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Random attractors for Ginzburg–Landau equations driven by difference noise of a Wiener-like process

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Abstract

We consider a Wong–Zakai process, which is the difference of a Wiener-like process. We then prove that there are random attractors for non-autonomous Ginzburg–Landau equations driven by linear multiplicative noise in terms of Wong–Zakai process and Wiener-like process, respectively. Moreover, we establish the upper semi-continuity of random attractors as the size of difference noise tends to zero.

MSC: 35B40; 35B41; 37L30

Keywords: Random attractors; Random dynamical systems; Ginzburg–Landau Equations; Upper semi-continuity; Wong–Zakai difference; Wiener-like process

1 Introduction

Given a Wiener process, its δ-difference is called a Wong–Zakai process [40, 41]. Such difference noise was often used to study stochastic equations as an approximation of white noise [15, 17, 19, 28, 30, 31].

In this paper, we consider a so-called *Wiener-like process*. Let

$$\varOmega = \mathcal{C}_0(\mathbb{R},\mathbb{R}) = \left\{\omega \in C(\mathbb{R},\mathbb{R}) : \omega(0) = 0, \lim_{t\to\pm\infty} \frac{\omega(t)}{t} = 0\right\} \tag{1.1}$$

and equip it with the Fréchet metric and the Borel σ-algebra $\mathcal{F} = \mathcal{B}(\varOmega)$. Then the shift $\theta_t : \varOmega \to \varOmega$, $\omega(\cdot) \mapsto \omega(\cdot + t) - \omega(t)$ is measurable for each $t \in \mathbb{R}$.

We then take an arbitrary probability measure P on the measurable space $(\varOmega, \mathcal{F})$. On the probability space $(\varOmega, \mathcal{F}, P)$, we obtain a stochastic process given by $W(t,\omega) = \omega(t)$, which is called a *Wiener-like process* [21]. If P is a Wiener measure, then the corresponding process is just the standard Wiener process; see [1, 3–5, 7, 24, 44].

In other words, a Wiener-like process only satisfies the properties on the right-hand side of (1.1). We do not require other properties (such as increment independence and Gauss distribution) of a Wiener process.

For each $\delta > 0$ (the case of $\delta < 0$ is similar), the δ-difference of the Wiener-like process determines the Wong–Zakai process given by

$$\mathcal{G}_\delta(t,\omega) := \mathcal{G}_\delta(\theta_t\omega) = \frac{1}{\delta}\big(\omega(t+\delta) - \omega(t)\big), \quad \forall t \in \mathbb{R}, \omega \in \Omega. \tag{1.2}$$

The difference process is not a Wiener-like process since $\mathcal{G}_\delta(0,\omega) = \omega(\delta)/\delta \neq 0$. However, by (1.1), we have $\mathcal{G}_\delta(\cdot,\omega) \in C(\mathbb{R},\mathbb{R})$ and $\mathcal{G}_\delta(\theta_t\omega)/t \to 0$ as $t \to \pm\infty$.

Recently, Lu and Wang [27] (see also [13, 14, 38]) have studied both the existence and approximation of random attractors for the reaction–diffusion equation driven by difference noise of a Wiener process.

In this paper, we consider the complex Ginzburg–Landau equation perturbed by difference noise of a Wiener-like process:

$$\frac{\partial u_\delta}{\partial t} - \big(\lambda + i\mu(t)\big)\Delta u_\delta = \gamma u_\delta - \big(\kappa + i\beta(t)\big)|u_\delta|^2 u_\delta + f(t,x) + u_\delta \mathcal{G}_\delta(\theta_t\omega), \tag{1.3}$$

$$u_\delta(t,0) = u_\delta(t,1) = 0, \qquad u_\delta(\tau,x) = u_{\delta,\tau}(x), \quad t \geq \tau, x \in \mathcal{I}, \tag{1.4}$$

where $\mathcal{I} = (0,1) \subset \mathbb{R}$, $\lambda, \gamma, \kappa > 0$, $\mu, \beta \in C_b(\mathbb{R},\mathbb{R})$ and $f \in L^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{L}^2(\mathcal{I}))$.

The first aim in this paper is to establish a random attractor $\mathcal{A}_\delta$ for the problem (1.3)–(1.4). In view of both the non-autonomous and the random nature, the attractor is actually a bi-parametric set formulated by $\mathcal{A}_\delta = \{\mathcal{A}_\delta(\tau,\omega)\}$ and called a pullback random attractor, which was first introduced by Crauel et al. [8] and by Wang [32] independently, with developments [2, 9, 10, 18, 20, 26, 36, 37, 42, 43, 45].

The second aim is to prove the upper semi-continuity of the attractors:

$$\lim_{\delta\to 0} \mathrm{dist}_{\mathbb{L}^2(\mathcal{O})}\big(\mathcal{A}_\delta(\tau,\omega), \mathcal{A}_0(\tau,\omega)\big) = 0, \quad \forall \tau \in \mathbb{R}, \omega \in \Omega, \tag{1.5}$$

where $\mathcal{A}_0$ is the random attractor for the following limiting equation perturbed by the Wiener-like process:

$$\frac{\partial u}{\partial t} - \big(\lambda + i\mu(t)\big)\Delta u = \gamma u - \big(\kappa + i\beta(t)\big)|u|^2 u + f(t,x) + u \circ \frac{dW}{dt} \tag{1.6}$$

with the same initial-boundary conditions as in (1.4).

By an abstract combined result on both existence and upper semi-continuity of random attractors, given by Li et al. [23] (also see [12]), we have to verify three aspects: (a) the convergence of the solution operators from Eqs. (1.3) to (1.6), (b) the equi-absorption of the systems for all small size δ of difference noise and (c) the equi-asymptotic compactness in small size.

It is worth pointing out that all uniform estimates depend on the convergence of $\mathcal{G}_\delta(\theta_t\omega)$ as $\delta \to 0$. However, since the Wiener-like process $\omega(\cdot)$ may be nowhere differential, it is easy from (1.2) to see that $\mathcal{G}_\delta(\theta_t\omega)$ generally diverges as $\delta \to 0$. Instead of this convergence, we must prove a convergence in the sense of the integrals of $\mathcal{G}_\delta(\theta_t\omega)$, which can be deduced from the convergence of the Wiener-like process as given in (1.1).

2 Uniform absorption in size for approximate equations

2.1 The cocycle generated from the approximate equation

A standard method can show the well-posed property of the problem (1.3)–(1.4) and the existence of a family of cocycles given by $\Phi_\delta : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times \mathbb{L}^2(\mathcal{I}) \to \mathbb{L}^2(\mathcal{I})$,

$$\Phi_\delta(t,\tau,\omega)u_{\delta,\tau} = u_\delta(t+\tau,\tau,\theta_{-\tau}\omega,u_{\delta,\tau}). \tag{2.1}$$

The same method as in [11] can show the measurability of Φ_δ in ω, and the cocycle property (see [32]) can be deduced from the uniqueness of solutions.

We consider a universe $\mathfrak{D}$ of all tempered bi-parametric sets in $\mathbb{L}^2(\mathcal{I})$, that is, for $\mathcal{D} = \{\mathcal{D}(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$, we have $\mathcal{D} \in \mathfrak{D}$ if and only if

$$\lim_{t\to\infty} e^{-\alpha t}\bigl\|\mathcal{D}(\tau-t,\theta_{-t}\omega)\bigr\|^2 = 0, \quad \forall\alpha>0, \tau\in\mathbb{R}, \omega\in\Omega, \tag{2.2}$$

where the norm of a set means the maximum of $\mathbb{L}^2$-norms of all elements.

In order to obtain a $\mathfrak{D}$-pullback absorption set, we make some assumptions.

Assumption F $f \in L^2_{\mathrm{loc}}(\mathbb{R},\mathbb{L}^2(\mathcal{I}))$ and there is a $\alpha_0 > 0$ such that

$$\int_{-\infty}^0 e^{\alpha_0 s}\bigl\|f(s,\cdot)\bigr\|^2\,ds < \infty, \tag{2.3}$$

$$\lim_{t\to\infty} e^{-\alpha t}\int_{-\infty}^0 e^{\alpha_0 s}\bigl(\bigl\|f(s-t,\cdot)\bigr\|^2\bigr)\,ds = 0, \quad \forall\alpha>0. \tag{2.4}$$

We also need the following convergence from the Wong–Zakai process to a Wiener-like process.

Lemma 2.1 *Let $\tau\in\mathbb{R}$, $\omega\in\Omega$ and $T>0$. Then*

$$\lim_{\delta\to 0}\sup_{t\in[\tau,\tau+T]}\left|\int_0^t \mathcal{G}_\delta(\theta_s\omega)\,ds - \omega(t)\right| = 0. \tag{2.5}$$

Moreover, for each $\varepsilon>0$, there exist $\delta_0(\varepsilon,\omega)>0$ and $C_0(\varepsilon,\omega)>0$ such that

$$\left|\int_0^t \mathcal{G}_\delta(\theta_s\omega)\,ds\right| \le \varepsilon|t| + C_0(\varepsilon,\omega), \quad \forall t\in\mathbb{R}, \delta\in(0,\delta_0]. \tag{2.6}$$

Proof By the mean value theorem, there is a $r_{t,\delta}\in[t,t+\delta]$ such that

$$\left|\frac{1}{\delta}\int_t^{t+\delta}\omega(s)\,ds - \omega(t)\right| = \bigl|\omega(r_{t,\delta})-\omega(t)\bigr|.$$

By (1.1), $\omega(\cdot)$ is continuous and thus uniformly continuous on $[\tau,\tau+T+1]$, which implies that

$$\lim_{\delta\to 0}\sup_{t\in[\tau,\tau+T]}\left|\frac{1}{\delta}\int_t^{t+\delta}\omega(s)\,ds - \omega(t)\right| = 0 \quad \text{and} \quad \lim_{\delta\to 0}\frac{1}{\delta}\left|\int_0^\delta \omega(s)\,ds\right| = 0$$

in view of $\omega(0) = 0$. Therefore, by the definition (1.2),

$$\sup_{t\in[\tau,\tau+T]}\left|\int_0^t \mathcal{G}_\delta(\theta_s\omega)\,ds - \omega(t)\right| \le \sup_{t\in[\tau,\tau+T]}\left|\frac{1}{\delta}\int_t^{t+\delta}\omega(s)\,ds - \omega(t)\right| + \frac{1}{\delta}\left|\int_0^\delta \omega(s)\,ds\right| \to 0$$

as $\delta \to 0$. Hence, (2.5) holds true.

Given now $\varepsilon > 0$, there is a $\delta_1 \in (0, 1]$ such that

$$\sup_{0<|\delta|\le\delta_1}\left|\frac{1}{\delta}\int_0^\delta \omega(s)\,ds\right| \le \epsilon.$$

By (1.1), $|\omega(s)/s| \le \varepsilon$ for all $|s| \ge s_0 - 1$ with a large $s_0(\epsilon)$. Then, for all $|t| \ge s_0$ and $\delta \in (0, \delta_1]$, there is a $r_{t,\delta}$ with $|r_{t,\delta} - t| \le |\delta|$ such that

$$\left|\frac{1}{\delta}\int_t^{t+\delta}\omega(s)\,ds\right| = \left|\omega(r_{t,\delta})\right| = \left|\frac{\omega(r_{t,\delta})}{r_{t,\delta}}\right||r_{t,\delta}| \le \varepsilon\left(|t| + 1\right).$$

By (1.2), we find that, for all $|t| \ge s_0$ and $0 < |\delta| \le \delta_1$,

$$\left|\int_0^t \mathcal{G}_\delta(\theta_r\omega)\,dr\right| \le \left|\frac{1}{\delta}\int_t^{t+\delta}\omega(s)\,ds\right| + \left|\frac{1}{\delta}\int_0^\delta \omega(s)\,ds\right| \le \varepsilon\left(|t| + 2\right).$$

By (2.5), there is $\delta_0 \in (0, \delta_1]$ such that, for all $|t| \le s_0$ and $\delta \in (0, \delta_0]$,

$$\left|\int_0^t \mathcal{G}_\delta(\theta_r\omega)\,dr\right| \le \sup_{|s|\le s_0}\left|\int_0^s \mathcal{G}_\delta(\theta_r\omega)\,dr - \omega(s)\right| + \sup_{|s|\le s_0}\left|\omega(s)\right| \le \varepsilon + C(\omega).$$

Therefore, (2.6) holds true for all $t \in \mathbb{R}$.

In order to prove that the absorption is uniform in size, we consider a change of variables:

$$v_\delta(t,\tau,\omega) = g_\delta^{-1}(t,\omega)u_\delta(t,\tau,\omega), \quad \text{where } g_\delta(t,\omega) = e^{\int_0^t \mathcal{G}_\delta(\theta_r\omega)\,dr}. \tag{2.7}$$

Then from (1.3) we obtain a random equation:

$$\frac{\partial v_\delta}{\partial t} - \left(\lambda + i\mu(t)\right)\Delta v_\delta = \gamma v_\delta - g_\delta^2(t,\omega)\left(\kappa + i\beta(t)\right)|v_\delta|^2 v_\delta + g_\delta^{-1}(t,\omega)f(t,\cdot) \tag{2.8}$$

with $v_\delta \equiv 0$ on $\partial\mathcal{I}$ and $v_\delta(\tau) = v_{\delta,\tau} = g_\delta(\tau,\omega)u_{\delta,\tau}$.

By Lemma 2.1 and the inequality $|e^a - e^b| \le e^{|a|+|b|}|b - a|$, we have

$$\lim_{\delta\to 0}\sup_{t\in[\tau,\tau+T]}\left(\left|g_\delta(t,\omega) - e^{\omega(t)}\right| + \left|g_\delta^{-1}(t,\omega) - e^{-\omega(t)}\right|\right) = 0. \tag{2.9}$$

□

2.2 Uniform absorption in size for approximate equations

Lemma 2.2 *For each $\delta > 0$, $\mathcal{D}_\delta \in \mathfrak{D}$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$, there are $T_\delta := T(\mathcal{D}_\delta, \tau, \omega) \ge 1$ such that, for all $t \ge T_\delta$ and $u_{\delta,\tau-t} \in \mathcal{D}_\delta(\tau - t, \theta_{-t}\omega)$,*

$$\left\|u_\delta(\tau,\tau - t,\theta_{-\tau}\omega,u_{\delta,\tau-t})\right\|^2 \le R_\delta(\tau,\omega) + 1, \tag{2.10}$$

where u_δ is a solution of the problem (1.3), *and for a positive constant* c_1,

$$R_\delta(\tau,\omega) := c_1 \int_{-\infty}^{0} e^{2\alpha_0 s - 2\int_0^s \mathcal{G}_\delta(\theta_r\omega)\,dr}\big(\|f(s+\tau)\|^2 + 1\big)\,ds. \tag{2.11}$$

Proof We multiply Eq. (2.8) with the conjugate function $\overline{v_\delta}$ and then take the real part to obtain

$$\frac{1}{2}\frac{d}{ds}\|v_\delta\|^2 + \lambda\|\nabla v_\delta\|^2 = \gamma\|v_\delta\|^2 - \kappa g_\delta^2(s,\omega)\|v_\delta\|_4^4 + g_\delta^{-1}(s,\omega)\operatorname{Re}(f,v_\delta), \tag{2.12}$$

where $\|\cdot\|_4$ denotes the norm in $\mathbb{L}^4(\mathcal{I})$. The Young inequality gives

$$g_\delta^{-1}(s,\omega)\big|\operatorname{Re}\big(f(s),v_\delta\big)\big| \le \alpha_0\|v_\delta\|^2 + c g_\delta^{-2}(s,\omega)\|f(s)\|^2,$$

where α_0 is the number in Assumption F. By the Young inequality again,

$$(\gamma + 2\alpha_0)\|v_\delta\|^2 - \frac{\kappa}{2} g_\delta^2(s,\omega)\|v_\delta\|_4^4 \le c\big(|\mathcal{O}|\big) g_\delta^{-2}(s,\omega).$$

So, we can rewrite (2.12) for the solution $v_\delta(s) = v_\delta(s,\tau-t,\theta_{-\tau}\omega, v_{\delta,\tau-t})$:

$$\begin{aligned} &\frac{d}{ds}\|v_\delta\|^2 + 2\alpha_0\|v_\delta\|^2 + \lambda\|\nabla v_\delta\|^2 + \kappa g_\delta^2(s,\theta_{-\tau}\omega)\|v_\delta\|_4^4 \\ &\quad\le c_1 g_\delta^{-2}(s,\theta_{-\tau}\omega)\big(\|f(s)\|^2 + 1\big). \end{aligned} \tag{2.13}$$

Multiplying (2.13) by $e^{2\alpha_0 s}$ and then integrating over $(\tau-t,\tau)$, we obtain

$$\begin{aligned} &\big\|v_\delta(\tau,\tau-t,\theta_{-\tau}\omega,v_{\delta,\tau-t})\big\|^2 + \lambda\int_{\tau-t}^{\tau} e^{2\alpha_0(s-\tau)}\big\|\nabla v_\delta(s,\tau-t,\theta_{-\tau}\omega,v_{\delta,\tau-t})\big\|^2\,ds \\ &\qquad + \kappa\int_{\tau-t}^{\tau} e^{2\alpha_0(s-\tau)} g_\delta^2(s,\theta_{-\tau}\omega)\big\|v_\delta(s,\tau-t,\theta_{-\tau}\omega,v_{\delta,\tau-t})\big\|_4^4\,ds \\ &\quad\le e^{-2\alpha_0 t}\|v_{\delta,\tau-t}\|^2 + c_1\int_{\tau-t}^{\tau} e^{2\alpha_0(s-\tau)} g_\delta^{-2}(s,\theta_{-\tau}\omega)\big(\|f(s)\|^2 + 1\big)\,ds. \end{aligned} \tag{2.14}$$

By the change of variables (2.7), we have $v_{\delta,\tau-t} = g_\delta^{-1}(\tau,\theta_{-\tau}\omega)u_{\delta,\tau-t}$ and

$$\begin{aligned} \big\|u_\delta(\tau,\tau-t,\theta_{-\tau}\omega,v_{\delta,\tau-t})\big\|^2 &= g_\delta^2(\tau,\theta_{-\tau}\omega)\big\|v_\delta(\tau,\tau-t,\theta_{-\tau}\omega,v_{\delta,\tau-t})\big\|^2 \\ &\le e^{-2\alpha_0 t}\|u_{\delta,\tau-t}\|^2 + I, \end{aligned} \tag{2.15}$$

where

$$\begin{aligned} I &:= c_1 g_\delta^2(\tau,\theta_{-\tau}\omega)\int_{\tau-t}^{\tau} e^{2\alpha_0(s-\tau)} g_\delta^{-2}(s,\theta_{-\tau}\omega)\big(\|f(s)\|^2 + 1\big)\,ds \\ &= c_1\int_{\tau-t}^{\tau} e^{2\alpha_0(s-\tau) + 2\int_0^\tau \mathcal{G}_\delta(\theta_{r-\tau}\omega)\,dr - 2\int_0^s \mathcal{G}_\delta(\theta_{r-\tau}\omega)\,dr}\big(\|f(s)\|^2 + 1\big)\,ds \\ &= c_1\int_{\tau-t}^{\tau} e^{2\alpha_0(s-\tau) - 2\int_0^{s-\tau} \mathcal{G}_\delta(\theta_r\omega)\,dr}\big(\|f(s)\|^2 + 1\big)\,ds \\ &\le c_1\int_{-\infty}^{0} e^{2\alpha_0 s - 2\int_0^s \mathcal{G}_\delta(\theta_r\omega)\,dr}\big(\|f(s+\tau)\|^2 + 1\big)\,ds =: R_\delta(\tau,\omega). \end{aligned}$$

On the other hand, since $u_{\delta,\tau-t} \in \mathcal{D}_\delta(\tau-t,\theta_{-\tau}\omega)$, there is $T_\delta = T(\mathcal{D}_\delta,\tau,\omega)$ such that, for all $t \geq T_\delta$,

$$e^{-2\alpha_0 t}\|u_{\delta,\tau-t}\|^2 \leq e^{-2\alpha_0 t}\left\|\mathcal{D}_\delta(\tau-t,\theta_{-\tau}\omega)\right\|^2 \leq 1.$$

Substituting the above estimates into (2.15), we obtain (2.10) as desired. □

In addition, by (2.14) and (2.15), we have, for all $t \geq T_\delta$ and $u_{\delta,\tau-t} \in \mathcal{D}_\delta(\tau-t,\theta_{-\tau}\omega)$,

$$\begin{aligned}&\int_{\tau-t}^{\tau} e^{2\alpha_0(s-\tau)}\left\|\nabla v_\delta(s,\tau-t,\theta_{-\tau}\omega,v_{\delta,\tau-t})\right\|^2 ds \\ &\quad\leq g_\delta^{-2}(\tau,\theta_{-\tau}\omega)\left(e^{-2\alpha_0 t}\|u_{\delta,\tau-t}\|^2 + I\right) \leq \frac{c(R_\delta(\tau,\omega)+1)}{g_\delta^2(\tau,\theta_{-\tau}\omega)}.\end{aligned} \tag{2.16}$$

Similarly, we have the following useful estimate:

$$\begin{aligned}&\int_{\tau-t}^{\tau} e^{2\alpha_0(s-\tau)} g_\delta^2(s,\theta_{-\tau}\omega)\left\|v_\delta(s,\tau-t,\theta_{-\tau}\omega,v_{\delta,\tau-t})\right\|_4^4 ds \\ &\quad\leq c\big(R_\delta(\tau,\omega)+1\big)g_\delta^{-2}(\tau,\theta_{-\tau}\omega).\end{aligned} \tag{2.17}$$

Proposition 2.3 *Under the Assumption F, for each $\delta > 0$, the cocycle Φ_δ has a closed, $\mathfrak{D}$-pullback random absorbing set $\mathcal{K}_\delta \in \mathfrak{D}$ in $\mathbb{L}^2(\mathcal{I})$, given by*

$$\mathcal{K}_\delta(\tau,\omega) := \left\{w \in \mathbb{L}^2(\mathcal{I}) : \|w\|^2 \leq R_\delta(\tau,\omega)+1\right\}, \quad \forall(\tau,\omega) \in \mathbb{R}\times\Omega, \tag{2.18}$$

where $R_\delta(\tau,\omega)$ is given in (2.11) and satisfies

$$\lim_{\delta\to 0} R_\delta(\tau,\omega) = c_1\int_{-\infty}^{0} e^{2\alpha_0 s - 2\omega(s)}\left(\left\|f(s+\tau)\right\|^2 + 1\right) ds =: R_0(\tau,\omega). \tag{2.19}$$

Proof We first prove that each $R_\delta(\tau,\omega)$ is finite. Notice that the formula (2.6) in Lemma 2.1 holds true for every $\delta > 0$. Hence, for each $\varepsilon > 0$ and $\omega \in \Omega$, there is a $C_\delta(\varepsilon,\omega) > 0$ such that

$$\left|\int_0^s \mathcal{G}_\delta(\theta_r\omega)\,dr\right| \leq \varepsilon|s| + C_\delta(\varepsilon,\omega), \quad \forall s \in \mathbb{R}. \tag{2.20}$$

By taking $\varepsilon = \frac{\alpha_0}{2}$, there is a $C_\delta(\omega)$ such that

$$\begin{aligned}R_\delta(\tau,\omega) &:= c_1\int_{-\infty}^{0} e^{2\alpha_0 s - 2\int_0^s \mathcal{G}_\delta(\theta_r\omega)\,dr}\left(\left\|f(s+\tau)\right\|^2+1\right) ds \\ &\leq c_1 e^{2C_\delta(\omega)}\int_{-\infty}^{0} e^{\alpha_0 s}\left(\left\|f(s+\tau)\right\|^2+1\right) ds.\end{aligned}$$

By (2.3) in Assumption F,

$$\begin{aligned}\int_{-\infty}^{0} e^{\alpha_0 s}\left\|f(s+\tau)\right\|^2 ds &= \int_{-\infty}^{\tau} e^{\alpha_0(s-\tau)}\left\|f(s)\right\|^2 ds \\ &= e^{-\alpha_0\tau}\int_{-\infty}^{0} e^{\alpha s}\left\|f(s)\right\|^2 ds + e^{-\alpha_0\tau}\int_0^{\tau} e^{\alpha_0 s}\left\|f(s)\right\|^2 ds < +\infty\end{aligned}$$

and thus $R_\delta(\tau,\omega)$ is finite.

The mapping $\omega \to R_\delta(\tau,\omega)$ is obviously measurable and thus $\mathcal{K}_\delta$ is a family of random sets. By Lemma 2.2, $\mathcal{K}_\delta$ is a $\mathfrak{D}$-pullback absorbing set for Φ_δ.

We then prove $\mathcal{K}_\delta \in \mathfrak{D}$. Indeed, for any $\alpha > 0$, we take $\varepsilon = \min\{\frac{\alpha}{5}, \frac{\alpha_0}{2}\}$ in (2.20), then, by (2.4) in Assumption F, as $t \to +\infty$, i.e. as $\tilde{t} = t - \tau \to +\infty$,

$$\begin{aligned} & e^{-\alpha t} R_\delta(\tau - t, \theta_{-t}\omega) \\ &= c_1 e^{-\alpha t} \int_{-\infty}^0 e^{2\alpha_0 s - 2\int_0^s \mathcal{G}_\delta(\theta_{r-t}\omega)\,dr} \bigl(\|f(s+\tau-t)\|^2 + 1 \bigr)\,ds \\ &\le c_1 e^{4C_\delta(\omega)} e^{-\alpha t} \int_{-\infty}^0 e^{2\alpha_0 s + 2\varepsilon(t-s) + 2\varepsilon t} \bigl(\|f(s+\tau-t)\|^2 + 1 \bigr)\,ds \\ &= c_1 e^{4C_\delta(\omega)} e^{-(\alpha - 4\varepsilon)(\tilde{t}+\tau)} \int_{-\infty}^0 e^{(2\alpha_0 - 2\varepsilon)s} \bigl(\|f(s-\tilde{t})\|^2 + 1 \bigr)\,ds \to 0 \end{aligned}$$

in view of the facts that $\alpha - 4\varepsilon > 0$ and $2\alpha_0 - 2\varepsilon \ge \alpha_0$.

Finally, we show the convergence (2.19). By (2.6) in Lemma 2.1, there are $\delta_0 > 0$ and $C_0(\omega) > 0$ (independent of δ) such that

$$\sup_{\delta \in (0,\delta_0]} \left| \int_0^s \mathcal{G}_\delta(\theta_r\omega)\,dr \right| \le \frac{\alpha}{2}|s| + C_0(\omega), \quad \forall s \in \mathbb{R}. \tag{2.21}$$

Hence, by taking the supremum of $R_\delta(\tau,\omega)$ on $\delta \in (0,\delta_0]$, we have

$$\begin{aligned} & \sup_{\delta \in (0,\delta_0]} \int_{-\infty}^0 e^{2\alpha s - 2\int_0^s \mathcal{G}_\delta(\theta_r\omega)\,dr} \bigl(\|f(s+\tau)\|^2 + 1 \bigr)\,ds \\ &\le e^{2C_0(\omega)} \int_{-\infty}^0 e^{\alpha s} \bigl(\|f(s+\tau)\|^2 + 1 \bigr)\,ds < +\infty. \end{aligned}$$

Hence, by (2.5) in Lemma 2.1, the Lebesgue controlled convergence theorem gives

$$\begin{aligned} \lim_{\delta \to 0} R_\delta(\tau,\omega) &= c_1 \int_{-\infty}^0 \lim_{\delta \to 0} e^{2\alpha_0 s - 2\int_0^s \mathcal{G}_\delta(\theta_r\omega)\,dr} \bigl(\|f(s+\tau)\|^2 + 1 \bigr)\,ds \\ &= c_1 \int_{-\infty}^0 e^{2\alpha_0 s - 2\omega(s)} \bigl(\|f(s+\tau)\|^2 + 1 \bigr)\,ds = R_0(\tau,\omega). \end{aligned}$$

□

3 Uniform compactness in size for approximate equations

3.1 Uniformly asymptotic compactness

Lemma 3.1 *For each $\mathcal{D}_\delta \in \mathfrak{D}$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$, let $T_\delta \ge 1$ be the entrance time in Lemma 2.2. Then there is a $\delta_0 > 0$ such that, for all $\delta \in (0,\delta_0]$, $t \ge T_\delta$ and $u_{\delta,\tau-t} \in \mathcal{D}_\delta(\tau - t, \theta_{-t}\omega)$,*

$$\bigl\| \nabla u_\delta(\tau, \tau - t, \theta_{-\tau}\omega, v_{\delta,\tau-t}) \bigr\|^2 \le e^{\Upsilon(M_0(\tau,\omega))(R_0(\tau,\omega)+2)} < +\infty, \tag{3.1}$$

where $\Upsilon(y) = a_4 y^4 + a_2 y^2 + a_0$ $(y > 0)$ with positive coefficients, $R_0(\tau,\omega)$ is given in (2.19) and

$$M_0(\tau,\omega) := \sup_{s \in [\tau-1,\tau]} e^{\pm(\omega(s-\tau) - \omega(-\tau))}. \tag{3.2}$$

Proof We multiply Eq. (2.8) by $-\overline{\Delta v_\delta} = -\Delta\overline{v_\delta}$ and then take the real part to find

$$\begin{aligned} &\frac{1}{2}\frac{d}{ds}\|\nabla v_\delta\|^2 + \lambda\|\Delta v_\delta\|^2 \\ &\quad = \gamma\|\nabla v_\delta\|^2 + g_\delta^2(s,\omega)\operatorname{Re}\bigl(\kappa + i\beta(t)\bigr)\bigl(|v_\delta|^2 v_\delta, \Delta v_\delta\bigr) - g_\delta^{-1}(s,\omega)\operatorname{Re}\bigl(f(s), \Delta v_\delta\bigr). \end{aligned} \tag{3.3}$$

By the Young inequality we obtain

$$g_\delta^{-1}(s,\omega)\bigl|\operatorname{Re}\bigl(f(s), \Delta v_\delta\bigr)\bigr| \le \frac{\lambda}{4}\|\Delta v_\delta\|^2 + c_3 g_\delta^{-2}(s,\omega)\bigl\|f(s)\bigr\|^2.$$

Since $\mathcal{I}$ is a 1D-domain, by the compactness of Sobolev embedding and the interpolation inequality, we have the following inequality (see Temam [29]):

$$\|\nabla w\|_4^2 \le c\|\nabla w\|\bigl(\|w\|^2 + \|\Delta w\|^2\bigr)^{\frac{1}{2}}, \quad \forall w \in \mathbb{H}_0^1(\mathcal{I}) \cap \mathbb{H}^2(\mathcal{I}).$$

By the initial assumption, $\beta \in C_b(\mathbb{R}, \mathbb{R})$ and thus $\beta_0 := \sup_{t\in\mathbb{R}}|\beta(t)| < +\infty$. Hence,

$$\begin{aligned} &\bigl|\operatorname{Re}\bigl(\kappa + i\beta(t)\bigr)\bigl(|v_\delta|^2 v_\delta, \Delta\overline{v_\delta}\bigr)\bigr| \\ &\quad = \left|\operatorname{Re}\bigl(\kappa + i\beta(t)\bigr)\int_{\mathcal{I}}\bigl(2|v_\delta|^2|\nabla v_\delta|^2 + v_\delta^2\nabla\overline{v_\delta}\cdot\nabla\overline{v_\delta}\bigr)\,dx\right| \\ &\quad \le 3(\kappa + \beta_0)\int_{\mathcal{I}}|v_\delta|^2|\nabla v_\delta|^2\,dx \\ &\quad \le c\|v_\delta\|_4^2\|\nabla v_\delta\|_4^2 \\ &\quad \le c\|v_\delta\|_4^2\|\nabla v_\delta\|\bigl(\|v_\delta\|^2 + \|\Delta v_\delta\|^2\bigr)^{\frac{1}{2}}, \end{aligned}$$

which together with the Poincaré inequality implies that

$$\begin{aligned} &g_\delta^2(s,\omega)\bigl|\operatorname{Re}\bigl(\kappa + i\beta(t)\bigr)\bigl(|v_\delta|^2 v_\delta, \Delta\overline{v_\delta}\bigr)\bigr| \\ &\quad \le \frac{\lambda}{4}\bigl(\|v_\delta\|^2 + \|\Delta v_\delta\|^2\bigr) + c g_\delta^4(s,\omega)\|v_\delta\|_4^4\|\nabla v_\delta\|^2 \\ &\quad \le \frac{\lambda}{4}\|\Delta v_\delta\|^2 + c\bigl(1 + g_\delta^4(s,\omega)\|v_\delta\|_4^4\bigr)\|\nabla v_\delta\|^2. \end{aligned}$$

Substituting it into (3.3) at the sample $\theta_{-\tau}\omega$, we obtain

$$\frac{d}{ds}\|\nabla v_\delta\|^2 \le c_4\bigl(1 + g_\delta^4(s,\theta_{-\tau}\omega)\|v_\delta\|_4^4\bigr)\|\nabla v_\delta\|^2 + c_3 g_\delta^{-2}(s,\theta_{-\tau}\omega)\bigl\|f(s)\bigr\|^2.$$

By the uniform Gronwall lemma [29] (also see [25, 36] for the non-autonomous version), we obtain

$$\bigl\|\nabla v_\delta(\tau, \tau - t, \theta_{-\tau}\omega, v_{\delta,\tau-t})\bigr\|^2 \le e^{I_1 + I_2(\delta)}\bigl(I_3(\delta) + I_4(\delta)\bigr), \tag{3.4}$$

where $I_1 := c_4 \int_{\tau-1}^{\tau} ds = c_4$ and

$$I_2(\delta) := c_4 \int_{\tau-1}^{\tau} g_\delta^4(s,\theta_{-\tau}\omega) \| v_\delta(s,\tau-t,\theta_{-\tau}\omega, v_{\delta,\tau-t}) \|_4^4 \, ds,$$
$$I_3(\delta) := \int_{\tau-1}^{\tau} \| \nabla v_\delta(s,\tau-t,\theta_{-\tau}\omega, v_{\delta,\tau-t}) \|^2 \, ds,$$
$$I_4(\delta) := c_3 \int_{\tau-1}^{\tau} g_\delta^{-2}(s,\theta_{-\tau}\omega) \| f(s) \|^2 \, ds.$$

We will use (2.17) to estimate $I_2(\delta)$. Indeed, by (2.9),

$$\begin{aligned} g_\delta(s,\theta_{-\tau}\omega) &= e^{\int_0^s \mathcal{G}_\delta(\theta_{r-\tau}\omega)\,dr} \\ &= e^{\int_0^{s-\tau} \mathcal{G}_\delta(\theta_r\omega)\,dr - \int_0^{-\tau} \mathcal{G}_\delta(\theta_r\omega)\,dr} \to e^{\omega(s-\tau)-\omega(-\tau)} \end{aligned}$$

as $\delta \to 0$ uniformly in $s \in [\tau-1,\tau]$. Hence, there is a $\delta_1 > 0$ such that

$$\sup_{\delta\in(0,\delta_1]} \sup_{s\in[\tau-1,\tau]} g_\delta(s,\theta_{-\tau}\omega) \le \sup_{s\in[\tau-1,\tau]} e^{\omega(s-\tau)-\omega(-\tau)} + 1 \le M_0(\tau,\omega) + 1, \tag{3.5}$$

where $M_0(\tau,\omega)$ is defined by (3.2). So, for all $\delta \in (0,\delta_1]$ and $t \ge 1$,

$$\begin{aligned} I_2(\delta) &\le c_4 e^{2\alpha_0} \sup_{s\in[\tau-1,\tau]} g_\delta^2(s,\theta_{-\tau}\omega) \\ &\quad \times \int_{\tau-1}^{\tau} e^{2\alpha_0(s-\tau)} g_\delta^2(s,\theta_{-\tau}\omega) \| v_\delta(s,\tau-t,\theta_{-\tau}\omega, v_{\delta,\tau-t}) \|_4^4 \, ds \\ &\le c\big(M_0^2(\tau,\omega)+1\big) \int_{\tau-t}^{\tau} e^{2\alpha(s-\tau)} g_\delta^2(s,\theta_{-\tau}\omega) \| v_\delta(s,\tau-t,\theta_{-\tau}\omega, v_{\delta,\tau-t}) \|_4^4 \, ds. \end{aligned}$$

By (2.17), for all $\delta \in (0,\delta_1]$, $t \ge T_\delta$ and $u_{\delta,\tau-t} \in \mathcal{D}_\delta(\tau-t,\theta_{-t}\omega)$,

$$I_2(\delta) \le c\big(M_0^2(\tau,\omega)+1\big)\big(R_\delta(\tau,\omega)+1\big)(\tau,\omega) g_\delta^{-2}(\tau,\theta_{-\tau}\omega).$$

By the convergence (2.19), $R_\delta(\tau,\omega) \le R_0(\tau,\omega)+1$ for all $\delta \in (0,\delta_2]$ with $\delta_2 \le \delta_1$. By the same method as in (3.5), there is a $\delta_3 \in (0,\delta_2]$ such that, for all $\delta \in (0,\delta_3]$,

$$\sup_{s\in[\tau-1,\tau]} g_\delta^{-1}(s,\theta_{-\tau}\omega) \le \sup_{s\in[\tau-1,\tau]} e^{-(\omega(s-\tau)-\omega(-\tau))} + 1 \le M_0(\tau,\omega) + 1. \tag{3.6}$$

Hence, for all $\delta \in (0,\delta_3]$, $t \ge T_\delta$ and $u_{\delta,\tau-t} \in \mathcal{D}_\delta(\tau-t,\theta_{-t}\omega)$,

$$I_2(\delta) \le \Upsilon\big(M_0(\tau,\omega)\big)\big(R_0(\tau,\omega)+2\big), \tag{3.7}$$

where $\Upsilon(\cdot)$ denotes the fourth-order polynomial with positive coefficients.

Similarly, by (2.16), there is a $\delta_4 \in (0,\delta_3]$ such that, for all $\delta \in (0,\delta_4]$, $t \ge T_\delta$ and $u_{\delta,\tau-t} \in \mathcal{D}_\delta(\tau-t,\theta_{-t}\omega)$,

$$I_3(\delta) \le \Upsilon\big(M_0(\tau,\omega)\big)\big(R_0(\tau,\omega)+1\big). \tag{3.8}$$

By (3.6) and the Assumption F, we have

$$\sup_{\delta\in(0,\delta_3]} I_4(\delta) \le \Upsilon\big(M_0(\tau,\omega)\big)\int_{\tau-1}^{\tau}\big\|f(s)\big\|^2\,ds \le \Upsilon\big(M_0(\tau,\omega)\big)\big(R_0(\tau,\omega)+1\big). \tag{3.9}$$

We substitute (3.7)–(3.9) into (3.4) to find that, for all $\delta\in(0,\delta_4]$, $t\ge T_\delta$ and $u_{\delta,\tau-t}\in\mathcal{D}_\delta(\tau-t,\theta_{-t}\omega)$,

$$\sup_{\delta\in(0,\delta_5]}\big\|\nabla v_\delta(\tau,\tau-t,\theta_{-\tau}\omega,v_{\delta,\tau-t})\big\|^2 \le e^{\Upsilon(M_0(\tau,\omega))(R_0(\tau,\omega)+2)}. \tag{3.10}$$

By using the relationship

$$u_\delta(\tau,\tau-t,\theta_{-\tau}\omega,u_{\delta,\tau-t}) = g_\delta^2(\tau,\theta_{-\tau}\omega)v_\delta(\tau,\tau-t,\theta_{-\tau}\omega,v_{\delta,\tau-t}),$$

we see from (3.5) and (3.10) that (3.3) holds true for all $\delta\in(0,\delta_4]$, $t\ge T_\delta$ and $u_{\delta,\tau-t}\in\mathcal{D}_\delta(\tau-t,\theta_{-t}\omega)$. □

3.2 Random attractors for the equation with difference noise

A bi-parametric set $\mathcal{A}_\delta=\{\mathcal{A}_\delta(\tau,\omega)\}\in\mathfrak{D}$ is called a $\mathfrak{D}$*-pullback random attractor* for the cocycle Φ_δ if $\mathcal{A}_\delta$ is random, compact, invariant and $\mathfrak{D}$-pullback attracting. The details and existence criteria can be found in [26, 32, 33].

Theorem 3.2 *Each Ginzburg–Landau equation with δ-difference noise possesses a unique $\mathfrak{D}$-pullback random attractor $\mathcal{A}_\delta=\{\mathcal{A}_\delta(\tau,\omega)\}$ in $\mathbb{L}^2(\mathcal{I})$.*

Proof By Proposition 2.3, the cocycle Φ_δ has a $\mathfrak{D}$-pullback random absorbing set $\mathcal{K}_\delta=\{\mathcal{K}_\delta(\tau,\omega)\}\in\mathfrak{D}$.

We prove that for each $\delta>0$ the cocycle Φ_δ is $\mathfrak{D}$-pullback asymptotically compact in $\mathbb{L}^2(\mathcal{I})$. Indeed, let $t_n\to+\infty$ and $u_{\delta,\tau-t_n}\in\mathcal{D}_\delta(\tau-t_n,\theta_{-t_n}\omega)$ with $\mathcal{D}_\delta\in\mathfrak{D}$, $\tau\in\mathbb{R}$ and $\omega\in\Omega$. Then, by the same method as in Lemma 3.1, there is a large $N\in\mathbb{N}$ such that, for all $n\ge N$,

$$\big\|\nabla u_\delta(\tau,\tau-t_n,\theta_{-\tau}\omega,u_{\delta,\tau-t_n})\big\|^2 \le e^{\Upsilon(M_\delta(\tau,\omega))(R_\delta(\tau,\omega)+1)} < +\infty,$$

where, by the continuity of $\mathcal{G}_\delta$,

$$M_\delta(\tau,\omega) := \sup_{s\in[\tau-1,\tau]} e^{\pm\int_0^s \mathcal{G}_\delta(\theta_{r-\tau}\omega)\,dr} < +\infty.$$

Therefore, the sequence

$$\big\{\Phi_\delta(t_n,\tau-t_n,\theta_{-t_n}\omega)u_{\delta,\tau-t_n}\big\} = \big\{u_{\delta_n}(\tau,\tau-t_n,\theta_{-\tau}\omega,u_{\delta,\tau-t_n})\big\} \tag{3.11}$$

is bounded in $\mathbb{H}_0^1(\mathcal{I})$. By the compactness of the Sobolev embedding $\mathbb{H}_0^1(\mathcal{I})\hookrightarrow\mathbb{L}^2(\mathcal{I})$, the sequence has a convergent subsequence in $\mathbb{L}^2(\mathcal{I})$. By the abstract result in [26, 32], there is a $\mathfrak{D}$-pullback random attractor such that $\{\mathcal{A}_\delta(\tau,\omega)\subset\{\mathcal{K}_\delta(\tau,\omega)$. □

3.3 Random attractors for the equation with Wiener-like noise

We now consider the Ginzburg–Landau equation (1.6) with Wiener-like noise. Let

$$v(t,\tau,\omega,v_\tau) = e^{-\omega(t)}u(t,\tau,\omega,u_\tau). \tag{3.12}$$

We obtain a random equation:

$$\frac{\partial v}{\partial t} - \bigl(\lambda + i\mu(t)\bigr)\Delta v = \gamma v - e^{2\omega(t)}\bigl(\kappa + i\beta(t)\bigr)|v|^2 v + e^{-\omega(t)}f(t,x), \tag{3.13}$$

with the initial-boundary conditions

$$v(t,0) = v(t,1) = 0, \qquad v(\tau,x) = v_\tau(x), \quad x \in \mathcal{I}, t \geq \tau, \tag{3.14}$$

where $v_\tau(x) = e^{-\omega(\tau)}u_\tau(x)$. As in [35], it is standard to show that problem (3.13)–(3.14) has a unique solution

$$v(\cdot,\tau,\omega,v_\tau) \in C\bigl([\tau,\infty),L^2(\mathcal{I})\bigr) \cap L^2_{\mathrm{loc}}\bigl([\tau,\infty),H^1_0(\mathcal{I})\bigr).$$

Passing to the variable u, we obtain a cocycle $\Phi_0 : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times \mathbb{L}^2(\mathcal{I}) \to \mathbb{L}^2(\mathcal{I})$ for the stochastic equation (1.6), given by

$$\Phi_0(t,\tau,\omega,u_\tau) = u(t+\tau,\tau,\theta_{-\tau}\omega,u_\tau) = e^{\omega(t)-\omega(-\tau)}v(t+\tau,\tau,\theta_{-\tau}\omega,v_\tau). \tag{3.15}$$

The same method as given in Proposition 2.3 shows that the cocycle Φ_0 has a $\mathfrak{D}$-pullback random absorbing set $\mathcal{K}_0 \in \mathfrak{D}$ in the space $\mathbb{L}^2(\mathcal{I})$, given by

$$\mathcal{K}_0(\tau,\omega) := \bigl\{w \in \mathbb{L}^2(\mathcal{I}) : \|w\|^2 \leq R_0(\tau,\omega) + 2\bigr\}, \quad \forall(\tau,\omega) \in \mathbb{R} \times \Omega, \tag{3.16}$$

where $R_0(\tau,\omega)$ is just the limit of $R_\delta(\tau,\omega)$ as given in (2.19).

By the same method as given in Lemma 3.1, one can show that the cocycle Φ_0 has another $\mathfrak{D}$-pullback absorbing set $\widetilde{\mathcal{K}_0}(\tau,\omega) \subset \mathbb{H}^1_0(\mathcal{I})$, given by

$$\widetilde{\mathcal{K}_0}(\tau,\omega) := \bigl\{w \in \mathbb{H}^1_0(\mathcal{I}) : \|\nabla w\|^2 \leq e^{\Upsilon(M_0(\tau,\omega))(R_0(\tau,\omega)+2)}\bigr\}. \tag{3.17}$$

By the compactness of the Sobolev embedding, Φ_0 is $\mathfrak{D}$-pullback asymptotically compact. So, we obtain

Theorem 3.3 *The Ginzburg–Landau equation with Wiener-like noise possesses a unique $\mathfrak{D}$-pullback random attractor $\mathcal{A}_0 = \{\mathcal{A}_0(\tau,\omega)\}$ in $\mathbb{L}^2(\mathcal{I})$.*

4 Upper semi-continuity of random attractors

We need to prove the convergence from Φ_δ to Φ_0 as $\delta \to 0$.

Lemma 4.1 *Let u_δ and u be the solutions of (1.3) and (1.6) with initial data $u_{\delta,\tau}, u_\tau \in \mathbb{L}^2(\mathcal{I})$, respectively. If $\|u_{\delta,\tau} - u_\tau\| \to 0$ as $\delta \to 0$, then*

$$\lim_{\delta\to 0} \sup_{t\in[\tau,\tau+T]} \bigl\|u_\delta(t,\tau,\omega,u_{\delta,\tau}) - u(t,\tau,\omega,u_\tau)\bigr\| = 0, \quad \forall T > 0. \tag{4.1}$$

Proof For each $\delta \in (0, \delta_0]$ with the positive number δ_0 in Lemma 3.1, we define

$$\xi_\delta(t) := v_\delta(t, \tau, \omega, v_{\delta,\tau}) - v(t, \tau, \omega, v_\tau), \quad t \in [\tau, \tau + T]. \tag{4.2}$$

By the difference between Eqs. (2.8) and (3.13), we obtain

$$\begin{aligned} &\frac{\partial \xi_\delta}{\partial t} - \big(\lambda + i\mu(t)\big)\Delta \xi_\delta \\ &\quad = \gamma \xi_\delta + \big(e^{-\int_0^t \mathcal{G}_\delta(\theta_r\omega)\,dr} - e^{-\omega(t)}\big) f(t, \cdot) \\ &\qquad - \big(\kappa + i\beta(t)\big)\big(e^{2\int_0^t \mathcal{G}_\delta(\theta_r\omega)\,dr}|v_\delta|^2 v_\delta - e^{2\omega(t)}|v|^2 v\big). \end{aligned} \tag{4.3}$$

Multiplying (4.3) with $\overline{\xi_\delta}$ and taking the real part, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\xi_\delta\|^2 + \lambda\|\nabla \xi_\delta\|^2 \\ &\quad = \gamma\|\xi_\delta\|^2 + \big(e^{-\int_0^t \mathcal{G}_\delta(\theta_r\omega)\,dr} - e^{-\omega(t)}\big)\big(f(t), \xi_\delta\big) \\ &\qquad - \mathrm{Re}\big[\big(\kappa + i\beta(t)\big)\big(e^{2\int_0^t \mathcal{G}_\delta(\theta_r\omega)\,dr}|v_\delta|^2 v_\delta - e^{2\omega(t)}|v|^2 v, \xi_\delta\big)\big]. \end{aligned} \tag{4.4}$$

We split the last term of (4.4) to obtain

$$\begin{aligned} &\big(e^{2\int_0^t \mathcal{G}_\delta(\theta_r\omega)\,dr}|v_\delta|^2 v_\delta - e^{2\omega(t)}|v|^2 v, \xi_\delta\big) \\ &\quad = \big(e^{2\int_0^t \mathcal{G}_\delta(\theta_r\omega)\,dr} - e^{2\omega(t)}\big)\big(|v_\delta|^2 v_\delta, \xi_\delta\big) + e^{2\omega(t)}\big(|v_\delta|^2 v_\delta - |v|^2 v, \xi_\delta\big). \end{aligned} \tag{4.5}$$

By the Gagliardo–Nirenberg inequality, $\|w\|_4^4 \le c\|w\|^2\|\nabla w\|^2$, we have

$$\big|\big(|v_\delta|^2 v_\delta, \xi_\delta\big)\big| \le c\|\xi_\delta\|_4^4 + \|v_\delta\|_4^4 \le c\big(\|\nabla v_\delta\|^2 + \|\nabla v\|^2\big)\|\xi_\delta\|^2 + \|v_\delta\|_4^4. \tag{4.6}$$

By Lemma 2.1 or (2.9), we have, as $\delta \to 0$,

$$\begin{aligned} &C_{\delta,1}(T) := \sup_{t\in[\tau,\tau+T]} \big|e^{2\int_0^t \mathcal{G}_\delta(\theta_r\omega)\,dr} - e^{2\omega(t)}\big| \to 0, \\ &C_{\delta,2}(T) := \sup_{t\in[\tau,\tau+T]} \big|e^{-\int_0^t \mathcal{G}_\delta(\theta_r\omega)\,dr} - e^{-\omega(t)}\big| \to 0, \end{aligned}$$

which further implies

$$\sup_{\delta\in(0,\delta_0]} \sup_{t\in[\tau,\tau+T]} \big(e^{2\int_0^t \mathcal{G}_\delta(\theta_r\omega)\,dr} + e^{-2\int_0^t \mathcal{G}_\delta(\theta_r\omega)\,dr}\big) \le C(T) < +\infty.$$

Hence, by (4.6) and $\beta \in C_b(\mathbb{R}, \mathbb{R})$,

$$\begin{aligned} &\big|\mathrm{Re}\big(\kappa + i\beta(t)\big)\big(e^{2\int_0^t \mathcal{G}_\delta(\theta_r\omega)\,dr} - e^{2\omega(t)}\big)\big(|v_\delta|^2 v_\delta, \xi_\delta\big)\big| \\ &\quad \le C(T)\big(\|\nabla v_\delta\|^2 + \|\nabla v\|^2\big)\|\xi_\delta\|^2 + C_{\delta,1}(T)\|v_\delta\|_4^4. \end{aligned} \tag{4.7}$$

Furthermore, on the 1D-domain, we have the Agmon inequality, $\|w\|_\infty^2 \le c\|w\|\|\nabla w\|$ for $w \in \mathbb{H}_0^1(\mathcal{I})$, and thus

$$\begin{aligned} \left|\left(|v_\delta|^2 v_\delta - |v|^2 v, \xi_\delta\right)\right| &= \left|\int_{\mathcal{O}} \left(|v_\delta|^2 v_\delta - |v|^2 v\right)\overline{\xi_\delta}\, dx\right| \\ &= \left|\int_{\mathcal{O}} |v_\delta|^2|\xi_\delta|^2 + \overline{v_\delta} v|\xi_\delta|^2 + v^2(\overline{\xi_\delta})^2\, dx\right| \\ &\le \int_{\mathcal{O}} \left(|v_\delta|^2 + |v_\delta||v| + |v|^2\right)|\xi_\delta|^2\, dx \\ &\le \frac{3}{2}\int_{\mathcal{O}} \left(|v_\delta|^2 + |v|^2\right)|\xi_\delta|^2\, dx \\ &\le 3\int_{\mathcal{O}} |\xi_\delta|^4\, dx + \frac{9}{2}\int_{\mathcal{O}} |v|^2|\xi_\delta|^2\, dx \\ &\le 3\|\xi_\delta\|_4^4 + \frac{9}{2}\|v\|_\infty^2\|\xi_\delta\|^2 \\ &\le c\left(\|\nabla v_\delta\|^2 + \|\nabla v\|^2\right)\|\xi_\delta\|^2 + c\|v\|\|\nabla v\|\|\xi_\delta\|^2 \\ &\le c\left(\|\nabla v_\delta\|^2 + \|\nabla v\|^2\right)\|\xi_\delta\|^2. \end{aligned} \tag{4.8}$$

Hence, for all $t \in [\tau, \tau + T]$,

$$\begin{aligned} &\left|\operatorname{Re}\left(\kappa + i\beta(t)\right)e^{2\omega(t)}\left(|v_\delta|^2 v_\delta - |v|^2 v, \xi_\delta\right)\right| \\ &\quad \le C(T)\left(\|\nabla v_\delta\|^2 + \|\nabla v\|^2\right)\|\xi_\delta\|^2. \end{aligned} \tag{4.9}$$

By (4.5), (4.7) and (4.9), we have

$$\begin{aligned} &\left|\operatorname{Re}\left[\left(\kappa + i\beta(t)\right)\left(e^{2\int_0^t \mathcal{G}_\delta(\theta_r\omega)\, dr}|v_\delta|^2 v_\delta - e^{2\omega(t)}|v|^2 v, \xi_\delta\right)\right]\right| \\ &\quad \le C(T)\left(\|\nabla v_\delta\|^2 + \|\nabla v\|^2\right)\|\xi_\delta\|^2 + C_{\delta,1}(T)\|v_\delta\|_4^4. \end{aligned} \tag{4.10}$$

On the other hand, the Young inequality gives

$$\left|\left(e^{-\int_0^t \mathcal{G}_\delta(\theta_r\omega)\, dr} - e^{-\omega(t)}\right)\left(f(t), \xi_\delta\right)\right| \le \frac{1}{4}\|\xi_\delta\|^2 + C_{\delta,2}^2(T)\left\|f(t)\right\|^2. \tag{4.11}$$

We substitute (4.10) and (4.11) into (4.4) to obtain

$$\frac{d}{dt}\|\xi_\delta\|^2 \le C_0\left(\|\nabla v_\delta\|^2 + \|\nabla v\|^2 + 1\right)\|\xi_\delta\|^2 + C_\delta\left(\|v_\delta\|_4^4 + \left\|f(t)\right\|^2\right), \tag{4.12}$$

where $C_\delta = C_{\delta,1}(T) + C_{\delta,2}^2(T) \to 0$ as $\delta \to 0$.

By applying the Gronwall inequality on (4.12), we obtain, for all $t \in [\tau, \tau + T]$,

$$\begin{aligned} \left\|\xi_\delta(t)\right\|^2 &\le e^{C_0\int_\tau^{\tau+T}(\|\nabla v_\delta(r)\|^2 + \|\nabla v(r)\|^2 + 1)\, dr} \\ &\quad \times \left(\|\xi_{\tau,\delta}\|^2 + C_\delta\int_\tau^{\tau+T}\left(\left\|v_\delta(s)\right\|_4^4 + \left\|f(s)\right\|^2\right) ds\right). \end{aligned} \tag{4.13}$$

By (2.16)–(2.17) in Lemma 2.2, there is a $\delta_0 > 0$ such that

$$\sup_{\delta\in(0,\delta_0]} \int_\tau^{\tau+T} \left(\left\|\nabla v_\delta(r,\tau,\omega,v_\tau)\right\|^2 + \left\|v_\delta(r)\right\|_4^4\right) dr \le C(T) < +\infty.$$

Since $v \in L^2(\tau,\tau+T,\mathbb{H}_0^1(\mathcal{I}))$, we have

$$\int_\tau^{\tau+T} \left\|\nabla v(r,\tau,\omega,v_\tau)\right\|_{\mathbb{H}^2}^2 ds \le C(T,\tau,\omega) < +\infty. \tag{4.14}$$

Noticing that f is locally integrable, we have, for all $\delta \in (0,\delta_0]$,

$$\sup_{t\in[\tau,\tau+T]} \left\|\xi_\delta(t)\right\|^2 \le C(T)\left(\|\xi_{\tau,\delta}\|^2 + C_\delta\right). \tag{4.15}$$

By Lemma 2.1 and $\|u_{\delta,\tau} - u_\tau\| \to 0$ as $\delta \to 0$, we have

$$\begin{aligned}\|\xi_{\tau,\delta}\|^2 = \|v_{\tau,\delta} - v_\tau\|^2 &\le e^{-2\int_0^\tau \mathcal{G}_\delta(\theta_r\omega)\,dr}\|u_{\tau,\delta} - u_\tau\|^2 \\ &\quad + e^{-2\int_0^\tau \mathcal{G}_\delta(\theta_r\omega)\,dr}\left(e^{2\int_0^\tau \mathcal{G}_\delta(\theta_r\omega)\,dr} - e^{2\omega(\tau)}\right)e^{-2\omega(\tau)}\|u_\tau\|^2 \to 0\end{aligned}$$

as $\delta \to 0$. On the other hand,

$$\begin{aligned}u_\delta(t,\tau,\omega,u_{\delta,\tau}) - u(t,\tau,\omega,u_\tau) &= e^{\int_0^t \mathcal{G}_\delta(\theta_r\omega)\,dr}v_\delta(t) - e^{\omega(t)}v(t) \\ &= e^{\int_0^t \mathcal{G}_\delta(\theta_r\omega)\,dr}\xi_\delta(t) + \left(e^{\int_0^t \mathcal{G}_\delta(\theta_r\omega)\,dr} - e^{\omega(t)}\right)v(t,\tau,\omega,v_\tau).\end{aligned}$$

Notice $C_\delta \to 0$ in (4.15), we finish the proof. □

Remark In a two-dimensional domain, the estimates in (4.8) may not be true and so we cannot prove the convergence of the system. This is the reason why we restrict the equation on the one-dimensional domain. In fact, the existence of a random attractor holds true in a two-dimensional domain.

Finally, we show the upper semi-continuity of attractors as the size of noise tends to zero, which is different from the case of varying density of noise [6, 16, 22, 39].

Theorem 4.2 *Let $\mathcal{A}_\delta$ and $\mathcal{A}_0$ be random attractors for Ginzburg–Landau equations with difference noise and Wiener-like noise, as given in Theorems 3.2 and 3.3, respectively. Then*

$$\lim_{\delta\to 0} \operatorname{dist}_{\mathbb{L}^2(\mathcal{I})}\left(\mathcal{A}_\delta(\tau,\omega), \mathcal{A}_0(\tau,\omega)\right) = 0, \quad \forall \tau \in \mathbb{R}, \omega \in \Omega. \tag{4.16}$$

Proof By all previous uniform estimates, the abstract results as given in [23, 34] seems to be applied. However, we give a direct proof for completeness.

Suppose (4.16) is not true, then there are $\varepsilon_0 > 0$, $\delta_n \to 0$ and $z_n \in \mathcal{A}_{\delta_n}(\tau,\omega)$ with $\tau \in \mathbb{R}$, $\omega \in \Omega$ such that

$$\operatorname{dist}_{\mathbb{L}^2(\mathcal{I})}\left(z_n, \mathcal{A}_0(\tau,\omega)\right) \ge \varepsilon_0, \quad \forall n \in \mathbb{N}. \tag{4.17}$$

We assume without loss of generality that $\delta_n \le \delta_0(\tau,\omega)$ for all $n \in \mathbb{N}$, where δ_0 is given in Lemma 3.1. For each fixed $n \in \mathbb{N}$, we have $\mathcal{A}_{\delta_n} \in \mathfrak{D}$, let $T_{\delta_n} = T(\mathcal{A}_{\delta_n},\tau,\omega)$ as given in Lemma 3.1. By the invariance of $\mathcal{A}_{\delta_n}$ and by Lemma 3.1,

$$z_n \in \mathcal{A}_{\delta_n}(\tau,\omega) = \Phi_{\delta_n}(T_{\delta_n},\tau - T_{\delta_n},\theta_{-T_{\delta_n}}\omega)\mathcal{A}_{\delta_n}(\tau - T_{\delta_n},\theta_{-T_{\delta_n}}\omega) \subset \widetilde{\mathcal{K}_0}(\tau,\omega),$$

where $\widetilde{\mathcal{K}_0}(\tau,\omega)$ is the bounded ball in $\mathbb{H}_0^1(\mathcal{I})$, as given in (3.17). By the compactness of the Sobolev embedding, $\widetilde{\mathcal{K}_0}(\tau,\omega)$ is pre-compact in $\mathbb{L}^2(\mathcal{I})$ and thus, passing to a subsequence, we can assume that $\|z_n - z_0\| \to 0$ for some $z_0 \in \mathbb{L}^2(\mathcal{I})$.

Next, we intend to prove $z_0 \in \mathcal{A}_0(\tau,\omega)$, which will be a contradiction with (4.17). For $m = 1$, the invariance shows that there are $y_n^1 \in \mathcal{A}_{\delta_n}(\tau - 1,\theta_{-1}\omega)$ such that

$$\Phi_{\delta_n}(1,\tau - 1,\theta_{-1}\omega)y_n^1 = z_n, \quad \forall n \in \mathbb{N}.$$

By the same method as above, there is a $N \in \mathbb{N}$ such that $\delta_n \le \delta_0(\tau - 1,\theta_{-1}\omega)$ for all $n \ge N$ and thus Lemma 3.1 gives

$$\{y_n^1 : n \ge N\} \subset \widetilde{\mathcal{K}_0}(\tau - 1,\theta_{-1}\omega).$$

By the compactness of the Sobolev embedding, the sequence $\{y_n^1\}$ has a convergent subsequence $\{y_{n1}^1\}$ such that

$$\|y_{n1}^1 - y^1\| \to 0, \quad \text{for some } y^1 \in \mathbb{L}^2(\mathcal{I}).$$

Repeating this process, there are $y_{n,m-1}^m \in \mathcal{A}_{\delta_{n,m-1}}(\tau - m,\theta_{-m}\omega)$ such that

$$\Phi_{\delta_{n,m-1}}(m,\tau - m,\theta_{-m}\omega)y_{n,m-1}^m = z_{n,m-1}, \quad \forall n \in \mathbb{N},$$

and, for an index subsequence $\{nm\}$ of $\{n,m-1\}$,

$$\|y_{nm}^m - y^m\| \to 0, \quad \text{for some } y^m \in \mathbb{L}^2(\mathcal{I}).$$

We consider the diagonal subsequence $\{nn\}$ of $\{n\}$ to obtain

$$\|y_{nn}^m - y^m\| \to 0, \quad \text{and} \quad \Phi_{\delta_{nn}}(m,\tau - m,\theta_{-m}\omega)y_{nn}^m = z_{nn}\ (\to z_0).$$

By the convergence (4.1) in Lemma 4.1, we have

$$z_{nn} \to \Phi_0(m,\tau - m,\theta_{-m}\omega)y^m \quad \text{and so} \quad \Phi_0(m,\tau - m,\theta_{-m}\omega)y^m = z_0.$$

On the other hand, by Proposition 2.3,

$$\begin{aligned} \|y^m\|^2 &\le \limsup_{n\to\infty}\|\mathcal{A}_{\delta_{nn}}(\tau - m,\theta_{-m}\omega)\|^2 \\ &\le \limsup_{n\to\infty} R_{\delta_{nn}}(\tau - m,\theta_{-m}\omega) + 1 \le R_0(\tau - m,\theta_{-m}\omega) + 2. \end{aligned}$$

Since $R_0(\tau,\omega)+2$ is tempered (i.e. $\mathcal{K}_0 \in \mathfrak{D}$), it follows from the attraction of $\mathcal{A}_0$ that

$$\operatorname{dist}\bigl(z_0, \mathcal{A}_0(\tau,\omega)\bigr) = \operatorname{dist}\bigl(\Phi_0(m, \tau - m, \theta_{-m}\omega)y^m, \mathcal{A}_0(\tau,\omega)\bigr) \to 0$$

as $m \to \infty$. Hence, $z_0 \in \mathcal{A}_0(\tau,\omega)$ as desired. □

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