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Analysis of coupled systems of implicit impulsive fractional differential equations involving Hadamard derivatives

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Abstract

We present some results on the existence, uniqueness and Hyers–Ulam stability to the solution of an implicit coupled system of impulsive fractional differential equations having Hadamard type fractional derivative. Using a fixed point theorem of Kransnoselskii's type, the existence and uniqueness results are obtained. Along these lines, different kinds of Hyers–Ulam stability are discussed. An example is given to illustrate the main theorems.

Keywords: Hadamard fractional derivative; Nonlinear implicit impulsive coupled system; Existence theory; Hyers–Ulam stability

1 Introduction

Fractional calculus is one of the emerging areas of investigation. The fractional differential operators are used to model several physical phenomena in a much better form than ordinary differential operators, which are local. Results obtained by fractional differential equations (FDEs) are much better and more accurate. For applications and details of fractional calculus, we refer the reader to [1, 2]. FDEs also serve as an excellent tool for the description of hereditary properties of various materials and processes [3]. The theory of FDEs, involving different kinds of boundary conditions, has been a field of interest in pure and applied sciences. Nonlocal conditions are used to describe certain features of physics and applied mathematics such as blood chemical engineering, flow problems, underground water flow, thermo–elasticity, population dynamics, and so on [4–21] and references cited therein. Our work is concerned with implicit impulsive coupled systems of FDEs. The impulsive FDEs are of great value. The said equations arise in business mathematics, management sciences and other managerial sciences and so forth. Some physical phenomena have sudden changes and discontinuous jumps. To model such problems, we impose impulsive conditions on the differential equations at discontinuity points; see for example [22–25] and the references cited therein.

In the classical text [26], it has been mentioned that Hadamard in 1892 [27] suggested a concept of fractional integro–differentiation in terms of the fractional power of the type $(x\frac{d}{dx})^\alpha$ in contrast to its Riemann–Liouville counterpart of the form $(\frac{d}{dx})^\alpha$. The kind of derivative, introduced by Hadamard, contains the logarithmic function of the arbitrary exponent in the kernel of the integral appearing in its definition. The Hadamard construction is invariant in relation to dilation and is well suited to the problems containing half-axes. Coupled systems of FDEs have also been investigated by many authors. Such systems appear naturally in many real–world situations. Some recent results on the topic can be found in a series of papers [28–45] and the references cited therein.

Another aspect of FDEs which has very recently got attention from the researchers is concerning to the Ulam type stability analysis of the aforesaid equations. The mentioned stability was first pointed out by Ulam [46] in 1940, which was further explained by Hyers [47], over Banach space. Later on, many researchers have done valuable work on the same task and interesting results were obtained for different functional equations; for details see[44, 48–50] and the references cited therein. This stability analysis is very useful in many applications, such as numerical analysis, optimization, etc., where finding the exact solution is quite difficult. For a detailed study of Ulam type stability with different approaches, we recommend [51–53] and the references cited therein.

The existence and uniqueness of Cauchy problems for fractional differential equations involving the Hadamard derivatives have been discussed by Kilbas *et al.* [54]. Using the contraction principle, the existence and uniqueness of the solution of sequential fractional differential equations with Hadamard derivative have been explored by Klimek [55]. Recently, Wang *et al.* [56] discussed the existence, blowing-up solutions and Ulam–Hyers stability of fractional differential equations with Hadamard derivative by using some classical methods. The area of research which has got tremendous attention from the researchers and these days is growing very fast is devoted to the existence and the Hyers–Ulam stability to the solution of implicit FDEs and coupled systems of implicit FDEs. The implicit FDEs represent a very important class of fractional differential equations. For details see [57–61] and the references cited therein.

In [62], the authors studied the existence and Hyers–Ulam stability of the following implicit FDEs involving Hadamard derivatives:

$$\begin{cases} {}_H\mathfrak{D}^{\alpha} p(t) - f(t, p(t), {}_H\mathfrak{D}^{\alpha} p(t)) = 0, \quad \alpha \in (0,1), \\ p(1) = p_1, \quad p_1 \in \mathbb{R}, \end{cases}$$

where $t \in [1, T]$, $T > 1$ and ${}_H\mathfrak{D}^{\alpha}$ denotes the Hadamard fractional derivative of order α.

In [63], the authors investigated the existence, uniqueness and different kinds of Hyers–Ulam stability for the considered coupled system involving the Caputo derivative:

$$\begin{cases} {}^c\mathfrak{D}^{\alpha} p(t) - f(t, q(t), {}^c\mathfrak{D}^{\alpha} p(t)) = 0; \quad t \in \mathcal{J}, \\ {}^c\mathfrak{D}^{\beta} q(t) - g(t, p(t), {}^c\mathfrak{D}^{\beta} q(t)) = 0; \quad t \in \mathcal{J}, \\ p'(0) = p''(0) = 0, \qquad p(1) = \lambda p(\eta), \quad \lambda, \eta \in (0,1), \\ q'(0) = q''(0) = 0, \qquad q(1) = \lambda q(\eta), \quad \lambda, \eta \in (0,1), \end{cases}$$

where $\mathcal{J} = [0, 1]$, $2 < \alpha, \beta \leq 3$ and $f, g : \mathcal{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

In [64], the authors proved the existence, uniqueness and different kinds of Hyers–Ulam stability for the following coupled system involving the Riemann–Liouville derivative:

$$\begin{cases} \mathfrak{D}^{\alpha}p(t) - f(t, q(t), \mathfrak{D}^{\alpha}p(t)) = 0; & t \in \mathcal{J}, \\ \mathfrak{D}^{\beta}q(t) - g(t, p(t), \mathfrak{D}^{\beta}q(t)) = 0; & t \in \mathcal{J}, \\ \mathfrak{D}^{\alpha-2}p(0^+) = \gamma_1 \mathfrak{D}^{\alpha-2}p(T^-), & \mathfrak{D}^{\alpha-1}p(0^+) = \beta_1 \mathfrak{D}^{\alpha-1}p(T^-), \\ \mathfrak{D}^{\beta-2}q(0^+) = \gamma_2 \mathfrak{D}^{\beta-2}q(T^-), & \mathfrak{D}^{\beta-1}q(0^+) = \beta_2 \mathfrak{D}^{\beta-1}q(T^-), \end{cases}$$

where $t \in \mathcal{J} = [0, T]$, $T > 0$, $1 < \alpha, \beta \leq 2$ and $\beta_1, \beta_2, \gamma_1, \gamma_2 \neq 1$. $\mathfrak{D}^{\alpha}$, $\mathfrak{D}^{\beta}$ are Riemann–Liouville derivatives of fractional order and $f, g : \mathcal{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions. For more recent work and details as regards implicit coupled systems, the reader may refer to [65, 66].

Motivated by the above work, we consider the following coupled impulsive implicit FDEs involving Hadamard derivatives:

$$\begin{cases} {}_H\mathfrak{D}^{\alpha}p(t) - f(t, p(t), {}_H\mathfrak{D}^{\beta}q(t)) = 0, & t \in \mathcal{J}, t \neq t_i, i = 1, 2, \ldots, m, \\ {}_H\mathfrak{D}^{\beta}q(t) - g(t, q(t), {}_H\mathfrak{D}^{\alpha}p(t)) = 0, & t \in \mathcal{J}, t \neq t_j, j = 1, 2, \ldots, n, \\ \Delta p(t_i) = \mathcal{I}_i p(t_i), \quad \Delta p'(t_i) = \widetilde{\mathcal{I}}_i p(t_i), & i = 1, 2, \ldots, m, \\ \Delta q(t_j) = \mathcal{I}_j q(t_j), \quad \Delta q'(t_j) = \widetilde{\mathcal{I}}_j q(t_j), & j = 1, 2, \ldots, n, \\ p(T) = \int_1^T \frac{(\ln \frac{T}{s})^{\alpha-1}}{\Gamma(\alpha)} \phi(s, p(s)) \frac{ds}{s}, & p'(T) = \varphi(p), \\ q(T) = \int_1^T \frac{(\ln \frac{T}{s})^{\beta-1}}{\Gamma(\beta)} \phi(s, q(s)) \frac{ds}{s}, & q'(T) = \varphi(q), \end{cases} \tag{1.1}$$

where $1 < \alpha, \beta \leq 2$, $f, g : \mathcal{J} \times \mathbb{R}^2 \to \mathbb{R}$ and $\phi, \varphi : \mathcal{C}(\mathcal{J}, \mathbb{R}) \to \mathbb{R}$ are continuous functions and

$$\Delta p(t_i) = p(t_i^+) - p(t_i^-), \qquad \Delta p'(t_i) = p'(t_i^+) - p'(t_i^-),$$
$$\Delta q(t_j) = q(t_j^+) - q(t_j^-), \qquad \Delta q'(t_j) = q'(t_j^+) - q'(t_j^-).$$

The notations $p(t_i^+), q(t_j^+)$ are right limits and $p(t_i^-), q(t_j^-)$ are left limits; $\mathcal{I}_i, \widetilde{\mathcal{I}}_i, \mathcal{I}_j, \widetilde{\mathcal{I}}_j : \mathbb{R} \to \mathbb{R}$ are continuous functions; ${}_H\mathfrak{D}^{\alpha}$, ${}_H\mathfrak{D}^{\beta}$ are the Hadamard derivative operators of order α and β, respectively.

For system (1.1), we discuss necessary and sufficient conditions for the existence and uniqueness of a positive solution by using the Kransnoselskii fixed point and the Banach contraction theorems. Further, we investigate various kinds of Hyers–Ulam, generalized Hyers–Ulam, Hyers–Ulam–Rassias and generalized Hyers–Ulam–Rassias stabilities.

2 Preliminaries

In this section, we introduce some fundamental descriptions and lemmas which are used throughout this paper; for details, reader should study [54, 67].

Endowed by the norms $\|p\| = \max\{|p(t)|, t \in \mathcal{J}\}$, $\|q\| = \max\{|q(t)|, t \in \mathcal{J}\}$, $\mathcal{PC}(\mathcal{J}, \mathbb{R}_+)$, which is a Banach space under these norms, and hence, the products of these are also Banach spaces, with norm $\|(p, q)\| = \|p\| + \|q\|$.

Let $\mathcal{E}_1$ and $\mathcal{E}_2$ denote spaces of the piecewise continuous functions defined as

$$\begin{aligned} \mathcal{E}_1 &= \mathcal{PC}_{2-\alpha,\ln}(\mathcal{J}, \mathbb{R}_+) \\ &= \{p : \mathcal{J} \to \mathbb{R}_+, p(t_i^+), p(t_i^-) \text{ and } p'(t_i^+), p'(t_i^-) \text{ exist for } i = 1, 2, \ldots, m\}, \end{aligned}$$

$$\begin{aligned}\mathcal{E}_2 &= \mathcal{PC}_{2-\beta,\ln}(\mathcal{J},\mathbb{R}_+)\\ &= \left\{q:\mathcal{J}\to\mathbb{R}_+, q\left(t_j^+\right), q\left(t_j^-\right) \text{ and } q'\left(t_j^+\right), q'\left(t_j^-\right) \text{ exist for } j=1,2,\dots,n\right\},\end{aligned}$$

with norms

$$\begin{aligned}\|p\|_{\mathcal{E}_1} &= \sup\left\{\left|p(t)(\ln t)^{2-\alpha}\right|, t\in\mathcal{J}\right\},\\ \|q\|_{\mathcal{E}_2} &= \sup\left\{\left|q(t)(\ln t)^{2-\beta}\right|, t\in\mathcal{J}\right\},\end{aligned}$$

respectively. Their product $\mathcal{E} = \mathcal{E}_1 \times \mathcal{E}_2$ is also a Banach space with norm $\|(p,q)\|_{\mathcal{E}} = \|p\|_{\mathcal{E}_1} + \|q\|_{\mathcal{E}_2}$.

We recall the following definitions from [68].

Definition 2.1 The Hadamard fractional integral of order $\alpha \in \mathbb{R}_+$ of the function $p(t)$ is defined by

$$_H\mathfrak{J}^\alpha p(t) = \frac{1}{\Gamma(\alpha)}\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1} p(s)\frac{ds}{s},$$

$1 < t \le T$, where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.2 The Hadamard fractional derivative of order $\alpha \in [\ell-1,\ell)$, $n \in \mathcal{Z}_+$, of the function $p(t)$ is defined by

$$_H\mathfrak{D}^\alpha p(t) = \frac{1}{\Gamma(\ell-\alpha)}\left(t\frac{d}{dt}\right)^\ell \int_a^t \left(\ln\frac{t}{s}\right)^{\ell-\alpha+1} p(s)\frac{ds}{s},$$

$1 < t \le T$, where $\Gamma(\cdot)$ is the Gamma function.

Lemma 2.3 ([69]) *Let $\alpha > 0$ and p be any function, then the homogeneous differential equation along with Hadamard fractional order $_H\mathfrak{D}^\alpha p(t) = 0$ has solutions*

$$p(t) = b_1(\ln t)^{\alpha-1} + b_2(\ln t)^{\alpha-2} + b_3(\ln t)^{\alpha-3} + \cdots + b_\ell(\ln t)^{\alpha-\ell},$$

and the following formula holds:

$$_H\mathfrak{J}^\alpha\, _H\mathfrak{D}^\alpha p(t) = p(t) + b_1(\ln t)^{\alpha-1} + b_2(\ln t)^{\alpha-2} + b_3(\ln t)^{\alpha-3} + \cdots + b_\ell(\ln t)^{\alpha-\ell},$$

where $b_j \in \mathbb{R}$, $j = 1,2,\dots,\ell$ and $\ell-1 < \alpha < \ell$.

Theorem 2.4 (Altman [70]) *Let $\mathcal{S} \neq \emptyset$ be a convex and closed subset of Banach space $\mathcal{E}$. Consider two operators $\mathbb{F}$, $\mathbb{G}$ such that*

(i) *$\mathbb{F}(p,q) + \mathbb{G}(p,q) \in \mathcal{S}$, where $(p,q) \in \mathcal{S}$.*

(ii) *$\mathbb{F}$ is contractive operator.*

(iii) *$\mathbb{G}$ is completely continuous operator.*

Then the operator system $\mathbb{F}(p,q) + \mathbb{G}(p,q) = (p,q) \in \mathcal{E}$ has a solution $(p,q) \in \mathcal{S}$.

2.1 Hyers–Ulam stability definitions and remarks

The following definitions and remarks are adopted from [65, 71].

Definition 2.5 The coupled system (1.1) is said to be Hyers–Ulam stable if there exist $K_{\alpha,\beta} = \max\{K_\alpha, K_\beta\} > 0$ such that, for $\varrho = \max\{\varrho_\alpha, \varrho_\beta\} > 0$ and for every solution $(p,q) \in \mathcal{E}$ of the inequality

$$\begin{cases} |{}_H\mathfrak{D}^\alpha p(t) - f(t, p(t), {}_H\mathfrak{D}^\beta q(t))| \le \varrho_\alpha, & t \in \mathcal{J}, \\ |\Delta p(t_i) - \mathcal{I}_i(p(t_i))| \le \varrho_\alpha, & i = 1, 2, \ldots, m, \\ |\Delta \widehat{p'}(t_i) - \widetilde{\mathcal{I}}_i(p(t_i))| \le \varrho_\alpha, & i = 1, 2, \ldots, m, \\ |{}_H\mathfrak{D}^\beta q(t) - g(t, q(t), {}_H\mathfrak{D}^\alpha p(t))| \le \varrho_\beta, & t \in \mathcal{J}, \\ |\Delta q(t_j) - \mathcal{I}_j(q(t_j))| \le \varrho_\beta, & j = 1, 2, \ldots, n, \\ |\Delta \widehat{q'}(t_j) - \widetilde{\mathcal{I}}_j(q(t_j))| \le \varrho_\beta, & j = 1, 2, \ldots, n, \end{cases} \tag{2.1}$$

there exists a unique solution $(\widehat{p}, \widehat{q}) \in \mathcal{E}$ with

$$\|(p,q) - (\widehat{p}, \widehat{q})\|_{\mathcal{E}} \le K_{\alpha,\beta}\varrho, \quad t \in \mathcal{J}.$$

Definition 2.6 The coupled system (1.1) is said to be generalized Hyers–Ulam stable if there exist $\Phi \in \mathcal{C}(\mathbb{R}^+, \mathbb{R}^+)$ with $\Phi(0) = 0$ such that, for any approximate solution $(p,q) \in \mathcal{E}$ of inequality (2.1), there exists a unique solution $(\widehat{p}, \widehat{q}) \in \mathcal{E}$ of (1.1) satisfying

$$\|(p,q) - (\widehat{p}, \widehat{q})\|_{\mathcal{E}} \le \Phi(\varrho), \quad t \in \mathcal{J}.$$

Denote $\Psi_{\alpha,\beta} = \max\{\Psi_\alpha, \Psi_\beta\} \in \mathcal{C}(\mathcal{J}, \mathbb{R})$ and $K_{\Psi_\alpha,\Psi_\beta} = \max\{K_{\Psi_\alpha}, K_{\Psi_\beta}\} > 0$.

Definition 2.7 ([71]) The coupled system (1.1) is said to be Hyers–Ulam–Rassias stable with respect to $\Psi_{\alpha,\beta}$ if there exists a constant $K_{\Psi_\alpha,\Psi_\beta}$ such that, for some $\varrho > 0$ and for any approximate solution $(p,q) \in \mathcal{E}$ of the inequality

$$\begin{cases} |{}_H\mathfrak{D}^\alpha p(t) - f(t, p(t), {}_H\mathfrak{D}^\beta q(t))| \le \Psi_\alpha(t)\varrho_\alpha, & t \in \mathcal{J}, \\ |{}_H\mathfrak{D}^\beta q(t) - g(t, q(t), {}_H\mathfrak{D}^\alpha p(t))| \le \Psi_\beta(t)\varrho_\beta, & t \in \mathcal{J}, \end{cases} \tag{2.2}$$

there exists a unique solution $(\widehat{p}, \widehat{q}) \in \mathcal{E}$ with

$$\|(p,q) - (\widehat{p}, \widehat{q})\|_{\mathcal{E}} \le K_{\Psi_\alpha,\Psi_\beta}\Psi_{\alpha,\beta}\varrho, \quad t \in \mathcal{J}. \tag{2.3}$$

Definition 2.8 The coupled system (1.1) is said to be generalized Hyers–Ulam–Rassias stable with respect to $\Psi_{\alpha,\beta}$ if there exists a constants $K_{\Psi_\alpha,\Psi_\beta}$ such that, for any approximate solution $(p,q) \in \mathcal{E}$ of inequality (2.2), there exists a unique solution $(\widehat{p}, \widehat{q}) \in \mathcal{E}$ of (1.1) satisfying

$$\|(p,q) - (\widehat{p}, \widehat{q})\|_{\mathcal{E}} \le K_{\Psi_\alpha,\Psi_\beta}\Psi_{\alpha,\beta}(t), \quad t \in \mathcal{J}. \tag{2.4}$$

Remark 2.9 We say that $(p,q) \in \mathcal{E}$ is a solution of the system of inequalities (2.1) if there exist functions Υ_f, $\Upsilon_g \in \mathcal{C}(\mathcal{J}, \mathcal{R})$ depending upon p, q, respectively, such that

(I) $|\Upsilon_f(t)| \le \varrho_\alpha$, $|\Upsilon_g(t)| \le \varrho_\beta$, $t \in \mathcal{J}$;

(II)

$$\begin{cases} {}_H\mathfrak{D}^\alpha p(t) = f(t, p(t), {}_H\mathfrak{D}^\beta q(t)) + \Upsilon_f(t), \\ \Delta p(t_i) = \mathcal{I}_i(p(t_i)) + \Upsilon_{f_i}, \\ \Delta p'(t_i) = \widetilde{\mathcal{I}}_i(p(t_i)) + \Upsilon_{f_i}, \\ {}_H\mathfrak{D}^\beta q(t) = g(t, q(t), {}_H\mathfrak{D}^\alpha p(t)) + \Upsilon_g(t), \\ \Delta q(t_j) = \mathcal{I}_j(q(t_j)) + \Upsilon_{g_j}, \\ \Delta q'(t_j) = \widetilde{\mathcal{I}}_j(q(t_j)) + \Upsilon_{g_j}. \end{cases}$$

3 Existence results

In the current section, we set up conditions for the existence and uniqueness of solutions to the proposed system (1.1).

Theorem 3.1 *Let f be a function; the subsequent linear impulsive boundary value problem*

$$\begin{cases} {}_H\mathfrak{D}^\alpha p(t) = f(t), \quad t \in \mathcal{J}, t \neq t_i, i = 1, 2, \ldots, m, \\ \Delta p(t_i) = \mathcal{I}_i(p(t_i)), \qquad \Delta p'(t_i) = \widetilde{\mathcal{I}}_i(p(t_i)), \quad t \neq t_i, i = 1, 2, \ldots, m, \\ p(T) = \int_1^T \frac{(\ln \frac{T}{s})^{\alpha-1}}{\Gamma(\alpha)} \phi(s, p(s)) \frac{ds}{s}, \quad p'(T) = \varphi(p), \end{cases} \tag{3.1}$$

has solutions

$$\begin{aligned} p(t) ={}& T\mathcal{A}_0(\alpha)\varphi(p)(\ln t)^{\alpha-2} + \sum_{i=1}^k \mathcal{A}_{1i}(\alpha)(\ln t)^{\alpha-2}\mathcal{I}_i(p_i) + \sum_{i=1}^k \mathcal{A}_{2i}(\alpha)(\ln t)^{\alpha-2}\widetilde{\mathcal{I}}_i(p_i) \\ &+ \frac{\mathcal{A}_3(\alpha)(\ln t)^{\alpha-2}}{\Gamma(\alpha)} \int_1^T \left(\ln \frac{T}{s}\right)^{\alpha-1} \phi\bigl(s, p(s)\bigr) \frac{ds}{s} \\ &+ \frac{\mathcal{A}_0(\alpha)(\ln t)^{\alpha-2}}{\Gamma(\alpha-1)} \int_{t_k}^T \left(\ln \frac{T}{s}\right)^{\alpha-2} f(s) \frac{ds}{s} \\ &+ \frac{\mathcal{A}_4(\alpha)(\ln t)^{\alpha-2}}{\Gamma(\alpha)} \int_{t_k}^T \left(\ln \frac{T}{s}\right)^{\alpha-1} f(s) \frac{ds}{s} \\ &+ \sum_{i=1}^k \frac{\mathcal{A}_{5i}(\alpha)(\ln t)^{\alpha-2}}{\Gamma(\alpha)} \int_{t_{i-1}}^{t_i} \left(\ln \frac{t_i}{s}\right)^{\alpha-1} f(s) \frac{ds}{s} \\ &+ \sum_{i=1}^k \frac{\ln t^{3-\alpha} (\log_{t_i} t)^{\alpha-2}}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_i} \left(\ln \frac{t_i}{s}\right)^{\alpha-2} f(s) \frac{ds}{s} \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^t \left(\ln \frac{t}{s}\right)^{\alpha-1} f(s) \frac{ds}{s}, \quad k = 1, 2, \ldots, m, \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} &\mathcal{A}_0(\alpha) = \ln \frac{t}{T} (\ln T)^{2-\alpha}, \\ &\mathcal{A}_{1i}(\alpha) = (\alpha-1)(\ln t - \alpha + 2)(\ln t_i)^{3-\alpha} - \frac{(\alpha-2)(\ln t^2 - \alpha + 1)(\ln t_i)^{2-\alpha}}{\ln t_i}, \end{aligned}$$

$$\mathcal{A}_{2i}(\alpha) = \ln t^{t_i(3-\alpha)}(\ln t_i)^{2-\alpha},$$

$$\mathcal{A}_3(\alpha) = \big(\alpha - 1 - \log_T t^{\alpha-2}\big)(\ln t)^{2-\alpha},$$

$$\mathcal{A}_4(\alpha) = \log_T \frac{t}{T^{\alpha-1}}(\ln T)^{2-\alpha} \quad \text{and}$$

$$\mathcal{A}_{5i}(\alpha) = \left[\ln \frac{t^{\alpha-1}}{T^{\alpha-2}} + \log_{t_i}\left(\frac{Tt_i}{t^2}\right)^{\alpha-2}\right](\ln t_i)^{2-\alpha}.$$

Proof Consider

$$_H\mathfrak{D}^\alpha p(t) = f(t), \quad 1 < \alpha \leq 2, t \in \mathcal{J}. \tag{3.3}$$

For $t \in (1, t_1]$, Lemma 2.3 gives

$$\begin{aligned} p(t) &= c_1(\ln t)^{\alpha-1} + c_2(\ln t)^{\alpha-2} + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1} f(s)\frac{ds}{s}, \\ p'(t) &= \frac{c_1(\alpha-1)}{t}(\ln t)^{\alpha-2} + \frac{c_2(\alpha-2)}{t}(\ln t)^{\alpha-3} + \frac{1}{\Gamma(\alpha-1)}\int_1^t \frac{1}{t}\left(\ln\frac{t}{s}\right)^{\alpha-2} f(s)\frac{ds}{s}. \end{aligned} \tag{3.4}$$

Again from Lemma 2.3, for $t \in (t_1, t_2]$

$$\begin{aligned} p(t) &= b_1(\ln t)^{\alpha-1} + b_2(\ln t)^{\alpha-2} + \frac{1}{\Gamma(\alpha)}\int_{t_1}^t \left(\ln\frac{t}{s}\right)^{\alpha-1} f(s)\frac{ds}{s}, \\ p'(t) &= \frac{b_1(\alpha-1)}{t}(\ln t)^{\alpha-2} + \frac{b_2(\alpha-2)}{t}(\ln t)^{\alpha-3} + \frac{1}{\Gamma(\alpha-1)}\int_{t_1}^t \frac{1}{t}\left(\ln\frac{t}{s}\right)^{\alpha-2} f(s)\frac{ds}{s}. \end{aligned} \tag{3.5}$$

Using initial impulses

$$\begin{aligned} b_1 &= c_1 - (\alpha-2)(\ln t_1)^{1-\alpha}\mathcal{I}_1\big(p(t_1)\big) + t_1(\ln t_1)^{2-\alpha}\widetilde{\mathcal{I}}_1\big(p(t_1)\big) \\ &\quad + \frac{(\ln t_1)^{2-\alpha}}{\Gamma(\alpha-1)}\int_1^{t_1}\left(\ln\frac{t_1}{s}\right)^{\alpha-2} f(s)\frac{ds}{s} \\ &\quad - \frac{(\alpha-2)(\ln t_1)^{1-\alpha}}{\Gamma(\alpha)}\int_1^{t_1}\left(\ln\frac{t_1}{s}\right)^{\alpha-1} f(s)\frac{ds}{s}, \\ b_2 &= c_2 + (\alpha-1)(\ln t_1)^{2-\alpha}\mathcal{I}_1\big(p(t_1)\big) - t_1(\ln t_1)^{3-\alpha}\widetilde{\mathcal{I}}_1\big(p(t_1)\big) \\ &\quad - \frac{(\ln t_1)^{3-\alpha}}{\Gamma(\alpha-1)}\int_1^{t_1}\left(\ln\frac{t_1}{s}\right)^{\alpha-2} f(s)\frac{ds}{s} \\ &\quad + \frac{(\alpha-1)(\ln t_1)^{2-\alpha}}{\Gamma(\alpha)}\int_1^{t_1}\left(\ln\frac{t_1}{s}\right)^{\alpha-1} f(s)\frac{ds}{s}. \end{aligned}$$

Substituting the values of b_1, b_2 in (3.5)

$$\begin{aligned} p(t) &= c_1(\ln t)^{\alpha-1} + c_2(\ln t)^{\alpha-2} + \big((\alpha-1) - (\alpha-2)\log_{t_1} t\big)(\log_{t_1} t)^{\alpha-2}\mathcal{I}_1\big(p(t_1)\big) \\ &\quad + t_1(\ln t - \ln t_1)(\log_{t_1} t)^{\alpha-2}\widetilde{\mathcal{I}}_1\big(p(t_1)\big) \\ &\quad + \frac{(\ln t - \ln t_1)(\log_{t_1} t)^{\alpha-2}}{\Gamma(\alpha-1)}\int_1^{t_1}\left(\ln\frac{t_1}{s}\right)^{\alpha-2} f(s)\frac{ds}{s} \end{aligned}$$

$$+\frac{((\alpha-1)-(\alpha-2)\log_{t_1}t)(\log_{t_1}t)^{\alpha-2}}{\Gamma(\alpha)}\int_1^{t_1}\left(\ln\frac{t_1}{s}\right)^{\alpha-1}f(s)\frac{ds}{s}$$
$$+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1}f(s)\frac{ds}{s}.$$

Similarly for $t\in(t_k,T)$

$$\begin{aligned}p(t)&=c_1(\ln t)^{\alpha-1}+c_2(\ln t)^{\alpha-2}+\sum_{i=1}^{k}\big((\alpha-1)-(\alpha-2)\log_{t_i}t\big)(\log_{t_i}t)^{\alpha-2}\mathcal{I}_i\big(p(t_i)\big)\\&+\sum_{i=1}^{k}t_i(\ln t-\ln t_i)(\log_{t_i}t)^{\alpha-2}\widetilde{\mathcal{I}}_i\big(p(t_i)\big)\\&+\sum_{i=1}^{k}\frac{(\ln t-\ln t_i)(\log_{t_i}t)^{\alpha-2}}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}\left(\ln\frac{t_i}{s}\right)^{\alpha-2}f(s)\frac{ds}{s}\\&+\sum_{i=1}^{k}\frac{((\alpha-1)-(\alpha-2)\log_{t_i}t)(\log_{t_i}t)^{\alpha-2}}{\Gamma(\alpha)}\int_{t_{i-1}}^{t_i}\left(\ln\frac{t_i}{s}\right)^{\alpha-1}f(s)\frac{ds}{s}\\&+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1}f(s)\frac{ds}{s}.\end{aligned}\tag{3.6}$$

Similarly for $t\in(t_k,T)$

$$\begin{aligned}p'(t)&=\frac{c_1(\alpha-1)(\ln t)^{\alpha-2}}{t}+\frac{c_2(\alpha-2)(\ln t)^{\alpha-3}}{t}\\&+\sum_{i=1}^{k}\frac{(\alpha-1)(\alpha-2)(\log_t e-\log_e t_i)(\log_{t_i}t)^{\alpha-2}}{t}\mathcal{I}_i\big(p(t_i)\big)\\&+\sum_{i=1}^{k}\frac{t_i((\alpha-1)-(\alpha-2)\log_t t_i)(\log_{t_i}t)^{\alpha-2}}{t}\widetilde{\mathcal{I}}_i\big(p(t_i)\big)\\&+\frac{1}{t\Gamma(\alpha-1)}\int_{t_k}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-2}f(s)\frac{ds}{s}\\&+\sum_{i=1}^{k}\frac{((\alpha-1)-(\alpha-2)\log_t t_i)(\log_{t_i}t)^{\alpha-2}}{t\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}\left(\ln\frac{t_i}{s}\right)^{\alpha-2}f(s)\frac{ds}{s}\\&+\sum_{i=1}^{k}\frac{(\alpha-1)(\alpha-2)(\log_t e-\log_e t_i)(\log_{t_i}t)^{\alpha-2}}{t\Gamma(\alpha)}\int_{t_{i-1}}^{t_i}\left(\ln\frac{t_i}{s}\right)^{\alpha-1}f(s)\frac{ds}{s}.\end{aligned}\tag{3.7}$$

Utilizing boundary conditions $p(T)=\int_1^T\frac{(\ln\frac{T}{s})^{\alpha-1}}{\Gamma(\alpha)}\phi(s,p(s))\frac{ds}{s}$ and $p'(T)=\varphi(p)$, we obtain

$$\begin{aligned}c_1&=T\varphi(p)(\ln T)^{2-\alpha}-\frac{(\ln T)^{1-\alpha}(\alpha-2)}{\Gamma(\alpha)}\int_1^T\left(\ln\frac{T}{s}\right)^{\alpha-1}\phi\big(s,p(s)\big)\frac{ds}{s}\\&+\frac{(\ln T)^{1-\alpha}}{\Gamma(\alpha)}\int_{t_k}^{T}\left(\ln\frac{T}{s}\right)^{\alpha-1}f(s)\frac{ds}{s}\\&+\sum_{i=1}^{k}\left(\ln t_i^{\alpha-1}-\frac{\alpha-2}{\ln t_i}\right)(\ln t_i)^{2-\alpha}\mathcal{I}_i\big(p(t_i)\big)-(\alpha-2)\sum_{i=1}^{k}t_i(\ln t_i)^{2-\alpha}\widetilde{\mathcal{I}}_i\big(p(t_i)\big)\end{aligned}$$

$$
\begin{aligned}
&-\frac{\alpha-2}{\Gamma(\alpha-1)}\sum_{i=1}^{k}(\ln t_i)^{2-\alpha}\int_{t_{i-1}}^{t_i}\left(\ln\frac{t_i}{s}\right)^{\alpha-2}f(s)\frac{ds}{s}-\frac{(\ln T)^{2-\alpha}}{\Gamma(\alpha-1)}\int_{t_k}^{T}\left(\ln\frac{T}{s}\right)^{\alpha-2}f(s)\frac{ds}{s}\\
&+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\left(\ln t_i^{\alpha-1}-\frac{\alpha-2}{\ln t_i}\right)(\ln t_i)^{2-\alpha}\int_{t_{i-1}}^{t_i}\left(\ln\frac{t_i}{s}\right)^{\alpha-1}f(s)\frac{ds}{s},\\
c_2 ={}& \frac{(\ln T)^{2-\alpha}}{\Gamma(\alpha-1)}\int_1^T\left(\ln\frac{T}{s}\right)^{\alpha-1}\phi\big(s,p(s)\big)\frac{ds}{s}-T\varphi(p)(\ln T)^{3-\alpha}+\sum_{i=1}^{k}t_i(\ln t_i)^{3-\alpha}\widetilde{\mathcal{I}}_i\big(p(t_i)\big)\\
&+(\alpha-1)\sum_{i=1}^{k}\big(\ln T^{(\alpha-2)(\log_{t_i}e-\log_e t_i)}-1\big)(\ln t_i)^{2-\alpha}\mathcal{I}_i\big(p(t_i)\big)\\
&+\frac{(\ln T)^{3-\alpha}}{\Gamma(\alpha-1)}\int_{t_k}^{T}\left(\ln\frac{T}{s}\right)^{\alpha-2}f(s)\frac{ds}{s}\\
&+\frac{1}{\Gamma(\alpha-1)}\sum_{i=1}^{k}\big(\ln T^{(\alpha-2)(\log_{t_i}e-\log_e t_i)}-1\big)(\ln t_i)^{2-\alpha}\int_{t_{i-1}}^{t_i}\left(\ln\frac{t_i}{s}\right)^{\alpha-1}f(s)\frac{ds}{s}\\
&+\frac{1}{\Gamma(\alpha-1)}\sum_{i=1}^{k}(\ln t_i)^{3-\alpha}\int_{t_{i-1}}^{t_i}\left(\ln\frac{t_i}{s}\right)^{\alpha-2}f(s)\frac{ds}{s}\\
&-\frac{(\ln T)^{2-\alpha}}{\Gamma(\alpha-1)}\int_{t_i}^{T}\left(\ln\frac{T}{s}\right)^{\alpha-1}f(s)\frac{ds}{s},\quad k=1,2,\dots,m.
\end{aligned}
$$

Substituting c_1 and c_2 in (3.6), we obtain (3.2). □

Corollary 3.2 *In view of Theorem 3.1, our coupled system (1.1) has the following solution:*

$$
\begin{cases}
p(t) = T\mathcal{A}_0(\alpha)\varphi(p)(\ln t)^{\alpha-2}\\
\quad+\sum_{i=1}^{k}\mathcal{A}_{1i}(\alpha)(\ln t)^{\alpha-2}\mathcal{I}_i(p_i)+\sum_{i=1}^{k}\mathcal{A}_{2i}(\alpha)(\ln t)^{\alpha-2}\widetilde{\mathcal{I}}_i(p_i)\\
\quad+\frac{\mathcal{A}_3(\alpha)(\ln t)^{\alpha-2}}{\Gamma(\alpha)}\int_1^T(\ln\frac{T}{s})^{\alpha-1}\phi(s,p(s))\frac{ds}{s}\\
\quad+\frac{\mathcal{A}_0(\alpha)(\ln t)^{\alpha-2}}{\Gamma(\alpha-1)}\int_{t_k}^T(\ln\frac{T}{s})^{\alpha-2}f(s,p(s),{}_H\mathfrak{D}^\beta q(s))\frac{ds}{s}\\
\quad+\frac{\mathcal{A}_4(\alpha)(\ln t)^{\alpha-2}}{\Gamma(\alpha)}\int_{t_k}^T(\ln\frac{T}{s})^{\alpha-1}f(s,p(s),{}_H\mathfrak{D}^\beta q(s))\frac{ds}{s}\\
\quad+\frac{1}{\Gamma(\alpha)}\int_{t_k}^t(\ln\frac{t}{s})^{\alpha-1}f(s,p(s),{}_H\mathfrak{D}^\beta q(s))\frac{ds}{s}\\
\quad+\sum_{i=1}^{k}\frac{\mathcal{A}_{5i}(\alpha)(\ln t)^{\alpha-2}}{\Gamma(\alpha)}\int_{t_{i-1}}^{t_i}(\ln\frac{t_i}{s})^{\alpha-1}f(s,p(s),{}_H\mathfrak{D}^\beta q(s))\frac{ds}{s}\\
\quad+\sum_{i=1}^{k}\frac{\ln t^{3-\alpha}(\log_{t_i}t)^{\alpha-2}}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(\ln\frac{t_i}{s})^{\alpha-2}f(s,p(s),{}_H\mathfrak{D}^\beta q(s))\frac{ds}{s},\\
\quad k=1,2,\dots,m,\\
q(t) = \mathcal{A}_0(\beta)\varphi(q)(\ln t)^{\beta-2}\\
\quad+\sum_{j=1}^{n}\mathcal{A}_{1j}(\beta)(\ln t)^{\beta-2}\mathcal{I}_j(q_j)+\sum_{j=1}^{k}\mathcal{A}_{2j}(\beta)(\ln t)^{\beta-2}\widetilde{\mathcal{I}}_j(q_j)\\
\quad+\frac{\mathcal{A}_3(\beta)(\ln t)^{\beta-2}}{\Gamma(\beta)}\int_1^T(\ln\frac{T}{s})^{\beta-1}\phi(s,q(s))\frac{ds}{s}\\
\quad+\frac{\mathcal{A}_0(\beta)(\ln t)^{\beta-2}}{\Gamma(\beta-1)}\int_{t_k}^T(\ln\frac{T}{s})^{\beta-2}g(s,{}_H\mathfrak{D}^\alpha p(s),q(s))\frac{ds}{s}\\
\quad+\frac{\mathcal{A}_4(\beta)(\ln t)^{\beta-2}}{\Gamma(\beta)}\int_{t_k}^T(\ln\frac{T}{s})^{\beta-1}g(s,{}_H\mathfrak{D}^\alpha p(s),q(s))\frac{ds}{s}\\
\quad+\frac{1}{\Gamma(\beta)}\int_{t_k}^t(\ln\frac{t}{s})^{\beta-1}g(s,{}_H\mathfrak{D}^\alpha p(s),q(s))\frac{ds}{s}\\
\quad+\sum_{j=1}^{k}\frac{\mathcal{A}_{5j}(\beta)(\ln t)^{\beta-2}}{\Gamma(\beta)}\int_{t_{j-1}}^{t_j}(\ln\frac{t_j}{s})^{\beta-1}g(s,{}_H\mathfrak{D}^\alpha p(s),q(s))\frac{ds}{s}\\
\quad+\sum_{j=1}^{k}\frac{\ln t^{3-\beta}(\log_{t_j}t)^{\beta-2}}{\Gamma(\beta-1)}\int_{t_{j-1}}^{t_j}(\ln\frac{t_j}{s})^{\beta-2}g(s,{}_H\mathfrak{D}^\alpha p(s),q(s))\frac{ds}{s},\\
\quad k=1,2,\dots,n.
\end{cases}
\tag{3.8}
$$

We use the following notations for convenience:

$$y(t) = f\big(t, p(t), z(t)\big), z(t) = g\big(t, q(t), y(t)\big).$$

Hence, for $t \in \mathcal{J}$, (3.8) becomes

$$\begin{cases} p(t) = T\mathcal{A}_0(\alpha)\varphi(p)(\ln t)^{\alpha-2} + \sum_{i=1}^{k} \mathcal{A}_{1i}(\alpha)(\ln t)^{\alpha-2}\mathcal{I}_i(p_i) + \sum_{i=1}^{k} \mathcal{A}_{2i}(\alpha)(\ln t)^{\alpha-2}\widetilde{\mathcal{I}}_i(p_i) \\ \quad + \frac{\mathcal{A}_3(\alpha)(\ln t)^{\alpha-2}}{\Gamma(\alpha)} \int_1^T (\ln \frac{T}{s})^{\alpha-1}\phi(s,p(s))\frac{ds}{s} + \frac{\mathcal{A}_0(\alpha)(\ln t)^{\alpha-2}}{\Gamma(\alpha-1)} \int_{t_k}^T (\ln \frac{T}{s})^{\alpha-2}y(s)\frac{ds}{s} \\ \quad + \frac{\mathcal{A}_4(\alpha)(\ln t)^{\alpha-2}}{\Gamma(\alpha)} \int_{t_k}^T (\ln \frac{T}{s})^{\alpha-1}y(s)\frac{ds}{s} + \sum_{i=1}^{k} \frac{\mathcal{A}_{5i}(\alpha)(\ln t)^{\alpha-2}}{\Gamma(\alpha)} \int_{t_{i-1}}^{t_i} (\ln \frac{t_i}{s})^{\alpha-1}y(s)\frac{ds}{s} \\ \quad + \sum_{i=1}^{k} \frac{\ln t^{3-\alpha}(\log_{t_i} t)^{\alpha-2}}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_i} (\ln \frac{t_i}{s})^{\alpha-2}y(s)\frac{ds}{s} + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (\ln \frac{t}{s})^{\alpha-1}y(s)\frac{ds}{s}, \\ \quad k = 1, 2, \dots, m, \\ q(t) = \mathcal{A}_0(\beta)\varphi(q)(\ln t)^{\beta-2} + \sum_{j=1}^{k} \mathcal{A}_{1j}(\beta)(\ln t)^{\beta-2}\mathcal{I}_j(q_j) + \sum_{j=1}^{k} \mathcal{A}_{2j}(\beta)(\ln t)^{\beta-2}\widetilde{\mathcal{I}}_j(q_j) \\ \quad + \frac{\mathcal{A}_3(\beta)(\ln t)^{\beta-2}}{\Gamma(\beta)} \int_1^T (\ln \frac{T}{s})^{\beta-1}\phi(s,q(s))\frac{ds}{s} + \frac{\mathcal{A}_0(\beta)(\ln t)^{\beta-2}}{\Gamma(\beta-1)} \int_{t_k}^T (\ln \frac{T}{s})^{\beta-2}z(s)\frac{ds}{s} \\ \quad + \frac{\mathcal{A}_4(\beta)(\ln t)^{\beta-2}}{\Gamma(\beta)} \int_{t_k}^T (\ln \frac{T}{s})^{\beta-1}z(s)\frac{ds}{s} + \sum_{j=1}^{k} \frac{\mathcal{A}_{5j}(\beta)(\ln t)^{\beta-2}}{\Gamma(\beta)} \int_{t_{j-1}}^{t_j} (\ln \frac{t_j}{s})^{\beta-1}z(s)\frac{ds}{s} \\ \quad + \sum_{j=1}^{k} \frac{\ln t^{3-\beta}(\log_{t_j} t)^{\beta-2}}{\Gamma(\beta-1)} \int_{t_{j-1}}^{t_j} (\ln \frac{t_j}{s})^{\beta-2}z(s)\frac{ds}{s} + \frac{1}{\Gamma(\beta)} \int_{t_k}^t (\ln \frac{t}{s})^{\beta-1}z(s)\frac{ds}{s}, \\ \quad k = 1, 2, \dots, n. \end{cases}$$

If p, q are the solutions of the proposed system (1.1) and $t \in \mathcal{J}$, then

$$\begin{aligned} p(t) &= T\mathcal{A}_0(\alpha)\varphi(p)(\ln t)^{\alpha-2} + \sum_{i=1}^{k} \mathcal{A}_{1i}(\alpha)(\ln t)^{\alpha-2}\mathcal{I}_i(p_i) + \sum_{i=1}^{k} \mathcal{A}_{2i}(\alpha)(\ln t)^{\alpha-2}\widetilde{\mathcal{I}}_i(p_i) \\ &\quad + \frac{\mathcal{A}_3(\alpha)(\ln t)^{\alpha-2}}{\Gamma(\alpha)} \int_1^T \left(\ln \frac{T}{s}\right)^{\alpha-1} \phi\big(s,p(s)\big)\frac{ds}{s} \\ &\quad + \frac{\mathcal{A}_0(\alpha)(\ln t)^{\alpha-2}}{\Gamma(\alpha-1)} \int_{t_k}^T \left(\ln \frac{T}{s}\right)^{\alpha-2} f\big(s,p(s),z(s)\big)\frac{ds}{s} \\ &\quad + \frac{\mathcal{A}_4(\alpha)(\ln t)^{\alpha-2}}{\Gamma(\alpha)} \int_{t_k}^T \left(\ln \frac{T}{s}\right)^{\alpha-1} f\big(s,p(s),z(s)\big)\frac{ds}{s} \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t \left(\ln \frac{t}{s}\right)^{\alpha-1} f\big(s,p(s),z(s)\big)\frac{ds}{s} \\ &\quad + \sum_{i=1}^{k} \frac{\mathcal{A}_{5i}(\alpha)(\ln t)^{\alpha-2}}{\Gamma(\alpha)} \int_{t_{i-1}}^{t_i} \left(\ln \frac{t_i}{s}\right)^{\alpha-1} f\big(s,p(s),z(s)\big)\frac{ds}{s} \\ &\quad + \sum_{i=1}^{k} \frac{\ln t^{3-\alpha}(\log_{t_i} t)^{\alpha-2}}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_i} \left(\ln \frac{t_i}{s}\right)^{\alpha-2} f\big(s,p(s),z(s)\big)\frac{ds}{s}, \quad k = 1, 2, \dots, m, \end{aligned}$$

and

$$\begin{aligned} q(t) &= \mathcal{A}_0(\beta)\varphi(q)(\ln t)^{\beta-2} + \sum_{j=1}^{k} \mathcal{A}_{1j}(\beta)(\ln t)^{\beta-2}\mathcal{I}_j(q_j) + \sum_{j=1}^{k} \mathcal{A}_{2j}(\beta)(\ln t)^{\beta-2}\widetilde{\mathcal{I}}_j(q_j) \\ &\quad + \frac{\mathcal{A}_3(\beta)(\ln t)^{\beta-2}}{\Gamma(\beta)} \int_1^T \left(\ln \frac{T}{s}\right)^{\beta-1} \phi\big(s,q(s)\big)\frac{ds}{s} \end{aligned}$$

$$
\begin{aligned}
&+ \frac{\mathcal{A}_0(\beta)(\ln t)^{\beta-2}}{\Gamma(\beta-1)} \int_{t_k}^{T} \left(\ln \frac{T}{s}\right)^{\beta-2} g\big(s, q(s), y(s)\big) \frac{ds}{s} \\
&+ \frac{\mathcal{A}_4(\beta)(\ln t)^{\beta-2}}{\Gamma(\beta)} \int_{t_k}^{T} \left(\ln \frac{T}{s}\right)^{\beta-1} g\big(s, q(s), y(s)\big) \frac{ds}{s} \\
&+ \frac{1}{\Gamma(\beta)} \int_{t_k}^{t} \left(\ln \frac{t}{s}\right)^{\beta-1} g\big(s, q(s), y(s)\big) \frac{ds}{s} \\
&+ \sum_{j=1}^{k} \frac{\mathcal{A}_{5j}(\beta)(\ln t)^{\beta-2}}{\Gamma(\beta)} \int_{t_{j-1}}^{t_j} \left(\ln \frac{t_j}{s}\right)^{\beta-1} g\big(s, q(s), y(s)\big) \frac{ds}{s} \\
&+ \sum_{j=1}^{k} \frac{\ln t^{3-\beta} (\log_{t_j} t)^{\beta-2}}{\Gamma(\beta-1)} \int_{t_{j-1}}^{t_j} \left(\ln \frac{t_j}{s}\right)^{\beta-2} g\big(s, q(s), y(s)\big) \frac{ds}{s}, \quad k = 1, 2, \ldots, n.
\end{aligned}
$$

Now, we transform the proposed system (1.1) into a fixed point problem. Let the operators $\mathbb{F}, \mathbb{G} : \mathcal{E} \to \mathcal{E}$ be defined as

$$
\begin{aligned}
&\mathbb{F}(p,q)(t) = \big(\mathbb{F}_1 p(t), \mathbb{F}_2 q(t)\big), \\
&\mathbb{G}(p,q)(t) = \big(\mathbb{G}_1(p,y)(t), \mathbb{G}_2(q,z)(t)\big), \\
&\mathbb{F}(p,q)(t) \\
&\quad = \begin{cases}
\mathbb{F}_1(p(t)) = T\mathcal{A}_0(\alpha)\varphi(p)(\ln t)^{\alpha-2} \\
\qquad + \sum_{i=1}^{k} \mathcal{A}_{1i}(\alpha)(\ln t)^{\alpha-2}\mathcal{I}_i(p_i) + \sum_{i=1}^{k} \mathcal{A}_{2i}(\alpha)(\ln t)^{\alpha-2}\widetilde{\mathcal{I}}_i(p_i) \\
\qquad + \frac{\mathcal{A}_3(\alpha)(\ln t)^{\alpha-2}}{\Gamma(\alpha)} \int_1^T (\ln \frac{T}{s})^{\alpha-1} \phi(s, p(s)) \frac{ds}{s}, \quad k = 1, 2, \ldots, m, \\
\mathbb{F}_2(q(t)) = \mathcal{A}_0(\beta)\varphi(q)(\ln t)^{\beta-2} \\
\qquad + \sum_{j=1}^{k} \mathcal{A}_{1j}(\beta)(\ln t)^{\beta-2}\mathcal{I}_j(q_j) + \sum_{j=1}^{k} \mathcal{A}_{2j}(\beta)(\ln t)^{\beta-2}\widetilde{\mathcal{I}}_j(q_j) \\
\qquad + \frac{\mathcal{A}_3(\beta)(\ln t)^{\beta-2}}{\Gamma(\beta)} \int_1^T (\ln \frac{T}{s})^{\beta-1} \phi(s, q(s)) \frac{ds}{s}, \quad k = 1, 2, \ldots, n,
\end{cases}
\end{aligned} \tag{3.9}
$$

and

$$
\begin{aligned}
&\mathbb{G}(p,q)(t) \\
&\quad = \begin{cases}
\mathbb{G}_1(p,q)(t) = \frac{\mathcal{A}_0(\alpha)(\ln t)^{\alpha-2}}{\Gamma(\alpha-1)} \int_{t_k}^{T} (\ln \frac{T}{s})^{\alpha-2} f(s, p(s), {}_H\mathfrak{D}^{\beta} q(s)) \frac{ds}{s} \\
\qquad + \frac{\mathcal{A}_4(\alpha)(\ln t)^{\alpha-2}}{\Gamma(\alpha)} \int_{t_k}^{T} (\ln \frac{T}{s})^{\alpha-1} f(s, p(s), {}_H\mathfrak{D}^{\beta} q(s)) \frac{ds}{s} \\
\qquad + \sum_{i=1}^{k} \frac{\mathcal{A}_{5i}(\alpha)(\ln t)^{\alpha-2}}{\Gamma(\alpha)} \int_{t_{i-1}}^{t_i} (\ln \frac{t_i}{s})^{\alpha-1} f(s, p(s), {}_H\mathfrak{D}^{\beta} q(s)) \frac{ds}{s} \\
\qquad + \sum_{i=1}^{k} \frac{\ln t^{3-\alpha} (\log_{t_i} t)^{\alpha-2}}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_i} (\ln \frac{t_i}{s})^{\alpha-2} f(s, p(s), {}_H\mathfrak{D}^{\beta} q(s)) \frac{ds}{s} \\
\qquad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (\ln \frac{t}{s})^{\alpha-1} f(s, p(s), {}_H\mathfrak{D}^{\beta} q(s)) \frac{ds}{s}, \quad k = 1, 2, \ldots, m, \\
\mathbb{G}_2(p,q)(t) = \frac{\mathcal{A}_0(\beta)(\ln t)^{\beta-2}}{\Gamma(\beta-1)} \int_{t_k}^{T} (\ln \frac{T}{s})^{\beta-2} g(s, q(s), {}_H\mathfrak{D}^{\alpha} p(s)) \frac{ds}{s} \\
\qquad + \frac{\mathcal{A}_4(\beta)(\ln t)^{\beta-2}}{\Gamma(\beta)} \int_{t_k}^{T} (\ln \frac{T}{s})^{\beta-1} g(s, q(s), {}_H\mathfrak{D}^{\alpha} p(s)) \frac{ds}{s} \\
\qquad + \sum_{j=1}^{k} \frac{\mathcal{A}_{5j}(\beta)(\ln t)^{\beta-2}}{\Gamma(\beta)} \int_{t_{j-1}}^{t_j} (\ln \frac{t_j}{s})^{\beta-1} g(s, q(s), {}_H\mathfrak{D}^{\alpha} p(s)) \frac{ds}{s} \\
\qquad + \sum_{j=1}^{k} \frac{\ln t^{3-\beta} (\log_{t_j} t)^{\beta-2}}{\Gamma(\beta-1)} \int_{t_{j-1}}^{t_j} (\ln \frac{t_j}{s})^{\beta-2} g(s, q(s), {}_H\mathfrak{D}^{\alpha} p(s)) \frac{ds}{s} \\
\qquad + \frac{1}{\Gamma(\beta)} \int_{t_j}^{t} (\ln \frac{t}{s})^{\beta-1} g(s, q(s), {}_H\mathfrak{D}^{\alpha} p(s)) \frac{ds}{s}, \quad k = 1, 2, \ldots, n.
\end{cases}
\end{aligned} \tag{3.10}
$$

For further analysis, the following assumptions need to hold:

(H$_1$) For $t \in \mathcal{J}$ and $p, y \in \mathbb{R}$, there are $o_1, \rho_1, \varrho_1 \in \mathcal{C}(\mathcal{J}, \mathbb{R}^+)$, such that

$$\left|f\left(t, p(t), y(t)\right)\right| \leq o_1(t) + \rho_1(t)\left|p(t)\right| + \varrho_1(t)\left|y(t)\right|$$

with $o_1^* = \sup_{t\in\mathcal{J}} o_1(t)$, $\rho_1^* = \sup_{t\in\mathcal{J}} \rho_1(t)$ and $\varrho_1^* = \sup_{t\in\mathcal{J}} \varrho_1(t) < 1$.

Similarly, for $t \in \mathcal{J}$ and $q, z \in \mathbb{R}$, there are $o_2, \rho_2, \varrho_2 \in \mathcal{C}(\mathcal{J}, \mathbb{R}^+)$, such that

$$\left|g\left(t, q(t), z(t)\right)\right| \leq o_2(t) + \rho_2(t)\left|q(t)\right| + \varrho_2(t)\left|z(t)\right|$$

with $o_2^* = \sup_{t\in\mathcal{J}} o_2(t)$, $\rho_2^* = \sup_{t\in\mathcal{J}} \rho_2(t)$ and $\varrho_2^* = \sup_{t\in\mathcal{J}} \varrho_2(t) < 1$.

(H$_2$) $\varphi, I_k, \widetilde{I}_k : \mathbb{R} \to \mathbb{R}$ is continuous and there exist constants $\mathcal{M}_\varphi, \mathcal{M}_\mathcal{I}, \mathcal{M}_{\widetilde{\mathcal{I}}}, \mathcal{M}'_\mathcal{I}, \mathcal{M}'_{\widetilde{\mathcal{I}}}$, $\widetilde{\mathcal{M}}_\varphi, \widetilde{\mathcal{M}}_\mathcal{I}, \widetilde{\mathcal{M}}_{\widetilde{\mathcal{I}}}, \widetilde{\mathcal{M}}'_\mathcal{I}, \widetilde{\mathcal{M}}'_{\widetilde{\mathcal{I}}} > 0$ such that for any $(p, q) \in \mathcal{E}$

$$\begin{aligned}
&\left|\varphi(p)\right| \leq \mathcal{M}_\varphi, \qquad \left|\varphi(q)\right| \leq \widetilde{\mathcal{M}}_\varphi,\\
&\left|\mathcal{I}_k\left(p(t)\right)\right| \leq \mathcal{M}_\mathcal{I}|p| + \mathcal{M}'_\mathcal{I}, \qquad \left|\mathcal{I}_k\left(q(t)\right)\right| \leq \widetilde{\mathcal{M}}_\mathcal{I}|q| + \widetilde{\mathcal{M}}'_\mathcal{I},\\
&\left|\widetilde{\mathcal{I}}_k\left(p(t)\right)\right| \leq \mathcal{M}_{\widetilde{\mathcal{I}}}|p| + \mathcal{M}'_{\widetilde{\mathcal{I}}}, \qquad \left|\widetilde{\mathcal{I}}_k\left(q(t)\right)\right| \leq \widetilde{\mathcal{M}}_{\widetilde{\mathcal{I}}}|q| + \widetilde{\mathcal{M}}'_{\widetilde{\mathcal{I}}},
\end{aligned}$$

where $k = \{0, 1, \ldots, m\}$.

(H$_3$) For $t \in \mathcal{J}$ and $q \in \mathbb{R}$, there are $\omega_1, \vartheta_1 \in \mathcal{C}(\mathcal{J}, \mathbb{R}^+)$, such that

$$\left|\phi\left(t, p(t)\right)\right| \leq \omega_1(t) + \vartheta_1(t)|p|$$

with $\omega_1^* = \sup_{t\in\mathcal{J}} \omega_1(t)$ and $\vartheta_1^* = \sup_{t\in\mathcal{J}} \vartheta_1(t) < 1$.

Similarly, for $t \in \mathcal{J}$ and $q \in \mathbb{R}$, there are $\omega_2, \vartheta_2 \in \mathcal{C}(\mathcal{J}, \mathbb{R}^+)$, such that

$$\left|\phi\left(t, q(t)\right)\right| \leq \omega_2(t) + \vartheta_2(t)|q|$$

with $\omega_2^* = \sup_{t\in\mathcal{J}} \omega_2(t)$ and $\vartheta_2^* = \sup_{t\in\mathcal{J}} \vartheta_2(t) < 1$.

(H$_4$) For all $p, y, \widetilde{p}, \widetilde{y} \in \mathbb{R}$ and for each $t \in \mathcal{J}$ there exists a constant $\mathcal{L}_f > 0$, $0 < \widetilde{\mathcal{L}}_f < 1$, such that

$$\left|f\left(t, p(t), y(t)\right) - f\left(t, \widetilde{p}(t), \widetilde{y}(t)\right)\right| \leq \mathcal{L}_f|p - \widetilde{p}| + \widetilde{\mathcal{L}}_f|y - \widetilde{y}|.$$

Similarly, for all $q, z, \widetilde{q}, \widetilde{z} \in \mathbb{R}$ and for each $t \in \mathcal{J}$ there exists a constant $\mathcal{L}_g > 0$, $0 < \widetilde{\mathcal{L}}_g < 1$, such that

$$\left|g\left(t, q(t), z(t)\right) - g\left(t, \widetilde{q}(t), \widetilde{z}(t)\right)\right| \leq \mathcal{L}_g|q - \widetilde{q}| + \widetilde{\mathcal{L}}_g|z - \widetilde{z}|.$$

(H$_5$) $I_k, \widetilde{I}_k : \mathbb{R} \to \mathbb{R}$ is continuous and there exist constants $\mathcal{L}_I, \mathcal{L}_{\widetilde{I}}, \widetilde{\mathcal{L}}_I, \widetilde{\mathcal{L}}_{\widetilde{I}} > 0$ such that for any $(p, q), (\widetilde{p}, \widetilde{q}) \in \mathcal{E}$

$$\begin{aligned}
&\left|I_k\left(p(t)\right) - I_k\left(\widetilde{p}(t)\right)\right| \leq \mathcal{L}_I|p - \widetilde{p}|, \qquad \left|I_k\left(q(t)\right) - I_k\left(\widetilde{q}(t)\right)\right| \leq \widetilde{\mathcal{L}}_I|q - \widetilde{q}|,\\
&\left|\widetilde{I}_k\left(p(t)\right) - \widetilde{I}_k\left(\widetilde{p}(t)\right)\right| \leq \mathcal{L}_{\widetilde{I}}|p - \widetilde{p}|, \qquad \left|\widetilde{I}_k\left(q(t)\right) - \widetilde{I}_k\left(\widetilde{q}(t)\right)\right| \leq \widetilde{\mathcal{L}}_{\widetilde{I}}|q - \widetilde{q}|.
\end{aligned}$$

($\mathbf{H}_6$) For all $p, \widetilde{p} \in \mathbb{R}$ and for each $t \in \mathcal{J}$ there exists a constant $\mathcal{L}_\phi, \mathcal{L}_\varphi > 0$, such that

$$\bigl|\phi\bigl(t,p(t)\bigr) - \phi\bigl(t,\widetilde{p}(t)\bigr)\bigr| \le \mathcal{L}_\phi |p - \widetilde{p}|, \qquad \bigl|\varphi(p) - \varphi(\widetilde{p})\bigr| \le \mathcal{L}_\varphi |p - \widetilde{p}|.$$

Similarly, for all $q, \widetilde{q} \in \mathbb{R}$ and for each $t \in \mathcal{J}$ there exists a constant $\widetilde{\mathcal{L}}_\phi, \widetilde{\mathcal{L}}_\varphi > 0$, such that

$$\bigl|\phi\bigl(t,q(t)\bigr) - \phi\bigl(t,\widetilde{q}(t)\bigr)\bigr| \le \widetilde{\mathcal{L}}_\phi |q - \widetilde{q}|, \qquad \bigl|\varphi(q) - \varphi(\widetilde{q})\bigr| \le \widetilde{\mathcal{L}}_\varphi |q - \widetilde{q}|.$$

Here we use Kransnoselskii's fixed point theorem to show that the operator $\mathbb{F} + \mathbb{G}$ has at least one fixed point. Therefore, we choose a closed ball

$$\mathcal{E}_r = \left\{ (p,q) \in \mathcal{E}, \bigl\|(p,q)\bigr\| \le r, \|p\| \le \frac{r}{2} \text{ and } \|q\| \le \frac{r}{2} \right\} \subset \mathcal{E},$$

where

$$r \ge \frac{\mathcal{M}_1^* + \mathcal{M}_1^{**} + \frac{(o_1^* + \varrho_1^* o_2^*)\mathcal{M}_3^* + (o_2^* + \varrho_2^* o_1^*)\mathcal{M}_3^{**}}{\varrho_1^* \varrho_2^* - 1}}{1 - \mathcal{M}_2^* - \mathcal{M}_2^{**} - \frac{\Upsilon_1^* \mathcal{M}_2^* + \Upsilon_2^* \mathcal{M}_2^{**}}{\varrho_1^* \varrho_2^* - 1}}.$$

Theorem 3.3 *If assumptions* ($\mathbf{H}_1$)–($\mathbf{H}_6$) *are true, then* (1.1) *has at least one solution.*

Proof For any $(p,q) \in \mathcal{E}_r$, we have

$$\bigl\|\mathbb{F}(p,q) + \mathbb{G}(p,q)\bigr\|_{\mathcal{E}} \le \bigl\|\mathbb{F}_1(p)\bigr\|_{\mathcal{E}_1} + \bigl\|\mathbb{F}_2(q)\bigr\|_{\mathcal{E}_2} + \bigl\|\mathbb{G}_1(p,q)\bigr\|_{\mathcal{E}_1} + \bigl\|\mathbb{G}_2(p,q)\bigr\|_{\mathcal{E}_2}. \tag{3.11}$$

From (3.9), we get

$$\begin{aligned} \bigl|\mathbb{F}_1 p(t)(\ln t)^{2-\alpha}\bigr| \le{}& T\bigl|\mathcal{A}_0(\alpha)\bigr|\bigl|\varphi(p)\bigr| + \sum_{i=1}^{k}\bigl|\mathcal{A}_{1i}(\alpha)\bigr|\bigl|\mathcal{I}_i\bigl(p(t_i)\bigr)\bigr| + \sum_{i=1}^{k}\bigl|\mathcal{A}_{2i}(\alpha)\bigr|\bigl|\widetilde{\mathcal{I}}_i\bigl(p(t_i)\bigr)\bigr| \\ &+ \frac{|\mathcal{A}_3(\alpha)|}{\Gamma(\alpha)}\int_1^T \left|\left(\ln\frac{T}{s}\right)^{\alpha-1}\right|\bigl|\phi\bigl(s,p(s)\bigr)\bigr|\frac{ds}{s}, \quad k = 1,2,\ldots,m. \end{aligned}$$

This implies that

$$\begin{aligned} \bigl\|\mathbb{F}_1(p)\bigr\|_{\mathcal{E}_1} &\le T\mathcal{M}_\varphi\bigl|\mathcal{A}_0(\alpha)\bigr| + k\mathcal{M}'_{\mathcal{I}}\bigl|\mathcal{A}_1(\alpha)\bigr| + k\mathcal{M}'_{\widetilde{\mathcal{I}}}\bigl|\mathcal{A}_2(\alpha)\bigr| + k\mathcal{M}_{\mathcal{I}}\bigl|\mathcal{A}_1(\alpha)\bigr|\|p\| \\ &\quad + k\mathcal{M}_{\widetilde{\mathcal{I}}}\bigl|\mathcal{A}_2(\alpha)\bigr|\|p\| - \frac{|\mathcal{A}_3(\alpha)|(\omega_1^* + \vartheta_1^*\|p\|)}{\Gamma(\alpha+1)}\bigl|(\ln T)^\alpha\bigr| \\ &\le \mathcal{M}_1^* + \mathcal{M}_2^*\|p\|. \end{aligned} \tag{3.12}$$

Similarly, we can obtain

$$\bigl\|\mathbb{F}_2(q)\bigr\|_{\mathcal{E}_2} \le \mathcal{M}_1^{**} + \mathcal{M}_2^{**}\|q\|, \tag{3.13}$$

where

$$\begin{aligned}\mathcal{M}_1^* &= T\mathcal{M}_\varphi\left|\mathcal{A}_0(\alpha)\right| + k\mathcal{M}'_{\mathcal{I}}\left|\mathcal{A}_1(\alpha)\right| + k\mathcal{M}'_{\widetilde{\mathcal{I}}}\left|\mathcal{A}_2(\alpha)\right| \\ &\quad - \frac{|\mathcal{A}_3(\alpha)|\omega_1^*}{\Gamma(\alpha+1)}\left|(\ln T)^\alpha\right|, \quad k = 1,2,\dots,m, \\ \mathcal{M}_2^* &= k\mathcal{M}_{\mathcal{I}}\left|\mathcal{A}_1(\alpha)\right| + k\mathcal{M}_{\widetilde{\mathcal{I}}}\left|\mathcal{A}_2(\alpha)\right| - \frac{|\mathcal{A}_3(\alpha)|\vartheta_1^*}{\Gamma(\alpha+1)}\left|(\ln T)^\alpha\right|, \quad k = 1,2,\dots,m, \\ \mathcal{M}_1^{**} &= T\widetilde{\mathcal{M}}_\varphi\left|\mathcal{A}_0(\beta)\right| + k\widetilde{\mathcal{M}}'_{\mathcal{I}}\left|\mathcal{A}_1(\beta)\right| + k\widetilde{\mathcal{M}}'_{\widetilde{\mathcal{I}}}\left|\mathcal{A}_2(\beta)\right| \\ &\quad - \frac{|\mathcal{A}_3(\beta)|\omega_2^*}{\Gamma(\beta+1)}\left|(\ln T)^\beta\right|, \quad k = 1,2,\dots,n, \\ \mathcal{M}_2^{**} &= k\widetilde{\mathcal{M}}_{\mathcal{I}}\left|\mathcal{A}_1(\beta)\right| + k\widetilde{\mathcal{M}}_{\widetilde{\mathcal{I}}}\left|\mathcal{A}_2(\beta)\right| - \frac{|\mathcal{A}_3(\beta)|\vartheta_2^*}{\Gamma(\beta+1)}\left|(\ln T)^\beta\right|, \quad k = 1,2,\dots,n.\end{aligned}$$

Also, we have

$$\begin{aligned}&\left|\mathbb{G}_1(p,q)(t)(\ln t)^{2-\alpha}\right| \\ &\quad\le \frac{|\mathcal{A}_0(\alpha)|}{\Gamma(\alpha-1)}\int_{t_k}^T\left|\left(\ln\frac{T}{s}\right)^{\alpha-2}\right|\left|y(s)\right|\frac{ds}{s} + \frac{|\mathcal{A}_4(\alpha)|}{\Gamma(\alpha)}\int_{t_k}^T\left|\left(\ln\frac{T}{s}\right)^{\alpha-1}\right|\left|y(s)\right|\frac{ds}{s} \\ &\qquad + \sum_{i=1}^k\frac{|\mathcal{A}_{5i}(\alpha)|}{\Gamma(\alpha)}\int_{t_{i-1}}^{t_i}\left|\left(\ln\frac{t_i}{s}\right)^{\alpha-1}\right|\left|y(s)\right|\frac{ds}{s} + \frac{|(\ln t)^{2-\alpha}|}{\Gamma(\alpha)}\int_{t_k}^t\left|\left(\ln\frac{t}{s}\right)^{\alpha-1}\right|\left|y(s)\right|\frac{ds}{s} \\ &\qquad + \sum_{i=1}^k\frac{|\ln t^{3-\alpha}(\ln t_i)^{2-\alpha}|}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}\left|\left(\ln\frac{t_i}{s}\right)^{\alpha-2}\right|\left|y(s)\right|\frac{ds}{s}, \quad k = 1,2,\dots,m.\end{aligned} \tag{3.14}$$

Now by $(\mathbf{H}_1)$, then

$$\begin{aligned}\left|y(t)\right| &= \left|f\left(t,p(t),z(t)\right)\right| \\ &\le o_1(t) + \rho_1(t)\left|p(t)\right| + \varrho_1(t)\left|z(t)\right| \\ &\le o_1(t) + \rho_1(t)\left|p(t)\right| + \varrho_1(t)\left|g\left(t,q(t),y(t)\right)\right| \\ &\le o_1(t) + \rho_1(t)\left|p(t)\right| + \varrho_1(t)\left(o_2(t) + \rho_2(t)\left|q(t)\right| + \varrho_2(t)\left|y(t)\right|\right) \\ &\le \frac{o_1(t) + \varrho_1(t)o_2(t)}{1-\varrho_1(t)\varrho_2(t)} + \frac{\rho_1(t)|p(t)| + \varrho_1(t)\rho_2(t)|q(t)|}{1-\varrho_1(t)\varrho_2(t)}.\end{aligned}$$

So, we obtain

$$\|y\| \le \frac{o_1^* + \varrho_1^* o_2^*}{1-\varrho_1^*\varrho_2^*} + \frac{\rho_1^*\|p\| + \varrho_1^*\rho_2^*\|q\|}{1-\varrho_1^*\varrho_2^*}. \tag{3.15}$$

Now by taking $\sup_{t\in\mathcal{J}}$ and using (3.15) in (3.14), we get

$$\begin{aligned}&\left\|\mathbb{G}_1(p,q)\right\|_{\mathcal{E}_1} \\ &\quad\le \left(\frac{o_1^* + \varrho_1^* o_2^*}{\varrho_1^*\varrho_2^* - 1} + \frac{\rho_1^*\|p\| + \varrho_1^*\rho_2^*\|q\|}{\varrho_1^*\varrho_2^* - 1}\right) \\ &\qquad\times\left(\frac{|\mathcal{A}_0(\alpha)||(\ln\frac{T}{t_k})^{\alpha-1}|}{\Gamma(\alpha)} + \frac{|\mathcal{A}_4(\alpha)||(\ln\frac{T}{t_k})^\alpha|}{\Gamma(\alpha+1)} + \frac{k|\mathcal{A}_5(\alpha)|}{\Gamma(\alpha+1)}\left|\left(\ln\frac{t_k}{t_{k-1}}\right)^\alpha\right|\right.\end{aligned}$$

$$+\frac{|(\ln t)^{2-\alpha}||(\ln\frac{t}{t_k})^{\alpha}|}{\Gamma(\alpha+1)}+\frac{k|\ln t^{3-\alpha}(\ln t_k)^{2-\alpha}||(\ln\frac{t_k}{t_{k-1}})^{\alpha-1}|}{\Gamma(\alpha)}\Big)$$
$$\leq\frac{(o_1^*+\varrho_1^*o_2^*)\mathcal{M}_3^*}{\varrho_1^*\varrho_2^*-1}+\frac{(\rho_1^*\|p\|+\varrho_1^*\rho_2^*\|q\|)\mathcal{M}_3^*}{\varrho_1^*\varrho_2^*-1}$$
$$\leq\frac{(o_1^*+\varrho_1^*o_2^*)\mathcal{M}_3^*}{\varrho_1^*\varrho_2^*-1}+\frac{\Upsilon_1^*\mathcal{M}_3^*}{\varrho_1^*\varrho_2^*-1}\|(p,q)\|. \tag{3.16}$$

Similarly

$$\|\mathbb{G}_2(p,q)\|_{\mathcal{E}_2}\leq\frac{(o_2^*+\varrho_2^*o_1^*)\mathcal{M}_3^{**}}{\varrho_1^*\varrho_2^*-1}+\frac{\Upsilon_2^*\mathcal{M}_3^{**}}{\varrho_1^*\varrho_2^*-1}\|(p,q)\|, \tag{3.17}$$

where

$$\Upsilon_1^*=\max\{\rho_1^*,\rho_2^*\varrho_1^*\},$$
$$\Upsilon_2^*=\max\{\rho_1^*\varrho_2^*,\rho_2^*\},$$

and

$$\mathcal{M}_3^*=\frac{|\mathcal{A}_0(\alpha)||(\ln\frac{T}{t_k})^{\alpha-1}|}{\Gamma(\alpha)}+\frac{|\mathcal{A}_4(\alpha)||(\ln\frac{T}{t_k})^{\alpha}|}{\Gamma(\alpha+1)}+\frac{k|\mathcal{A}_5(\alpha)||(\ln\frac{t_k}{t_{k-1}})^{\alpha}|}{\Gamma(\alpha+1)}$$
$$+\frac{|(\ln t)^{2-\alpha}||(\ln\frac{t}{t_k})^{\alpha}|}{\Gamma(\alpha+1)}+\frac{k|\ln t^{3-\alpha}(\ln t_k)^{2-\alpha}||(\ln\frac{t_k}{t_{k-1}})^{\alpha-1}|}{\Gamma(\alpha)},\quad k=1,2,\ldots,m,$$
$$\mathcal{M}_3^{**}=\frac{|\mathcal{A}_0(\beta)||(\ln\frac{T}{t_k})^{\beta-1}|}{\Gamma(\beta)}+\frac{|\mathcal{A}_4(\beta)||(\ln\frac{T}{t_k})^{\beta}|}{\Gamma(\beta+1)}+\frac{k|\mathcal{A}_5(\beta)||(\ln\frac{t_k}{t_{k-1}})^{\beta}|}{\Gamma(\beta+1)}$$
$$+\frac{|(\ln t)^{2-\beta}||(\ln\frac{t}{t_k})^{\beta}|}{\Gamma(\beta+1)}+\frac{k|\ln t^{3-\beta}(\ln t_k)^{2-\beta}||(\ln\frac{t_k}{t_{k-1}})^{\beta-1}|}{\Gamma(\beta)},\quad k=1,2,\ldots,n.$$

Putting (3.12), (3.13), (3.16) and (3.17) in (3.11), we get

$$\|\mathbb{F}(p,q)+\mathbb{G}(p,q)\|_{\mathcal{E}}$$
$$\leq\mathcal{M}_1^*+\mathcal{M}_1^{**}+\frac{(o_1^*+\varrho_1^*o_2^*)\mathcal{M}_3^*+(o_2^*+\varrho_2^*o_1^*)\mathcal{M}_3^{**}}{\varrho_1^*\varrho_2^*-1}+\left(\frac{\Upsilon_1^*\mathcal{M}_3^*+\Upsilon_2^*\mathcal{M}_3^{**}}{\varrho_1^*\varrho_2^*-1}\right)\|(p,q)\|$$
$$+\mathcal{M}_2^*\|p\|+\mathcal{M}_2^{**}\|q\|$$
$$\leq\mathcal{M}_1^*+\mathcal{M}_1^{**}+\frac{(o_1^*+\varrho_1^*o_2^*)\mathcal{M}_3^*+(o_2^*+\varrho_2^*o_1^*)\mathcal{M}_3^{**}}{\varrho_1^*\varrho_2^*-1}$$
$$+\left(\mathcal{M}_2^*+\mathcal{M}_2^{**}+\frac{\Upsilon_1^*\mathcal{M}_3^*+\Upsilon_2^*\mathcal{M}_3^{**}}{\varrho_1^*\varrho_2^*-1}\right)\|(p,q)\|$$
$$\leq r.$$

Hence, $\mathbb{F}(p,q)+\mathbb{G}(p,q)\in\mathcal{E}_r$.

Next, for any $t\in\mathcal{J}$, $(p,q),(\widetilde{p},\widetilde{q})\in\mathcal{E}$

$$\|\mathbb{F}(p,q)-\mathbb{F}(\widetilde{p},\widetilde{q})\|_{\mathcal{E}}$$
$$\leq\|\mathbb{F}_1(p)-\mathbb{F}_1(\widetilde{p})\|_{\mathcal{E}_1}+\|\mathbb{F}_2(q)-\mathbb{F}_2(\widetilde{q})\|_{\mathcal{E}_2}$$

$$
\begin{aligned}
&\leq T\big|\mathcal{A}_0(\alpha)\big|\big|\varphi(p)-\varphi(\widetilde{p})\big| + \sum_{i=1}^{k}\big|\mathcal{A}_{1i}(\alpha)\big|\big|\mathcal{I}_i(p_i)-\mathcal{I}_i(\widetilde{p}_i)\big| + \sum_{i=1}^{k}\big|\mathcal{A}_{2i}(\alpha)\big| \\
&\quad \times \big|\widetilde{\mathcal{I}}_i(p_i)-\widetilde{\mathcal{I}}_i(\widetilde{p}_i)\big| + \frac{|\mathcal{A}_3(\alpha)|}{\Gamma(\alpha)}\int_1^T\bigg|\bigg(\ln\frac{T}{s}\bigg)^{\alpha-1}\bigg|\big|\phi\big(s,p(s)\big)-\phi\big(s,\widetilde{p}(s)\big)\big|\frac{ds}{s} \\
&\quad + T\big|\mathcal{A}_0(\beta)\big|\big|\varphi(q)-\varphi(\widetilde{q})\big| + \sum_{j=1}^{k}\big|\mathcal{A}_{1j}(\beta)\big|\big|\mathcal{I}_j(q_j)-\mathcal{I}_j(\widetilde{q}_j)\big| + \sum_{j=1}^{k}\big|\mathcal{A}_{2j}(\beta)\big| \\
&\quad \times \big|\widetilde{\mathcal{I}}_j(q_j)-\widetilde{\mathcal{I}}_j(\widetilde{q}_j)\big| + \frac{|\mathcal{A}_3(\beta)|}{\Gamma(\beta)}\int_1^T\bigg|\bigg(\ln\frac{T}{s}\bigg)^{\beta-1}\bigg|\big|\phi\big(s,q(s)\big)-\phi\big(s,\widetilde{q}(s)\big)\big|\frac{ds}{s} \\
&\leq \bigg(T\mathcal{L}_\varphi\big|\mathcal{A}_0(\alpha)\big| + k\mathcal{L}_{\mathcal{I}}\big|\mathcal{A}_1(\alpha)\big| + k\mathcal{L}_{\widetilde{\mathcal{I}}}\big|\mathcal{A}_2(\alpha)\big| - \frac{\mathcal{L}_\phi|\mathcal{A}_3(\alpha)||(\ln T)^\alpha|}{\Gamma(\alpha+1)}\bigg)\|p-\widetilde{p}\| \\
&\quad + \bigg(T\widetilde{\mathcal{L}}_\varphi\big|\mathcal{A}_0(\beta)\big| + k\widetilde{\mathcal{L}}_{\mathcal{I}}\big|\mathcal{A}_1(\beta)\big| + k\widetilde{\mathcal{L}}_{\widetilde{\mathcal{I}}}\big|\mathcal{A}_2(\beta)\big| - \frac{\widetilde{\mathcal{L}}_\phi|\mathcal{A}_3(\beta)||(\ln T)^\beta|}{\Gamma(\beta+1)}\bigg) \\
&\quad \times \|q-\widetilde{q}\| \\
&\leq \mathcal{L}(\xi_1+\xi_2)\big\|(p-\widetilde{p},q-\widetilde{q})\big\|.
\end{aligned}
$$

Here $\mathcal{L} = \max\{\mathcal{L}_\varphi, \mathcal{L}_{\mathcal{I}}, \mathcal{L}_{\widetilde{\mathcal{I}}}, \mathcal{L}_\phi, \widetilde{\mathcal{L}}_\varphi, \widetilde{\mathcal{L}}_{\mathcal{I}}, \widetilde{\mathcal{L}}_{\widetilde{\mathcal{I}}}, \widetilde{\mathcal{L}}_\phi\}$,

$$
\xi_1 = T\big|\mathcal{A}_0(\alpha)\big| + k\big|\mathcal{A}_1(\alpha)\big| + k\big|\mathcal{A}_2(\alpha)\big| - \frac{|\mathcal{A}_3(\alpha)||(\ln T)^\alpha|}{\Gamma(\alpha+1)}, \quad k = 1,2,\ldots,m,
$$

and

$$
\xi_2 = T\big|\mathcal{A}_0(\beta)\big| + k\big|\mathcal{A}_1(\beta)\big| + k\big|\mathcal{A}_2(\beta)\big| - \frac{|\mathcal{A}_3(\beta)||(\ln T)^\beta|}{\Gamma(\beta+1)}, \quad k = 1,2,\ldots,n.
$$

Therefore, $\mathbb{F}$ is a contraction mapping.

Now, we prove the continuity and compactness of $\mathbb{G}$; we construct a sequence $T_s = (p_s,q_s)$ in $\mathcal{E}_r$ such that $(p_s,q_s) \to (p,q)$ for $n \to \infty$ in $\mathcal{E}_r$. Thus, we have

$$
\begin{aligned}
&\big\|\mathbb{G}(p_s,q_s)-\mathbb{G}(p,q)\big\|_{\mathcal{E}} \\
&\quad \leq \big\|\mathbb{G}_1(p_s,q_s)-\mathbb{G}_1(p,q)\big\|_{\mathcal{E}_1} + \big\|\mathbb{G}_2(p_s,q_s)-\mathbb{G}_2(p,q)\big\|_{\mathcal{E}_2} \\
&\quad \leq \bigg(\frac{|\mathcal{A}_0(\alpha)||(\ln\frac{T}{t_k})^{\alpha-1}|}{\Gamma(\alpha)} + \frac{|\mathcal{A}_4(\alpha)||(\ln\frac{T}{t_k})^{\alpha}|}{\Gamma(\alpha+1)} + \frac{|(\ln\frac{t}{t_k})^\alpha||(\ln t)^{2-\alpha}|}{\Gamma(\alpha+1)} + \frac{k|\mathcal{A}_5||(\ln\frac{t_k}{t_{k-1}})^\alpha|}{\Gamma(\alpha+1)} \\
&\qquad + \frac{k|\ln t^{3-\alpha}(\ln t_k)^{2-\alpha}||(\ln\frac{t_k}{t_{k-1}})^{\alpha-1}|}{\Gamma(\alpha)}\bigg)\bigg(\frac{\mathcal{L}_f\|p_s-p\| + \widetilde{\mathcal{L}}_f\mathcal{L}_g\|q_s-q\|}{\widetilde{\mathcal{L}}_f\widetilde{\mathcal{L}}_g-1}\bigg) \\
&\qquad + \bigg(\frac{|\mathcal{A}_0(\beta)||(\ln\frac{T}{t_k})^{\beta-1}|}{\Gamma(\beta)} + \frac{|\mathcal{A}_4(\beta)||(\ln\frac{T}{t_k})^{\beta}|}{\Gamma(\beta+1)} \\
&\qquad + \frac{|(\ln\frac{t}{t_k})^\beta||(\ln t)^{2-\beta}|}{\Gamma(\beta+1)} + \frac{k|\mathcal{A}_5(\beta)||(\ln\frac{t_k}{t_{k-1}})^\beta|}{\Gamma(\beta+1)} \\
&\qquad + \frac{k|\ln t^{3-\beta}(\ln t_k)^{2-\beta}||(\ln\frac{t_k}{t_{k-1}})^{\beta-1}|}{\Gamma(\beta)}\bigg)\bigg(\frac{\mathcal{L}_f\widetilde{\mathcal{L}}_g\|p_s-p\| + \mathcal{L}_g\|q_s-q\|}{\widetilde{\mathcal{L}}_f\widetilde{\mathcal{L}}_g-1}\bigg)
\end{aligned}
$$

$$\leq \mathcal{M}_3^*\left(\frac{\mathcal{L}_f\|p_s-p\|+\widetilde{\mathcal{L}}_f\mathcal{L}_g\|q_s-q\|}{\widetilde{\mathcal{L}}_f\widetilde{\mathcal{L}}_g-1}\right)+\mathcal{M}_3^{**}\left(\frac{\mathcal{L}_f\widetilde{\mathcal{L}}_g\|p_s-p\|+\mathcal{L}_g\|q_s-q\|}{\widetilde{\mathcal{L}}_f\widetilde{\mathcal{L}}_g-1}\right).$$

This implies $\|\mathbb{G}(p_s,q_s)-\mathbb{G}(p,q)\|_{\mathcal{E}}\to 0$ as $s\to\infty$, therefore $\mathbb{G}$ is continuous.

Next, we show that $\mathbb{G}$ is uniformly bounded on $\mathcal{E}_r$. From (3.16) and (3.17), we have

$$\begin{aligned}\|\mathbb{G}(p,q)(t)\|_{\mathcal{E}} &\leq \|\mathbb{G}_1(p,q)(t)\|_{\mathcal{E}_1}+\|\mathbb{G}_2(p,q)(t)\|_{\mathcal{E}_2}\\ &\leq \frac{(o_1^*+\varrho_1^*o_2^*)\mathcal{M}_3^*}{\varrho_1^*\varrho_2^*-1}+\frac{(o_2^*+\varrho_2^*o_1^*)\mathcal{M}_3^{**}}{\varrho_1^*\varrho_2^*-1}+\left(\frac{\Upsilon_1^*\mathcal{M}_3^*}{\varrho_1^*\varrho_2^*-1}+\frac{\Upsilon_2^*\mathcal{M}_3^{**}}{\varrho_1^*\varrho_2^*-1}\right)\|(p,q)\|\\ &\leq r.\end{aligned}$$

Thus, $\mathbb{G}$ is uniformly bounded on $\mathcal{E}_r$.

For equi-continuity, take $\tau_1,\tau_2\in\mathcal{J}$ with $\tau_1<\tau_2$ and for any $(p,q)\in\mathcal{E}_r\subset\mathcal{E}$, where $\mathcal{E}_r$ is clearly bounded, we have

$$\begin{aligned}&\|\mathbb{G}_1(p,q)(\tau_1)-\mathbb{G}_1(p,q)(\tau_2)\|_{\mathcal{E}}\\ &\quad=\max\left|\left(\mathbb{G}_1(p,q)(\tau_1)-\mathbb{G}_1(p,q)(\tau_2)\right)(\ln t)^{2-\alpha}\right|\\ &\quad\leq\Bigg[\left(\frac{|\mathcal{A}_0(\alpha)||(\ln\frac{T}{t_k})^{\alpha-1}|}{\Gamma(\alpha)}+\frac{|\mathcal{A}_4(\alpha)||(\ln\frac{T}{t_k})^{\alpha}|+k|\mathcal{A}_5(\alpha)||(\ln\frac{t_k}{t_{k-1}})^{\alpha}|}{\Gamma(\alpha+1)}\right)\\ &\qquad\times\left|(\ln t)^{2-\alpha}\right|\left|(\ln\tau_1)^{\alpha-2}-(\ln\tau_2)^{\alpha-2}\right|\\ &\qquad+\frac{k|(\ln t)^{2-\alpha}||(\ln\frac{t_k}{t_{k-1}})^{\alpha-1}||\ln\tau_1^{3-\alpha}(\log_{t_k}\tau_1)^{\alpha-2}-\ln\tau_2^{3-\alpha}(\log_{t_k}\tau_2)^{\alpha-2}|}{\Gamma(\alpha)}\Bigg]\\ &\qquad\times\left(\frac{o_1^*+\varrho_1^*o_2^*}{1-\varrho_1^*\varrho_2^*}+\frac{\rho_1^*\|p\|+\varrho_1^*\rho_2^*\|q\|}{1-\varrho_1^*\varrho_2^*}\right)\\ &\qquad+\frac{|(\ln t)^{2-\alpha}|}{\Gamma(\alpha)}\left|\int_{t_k}^{\tau_1}\left(\ln\frac{\tau_1}{s}\right)^{\alpha-1}f\left(s,p(s),{}_H\mathfrak{D}^{\beta}q(s)\right)\frac{ds}{s}\right.\\ &\qquad\left.-\int_{t_k}^{\tau_1}\left(\ln\frac{\tau_1}{s}\right)^{\alpha-1}f\left(s,p(s),{}_H\mathfrak{D}^{\beta}q(s)\right)\frac{ds}{s}\right|.\end{aligned}$$

This implies that

$$\|\mathbb{G}_1(p,q)(\tau_1)-\mathbb{G}_1(p,q)(\tau_2)\|_{\mathcal{E}_1}\to 0\quad\text{as }\tau_1\to\tau_2.$$

In the same way, we have

$$\|\mathbb{G}_2(p,q)(\tau_1)-\mathbb{G}_2(p,q)(\tau_2)\|_{\mathcal{E}_2}\to 0\quad\text{as }\tau_1\to\tau_2.$$

Hence

$$\|\mathbb{G}(p,q)(\tau_1)-\mathbb{G}(p,q)(\tau_2)\|_{\mathcal{E}}\to 0\quad\text{as }\tau_1\to\tau_2.$$

Therefore, $\mathbb{G}$ is relatively compact on $\mathcal{E}_r$. By the Arzelà–Ascoli theorem, $\mathbb{G}$ is compact and hence is a completely continuous operator. So (1.1) has at least one solution. □

Theorem 3.4 *Let the hypotheses* ($\mathbf{H}_4$)–($\mathbf{H}_6$) *be satisfied with*

$$\Lambda_1 + \Lambda_3 + \frac{\Lambda_2(\mathcal{L}_f + \widetilde{\mathcal{L}}_f \mathcal{L}_g) + \Lambda_4(\mathcal{L}_g + \mathcal{L}_f \widetilde{\mathcal{L}}_g)}{(\widetilde{\mathcal{L}}_f \widetilde{\mathcal{L}}_g - 1)} < 1, \tag{3.18}$$

then (1.1) *has unique solution.*

Proof Define an operator $\Phi = (\Phi_1, \Phi_2) : \mathcal{E} \to \mathcal{E}$, i.e., $\Phi(p,q)(t) = (\Phi_1(p,q), \Phi_2(p,q))(t)$, where

$$\begin{aligned}
\Phi_1(p,q)(t) = {} & T\mathcal{A}_0(\alpha)\varphi(p)(\ln t)^{\alpha-2} \\
& + \sum_{i=1}^{k} \mathcal{A}_{1i}(\alpha)(\ln t)^{\alpha-2}\mathcal{I}_i\big(p(t_i)\big) + \sum_{i=1}^{k} \mathcal{A}_{2i}(\alpha)(\ln t)^{\alpha-2}\widetilde{\mathcal{I}}_i\big(p(t_i)\big) \\
& + \frac{\mathcal{A}_3(\alpha)(\ln t)^{\alpha-2}}{\Gamma(\alpha)} \int_1^T \left(\ln\frac{T}{s}\right)^{\alpha-1} \phi\big(s,p(s)\big)\frac{ds}{s} \\
& + \frac{\mathcal{A}_0(\alpha)(\ln t)^{\alpha-2}}{\Gamma(\alpha-1)} \int_{t_k}^T \left(\ln\frac{T}{s}\right)^{\alpha-2} f\big(s,p(s),{}_H\mathfrak{D}^\beta q(s)\big)\frac{ds}{s} \\
& + \frac{\mathcal{A}_4(\alpha)(\ln t)^{\alpha-2}}{\Gamma(\alpha)} \int_{t_k}^T \left(\ln\frac{T}{s}\right)^{\alpha-1} f\big(s,p(s),{}_H\mathfrak{D}^\beta q(s)\big)\frac{ds}{s} \\
& + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t \left(\ln\frac{t}{s}\right)^{\alpha-1} f\big(s,p(s),{}_H\mathfrak{D}^\beta q(s)\big)\frac{ds}{s} \\
& + \sum_{i=1}^{k} \frac{\mathcal{A}_{5i}(\alpha)(\ln t)^{\alpha-2}}{\Gamma(\alpha)} \int_{t_{i-1}}^{t_i} \left(\ln\frac{t_i}{s}\right)^{\alpha-1} f\big(s,p(s),{}_H\mathfrak{D}^\beta q(s)\big)\frac{ds}{s} \\
& + \sum_{i=1}^{k} \frac{\ln t^{3-\alpha}(\log_{t_i} t)^{\alpha-2}}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_i} \left(\ln\frac{t_i}{s}\right)^{\alpha-2} f\big(s,p(s),{}_H\mathfrak{D}^\beta q(s)\big)\frac{ds}{s}, \\
& k = 1,2,\dots,m,
\end{aligned}$$

and

$$\begin{aligned}
\Phi_2(p,q)(t) = {} & T\mathcal{A}_0(\beta)\varphi(q)(\ln t)^{\beta-2} \\
& + \sum_{j=1}^{k} \mathcal{A}_{1j}(\beta)(\ln t)^{\beta-2}\mathcal{I}_j\big(q(t_j)\big) + \sum_{j=1}^{k} \mathcal{A}_{2j}(\beta)(\ln t)^{\beta-2}\widetilde{\mathcal{I}}_j\big(q(t_j)\big) \\
& + \frac{\mathcal{A}_3(\beta)(\ln t)^{\beta-2}}{\Gamma(\beta)} \int_1^T \left(\ln\frac{T}{s}\right)^{\beta-1} \phi\big(s,q(s)\big)\frac{ds}{s} \\
& + \frac{\mathcal{A}_0(\beta)(\ln t)^{\beta-2}}{\Gamma(\beta-1)} \int_{t_k}^T \left(\ln\frac{T}{s}\right)^{\beta-2} g\big(s,q(s),{}_H\mathfrak{D}^\alpha p(s)\big)\frac{ds}{s} \\
& + \frac{\mathcal{A}_4(\beta)(\ln t)^{\beta-2}}{\Gamma(\beta)} \int_{t_k}^T \left(\ln\frac{T}{s}\right)^{\beta-1} g\big(s,q(s),{}_H\mathfrak{D}^\alpha p(s)\big)\frac{ds}{s} \\
& + \frac{1}{\Gamma(\beta)} \int_{t_k}^t \left(\ln\frac{t}{s}\right)^{\beta-1} g\big(s,q(s),{}_H\mathfrak{D}^\alpha p(s)\big)\frac{ds}{s}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{k}\frac{\mathcal{A}_{5j}(\beta)(\ln t)^{\beta-2}}{\Gamma(\beta)}\int_{t_{j-1}}^{t_j}\left(\ln\frac{t_j}{s}\right)^{\beta-1}g\big(s,q(s),{}_H\mathfrak{D}^{\alpha}p(s)\big)\frac{ds}{s}\\
&+\sum_{j=1}^{k}\frac{\ln t^{3-\beta}(\log_{t_j}t)^{\beta-2}}{\Gamma(\beta-1)}\int_{t_{j-1}}^{t_j}\left(\ln\frac{t_j}{s}\right)^{\beta-2}g\big(s,q(s),{}_H\mathfrak{D}^{\alpha}p(s)\big)\frac{ds}{s},\\
&k=1,2,\ldots,n.
\end{aligned}
$$

In view of Theorem 3.3, we have

$$
\begin{aligned}
&\big|\big(\Phi_1(p,q)-\Phi_1(\widetilde{p},\widetilde{q})\big)(\ln t)^{2-\alpha}\big|\\
&\leq\Bigg[T\big|\mathcal{A}_0(\alpha)\big|\mathcal{L}_{\varphi}+k\big|\mathcal{A}_1(\alpha)\big|\mathcal{L}_{\mathcal{I}}+k\big|\mathcal{A}_2(\alpha)\big|\mathcal{L}_{\widetilde{\mathcal{I}}}-\frac{|\mathcal{A}_3(\alpha)||(\ln T)^{\alpha}|\mathcal{L}_{\phi}}{\Gamma(\alpha+1)}\\
&\quad+\Bigg(\frac{|\mathcal{A}_0(\alpha)||(\ln\frac{T}{t_k})^{\alpha-1}|}{\Gamma(\alpha)}+\frac{|\mathcal{A}_4(\alpha)||(\ln\frac{T}{t_k})^{\alpha}|}{\Gamma(\alpha+1)}+\frac{|(\ln t)^{2-\alpha}||(\ln\frac{t}{t_k})^{\alpha}|}{\Gamma(\alpha+1)}\\
&\quad+\frac{k|\mathcal{A}_5(\alpha)||(\ln\frac{t_k}{t_{k-1}})^{\alpha}|}{\Gamma(\alpha+1)}+\frac{m|\ln t^{3-\alpha}||(\ln t_k)^{2-\alpha}||(\ln\frac{t_k}{t_{k-1}})^{\alpha-1}|}{\Gamma(\alpha)}\Bigg)\frac{\mathcal{L}_f}{(\widetilde{\mathcal{L}}_f\widetilde{\mathcal{L}}_g-1)}\Bigg]\\
&\quad\times\big|p(t)-\widetilde{p}(t)\big|+\Bigg[\frac{|\mathcal{A}_0(\alpha)||(\ln\frac{T}{t_k})^{\alpha-1}|}{\Gamma(\alpha)}+\frac{|\mathcal{A}_4(\alpha)||(\ln\frac{T}{t_k})^{\alpha}|}{\Gamma(\alpha+1)}+\frac{|(\ln t)^{2-\alpha}||(\ln\frac{t}{t_k})^{\alpha}|}{\Gamma(\alpha+1)}\\
&\quad+\frac{k|\mathcal{A}_5(\alpha)||(\ln\frac{t_k}{t_{k-1}})^{\alpha}|}{\Gamma(\alpha+1)}+\frac{k|\ln t^{3-\alpha}||(\ln t_k)^{2-\alpha}||(\ln\frac{t_k}{t_{k-1}})^{\alpha-1}|}{\Gamma(\alpha)}\Bigg]\frac{\widetilde{\mathcal{L}}_f\mathcal{L}_g|q-\widetilde{q}|}{(\widetilde{\mathcal{L}}_f\widetilde{\mathcal{L}}_g-1)},\\
&k=1,2,\ldots,m.
\end{aligned}
$$

Taking $\sup_{t\in\mathcal{J}}$, we get

$$
\big\|\Phi_1(p,q)-\Phi_1(\widetilde{p},\widetilde{q})\big\|_{\mathcal{E}_1}\leq\left[\Lambda_1+\frac{\Lambda_2(\mathcal{L}_f+\widetilde{\mathcal{L}}_f\mathcal{L}_g)}{(\widetilde{\mathcal{L}}_f\widetilde{\mathcal{L}}_g-1)}\right]\big\|(p,q)-(\widetilde{p},\widetilde{q})\big\|,
$$

for $k=1,2,\ldots,m$, where

$$
\begin{aligned}
\Lambda_1&=T\big|\mathcal{A}_0(\alpha)\big|\mathcal{L}_{\varphi}+k\big|\mathcal{A}_1(\alpha)\big|\mathcal{L}_{\mathcal{I}}+k\big|\mathcal{A}_2(\alpha)\big|\mathcal{L}_{\widetilde{\mathcal{I}}}-\frac{|\mathcal{A}_3(\alpha)||(\ln T)^{\alpha}|\mathcal{L}_{\phi}}{\Gamma(\alpha+1)},\\
\Lambda_2&=\frac{|\mathcal{A}_0(\alpha)||(\ln\frac{T}{t_k})^{\alpha-1}|}{\Gamma(\alpha)}+\frac{|\mathcal{A}_4(\alpha)||(\ln\frac{T}{t_k})^{\alpha}|}{\Gamma(\alpha+1)}+\frac{|(\ln t)^{2-\alpha}||(\ln\frac{t}{t_k})^{\alpha}|}{\Gamma(\alpha+1)}\\
&\quad+\frac{k|\mathcal{A}_5(\alpha)||(\ln\frac{t_k}{t_{k-1}})^{\alpha}|}{\Gamma(\alpha+1)}+\frac{k|\ln t^{3-\alpha}||(\ln t_k)^{2-\alpha}||(\ln\frac{t_k}{t_{k-1}})^{\alpha-1}|}{\Gamma(\alpha)}.
\end{aligned}
$$

Similarly

$$
\big\|\Phi_2(p,q)-\Phi_2(\widetilde{p},\widetilde{q})\big\|_{\mathcal{E}_2}\leq\left[\Lambda_3+\frac{\Lambda_4(\mathcal{L}_g+\mathcal{L}_f\widetilde{\mathcal{L}}_g)}{(\widetilde{\mathcal{L}}_f\widetilde{\mathcal{L}}_g-1)}\right]\big\|(p,q)-(\widetilde{p},\widetilde{q})\big\|,
$$

for $k=1,2,\ldots,n$, where

$$
\Lambda_3=T\big|\mathcal{A}_0(\beta)\big|\widetilde{\mathcal{L}}_{\varphi}+k\big|\mathcal{A}_1(\beta)\big|\widetilde{\mathcal{L}}_{\mathcal{I}}+k\big|\mathcal{A}_2(\beta)\big|\widetilde{\mathcal{L}}_{\widetilde{\mathcal{I}}}-\frac{|\mathcal{A}_3(\beta)||(\ln T)^{\beta}|\widetilde{\mathcal{L}}_{\phi}}{\Gamma(\beta+1)},
$$

$$\Lambda_4 = \frac{|\mathcal{A}_0(\beta)||(\ln \frac{T}{t_k})^{\beta-1}|}{\Gamma(\beta)} + \frac{|\mathcal{A}_4(\beta)||(\ln \frac{T}{t_k})^{\beta}|}{\Gamma(\beta+1)} + \frac{|(\ln t)^{2-\beta}||(\ln \frac{t}{t_k})^{\beta}|}{\Gamma(\beta+1)} + \frac{k|\mathcal{A}_5(\beta)||(\ln \frac{t_k}{t_{k-1}})^{\beta}|}{\Gamma(\beta+1)} + \frac{k|\ln t^{3-\beta}||(\ln t_k)^{2-\beta}||(\ln \frac{t_k}{t_{k-1}})^{\beta-1}|}{\Gamma(\beta)}.$$

Hence,

$$\begin{aligned}&\|\Phi(p,q) - \Phi(\widetilde{p},\widetilde{q})\|_{\mathcal{E}}\\ &\quad\leq \left[\Lambda_1 + \Lambda_3 + \frac{\Lambda_2(\mathcal{L}_f + \widetilde{\mathcal{L}}_f\mathcal{L}_g) + \Lambda_4(\mathcal{L}_g + \mathcal{L}_f\widetilde{\mathcal{L}}_g)}{(\widetilde{\mathcal{L}}_f\widetilde{\mathcal{L}}_g - 1)}\right]\|(p,q) - (\widetilde{p},\widetilde{q})\|.\end{aligned}$$

This implies that the operator Φ is a contraction. Therefore, (1.1) has a unique solution. □

4 Ulam stability analysis

In this portion, we analyze different kinds of stability, like the Hyers–Ulam, generalized Hyers–Ulam, Hyers–Ulam–Rassias and generalized Hyers–Ulam–Rassias stability of the proposed system.

Theorem 4.1 *If assumptions* $(\mathbf{H}_1)$–$(\mathbf{H}_3)$ *and inequality* (3.18) *are satisfied and*

$$F = 1 - \frac{\mathcal{L}_f\widetilde{\mathcal{L}}_f\mathcal{L}_g\widetilde{\mathcal{L}}_g\Lambda_2\Lambda_4}{((\widetilde{\mathcal{L}}_f\widetilde{\mathcal{L}}_g - 1)((\ln t)^{\alpha-2} - \Lambda_1) - \Lambda_2\mathcal{L}_f)((\widetilde{\mathcal{L}}_f\widetilde{\mathcal{L}}_g - 1)((\ln t)^{\beta-2} - \Lambda_3) - \Lambda_4\mathcal{L}_g)} > 0,$$

then the unique solution of the coupled system (1.1) *is Hyers–Ulam stable and consequently generalized Hyers–Ulam stable.*

Proof Consider $(p,q) \in \mathcal{E}$ be an approximate solution of inequality (2.1) and let $(\widehat{p},\widehat{q}) \in \mathcal{E}$ be the unique solution of the coupled system given by

$$\begin{cases} {}_H\mathfrak{D}^{\alpha}\widehat{p}(t) = f(t,\widehat{p}(t), {}_H\mathfrak{D}^{\beta}\widehat{q}(t)), & t \in \mathcal{J}, t \neq t_i, i = 1,2,\ldots,m,\\ {}_H\mathfrak{D}^{\beta}\widehat{q}(t) = g(t, {}_H\mathfrak{D}^{\alpha}\widehat{p}(t),\widehat{q}(t)), & t \in \mathcal{J}, t \neq t_j, j = 1,2,\ldots,n,\\ \Delta\widehat{p}(t_i) = \mathcal{I}_i(\widehat{p}(t_i)), \quad \Delta\widehat{p}'(t_i) = \widetilde{\mathcal{I}}_i(\widehat{p}(t_i)), & i = 1,2,\ldots,m,\\ \Delta\widehat{q}(t_j) = \mathcal{I}_j(\widehat{q}(t_j)), \quad \Delta\widehat{q}'(t_j) = \widetilde{\mathcal{I}}_j(\widehat{q}(t_j)), & j = 1,2,\ldots,n,\\ \widehat{p}(T) = \int_1^T \frac{(\ln\frac{T}{s})^{\alpha-1}}{\Gamma(\alpha)}\phi(s,\widehat{p}(s))\frac{ds}{s}, & \widehat{p}'(T) = \varphi(\widehat{p}),\\ \widehat{q}(T) = \int_1^T \frac{(\ln\frac{T}{s})^{\beta-1}}{\Gamma(\beta)}\phi(s,\widehat{q}(s))\frac{ds}{s}, & \widehat{q}'(T) = \varphi(\widehat{q}). \end{cases} \tag{4.1}$$

By Remark 2.9 we have

$$\begin{cases} {}_H\mathfrak{D}^{\alpha}p(t) = f(t,p(t), {}_H\mathfrak{D}^{\beta}q(t)) + \Upsilon_f(t), & t \in \mathcal{J}, t \neq t_i, i = 1,2,\ldots,m,\\ \Delta p(t_i) = \mathcal{I}_i(p(t_i)) + \Upsilon_{f_i}, \quad \Delta p'(t_i) = \widetilde{\mathcal{I}}_i(p(t_i)) + \Upsilon_{f_i}, & i = 1,2,\ldots,m,\\ {}_H\mathfrak{D}^{\beta}q(t) = g(t, {}_H\mathfrak{D}^{\alpha}p(t), q(t)) + \Upsilon_g(t), & t \in \mathcal{J}, t \neq t_j, j = 1,2,\ldots,n,\\ \Delta q(t_j) = \mathcal{I}_j(q(t_j)) + \Upsilon_{g_j}, \quad \Delta q'(t_j) = \widetilde{\mathcal{I}}_j(q(t_j)) + \Upsilon_{g_j}, & j = 1,2,\ldots,n. \end{cases} \tag{4.2}$$

By Corollary 3.2, the solution of problem (4.2) is

$$\begin{cases} p(t) = T\mathcal{A}_0(\alpha)\varphi(p)(\ln t)^{\alpha-2} + \sum_{i=1}^{k}\mathcal{A}_{1i}(\alpha)(\ln t)^{\alpha-2}(\mathcal{I}_i(p(t_i)) + \Upsilon_{f_i}) \\ \quad + \sum_{i=1}^{k}\mathcal{A}_{2i}(\alpha)(\ln t)^{\alpha-2}(\widetilde{\mathcal{I}}_i(p(t_i)) + \Upsilon_{f_i}) \\ \quad + \frac{\mathcal{A}_0(\alpha)(\ln t)^{\alpha-2}}{\Gamma(\alpha-1)}\int_{t_k}^{T}(\ln\frac{T}{s})^{\alpha-2}(f(s,p(s),{}_H\mathfrak{D}^{\beta}q(s)) + \Upsilon_f(t))\frac{ds}{s} \\ \quad + \frac{\mathcal{A}_4(\alpha)(\ln t)^{\alpha-2}}{\Gamma(\alpha)}\int_{t_k}^{T}(\ln\frac{T}{s})^{\alpha-1}(f(s,p(s),{}_H\mathfrak{D}^{\beta}q(s)) + \Upsilon_f(t))\frac{ds}{s} \\ \quad + \sum_{i=1}^{k}\frac{\mathcal{A}_{5i}(\alpha)(\ln t)^{\alpha-2}}{\Gamma(\alpha)}\int_{t_{i-1}}^{t_i}(\ln\frac{t_i}{s})^{\alpha-1}(f(s,p(s),{}_H\mathfrak{D}^{\beta}q(s)) + \Upsilon_f(t))\frac{ds}{s} \\ \quad + \sum_{i=1}^{k}\frac{\ln t^{3-\alpha}(\log_{t_i}t)^{\alpha-2}}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(\ln\frac{t_i}{s})^{\alpha-2}(f(s,p(s),{}_H\mathfrak{D}^{\beta}q(s)) + \Upsilon_f(t))\frac{ds}{s} \\ \quad + \frac{\mathcal{A}_3(\alpha)(\ln t)^{\alpha-2}}{\Gamma(\alpha)}\int_1^{T}(\ln\frac{T}{s})^{\alpha-1}\phi(s,p(s))\frac{ds}{s} \\ \quad + \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(\ln\frac{t}{s})^{\alpha-1}(f(s,p(s),{}_H\mathfrak{D}^{\beta}q(s)) + \Upsilon_f(t))\frac{ds}{s}, \\ \quad k = 1,2,\ldots,m, \\ q(t) = T\mathcal{A}_0(\beta)\varphi(q)(\ln t)^{\beta-2} + \sum_{j=1}^{k}\mathcal{A}_{1j}(\beta)(\ln t)^{\beta-2}(\mathcal{I}_j(q(t_j)) + \Upsilon_{g_i}) \\ \quad + \sum_{j=1}^{k}\mathcal{A}_{2j}(\beta)(\ln t)^{\beta-2}(\widetilde{\mathcal{I}}_j(q(t_j)) + \Upsilon_{g_i}) \\ \quad + \frac{\mathcal{A}_0(\beta)(\ln t)^{\beta-2}}{\Gamma(\beta-1)}\int_{t_k}^{T}(\ln\frac{T}{s})^{\beta-2}(g(s,q(s),{}_H\mathfrak{D}^{\alpha}p(s)) + \Upsilon_g(t))\frac{ds}{s} \\ \quad + \frac{\mathcal{A}_4(\beta)(\ln t)^{\beta-2}}{\Gamma(\beta)}\int_{t_k}^{T}(\ln\frac{T}{s})^{\beta-1}(g(s,q(s),{}_H\mathfrak{D}^{\alpha}p(s)) + \Upsilon_g(t))\frac{ds}{s} \\ \quad + \sum_{j=1}^{k}\frac{\mathcal{A}_{5j}(\beta)(\ln t)^{\beta-2}}{\Gamma(\beta)}\int_{t_{j-1}}^{t_j}(\ln\frac{t_j}{s})^{\beta-1}(g(s,q(s),{}_H\mathfrak{D}^{\alpha}p(s)) + \Upsilon_g(t))\frac{ds}{s} \\ \quad + \sum_{j=1}^{k}\frac{\ln t^{3-\beta}(\log_{t_j}t)^{\beta-2}}{\Gamma(\beta-1)}\int_{t_{j-1}}^{t_j}(\ln\frac{t_j}{s})^{\beta-2}(g(s,q(s),{}_H\mathfrak{D}^{\alpha}p(s)) + \Upsilon_g(t))\frac{ds}{s} \\ \quad + \frac{\mathcal{A}_3(\beta)(\ln t)^{\beta-2}}{\Gamma(\beta)}\int_1^{T}(\ln\frac{T}{s})^{\beta-1}\phi(s,q(s))\frac{ds}{s} \\ \quad + \frac{1}{\Gamma(\beta)}\int_{t_k}^{t}(\ln\frac{t}{s})^{\beta-1}(g(s,q(s),{}_H\mathfrak{D}^{\alpha}p(s)) + \Upsilon_g(t))\frac{ds}{s}, \\ \quad k = 1,2,\ldots,n. \end{cases} \tag{4.3}$$

We consider

$$\begin{aligned} &\bigl|\bigl(p(t)-\widehat{p}(t)\bigr)(\ln t)^{2-\alpha}\bigr| \\ &\quad\leq T\bigl|\mathcal{A}_0(\alpha)\bigr|\bigl|\varphi(p)-\varphi(\widehat{p})\bigr| \\ &\quad+\sum_{i=1}^{k}\bigl|\mathcal{A}_{1i}(\alpha)\bigr|\bigl|\mathcal{I}_i\bigl(p(t_i)\bigr)-\mathcal{I}_i\bigl(\widehat{p}(t_i)\bigr)\bigr| + \sum_{i=1}^{k}\bigl|\mathcal{A}_{2i}(\alpha)\bigr|\bigl|\widetilde{\mathcal{I}}_i\bigl(p(t_i)\bigr)-\widetilde{\mathcal{I}}_i\bigl(\widehat{p}(t_i)\bigr)\bigr| \\ &\quad+\frac{|\mathcal{A}_0(\alpha)|}{\Gamma(\alpha-1)}\int_{t_k}^{T}\biggl|\biggl(\ln\frac{T}{s}\biggr)^{\alpha-2}\biggr|\bigl|f\bigl(s,p(s),{}_H\mathfrak{D}^{\beta}q(s)\bigr)-f\bigl(s,\widehat{p}(s),{}_H\mathfrak{D}^{\beta}\widehat{q}(s)\bigr)\bigr|\frac{ds}{s} \\ &\quad+\frac{|\mathcal{A}_4(\alpha)|}{\Gamma(\alpha)}\int_{t_k}^{T}\biggl|\biggl(\ln\frac{T}{s}\biggr)^{\alpha-1}\biggr|\bigl|f\bigl(s,p(s),{}_H\mathfrak{D}^{\beta}q(s)\bigr)-f\bigl(s,\widehat{p}(s),{}_H\mathfrak{D}^{\beta}\widehat{q}(s)\bigr)\bigr|\frac{ds}{s} \\ &\quad+\sum_{i=1}^{k}\frac{|\mathcal{A}_{5i}(\alpha)|}{\Gamma(\alpha)}\int_{t_{i-1}}^{t_i}\biggl|\biggl(\ln\frac{t_i}{s}\biggr)^{\alpha-1}\biggr|\bigl|f\bigl(s,p(s),{}_H\mathfrak{D}^{\beta}q(s)\bigr)-f\bigl(s,\widehat{p}(s),{}_H\mathfrak{D}^{\beta}\widehat{q}(s)\bigr)\bigr|\frac{ds}{s} \\ &\quad+\sum_{i=1}^{k}\frac{|\ln t^{3-\alpha}||(\ln t_i)^{2-\alpha}|}{\Gamma(\alpha-1)} \\ &\quad\times\int_{t_{i-1}}^{t_i}\biggl|\biggl(\ln\frac{t_i}{s}\biggr)^{\alpha-2}\biggr|\bigl|f\bigl(s,p(s),{}_H\mathfrak{D}^{\beta}q(s)\bigr)-f\bigl(s,\widehat{p}(s),{}_H\mathfrak{D}^{\beta}\widehat{q}(s)\bigr)\bigr|\frac{ds}{s} \end{aligned}$$

$$+\frac{|(\ln t)^{2-\alpha}|}{\Gamma(\alpha)}\int_{t_k}^{t}\left|\left(\ln\frac{t}{s}\right)^{\alpha-1}\right|\left|f\left(s,p(s),{}_H\mathfrak{D}^{\beta}q(s)\right)-f\left(s,\widehat{p}(s),{}_H\mathfrak{D}^{\beta}\widehat{q}(s)\right)\right|\frac{ds}{s}$$
$$+\frac{|\mathcal{A}_3(\alpha)|}{\Gamma(\alpha)}\int_{1}^{T}\left|\left(\ln\frac{T}{s}\right)^{\alpha-1}\right|\left|\phi\left(s,p(s)\right)-\phi\left(s,\widehat{p}(s)\right)\right|\frac{ds}{s}$$
$$+\sum_{i=1}^{k}\left|\mathcal{A}_{2i}(\alpha)\right||\Upsilon_{f_i}|+\frac{|\mathcal{A}_0(\alpha)|}{\Gamma(\alpha-1)}\int_{t_k}^{T}\left|\left(\ln\frac{T}{s}\right)^{\alpha-2}\right|\left|\Upsilon_f(s)\right|\frac{ds}{s}$$
$$+\sum_{i=1}^{k}\left|\mathcal{A}_{1i}(\alpha)\right||\Upsilon_{f_i}|+\frac{|\mathcal{A}_4(\alpha)|}{\Gamma(\alpha)}\int_{t_k}^{T}\left|\left(\ln\frac{T}{s}\right)^{\alpha-1}\right|\left|\Upsilon_f(s)\right|\frac{ds}{s}$$
$$+\sum_{i=1}^{k}\frac{|\mathcal{A}_{5i}(\alpha)|}{\Gamma(\alpha)}\int_{t_{i-1}}^{t_i}\left|\left(\ln\frac{t_i}{s}\right)^{\alpha-1}\right|\left|\Upsilon_f(s)\right|\frac{ds}{s}$$
$$+\sum_{i=1}^{k}\frac{|\ln t^{3-\alpha}||(\ln t_i)^{2-\alpha}|}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}\left|\left(\ln\frac{t_i}{s}\right)^{\alpha-2}\right|\left|\Upsilon_f(s)\right|\frac{ds}{s}$$
$$+\frac{|(\ln t)^{2-\alpha}|}{\Gamma(\alpha)}\int_{t_k}^{t}\left|\left(\ln\frac{t}{s}\right)^{\alpha-1}\right|\left|\Upsilon_f(s)\right|\frac{ds}{s}.$$

As in Theorem 3.4, we get

$$\|p-\widehat{p}\|_{\mathcal{E}_1}\le\left(\Lambda_1+\frac{\Lambda_2\mathcal{L}_f}{\widetilde{\mathcal{L}}_f\widetilde{\mathcal{L}}_g-1}\right)(\ln t)^{2-\alpha}\|p-\widehat{p}\|_{\mathcal{E}_1}+\frac{\Lambda_2\widetilde{\mathcal{L}}_f\mathcal{L}_g}{\widetilde{\mathcal{L}}_f\widetilde{\mathcal{L}}_g-1}(\ln t)^{2-\alpha}\|q-\widehat{q}\|_{\mathcal{E}_1}$$
$$+\left(\Lambda_2+k\left|\mathcal{A}_1(\alpha)\right|+k\left|\mathcal{A}_2(\alpha)\right|\right)\varrho_\alpha,\quad k=1,2,\ldots,m, \tag{4.4}$$

and

$$\|q-\widehat{q}\|_{\mathcal{E}_2}\le\frac{\Lambda_4\mathcal{L}_f\widetilde{\mathcal{L}}_g}{\widetilde{\mathcal{L}}_f\widetilde{\mathcal{L}}_g-1}(\ln t)^{2-\beta}\|p-\widehat{p}\|_{\mathcal{E}_2}+\left(\Lambda_3+\frac{\Lambda_4\mathcal{L}_g}{\widetilde{\mathcal{L}}_f\widetilde{\mathcal{L}}_g-1}\right)(\ln t)^{2-\beta}\|q-\widehat{q}\|_{\mathcal{E}_2}$$
$$+\left(\Lambda_4+k\left|\mathcal{A}_1(\beta)\right|+k\left|\mathcal{A}_2(\beta)\right|\right)\varrho_\beta,\quad k=1,2,\ldots,n. \tag{4.5}$$

From (4.4) and (4.5) we have

$$\|p-\widehat{p}\|_{\mathcal{E}_1}-\frac{\Lambda_2\widetilde{\mathcal{L}}_f\mathcal{L}_g}{(\widetilde{\mathcal{L}}_f\widetilde{\mathcal{L}}_g-1)((\ln t)^{\alpha-2}-\Lambda_1)-\Lambda_2\mathcal{L}_f}\|q-\widehat{q}\|_{\mathcal{E}_1}$$
$$\le\frac{\Lambda_2+k|\mathcal{A}_1(\alpha)|+k|\mathcal{A}_2(\alpha)|}{1-(\Lambda_1+\frac{\Lambda_2\mathcal{L}_f}{\widetilde{\mathcal{L}}_f\widetilde{\mathcal{L}}_g-1})(\ln t)^{2-\alpha}}\varrho_\alpha$$

and

$$\|q-\widehat{q}\|_{\mathcal{E}_2}-\frac{\Lambda_4\mathcal{L}_f\widetilde{\mathcal{L}}_g}{(\widetilde{\mathcal{L}}_f\widetilde{\mathcal{L}}_g-1)((\ln t)^{\beta-2}-\Lambda_3)-\Lambda_4\mathcal{L}_g}\|p-\widehat{p}\|_{\mathcal{E}_2}$$
$$\le\frac{\Lambda_4+k|\mathcal{A}_1(\beta)|+k|\mathcal{A}_2(\beta)|}{1-(\Lambda_3+\frac{\Lambda_4\mathcal{L}_g}{\widetilde{\mathcal{L}}_f\widetilde{\mathcal{L}}_g-1})(\ln t)^{2-\beta}}\varrho_\beta,$$

respectively. Let $\mathcal{G}_\alpha = \frac{\Lambda_2+k|\mathcal{A}_1(\alpha)|+k|\mathcal{A}_2(\alpha)|}{1-(\Lambda_1+\frac{\Lambda_2\mathcal{L}_f}{\widetilde{\mathcal{L}}_f\widetilde{\mathcal{L}}_g-1})(\ln t)^{2-\alpha}}$ and $\mathcal{G}_\beta = \frac{\Lambda_4+k|\mathcal{A}_1(\beta)|+k|\mathcal{A}_2(\beta)|}{1-(\Lambda_3+\frac{\Lambda_4\mathcal{L}_g}{\widetilde{\mathcal{L}}_f\widetilde{\mathcal{L}}_g-1})(\ln t)^{2-\beta}}$. Then the last two inequalities can be written in matrix form as

$$\begin{aligned}
&\begin{bmatrix} 1 & -\frac{\Lambda_2\widetilde{\mathcal{L}}_f\mathcal{L}_g}{(\widetilde{\mathcal{L}}_f\widetilde{\mathcal{L}}_g-1)((\ln t)^{\alpha-2}-\Lambda_1)-\Lambda_2\mathcal{L}_f} \\ -\frac{\Lambda_4\mathcal{L}_f\widetilde{\mathcal{L}}_g}{(\widetilde{\mathcal{L}}_f\widetilde{\mathcal{L}}_g-1)((\ln t)^{\beta-2}-\Lambda_3)-\Lambda_4\mathcal{L}_g} & 1 \end{bmatrix} \\
&\quad\times \begin{bmatrix} \|p-\widehat{p}\|_{\mathcal{E}_1} \\ \|q-\widehat{q}\|_{\mathcal{E}_2} \end{bmatrix} \le \begin{bmatrix} \mathcal{G}_\alpha\varrho_\alpha \\ \mathcal{G}_\beta\varrho_\beta \end{bmatrix}, \\
&\begin{bmatrix} \|p-\widehat{p}\|_{\mathcal{E}_1} \\ \|q-\widehat{q}\|_{\mathcal{E}_2} \end{bmatrix} \\
&\quad\le \begin{bmatrix} \frac{1}{F} & \frac{\Lambda_2\widetilde{\mathcal{L}}_f\mathcal{L}_g}{(\widetilde{\mathcal{L}}_f\widetilde{\mathcal{L}}_g-1)((\ln t)^{\alpha-2}-\Lambda_1)-\Lambda_2\mathcal{L}_f}\frac{1}{F} \\ \frac{\Lambda_4\mathcal{L}_f\widetilde{\mathcal{L}}_g}{(\widetilde{\mathcal{L}}_f\widetilde{\mathcal{L}}_g-1)((\ln t)^{\beta-2}-\Lambda_3)-\Lambda_4\mathcal{L}_g}\frac{1}{F} & \frac{1}{F} \end{bmatrix} \begin{bmatrix} \mathcal{G}_\alpha\varrho_\alpha \\ \mathcal{G}_\beta\varrho_\beta \end{bmatrix},
\end{aligned} \tag{4.6}$$

where

$$F = 1 - \frac{\mathcal{L}_f\widetilde{\mathcal{L}}_f\mathcal{L}_g\widetilde{\mathcal{L}}_g\Lambda_2\Lambda_4}{((\widetilde{\mathcal{L}}_f\widetilde{\mathcal{L}}_g-1)((\ln t)^{\alpha-2}-\Lambda_1)-\Lambda_2\mathcal{L}_f)((\widetilde{\mathcal{L}}_f\widetilde{\mathcal{L}}_g-1)((\ln t)^{\beta-2}-\Lambda_3)-\Lambda_4\mathcal{L}_g)} > 0.$$

From system (4.6) we have

$$\begin{aligned}
\|p-\widehat{p}\|_{\mathcal{E}_1} &\le \frac{\mathcal{G}_\alpha\varrho_\alpha}{F} + \frac{\Lambda_2\widetilde{\mathcal{L}}_f\mathcal{L}_g\mathcal{G}_\beta\varrho_\beta}{(\widetilde{\mathcal{L}}_f\widetilde{\mathcal{L}}_g-1)((\ln t)^{\alpha-2}-\Lambda_1)-\Lambda_2\mathcal{L}_f}\frac{1}{F}, \\
\|q-\widehat{q}\|_{\mathcal{E}_2} &\le \frac{\mathcal{G}_\beta\varrho_\beta}{F} + \frac{\Lambda_4\mathcal{L}_f\widetilde{\mathcal{L}}_g\mathcal{G}_\alpha\varrho_\alpha}{(\widetilde{\mathcal{L}}_f\widetilde{\mathcal{L}}_g-1)((\ln t)^{\beta-2}-\Lambda_3)-\Lambda_4\mathcal{L}_g}\frac{1}{F},
\end{aligned}$$

which implies that

$$\begin{aligned}
\|p-\widehat{p}\|_{\mathcal{E}_1} + \|q-\widehat{q}\|_{\mathcal{E}_2} &\le \frac{\mathcal{G}_\alpha\varrho_\alpha}{F} + \frac{\mathcal{G}_\beta\varrho_\beta}{F} + \frac{\Lambda_2\widetilde{\mathcal{L}}_f\mathcal{L}_g\mathcal{G}_\beta\varrho_\beta}{(\widetilde{\mathcal{L}}_f\widetilde{\mathcal{L}}_g-1)((\ln t)^{\alpha-2}-\Lambda_1)-\Lambda_2\mathcal{L}_f}\frac{1}{F} \\
&\quad + \frac{\Lambda_4\mathcal{L}_f\widetilde{\mathcal{L}}_g\mathcal{G}_\alpha\varrho_\alpha}{(\widetilde{\mathcal{L}}_f\widetilde{\mathcal{L}}_g-1)((\ln t)^{\beta-2}-\Lambda_3)-\Lambda_4\mathcal{L}_g}\frac{1}{F}.
\end{aligned}$$

If $\max\{\varrho_\alpha,\varrho_\beta\} = \varrho$ and $\frac{\mathcal{G}_\alpha}{F} + \frac{\mathcal{G}_\beta}{F} + \frac{\Lambda_2\widetilde{\mathcal{L}}_f\mathcal{L}_g\mathcal{G}_\beta}{(\widetilde{\mathcal{L}}_f\widetilde{\mathcal{L}}_g-1)((\ln t)^{\alpha-2}-\Lambda_1)-\Lambda_2\mathcal{L}_f}\frac{1}{F} + \frac{\Lambda_4\mathcal{L}_f\widetilde{\mathcal{L}}_g\mathcal{G}_\alpha}{(\widetilde{\mathcal{L}}_f\widetilde{\mathcal{L}}_g-1)((\ln t)^{\beta-2}-\Lambda_3)-\Lambda_4\mathcal{L}_g}\frac{1}{F} = \mathcal{G}_{\alpha,\beta}$, then

$$\|(p,q)-(\widehat{p},\widehat{q})\|_{\mathcal{E}} \le \mathcal{G}_{\alpha,\beta}\varrho.$$

This shows that system (1.1) is Hyers–Ulam stable. Also, if

$$\|(p,q)-(\widehat{p},\widehat{q})\|_{\mathcal{E}} \le \mathcal{G}_{\alpha,\beta}\Phi(\varrho),$$

with $\Phi(0) = 0$, then the solution of system (1.1) is generalized Hyers–Ulam stable. □

For the upcoming result, we suppose that:

($\mathbf{H}_6$) There exist two nondecreasing functions $\bar{w}_\alpha, \bar{w}_\beta \in \mathcal{C}(\mathcal{J}, \mathcal{R}^+)$ such that

$$ {}_H\mathfrak{I}^\alpha \bar{w}_\alpha(t) \le \mathcal{L}_\alpha \bar{w}_\alpha(t) \quad \text{and} \quad {}_H\mathfrak{I}^\beta \bar{w}_\beta(t) \le \mathcal{L}_\beta \bar{w}_\beta(t), \quad \text{where } \mathcal{L}_\alpha, \mathcal{L}_\beta > 0. $$

Theorem 4.2 *If assumptions* ($\mathbf{H}_1$)–($\mathbf{H}_3$) *and* ($\mathbf{H}_6$) *and inequality* (3.18) *are satisfied and*

$$ \digamma = 1 - \frac{\mathcal{L}_f \widetilde{\mathcal{L}}_f \mathcal{L}_g \widetilde{\mathcal{L}}_g \Lambda_2 \Lambda_4}{((\widetilde{\mathcal{L}}_f \widetilde{\mathcal{L}}_g - 1)((\ln t)^{\alpha-2} - \Lambda_1) - \Lambda_2 \mathcal{L}_f)((\widetilde{\mathcal{L}}_f \widetilde{\mathcal{L}}_g - 1)((\ln t)^{\beta-2} - \Lambda_3) - \Lambda_4 \mathcal{L}_g)} > 0, $$

then the unique solution of the coupled system (1.1) *is Hyers–Ulam–Rassias stable and consequently generalized Hyers–Ulam–Rassias stable.*

Proof By using Definitions 2.7 and 2.8, we can obtain our result performing the same steps as in Theorem 4.1. □

5 Example

Example 5.1 Consider

$$ \begin{cases} {}_H\mathfrak{D}^{\frac{6}{5}} p(t) - \frac{2+p(t)+{}_H\mathfrak{D}^{\frac{5}{4}} q(t)}{70e^{t+20}(1+p(t)+{}_H\mathfrak{D}^{\frac{5}{4}} q(t))} = 0, \quad t \neq \frac{3}{2}, \\ {}_H\mathfrak{D}^{\frac{5}{4}} q(t) - \frac{t\cos(p(t)) - q(t)\sin(t)}{50} - \frac{{}_H\mathfrak{D}^{\frac{6}{5}} p(t)}{25+{}_H\mathfrak{D}^{\frac{6}{5}} p(t)} = 0, \quad t \neq \frac{3}{2}, \\ \Delta p(\frac{3}{2}) = \mathcal{I}_1(p(\frac{3}{2})) = \frac{|p(\frac{3}{2})|}{2+|p(\frac{3}{2})|} \quad \text{and} \quad \Delta p'(\frac{3}{2}) = \widetilde{\mathcal{I}}_1(p(\frac{3}{2})) = \frac{|p(\frac{3}{2})|}{25+|p(\frac{3}{2})|}, \\ \Delta q(\frac{3}{2}) = \mathcal{I}_1(q(\frac{3}{2})) = \frac{|q(\frac{3}{2})|}{2+|q(\frac{3}{2})|} \quad \text{and} \quad \Delta q'(\frac{3}{2}) = \widetilde{\mathcal{I}}_1(q(\frac{3}{2})) = \frac{|q(\frac{3}{2})|}{25+|q(\frac{3}{2})|}, \quad t_1 = \frac{3}{2}, \\ p(e) = \int_1^e \frac{(\ln \frac{e}{s})^{\frac{1}{5}}}{\Gamma(\frac{6}{5})} \frac{s^2+p(s)}{60} \frac{ds}{s} \quad \text{and} \quad p'(e) = \sum_{k=1}^{10} \frac{1}{\wp_k} |p(\zeta_k)|, \quad 1 < \zeta_k < 2\hbar_k > 0, \\ q(e) = \int_1^e \frac{(\ln \frac{e}{s})^{\frac{1}{4}}}{\Gamma(\frac{5}{4})} \frac{s^2+q(s)}{60} \frac{ds}{s} \quad \text{and} \quad q'(e) = \sum_{k=1}^{10} \frac{1}{\wp_k} |q(\eta_k)|, \quad 1 < \zeta_k < 2\wp_k > 0, \end{cases} \tag{5.1} $$

where $\sum_{k=1}^{10} \frac{1}{\wp_k} < \frac{1}{2}$ for $t \in [1, e]$. From the system (5.1), we can see $\alpha = \frac{6}{5}$, $\beta = \frac{5}{4}$, $T = e$, $m = 1$ and $t_1 = \frac{3}{2}$. Also, we can easily find $\mathcal{L}_\varphi = \widetilde{\mathcal{L}}_\varphi = \frac{1}{2}$, $\mathcal{L}_\phi = \widetilde{\mathcal{L}}_\phi = \frac{1}{60}$, $\mathcal{L}_\mathcal{I} = \widetilde{\mathcal{L}}_\mathcal{I} = \frac{1}{2}$, $\mathcal{L}_{\widetilde{\mathcal{I}}} = \widetilde{\mathcal{L}}_{\widetilde{\mathcal{I}}} = \frac{1}{25}$, $\mathcal{L}_f = \widetilde{\mathcal{L}}_f = \frac{1}{70e^{20}}$ and $\mathcal{L}_g = \widetilde{\mathcal{L}}_g = \frac{1}{25}$. With the help of Theorem 3.4, the following inequality is found:

$$ \Lambda_1 + \Lambda_3 + \frac{\Lambda_2(\mathcal{L}_f + \widetilde{\mathcal{L}}_f \mathcal{L}_g) + \Lambda_4(\mathcal{L}_g + \mathcal{L}_f \widetilde{\mathcal{L}}_g)}{(\widetilde{\mathcal{L}}_f \widetilde{\mathcal{L}}_g - 1)} \approx 0.5366 > 0, $$

hence (5.1) has a unique solution. Also,

$$ \begin{aligned} \digamma &= 1 - \frac{\mathcal{L}_f \widetilde{\mathcal{L}}_f \mathcal{L}_g \widetilde{\mathcal{L}}_g \Lambda_2 \Lambda_4}{((\widetilde{\mathcal{L}}_f \widetilde{\mathcal{L}}_g - 1)((\ln t)^{\alpha-2} - \Lambda_1) - \Lambda_2 \mathcal{L}_f)((\widetilde{\mathcal{L}}_f \widetilde{\mathcal{L}}_g - 1)((\ln t)^{\beta-2} - \Lambda_3) - \Lambda_4 \mathcal{L}_g)} \\ &\approx 0.02280 > 0, \end{aligned} $$

hence by Theorem 4.1 the coupled system (5.1) is Hyers–Ulam stable and thus generalized Hyers–Ulam stable. Similarly, we can verify the condition of Theorems 3.3 and 4.2.

6 Conclusion

In this paper, we have used the Krasnoselskii fixed point theorem to achieve the necessary criteria for the existence and uniqueness of the solution of considered implicit coupled impulsive fractional differential systems given in (1.1). Additionally, under particular assumptions and conditions, we have established the Hyers–Ulam stability results for the solution of the considered problem (1.1). From the obtained results, we conclude that such a method is very powerful, effectual and suitable for the solution of nonlinear implicit coupled impulsive fractional differential equations.

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