameter estimation for continuous-time short-term interest rates model, Overback [21], Rossi [24] and Wei et al. [29] investigated the parameter estimation problem for the Cox–Ingersoll–Ross model by applying the maximum likelihood method, the least-square method and the Gaussian method, respectively. Moreover, some methods have been used to estimate parameters in general non-linear stochastic differential equation from continuous-time observations. For instance, we have Bayes estimations [16, 22], maximum likelihood estimations [3, 30, 31] and M -estimations [25, 32]. However, in fact, it is impossible to observe a process continuously in time. Therefore, parametric inference based on sampled data is important in dealing

with practical problems, such as least-square estimations [17, 18], approximate transition densities [2, 7] and adaptive maximum likelihood type estimations [26].

A variety of stochastic systems are described by stochastic differential equations [23], and sometimes the stochastic systems have incomplete information. Many authors studied the state estimation problem for incomplete information stochastic systems by using Kalman filtering or extended Kalman filtering [13, 15, 19, 28]. Furthermore, sometimes the parameters and states of a stochastic system are unknown at the same time. Therefore, the parameter estimation and state estimation need to be solved simultaneously. In recent years, some authors investigated the parameter estimation problem for incomplete information linear stochastic systems. For example, Deck and Theting [12] used Kalman filtering and the Bayes method to study the linear homogeneous stochastic systems. Kan et al. [14] discussed the linear nonhomogeneous stochastic systems based on the methods used in [12]. Mbalawata et al. [20] applied Kalman filtering and a maximum likelihood estimation to investigate the parameter and state estimation for linear stochastic systems.

As is well known, parameter estimation for incomplete information linear stochastic systems has been studied by some authors [12, 14, 20]. However, the asymptotic property of the parameter estimator has not been discussed in [20], and in [12, 14], only the drift parameter has been considered. In this paper, the parameter estimation problem for incomplete information nonlinear stochastic system is investigated from discrete observations. Firstly, the suboptimal estimation of the state is obtained by constructing the extended Kalman filtering. Secondly, the approximate likelihood function is given by using a Riemann sum and an Itô sum to approximate the integrals in the continuous-time likelihood function. In the approximate likelihood function, both drift and diffusion parameter are unknown. Finally, the consistency of the maximum likelihood estimator and the asymptotic normality of the error of estimation are proved by applying the martingale moment inequality, Hölder's inequality, the Chebyshev inequality, the Burkholder–Davis–Gundy inequality and the uniform ergodic theorem.

This paper is organized as follows. In Sect. 2, the state estimation is derived and the approximate likelihood function is given. In Sect. 3, some important lemmas are proved, the consistency of estimator and asymptotic normality of the error of estimation are discussed. In Sect. 4, an example is provided. The conclusion is given in Sect. 5.

2 Problem formulation and preliminaries

In this paper, the estimation problem for incomplete information stochastic system is investigated. The stochastic system is described as follows:

$$\begin{cases} dX_t = f(X_t, \theta) dt + dW_t, & X_0 \sim u_\theta, \\ dY_t = h(X_t) dt + dV_t, & Y_0 = Y_0, \end{cases} \quad (1)$$

where $\theta \in \Theta = \overline{\Theta_0}$ (the closure of Θ_0) with Θ_0 being an open bounded convex subset of \mathbb{R} is an unknown parameter, $(W_t, t \geq 0)$ and $(V_t, t \geq 0)$ are independent Wiener processes, X_t is ergodic, u_θ is the invariant measure, $\{Y_t\}$ is observable, while $\{X_t\}$ is unobservable.

From now on the work is under the assumptions below.

Assumption 1 $|f(x, \theta) - f(y, \theta)| \leq K(\theta)|x - y|$, $|f(x, \theta)| \leq K_1(\theta)(1 + |x|)$, $\sup\{K(\theta), K_1(\theta)\} < \infty$, $\theta \in \Theta$, $x, y \in \mathbb{R}$.

Assumption 2 $|h(x) - h(y)| \leq |x - y|$.

Assumption 3 $\mathbb{E}|X_0|^p < \infty$ for each $p > 0$.

Assumption 4 $\mathbb{E}[f(X_0, \theta)(f(X_0, \theta_0) - \frac{1}{2}f(X_0, \theta))]$ has the unique maximal value at $\theta = \theta_0$, where θ_0 is the true parameter.

The likelihood function cannot be given directly due to the unobservability of $\{X_t\}$. Therefore, we should solve the estimation problem of $\{X_t\}$ firstly.

The state estimator is designed as follows:

$$\begin{cases} d\widehat{X}_t = f(\widehat{X}_t, \theta) dt + K_t(dY_t - h(\widehat{X}_t) dt), \\ \widehat{X}_0 = X_0. \end{cases} \tag{2}$$

According to (1) and (2), we have

$$d(X_t - \widehat{X}_t) = (f(X_t, \theta) - f(\widehat{X}_t, \theta) - K_t(h(X_t) - h(\widehat{X}_t))) dt + dW_t - K_t dV_t. \tag{3}$$

From the Itô lemma and (3), it can be checked that

$$\begin{aligned} & d(X_t - \widehat{X}_t)^2 \\ &= 2(X_t - \widehat{X}_t)(f(X_t, \theta) - f(\widehat{X}_t, \theta) - K_t(h(X_t) - h(\widehat{X}_t))) dt \\ &\quad + 2(X_t - \widehat{X}_t)(dW_t - K_t dV_t) + (1 + K_t^2) dt \\ &= [2(X_t - \widehat{X}_t)(f(X_t, \theta) - f(\widehat{X}_t, \theta)) \\ &\quad - 2K_t(X_t - \widehat{X}_t)(h(X_t) - h(\widehat{X}_t)) + 1 + K_t^2] dt \\ &\quad + 2(X_t - \widehat{X}_t)(dW_t - K_t dV_t). \end{aligned} \tag{4}$$

Taking the expectation from both sides of (4), from Assumption 2, we obtain

$$\begin{aligned} & d\mathbb{E}(X_t - \widehat{X}_t)^2 \\ &= [2\mathbb{E}(X_t - \widehat{X}_t)(f(X_t, \theta) - f(\widehat{X}_t, \theta)) \\ &\quad - 2K_t\mathbb{E}(X_t - \widehat{X}_t)(h(X_t) - h(\widehat{X}_t)) + 1 + K_t^2] dt \\ &= [K_t^2 - 2K_t\mathbb{E}(X_t - \widehat{X}_t)^2 + 1 + 2K_t\mathbb{E}(X_t - \widehat{X}_t)^2 \\ &\quad + 2\mathbb{E}(X_t - \widehat{X}_t)(f(X_t, \theta) - f(\widehat{X}_t, \theta))] dt \\ &\quad - 2K_t\mathbb{E}(X_t - \widehat{X}_t)(h(X_t) - h(\widehat{X}_t)) dt \\ &= [(K_t - \mathbb{E}(X_t - \widehat{X}_t)^2)^2 - (\mathbb{E}(X_t - \widehat{X}_t)^2)^2 + 1] dt \\ &\quad + 2K_t\mathbb{E}(X_t - \widehat{X}_t)^2 dt - 2K_t\mathbb{E}(X_t - \widehat{X}_t)(h(X_t) - h(\widehat{X}_t)) dt \\ &\quad + 2\mathbb{E}(X_t - \widehat{X}_t)(f(X_t, \theta) - f(\widehat{X}_t, \theta)) dt \\ &\geq [-(\mathbb{E}(X_t - \widehat{X}_t)^2)^2 + 1 + 2\mathbb{E}(X_t - \widehat{X}_t)(f(X_t, \theta) - f(\widehat{X}_t, \theta))] dt. \end{aligned} \tag{5}$$

Therefore, when $K_t = \mathbb{E}(X_t - \widehat{X}_t)^2$, (5) has the minimum value

$$d\mathbb{E}(X_t - \widehat{X}_t)^2 = [-(\mathbb{E}(X_t - \widehat{X}_t)^2)^2 + 1 + 2\mathbb{E}(X_t - \widehat{X}_t)(f(X_t, \theta) - f(\widehat{X}_t, \theta))] dt. \tag{6}$$

From Assumption 1, one has

$$\begin{aligned} & [-(\mathbb{E}(X_t - \widehat{X}_t)^2)^2 + 1 + 2\mathbb{E}(X_t - \widehat{X}_t)(f(X_t, \theta) - f(\widehat{X}_t, \theta))] dt \\ & \leq [-(\mathbb{E}(X_t - \widehat{X}_t)^2)^2 + 1 + 2K(\theta)\mathbb{E}(X_t - \widehat{X}_t)^2] dt. \end{aligned}$$

Since $f(X_t, \theta)$ is nonlinear, we cannot obtain the optimal state estimation of X_t , the sub-optimal state estimation is considered.

Consider

$$d\mathbb{E}(X_t - \widehat{X}_t)^2 = (2K(\theta)\mathbb{E}(X_t - \widehat{X}_t)^2 - (\mathbb{E}(X_t - \widehat{X}_t)^2)^2 + 1) dt. \tag{7}$$

Let

$$\mathbb{E}(X_t - \widehat{X}_t)^2 = \gamma_t, \tag{8}$$

one has

$$d\gamma_t = (2K(\theta)\gamma_t - \gamma_t^2 + 1) dt. \tag{9}$$

It is easy to check that

$$\gamma_t = \frac{(\sqrt{K^2(\theta) + 1} + K(\theta))(1 - e^{-2t\sqrt{K^2(\theta)+1}})}{1 + \frac{K(\theta) + \sqrt{K^2(\theta)+1}}{\sqrt{K^2(\theta)+1-K(\theta)}}e^{-2t\sqrt{K^2(\theta)+1}}}. \tag{10}$$

Then we obtain

$$\gamma_t \rightarrow K(\theta) + \sqrt{K^2(\theta) + 1} = \gamma(\theta). \tag{11}$$

Therefore, it is obvious that

$$\begin{cases} d\widehat{X}_t = f(\widehat{X}_t, \theta) dt + \gamma_t(dY_t - \widehat{X}_t dt), \\ \widehat{X}_0 = X_0. \end{cases} \tag{12}$$

Let

$$dV_t^* = dY_t - \widehat{X}_t dt, \tag{13}$$

where $(V_t^*, t \geq 0)$ is assumed to be a standard Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$.

Hence

$$\begin{cases} d\widehat{X}_t = f(\widehat{X}_t, \theta) dt + \gamma_t dV_t^*, \\ \widehat{X}_0 = X_0. \end{cases} \tag{14}$$

It is assumed that the system (14) reaches the steady state, which means that

$$\begin{cases} d\widehat{X}_t = f(\widehat{X}_t, \theta) dt + \gamma(\theta) dV_t^*, \\ \widehat{X}_0 = X_0. \end{cases} \tag{15}$$

In summary, the suboptimal state estimation of X_t is (15).

The likelihood function obeys the expression

$$\ell_t(\theta) = \int_0^t \frac{f(\widehat{X}_s, \theta)}{\gamma^2(\theta)} d\widehat{X}_s - \frac{1}{2} \int_0^t \frac{f^2(\widehat{X}_s, \theta)}{\gamma^2(\theta)} ds. \tag{16}$$

Then the approximate likelihood function can be written as

$$\ell_n(\theta) = \sum_{i=1}^n \frac{f(\widehat{X}_{t_{i-1}}, \theta)}{\gamma^2(\theta)} (\widehat{X}_{t_i} - \widehat{X}_{t_{i-1}}) - \frac{\Delta}{2} \sum_{i=1}^n \frac{f^2(\widehat{X}_{t_{i-1}}, \theta)}{\gamma^2(\theta)}. \tag{17}$$

Remark 1 That system (14) reaches the steady state means that the Riccati equation satisfies $\frac{d\gamma_t}{dt} = 0$. Hence, we obtain $\gamma_t = \gamma(\theta)$.

Remark 2 In (15), both drift item and diffusion item have a parameter. Thus, it is difficult to discuss the asymptotic property of the estimator. In the next section, the problem is solved.

3 Main results and proofs

The following lemmas are useful to derive our results.

Lemma 1 Assume that $\{\widehat{X}_t\}$ is a solution of the stochastic differential (14) and Assumptions 1–4 hold. Then, for any integer $n \geq 1$ and $0 \leq s \leq t$,

$$\mathbb{E}|\widehat{X}_t - \widehat{X}_s|^{2p} = O(|t - s|^p).$$

Proof Suppose θ_0 is the true parameter value; by applying Hölder’s inequality, it follows that

$$\begin{aligned} & |\widehat{X}_t - \widehat{X}_s|^{2p} \\ &= \left| \int_s^t f(\widehat{X}_u, \theta_0) du + \gamma(\theta_0) \int_s^t dV_u^* \right|^{2p} \\ &\leq 2^{2p-1} \left(\left| \int_s^t f(\widehat{X}_u, \theta_0) du \right|^{2p} + (\gamma(\theta_0))^{2p} \left| \int_s^t dV_u^* \right|^{2p} \right) \\ &\leq 2^{2p-1} \left((t - s)^{2p-1} \int_s^t |f(\widehat{X}_u, \theta_0)|^{2p} du + (\gamma(\theta_0))^{2p} \left| \int_s^t dV_u^* \right|^{2p} \right). \end{aligned}$$

Since

$$|f(\widehat{X}_u, \theta_0)|^{2p} \leq K_1(\theta_0)^p 2^{p-1} (1 + |\widehat{X}_u|^{2p}), \tag{18}$$

from Assumption 3 together with the stationarity of the process, one has

$$\mathbb{E} \left[\int_s^t |f(\widehat{X}_u, \theta_0)|^{2p} du \right] = O(|t - s|). \tag{19}$$

From the Burkholder–Davis–Gundy inequality, it can be checked that

$$\mathbb{E} \left[\left| \int_s^t dV_u^* \right|^{2p} \right] \leq C_p \mathbb{E} \left| \int_s^t du \right|^p = C_p (t - s)^{2p}, \tag{20}$$

where C_p is a positive constant depending only on p .

Then we have

$$\mathbb{E} \left[\left| \int_s^t dV_u^* \right|^{2p} \right] = O(|t - s|^p). \tag{21}$$

From the above analysis, it follows that

$$\mathbb{E} |\widehat{X}_t - \widehat{X}_s|^{2p} = O(|t - s|^p). \tag{22}$$

The proof is complete. □

Lemma 2 *Under Assumptions 1 and 3, when $\Delta \rightarrow 0$, one has*

$$\mathbb{E} \left| \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{f(\widehat{X}_{t_{i-1}}, \theta) f(\widehat{X}_s, \theta_0)}{\gamma^2(\theta)} ds - \sum_{i=1}^n \frac{f(\widehat{X}_{t_{i-1}}, \theta) f(\widehat{X}_{t_{i-1}}, \theta_0)}{\gamma^2(\theta)} \Delta \right| \rightarrow 0.$$

Proof By applying Hölder’s inequality, it can be checked that

$$\begin{aligned} & \mathbb{E} \left| \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{f(\widehat{X}_{t_{i-1}}, \theta) f(\widehat{X}_s, \theta_0)}{\gamma^2(\theta)} ds - \sum_{i=1}^n \frac{f(\widehat{X}_{t_{i-1}}, \theta) f(\widehat{X}_{t_{i-1}}, \theta_0)}{\gamma^2(\theta)} \Delta \right| \\ &= \mathbb{E} \left| \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{f(\widehat{X}_{t_{i-1}}, \theta) (f(\widehat{X}_s, \theta_0) - f(\widehat{X}_{t_{i-1}}, \theta_0))}{\gamma^2(\theta)} ds \right| \\ &\leq \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \mathbb{E} \left| \frac{f(\widehat{X}_{t_{i-1}}, \theta) (f(\widehat{X}_s, \theta_0) - f(\widehat{X}_{t_{i-1}}, \theta_0))}{\gamma^2(\theta)} \right| ds \\ &\leq \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \left(\mathbb{E} \left[\frac{f(\widehat{X}_{t_{i-1}}, \theta)^2}{\gamma^2(\theta)} \right] \right)^{\frac{1}{2}} \left(\mathbb{E} [f(\widehat{X}_s, \theta_0) - f(\widehat{X}_{t_{i-1}}, \theta_0)]^2 \right)^{\frac{1}{2}} ds. \end{aligned}$$

From Lemma 1 together with Assumptions 1 and 3, we have

$$\mathbb{E} [f(\widehat{X}_s, \theta_0) - f(\widehat{X}_{t_{i-1}}, \theta_0)]^2 = O(\Delta), \tag{23}$$

and $\mathbb{E} \left[\frac{f(\widehat{X}_{t_{i-1}}, \theta)}{\gamma^2(\theta)} \right]^2$ is bounded.

From the above analysis, it follows that

$$\mathbb{E} \left| \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{f(\widehat{X}_{t_{i-1}}, \theta) f(\widehat{X}_s, \theta)}{\gamma^2(\theta)} ds - \sum_{i=1}^n \frac{f(\widehat{X}_{t_{i-1}}, \theta) f(\widehat{X}_{t_{i-1}}, \theta)}{\gamma^2(\theta)} \Delta \right| \rightarrow 0, \tag{24}$$

as $\Delta \rightarrow 0$.

The proof is complete. □

Lemma 3 *Under Assumptions 1 and 3, we have*

$$\mathbb{E} \left| \sum_{i=1}^n \frac{f^2(\widehat{X}_{t_{i-1}}, \theta)}{\gamma^2(\theta)} \Delta - \int_0^t \frac{f^2(\widehat{X}_s, \theta)}{\gamma^2(\theta)} ds \right| \rightarrow 0,$$

as $\Delta \rightarrow 0$.

Proof From the Hölder inequality and Assumption 1 together with the stationarity of the process, one has

$$\begin{aligned} & \mathbb{E} \left| \sum_{i=1}^n \frac{f^2(\widehat{X}_{t_{i-1}}, \theta)}{\gamma^2(\theta)} \Delta - \int_0^t \frac{f^2(\widehat{X}_s, \theta)}{\gamma^2(\theta)} ds \right| \\ &= \mathbb{E} \left| \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \left[\frac{f^2(\widehat{X}_{t_{i-1}}, \theta)}{\gamma^2(\theta)} - \frac{f^2(\widehat{X}_s, \theta)}{\gamma^2(\theta)} \right] ds \right| \\ &= \mathbb{E} \left| \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{f^2(\widehat{X}_{t_{i-1}}, \theta) - f^2(\widehat{X}_s, \theta)}{\gamma^2(\theta)} ds \right| \\ &\leq \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \mathbb{E} \left[\frac{|f(\widehat{X}_{t_{i-1}}, \theta) + f(\widehat{X}_s, \theta)| |f(\widehat{X}_{t_{i-1}}, \theta) - f(\widehat{X}_s, \theta)|}{\gamma^2(\theta)} \right] ds \\ &\leq \frac{K_2(\theta)}{\gamma^2(\theta)} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} (\mathbb{E} |\widehat{X}_{t_{i-1}} - \widehat{X}_s|^2)^{\frac{1}{2}} ds, \end{aligned}$$

where $\sup\{K_2(\theta)\} < \infty$.

According to the above analysis together with Lemma 1, it follows that

$$\mathbb{E} \left| \sum_{i=1}^n \frac{f^2(\widehat{X}_{t_{i-1}}, \theta)}{\gamma^2(\theta)} \Delta - \int_0^t \frac{f^2(\widehat{X}_s, \theta)}{\gamma^2(\theta)} ds \right| \rightarrow 0,$$

as $\Delta \rightarrow 0$.

The proof is complete. □

Remark 3 By employing the Hölder inequality, the Burkholder–Davis–Gundy inequality and the stationarity of the process, the above lemmas have been proved. These lemmas play a key role in the proof of the following main results.

In the following theorem, the consistency in probability of the maximum likelihood estimator is proved by applying a martingale moment inequality, the Chebyshev inequality, the uniform ergodic theorem and the results of Lemmas 1–3.

Theorem 1 *When $\Delta \rightarrow 0, n \rightarrow \infty$ and $n\Delta \rightarrow \infty,$*

$$\widehat{\theta}_0 \xrightarrow{P} \theta_0.$$

Proof According to the expression of the approximate likelihood function and Eq. (1), it follows that

$$\begin{aligned} \ell_n(\theta) &= \sum_{i=1}^n \frac{f(\widehat{X}_{t_{i-1}}, \theta)}{\gamma^2(\theta)} \left(\int_{t_{i-1}}^{t_i} f(\widehat{X}_s, \theta_0) ds + \gamma(\theta_0) \int_{t_{i-1}}^{t_i} dV_s^* \right) - \frac{\Delta}{2} \sum_{i=1}^n \frac{f^2(\widehat{X}_{t_{i-1}}, \theta)}{\gamma^2(\theta)} \\ &= \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{f(\widehat{X}_{t_{i-1}}, \theta)f(\widehat{X}_s, \theta_0)}{\gamma^2(\theta)} ds + \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{f(\widehat{X}_{t_{i-1}}, \theta)\gamma(\theta_0)}{\gamma^2(\theta)} dV_s^* \\ &\quad - \frac{1}{2} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{f^2(\widehat{X}_{t_{i-1}}, \theta)}{\gamma^2(\theta)} ds \\ &= \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{f(\widehat{X}_{t_{i-1}}, \theta)(f(\widehat{X}_s, \theta_0) - \frac{1}{2}f(\widehat{X}_{t_{i-1}}, \theta))}{\gamma^2(\theta)} ds \\ &\quad + \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{f(\widehat{X}_{t_{i-1}}, \theta)\gamma(\theta_0)}{\gamma^2(\theta)} dV_s^*. \end{aligned}$$

Then we have

$$\begin{aligned} &\frac{1}{n\Delta} \ell_n(\theta) \\ &= \frac{1}{n\Delta} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{f(\widehat{X}_{t_{i-1}}, \theta)(f(\widehat{X}_s, \theta_0) - \frac{1}{2}f(\widehat{X}_{t_{i-1}}, \theta))}{\gamma^2(\theta)} ds \\ &\quad + \frac{1}{n\Delta} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{f(\widehat{X}_{t_{i-1}}, \theta)\gamma(\theta_0)}{\gamma^2(\theta)} dV_s^*. \end{aligned} \tag{25}$$

From the martingale moment inequality, it can be checked that

$$\begin{aligned} &\mathbb{E} \left| \frac{1}{n\Delta} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{f(\widehat{X}_{t_{i-1}}, \theta)\gamma(\theta_0)}{\gamma^2(\theta)} dV_s^* \right|^2 \\ &\leq \frac{1}{(n\Delta)^2} C \frac{\gamma^2(\theta_0)}{\gamma^4(\theta)} \mathbb{E} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} (f(\widehat{X}_{t_{i-1}}, \theta))^2 ds \\ &\leq \frac{1}{n\Delta} C_1 \\ &\rightarrow 0, \end{aligned}$$

where C and C_1 are constants.

By applying the Chebyshev inequality, it can be found that

$$\frac{1}{n\Delta} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{f(\widehat{X}_{t_{i-1}}, \theta)\gamma(\theta_0)}{\gamma^2(\theta)} dV_s^* \xrightarrow{P} 0, \tag{26}$$

when $\Delta \rightarrow 0, n \rightarrow \infty$ and $n\Delta \rightarrow \infty.$

From Lemmas 2–3 together with the uniform ergodic theorem (see e.g. [22]), one has

$$\begin{aligned} & \frac{1}{n\Delta} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{f(\widehat{X}_{t_{i-1}}, \theta)(f(\widehat{X}_s, \theta) - \frac{1}{2}f(\widehat{X}_{t_{i-1}}, \theta))}{\gamma^2(\theta)} ds \\ & \xrightarrow{P} \mathbb{E} \left[\frac{f(x_0, \theta)(f(x_0, \theta) - \frac{1}{2}f(x_0, \theta))}{\gamma^2(\theta)} \right], \end{aligned}$$

when $\Delta \rightarrow 0, n \rightarrow \infty$ and $n\Delta \rightarrow \infty$.

Hence, it leads to the relation

$$\frac{1}{n\Delta} \ell_n(\theta) \xrightarrow{P} \mathbb{E} \left[\frac{f(x_0, \theta)(f(x_0, \theta) - \frac{1}{2}f(x_0, \theta))}{\gamma^2(\theta)} \right], \tag{27}$$

when $\Delta \rightarrow 0, n \rightarrow \infty$ and $n\Delta \rightarrow \infty$.

From Assumption 4, it is easy to check that

$$\widehat{\theta}_0 \xrightarrow{P} \theta_0, \tag{28}$$

when $\Delta \rightarrow 0, n \rightarrow \infty$ and $n\Delta \rightarrow \infty$.

The proof is complete. □

In the following theorem, the asymptotic normality of the error of estimation is proved by employing the martingale moment inequality, the Chebyshev inequality, the uniform ergodic theorem and the dominated convergence theorem.

Theorem 2 *When $\Delta \rightarrow 0, n^{\frac{1}{2}} \Delta \rightarrow 0$ and $n\Delta \rightarrow \infty$ as $n \rightarrow \infty$,*

$$\sqrt{n\Delta}(\theta_0 - \widehat{\theta}_0) \xrightarrow{d} N \left(0, \frac{\gamma^2(\theta_0)}{\mathbb{E}[f'(x_0, \theta_0)\gamma(\theta_0) - 2f(x_0, \theta_0)\gamma'(\theta_0)]^2} \right).$$

Proof Expanding $\ell'_n(\theta_0)$ about $\widehat{\theta}_0$, it follows that

$$\ell'_n(\theta_0) = \ell'_n(\widehat{\theta}_0) + \ell''_n(\widetilde{\theta})(\theta_0 - \widehat{\theta}_0), \tag{29}$$

where $\widetilde{\theta}$ is between $\widehat{\theta}_0$ and θ_0 .

In view of Theorem 1, it is well known that $\ell'_n(\widehat{\theta}_0) = 0$; then

$$\ell'_n(\theta_0) = \ell''_n(\widetilde{\theta})(\theta_0 - \widehat{\theta}_0). \tag{30}$$

Since

$$\begin{aligned} & \ell''_n(\theta) \\ & = \sum_{i=1}^n \frac{f''(\widehat{X}_{t_{i-1}}, \theta)\gamma^2(\theta) - 4\gamma(\theta)\gamma'(\theta)f'(\widehat{X}_{t_{i-1}}, \theta)}{\gamma^4(\theta)} (X_{t_i} - X_{t_{i-1}}) \end{aligned}$$

$$\begin{aligned}
 & + \sum_{i=1}^n \frac{6f(\widehat{X}_{t_{i-1}}, \theta)(\gamma'(\theta))^2 - 2\gamma(\theta)\gamma''(\theta)f(\widehat{X}_{t_{i-1}}, \theta)}{\gamma^4(\theta)} (X_{t_i} - X_{t_{i-1}}) \\
 & + \Delta \sum_{i=1}^n \frac{4\gamma(\theta)\gamma'(\theta)f'(\widehat{X}_{t_{i-1}}, \theta)f(\widehat{X}_{t_{i-1}}, \theta)}{\gamma^4(\theta)} \\
 & - \Delta \sum_{i=1}^n \frac{f''(\widehat{X}_{t_{i-1}}, \theta)f(\widehat{X}_{t_{i-1}}, \theta)\gamma^2(\theta) - 3f^2(\widehat{X}_{t_{i-1}}, \theta)(\gamma'(\theta))^2}{\gamma^4(\theta)} \\
 & + \Delta \sum_{i=1}^n \frac{\gamma(\theta)\gamma''(\theta)f^2(\widehat{X}_{t_{i-1}}, \theta) - \gamma^2(\theta)(f'(\widehat{X}_{t_{i-1}}, \theta))^2}{\gamma^4(\theta)} \\
 = & \sum_{i=1}^n \frac{f''(\widehat{X}_{t_{i-1}}, \theta)\gamma^2(\theta) - 4\gamma(\theta)\gamma'(\theta)f'(\widehat{X}_{t_{i-1}}, \theta)}{\gamma^4(\theta)} \int_{t_{i-1}}^{t_i} f(\widehat{X}_s, \theta_0) ds \\
 & + \sum_{i=1}^n \frac{f''(\widehat{X}_{t_{i-1}}, \theta)\gamma^2(\theta) - 4\gamma(\theta)\gamma'(\theta)f'(\widehat{X}_{t_{i-1}}, \theta)}{\gamma^4(\theta)} \gamma(\theta_0) \int_{t_{i-1}}^{t_i} dV_s^* \\
 & + \sum_{i=1}^n \frac{6f(\widehat{X}_{t_{i-1}}, \theta)(\gamma'(\theta))^2 - 2\gamma(\theta)\gamma''(\theta)f(\widehat{X}_{t_{i-1}}, \theta)}{\gamma^4(\theta)} \int_{t_{i-1}}^{t_i} f(\widehat{X}_s, \theta_0) ds \\
 & + \sum_{i=1}^n \frac{6f(\widehat{X}_{t_{i-1}}, \theta)(\gamma'(\theta))^2 - 2\gamma(\theta)\gamma''(\theta)f(\widehat{X}_{t_{i-1}}, \theta)}{\gamma^4(\theta)} \gamma(\theta_0) \int_{t_{i-1}}^{t_i} dV_s^* \\
 & + \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{4\gamma(\theta)\gamma'(\theta)f'(\widehat{X}_{t_{i-1}}, \theta)f(\widehat{X}_{t_{i-1}}, \theta)}{\gamma^4(\theta)} ds \\
 & - \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{f''(\widehat{X}_{t_{i-1}}, \theta)f(\widehat{X}_{t_{i-1}}, \theta)\gamma^2(\theta) - 3f^2(\widehat{X}_{t_{i-1}}, \theta)(\gamma'(\theta))^2}{\gamma^4(\theta)} ds \\
 & + \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{\gamma(\theta)\gamma''(\theta)f^2(\widehat{X}_{t_{i-1}}, \theta) - \gamma^2(\theta)(f'(\widehat{X}_{t_{i-1}}, \theta))^2}{\gamma^4(\theta)} ds \\
 = & \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{f''(\widehat{X}_{t_{i-1}}, \theta)(f(\widehat{X}_s, \theta_0) - f(\widehat{X}_{t_{i-1}}, \theta))}{\gamma^2(\theta)} ds \\
 & + \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{4f'(\widehat{X}_{t_{i-1}}, \theta)\gamma'(\theta)(f(\widehat{X}_{t_{i-1}}, \theta) - f(\widehat{X}_s, \theta_0))}{\gamma^3(\theta)} ds \\
 & + \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{3f(\widehat{X}_{t_{i-1}}, \theta)(\gamma'(\theta))^2(2f(\widehat{X}_s, \theta_0) - f(\widehat{X}_{t_{i-1}}, \theta))}{\gamma^4(\theta)} ds \\
 & + \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{f(\widehat{X}_{t_{i-1}}, \theta)\gamma(\theta)\gamma''(\theta)(f(\widehat{X}_{t_{i-1}}, \theta) - 2f(\widehat{X}_s, \theta_0))}{\gamma^3(\theta)} ds \\
 & - \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{(f'(\widehat{X}_{t_{i-1}}, \theta))^2}{\gamma^2(\theta)} ds \\
 & + \gamma(\theta_0) \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{f''(\widehat{X}_{t_{i-1}}, \theta)\gamma^2(\theta) - 4\gamma(\theta)\gamma'(\theta)f'(\widehat{X}_{t_{i-1}}, \theta)}{\gamma^4(\theta)} dV_s^* \\
 & + \gamma(\theta_0) \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{6f(\widehat{X}_{t_{i-1}}, \theta)(\gamma'(\theta))^2 - 2\gamma(\theta)\gamma''(\theta)f(\widehat{X}_{t_{i-1}}, \theta)}{\gamma^4(\theta)} dV_s^*
 \end{aligned}$$

we have

$$\begin{aligned}
 \ell_n''(\theta_0) &= \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{f''(\widehat{X}_{t_{i-1}}, \theta_0)(f(\widehat{X}_s, \theta_0) - f(\widehat{X}_{t_{i-1}}, \theta_0))}{\gamma^2(\theta_0)} ds \\
 &\quad + \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{4f'(\widehat{X}_{t_{i-1}}, \theta_0)\gamma'(\theta_0)(f(\widehat{X}_{t_{i-1}}, \theta_0) - f(\widehat{X}_s, \theta_0))}{\gamma^3(\theta_0)} ds \\
 &\quad + \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{3f(\widehat{X}_{t_{i-1}}, \theta_0)(\gamma'(\theta_0))^2(2f(\widehat{X}_s, \theta_0) - f(\widehat{X}_{t_{i-1}}, \theta_0))}{\gamma^4(\theta_0)} ds \\
 &\quad + \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{f(\widehat{X}_{t_{i-1}}, \theta_0)\gamma''(\theta_0)(f(\widehat{X}_{t_{i-1}}, \theta_0) - 2f(\widehat{X}_s, \theta_0))}{\gamma^2(\theta_0)} ds \\
 &\quad - \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{(f'(\widehat{X}_{t_{i-1}}, \theta_0))^2}{\gamma^2(\theta_0)} ds \\
 &\quad + \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{f''(\widehat{X}_{t_{i-1}}, \theta_0)\gamma(\theta_0) - 4\gamma'(\theta_0)f'(\widehat{X}_{t_{i-1}}, \theta_0)}{\gamma^2(\theta_0)} dV_s^* \\
 &\quad + \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{6f(\widehat{X}_{t_{i-1}}, \theta_0)(\gamma'(\theta_0))^2 - 2\gamma(\theta_0)\gamma''(\theta_0)f(\widehat{X}_{t_{i-1}}, \theta_0)}{\gamma^3(\theta_0)} dV_s^*.
 \end{aligned}$$

From the same method used in Theorem 1, it is easy to check that

$$\frac{1}{n\Delta} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{f''(\widehat{X}_{t_{i-1}}, \theta_0)\gamma(\theta_0) - 4\gamma'(\theta_0)f'(\widehat{X}_{t_{i-1}}, \theta_0)}{\gamma^2(\theta_0)} dV_s^* \xrightarrow{P} 0 \tag{31}$$

and

$$\frac{1}{n\Delta} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{6f(\widehat{X}_{t_{i-1}}, \theta_0)(\gamma'(\theta_0))^2 - 2\gamma(\theta_0)\gamma''(\theta_0)f(\widehat{X}_{t_{i-1}}, \theta_0)}{\gamma^3(\theta_0)} dV_s^* \xrightarrow{P} 0. \tag{32}$$

By applying the results of Lemmas 2–3 and the uniform ergodic theorem, it follows that

$$\frac{1}{n\Delta} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{f''(\widehat{X}_{t_{i-1}}, \theta_0)(f(\widehat{X}_s, \theta_0) - f(\widehat{X}_{t_{i-1}}, \theta_0))}{\gamma^2(\theta_0)} ds \xrightarrow{P} 0, \tag{33}$$

$$\frac{1}{n\Delta} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{4f'(\widehat{X}_{t_{i-1}}, \theta_0)\gamma'(\theta_0)(f(\widehat{X}_{t_{i-1}}, \theta_0) - f(\widehat{X}_s, \theta_0))}{\gamma^3(\theta_0)} ds \xrightarrow{P} 0, \tag{34}$$

$$\begin{aligned}
 &\frac{1}{n\Delta} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{3f(\widehat{X}_{t_{i-1}}, \theta_0)(\gamma'(\theta_0))^2(2f(\widehat{X}_s, \theta_0) - f(\widehat{X}_{t_{i-1}}, \theta_0))}{\gamma^4(\theta_0)} ds \\
 &\xrightarrow{P} 3 \frac{(\gamma'(\theta_0))^2}{\gamma^4(\theta_0)} \mathbb{E}[f(x_0, \theta_0)]^2,
 \end{aligned}$$

$$\begin{aligned} & \frac{1}{n\Delta} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{f(\widehat{X}_{t_{i-1}}, \theta_0) \gamma''(\theta_0) (f(\widehat{X}_{t_{i-1}}, \theta_0) - 2f(\widehat{X}_s, \theta_0))}{\gamma^2(\theta_0)} ds \\ & \xrightarrow{P} -\frac{(\gamma''(\theta_0))^2}{\gamma^2(\theta_0)} \mathbb{E}[f(x_0, \theta_0)]^2, \end{aligned}$$

and

$$\frac{1}{n\Delta} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{(f'(\widehat{X}_{t_{i-1}}, \theta_0))^2}{\gamma^2(\theta_0)} ds \xrightarrow{P} \frac{1}{\gamma^2(\theta_0)} \mathbb{E}[f'(x_0, \theta_0)]^2. \tag{35}$$

Therefore, we have

$$\begin{aligned} & \frac{1}{n\Delta} \ell''_n(\theta_0) \\ & \xrightarrow{P} \mathbb{E} \left[\frac{3(\gamma'(\theta_0))^2 - \gamma^2(\theta_0)(\gamma''(\theta_0))^2}{\gamma^4(\theta_0)} (f(x_0, \theta_0))^2 + \frac{1}{\gamma^2(\theta_0)} (f'(x_0, \theta_0))^2 \right]. \end{aligned}$$

According to the expression of $\ell''_n(\theta)$ and by employing the martingale moment inequality, the Chebyshev inequality, the uniform ergodic theorem and the dominated convergence theorem, it follows that

$$\frac{1}{n\Delta} (\ell''_n(\tilde{\theta}) - \ell''_n(\theta_0)) \xrightarrow{P} 0. \tag{36}$$

Hence, it can be found that

$$\frac{1}{n\Delta} \ell''_n(\tilde{\theta}) \xrightarrow{P} \mathbb{E} \left[\frac{3(\gamma'(\theta_0))^2 - \gamma^2(\theta_0)(\gamma''(\theta_0))^2}{\gamma^4(\theta_0)} (f(x_0, \theta_0))^2 + \frac{1}{\gamma^2(\theta_0)} (f'(x_0, \theta_0))^2 \right]. \tag{37}$$

Since

$$\begin{aligned} & \ell'_n(\theta) \\ & = \sum_{i=1}^n \frac{f'(\widehat{X}_{t_{i-1}}, \theta) \gamma(\theta) - 2f(\widehat{X}_{t_{i-1}}, \theta) \gamma'(\theta)}{\gamma^3(\theta)} (\widehat{X}_{t_i} - \widehat{X}_{t_{i-1}}) \\ & \quad - \Delta \sum_{i=1}^n \frac{f(\widehat{X}_{t_{i-1}}, \theta) f'(\widehat{X}_{t_{i-1}}, \theta) \gamma(\theta) - f^2(\widehat{X}_{t_{i-1}}, \theta) \gamma'(\theta)}{\gamma^3(\theta)} \\ & = \sum_{i=1}^n \frac{f'(\widehat{X}_{t_{i-1}}, \theta) \gamma(\theta) - 2f(\widehat{X}_{t_{i-1}}, \theta) \gamma'(\theta)}{\gamma^3(\theta)} \left(\int_{t_{i-1}}^{t_i} f(\widehat{X}_s, \theta_0) ds + \gamma(\theta_0) \int_{t_{i-1}}^{t_i} dV_s^* \right) \\ & \quad - \Delta \sum_{i=1}^n \frac{f(\widehat{X}_{t_{i-1}}, \theta) f'(\widehat{X}_{t_{i-1}}, \theta) \gamma(\theta) - f^2(\widehat{X}_{t_{i-1}}, \theta) \gamma'(\theta)}{\gamma^3(\theta)} \\ & = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{f'(\widehat{X}_{t_{i-1}}, \theta) (f(\widehat{X}_s, \theta_0) - f(\widehat{X}_{t_{i-1}}, \theta))}{\gamma^2(\theta)} ds \\ & \quad + \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{f(\widehat{X}_{t_{i-1}}, \theta) \gamma'(\theta) (f(\widehat{X}_{t_{i-1}}, \theta) - 2f(\widehat{X}_s, \theta_0))}{\gamma^3(\theta)} ds \\ & \quad + \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \gamma(\theta_0) \frac{f'(\widehat{X}_{t_{i-1}}, \theta) \gamma(\theta) - 2f(\widehat{X}_{t_{i-1}}, \theta) \gamma'(\theta)}{\gamma^3(\theta)} dV_s^*, \end{aligned}$$

it follows that

$$\begin{aligned} & \frac{1}{\sqrt{n\Delta}} \ell'_n(\theta_0) \\ &= \frac{1}{\sqrt{n\Delta}} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{f'(\widehat{X}_{t_{i-1}}, \theta_0)(f(\widehat{X}_s, \theta_0) - f(\widehat{X}_{t_{i-1}}, \theta_0))}{\gamma^2(\theta_0)} ds \\ & \quad + \frac{1}{\sqrt{n\Delta}} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{f(\widehat{X}_{t_{i-1}}, \theta_0)\gamma'(\theta_0)(f(X_{t_{i-1}}, \theta_0) - 2f(\widehat{X}_s, \theta_0))}{\gamma^3(\theta_0)} ds \\ & \quad + \frac{1}{\sqrt{n\Delta}} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{f'(\widehat{X}_{t_{i-1}}, \theta_0)\gamma(\theta_0) - 2f(\widehat{X}_{t_{i-1}}, \theta_0)\gamma'(\theta_0)}{\gamma^2(\theta_0)} dV_s^*. \end{aligned}$$

As

$$\begin{aligned} & \mathbb{E} \left| \frac{1}{\sqrt{n\Delta}} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{f'(\widehat{X}_{t_{i-1}}, \theta_0)(f(\widehat{X}_s, \theta_0) - f(\widehat{X}_{t_{i-1}}, \theta_0))}{\gamma^2(\theta_0)} ds \right| \\ & \leq \frac{1}{\sqrt{n\Delta}} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \mathbb{E} \left| \frac{f'(\widehat{X}_{t_{i-1}}, \theta_0)(f(\widehat{X}_s, \theta_0) - f(\widehat{X}_{t_{i-1}}, \theta_0))}{\gamma^2(\theta_0)} \right| ds \\ & \leq \frac{\sum_{i=1}^n \int_{t_{i-1}}^{t_i} (\mathbb{E}[f'(\widehat{X}_{t_{i-1}}, \theta_0)]^2)^{\frac{1}{2}} (\mathbb{E}[f(\widehat{X}_s, \theta_0) - f(\widehat{X}_{t_{i-1}}, \theta_0)]^2)^{\frac{1}{2}} ds}{\gamma^2(\theta_0)\sqrt{n\Delta}}, \end{aligned}$$

it is easy to check that $\mathbb{E}[f'(\widehat{X}_{t_{i-1}}, \theta_0)]^2$ is bounded.

From Lemma 1 and Assumption 1, we have

$$\mathbb{E}[f(\widehat{X}_s, \theta_0) - f(\widehat{X}_{t_{i-1}}, \theta_0)]^2 = O(\Delta). \tag{38}$$

Then it follows that

$$\mathbb{E} \left| \frac{1}{\sqrt{n\Delta}} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{f'(\widehat{X}_{t_{i-1}}, \theta_0)(f(\widehat{X}_s, \theta_0) - f(\widehat{X}_{t_{i-1}}, \theta_0))}{\gamma^2(\theta_0)} ds \right| \rightarrow 0, \tag{39}$$

when $\Delta \rightarrow 0$, $n^{\frac{1}{2}}\Delta \rightarrow 0$ and $n\Delta \rightarrow \infty$ as $n \rightarrow \infty$.

By applying the Chebyshev inequality, it can be found that

$$\frac{1}{\sqrt{n\Delta}} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{f'(\widehat{X}_{t_{i-1}}, \theta_0)(f(\widehat{X}_s, \theta_0) - f(\widehat{X}_{t_{i-1}}, \theta_0))}{\gamma^2(\theta_0)} ds \xrightarrow{P} 0. \tag{40}$$

By applying the same methods, we have

$$\frac{1}{\sqrt{n\Delta}} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{f(\widehat{X}_{t_{i-1}}, \theta_0)\gamma'(\theta_0)(f(X_{t_{i-1}}, \theta_0) - 2f(\widehat{X}_s, \theta_0))}{\gamma^3(\theta_0)} ds \xrightarrow{P} 0. \tag{41}$$

It is obvious that

$$\begin{aligned} & \frac{1}{\sqrt{n\Delta}} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{f'(\widehat{X}_{t_{i-1}}, \theta_0)\gamma(\theta_0) - 2f(\widehat{X}_{t_{i-1}}, \theta_0)\gamma'(\theta_0)}{\gamma^2(\theta_0)} dV_s^* \\ & \xrightarrow{d} N\left(0, \frac{1}{\gamma^2(\theta_0)} \mathbb{E}[f'(x_0, \theta_0)\gamma(\theta_0) - 2f(x_0, \theta_0)\gamma'(\theta_0)]^2\right). \end{aligned}$$

Hence, we have

$$\frac{1}{\sqrt{n\Delta}} \ell'_n(\theta_0) \xrightarrow{d} N\left(0, \frac{1}{\gamma^2(\theta_0)} \mathbb{E}[f'(x_0, \theta_0)\gamma(\theta_0) - 2f(x_0, \theta_0)\gamma'(\theta_0)]^2\right), \tag{42}$$

when $\Delta \rightarrow 0, n^{\frac{1}{2}}\Delta \rightarrow 0$ and $n\Delta \rightarrow \infty$ as $n \rightarrow \infty$.

From the above analysis, it can be checked that

$$\sqrt{n\Delta}(\theta_0 - \widehat{\theta}_0) \xrightarrow{d} N\left(0, \frac{\gamma^2(\theta_0)}{\mathbb{E}[f'(x_0, \theta_0)\gamma(\theta_0) - 2f(x_0, \theta_0)\gamma'(\theta_0)]^2}\right), \tag{43}$$

when $\Delta \rightarrow 0, n^{\frac{1}{2}}\Delta \rightarrow 0$ and $n\Delta \rightarrow \infty$ as $n \rightarrow \infty$.

The proof is complete. □

4 Example

Consider the incomplete information nonlinear stochastic system described as follows:

$$\begin{cases} dX_t = \theta \frac{X_t}{\sqrt{1+X_t^2}} dt + dW_t, & X_0 \sim u_\theta, \\ dY_t = X_t dt + dV_t, \end{cases}$$

where θ is an unknown parameter, $(W_t, t \geq 0)$ and $(V_t, t \geq 0)$ are independent Wiener processes, u_θ is the invariant measure, $\{Y_t\}$ is observable, while $\{X_t\}$ is unobservable, the equation of X_t is called a hyperbolic diffusion equation.

Firstly, it is easy to verify that X_t is an ergodic diffusion process.

Then, since

$$\begin{aligned} & \left| \theta \frac{x}{\sqrt{1+x^2}} - \theta \frac{y}{\sqrt{1+y^2}} \right| \leq 2\theta|x-y|, \\ & \left| \theta \frac{x}{\sqrt{1+x^2}} \right| \leq \theta|1+x|, \end{aligned}$$

$\mathbb{E}\left[\theta \frac{X_0}{\sqrt{1+X_0^2}} (\theta \frac{X_0}{\sqrt{1+X_0^2}} - \frac{1}{2}\theta \frac{X_0}{\sqrt{1+X_0^2}})\right]$ attains the unique maximum at $\theta = \theta_0$, it is easy to check that the coefficients satisfy Assumptions 1–4.

Therefore, it is easy to check that

$$\widehat{\theta}_0 \xrightarrow{P} \theta_0,$$

when $\Delta \rightarrow 0, n \rightarrow \infty$ and $n\Delta \rightarrow \infty$.

5 Conclusion

The aim of this paper is to estimate the parameter for incomplete information stochastic system from discrete observations. The suboptimal estimation of the state has been obtained by constructing the extended Kalman filtering and the approximate likelihood function has been given. The consistency of the maximum likelihood estimator and the asymptotic normality of the error of estimation have been proved by applying the martingale moment inequality, Hölder's inequality, the Chebyshev inequality, the Burkholder–Davis–Gundy inequality and the uniform ergodic theorem. Further research topics will include the parameter estimation for incomplete information stochastic systems driven by a Lévy process.

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