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Positive periodic solution for indefinite singular Liénard equation with p-Laplacian

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Abstract

The efficient conditions guaranteeing the existence of positive T-periodic solution to the p-Laplacian–Liénard equation

$$(\phi_p(x'(t)))' + f(x(t))x'(t) + \alpha_1(t)g(x(t)) = \frac{\alpha_2(t)}{x^{\mu}(t)},$$

are established in this paper. Here $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\alpha_1, \alpha_2 \in L([0, T], R)$, $f \in C(R_+, R)$ (R_+ stands for positive real numbers) with a singularity at $x = 0$, $g(x)$ is continuous on $(0; +\infty)$, μ is a constant with $\mu > 0$, the signs of α_1 and α_2 are allowed to change. The approach is based on the continuation theorem for p-Laplacian-like nonlinear systems obtained by Manásevich and Mawhin in (J. Differ. Equ. 145:367–393, 1998).

MSC: 34B15

Keywords: Singularity; Continuation theorem; Periodic solution

1 Introduction

This paper is devoted to investigating the existence of positive T-periodic solutions to the following equation with an indefinite singularity:

$$\big(\phi_p\big(x'(t)\big)\big)' + f\big(x(t)\big)x'(t) + \alpha_1(t)g\big(x(t)\big) = \frac{\alpha_2(t)}{x^{\mu}(t)}, \tag{1.1}$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\alpha_1, \alpha_2 \in L([0, T], R)$ with period T, $f \in C(R_+, R)$ (R_+ stands for positive real numbers) with a singularity at $x = 0$, $g(x)$ is continuous on $(0; +\infty)$, μ is a constant with $\mu > 0$. In this equation, the signs of α_1 and α_2 are allowed to change.

Let us recall the early work about second-order singular equations. In 1987, Lazer and Solimini [2] considered the following equations:

$$x'' + \frac{1}{x^{\gamma}} = p(t) \tag{1.2}$$

and

$$x'' - \frac{1}{x^{\gamma}} = p(t), \tag{1.3}$$

where $\gamma > 0$ and $p(t)$ is a periodic function. Equations (1.2) and (1.3) may be the simplest examples combining singular nonlinearity and periodic dependence of coefficients. In the related literature, it is said that the nonlinearity g has an attractive singularity (resp. repulsive singularity) at zero if $\lim_{x\to 0^+} g(x) = +\infty$ (resp. $\lim_{x\to 0^+} g(x) = -\infty$). Obviously, (1.2) has an attractive singularity and (1.3) has a repulsive singularity. After that, a lot of results have been obtained for second-order singular differential equations. Hakl and Zamora [3] answered an open problem presented by Lazer and Solimini [2]. Hakl, Torres, and Zamora [4] studied the existence of periodic solutions to the second-order differential equation with repulsive singularity and, based on Schaefer's fixed point theorem, new conditions for the existence of periodic solutions were obtained. Jiang, Chu, and Zhang [5] considered positive periodic solutions to the repulsive singular perturbations of the Hill equations and proved that such a perturbation problem has at least two positive periodic solutions when the anti-maximum principle holds for the Hill operator and the perturbation is superlinear at infinity. In [6], the authors considered the following singular equation:

$$x''(t) + f\bigl(x(t)\bigr)x'(t) + \phi(t)x^m(t) - \frac{\alpha(t)}{x^\mu(t)} = 0,$$

where $f : (0, +\infty) \to R$ is a continuous function which may have a singularity at the origin, the signs of ϕ and α are allowed to change, m is a non-negative constant, and μ is a positive constant. Obviously, when $p = 2$ and $g(x) = x^m(t)$, (1.1) is changed into the above equation. Hence, the above equation is a special case of (1.1), and (1.1) has a more general form. Since (1.1) contains a p-Laplacian operator and stronger nonlinearity, we will develop some new technique for overcoming the above difficulties in the present paper. For more details about second-order singular equations, see e.g. [7–13].

For the singular Liénard equation, Habets and Sanchez [14] considered the forced Liénard equation

$$\begin{cases} u'' + f(u)u' + g(t,u) = h(t), \\ u(0) = u(T), \qquad u'(0) = u'(T), \end{cases}$$

where g is continuous on $R \times (0, +\infty)$ and becomes infinite at $u = 0$. Based on upper and lower solutions and degree theory, they obtained some existence results for the above equation. In 1996, Zhang [15] studied the existence of positive T-periodic solutions of the singular Liénard equation

$$\begin{cases} u'' + f(u)u' + g(t,u) = 0, \\ u(0) = u(T), \qquad u'(0) = u'(T), \end{cases}$$

where g is an L^1-Carathéodory function and has a repulsive singularity at u. Hakl, Torres, and Zamora [16] studied the periodic problem for the second-order equation

$$u'' + f(u)u' + g(t,u) = h(t,u),$$

where h is a Carathéodory function and f, g are continuous functions on $(0, +\infty)$ which may have singularities at zero. Both attractive and repulsive singularities are considered.

Using a novel technique of construction of lower and upper functions, some existence results of periodic solutions have been obtained for the above equation. In 2013, Hakl and Zamora [17] investigated the following singular Liénard equation:

$$u'' + \frac{cu'}{u^{\mu}} + \frac{g_1}{u^{\nu}} - \frac{g_2}{u^{\gamma}} = h_0(t)u^{\delta}(t) \quad \text{for a.e. } t \in [0,\omega],$$

where g_1, g_2, δ are non-negative constants, c, μ, ν, γ are real numbers, $h_0 \in L([0,\omega],R)$. When μ or ν is sufficiently large, they obtained the existence of positive periodic solution for the above equation.

On the other hand, second-order indefinite singular equations have received great attention of many researchers. Hakl and Zamora [18] considered the existence of a T-periodic solution to the second-order differential equation

$$u'' = h(t)g(u),$$

where $g(u)$ is a positive and decreasing function which has a strong singularity at the origin, and the weight $h \in L(R/TZ)$ is a sign-changing function. By using Leray–Schauder degree theory, they obtained some efficient conditions guaranteeing the existence of a T-periodic solution to the above equation and proved that no T-periodic solution of certain homotopy appears on the boundary of an unbounded open set during the deformation to an autonomous problem. Bravo and Torres [19] investigated the existence of T-periodic solutions for

$$x'' = \frac{a(t)}{x^3},$$

where a is a piecewise constant. In that case, the dynamic is ruled by two alternating autonomous planar systems. Boscaggin and Zanolin [20] studied the problem of existence and multiplicity of positive periodic solutions to the scalar ODE

$$u'' + \lambda a(t)g(u) = 0, \quad \lambda > 0,$$

where $g(x)$ is a positive function on rR^+, superlinear at zero and sublinear at infinity, and $a(t)$ is a T-periodic and sign-indefinite weight with negative mean value. Using critical point theory, they proved the existence of at least two positive T-periodic solutions for λ large. In [16], Hakl, Torres, and Zamora studied the following singular second-order differential equation:

$$u''(t) + f\big(u(t)\big)u'(t) + \frac{g_1}{u^{\nu}} - \frac{g_2}{u^{\gamma}} = h_0(t)u^{\delta}(t) \quad \text{for a.e. } t \in [0,\omega], \tag{1.4}$$

where $g_1, g_2, \delta \geq 0$, $\nu > 0$, $\gamma \in R$, $h_0 \in L([0,\omega];R), f \in C(R^+;rR)$. For broad category parameters g_1, g_2, δ, ν, γ, the conditions guaranteeing solvability of problem (1.4) have been obtained. For more works about superlinear/sublinear problems with a weight function having an indefinite sign, see e.g. [21–26].

The distinctive contributions of this paper are outlined as follows:

(1) Problem (1.1) is a more general form compared with existing problems (see [3, 4, 10–13]). Hence, the results of this paper can be extended to other more specific problems;

(2) Due to indefinite singularity, it is very difficult to estimate a priori bound. In order to overcome this difficulty, we develop a new technique introduced in [1] for continuation theorem;
(3) A unified framework is established to handle singular equations with indefinite weight and p-Laplacian.

The following sections are organized as follows. In Sect. 2, some useful lemmas and notations are given. In Sect. 3, sufficient conditions are established for the existence of positive periodic solutions of (1.1). In Sect. 4, some applications are given to show the feasibility of our results. Finally, Sect. 5 concludes the paper.

2 Preliminary and some lemmas

In this section, we give some notations and lemmas which will be used in this paper. Let

$$C_T = \{x \mid x \in C(R,R), x(t+T) \equiv x(t), \forall t \in R\}$$

with the norm

$$|\varphi|_0 = \max_{t\in[0,T]} |\varphi(t)|, \quad \forall \varphi \in C_T$$

and

$$C_T^1 = \{x \mid x \in C^1(R,R), x(t+T) \equiv x(t), \forall t \in R\}$$

with the norm

$$|\varphi|_\infty = \max_{t\in[0,T]} \{|\varphi|_0, |\varphi'|_0\}, \quad \forall \varphi \in C_T^1.$$

Clearly, C_T and C_T^1 are Banach spaces. For each $\phi \in C_T$ with $y \in L([0,T],R)$, let

$$\phi_+(t) = \max\{\phi(t), 0\}, \qquad \phi_-(t) = \max\{-\phi(t), 0\}, \qquad \overline{\phi} = \frac{1}{T}\int_0^T \phi(s)\,ds.$$

Clearly, for $t \in R$, $\phi(t) = \phi_+(t) - \phi_-(t)$, $\overline{\phi} = \overline{\phi_+} - \overline{\phi_-}$.

Since p-Laplacian $(\phi_p(s'))$ $(p \neq 2)$ in (1.1) is a nonlinear operator, the famous Mawhin's continuation theorem [27] cannot be directly applied to (1.1). Fortunately, Manásevich and Mawhin [1] obtained the following continuation theorem for nonlinear systems with p-Laplacian-like operators.

Theorem 2.1 *Assume that Ω is an open bounded set in C_T such that the following conditions hold.*

(1) *For each $\lambda \in (0,1)$, the problem*

$$(\phi_p(u'))' = \lambda f(t,u,u'), \qquad u(0) = u(T), \qquad u'(0) = u'(T),$$

has no solution on $\partial\Omega$.

(2) *The equation*

$$\mathcal{F}(a) = \frac{1}{T}\int_0^T f(t,a,0)\,dt = 0$$

has no solution on $\partial\Omega \cap R^N$.

(3) *The Brouwer degree*

$$d_B(\mathcal{F}, \Omega \cap R^N, 0) \neq 0.$$

Then problem

$$(\phi_p(u')) = f(t,u,u'), \qquad u(0) = u(T), \qquad u'(0) = u'(T),$$

has a solution in $\overline{\Omega}$.

Lemma 2.1 ([4]) *Let* $u \in C([0,\omega], R)$ *be an arbitrary absolutely continuous function with* $u(0) = u(\omega)$. *Then the inequality*

$$\left(\max u(t) - \min u(t)\right)^2 \leq \frac{\omega}{4}\int_0^\omega \left|u'(s)\right|^2 ds$$

holds. Throughout this paper, assume that

$$\overline{\alpha_1}, \overline{\alpha_2} > 0.$$

Now, consider the equation

$$\left(\phi_p\left(x'(t)\right)\right)' + \lambda\left[f\left(x(t)\right)x'(t) + \alpha_1(t)g\left(x(t)\right)\right] = \lambda\frac{\alpha_2(t)}{x^\mu}, \quad \lambda \in (0,1]. \tag{2.1}$$

Let

$$\Omega = \left\{x \in C_T^1 : \left(\phi_p\left(x'(t)\right)\right)' + \lambda\left[f\left(x(t)\right)x'(t) + \alpha_1(t)g\left(x(t)\right)\right] = \lambda\frac{\alpha_2(t)}{x^\mu},\right.$$
$$\left.\lambda \in (0,1], x(t) > 0, t \in [0,T]\right\}.$$

Lemma 2.2 *Assume that there exist positive constants* g_L *and* g_M *such that*

$$g_L \leq g(u) \leq g_M, \quad \forall u \in R.$$

Furthermore, assume $\overline{(\alpha_2)_+} > 0$, $g_L\overline{(\alpha_1)_+} - g_M\overline{(\alpha_1)_-} > 0$. *Then, for each* $u \in \Omega$, *there are constants* $\eta_1, \eta_2 \in [0,T]$ *such that*

$$u(\eta_1) \leq \left(\frac{\overline{(\alpha_2)_+}}{g_L\overline{(\alpha_1)_+} - g_M\overline{(\alpha_1)_-}}\right)^{\frac{1}{\mu}} := A_1$$

and

$$u(\eta_2) \geq \left(\frac{\overline{\alpha_2}}{g_M\overline{(\alpha_1)_+}}\right)^{\frac{1}{\mu}} := A_2.$$

Proof Let $u \in \Omega$, we have

$$\left(\phi_p\left(u'(t)\right)\right)' + \lambda\left[f(u)u' + \alpha_1(t)g\left(u(t)\right)\right] = \lambda\frac{\alpha_2(t)}{u^\mu}. \tag{2.2}$$

Integrating (2.2) over $[0, T]$, we have

$$\int_0^T \alpha_1(t)g\left(u(t)\right)dt = \int_0^T \frac{\alpha_2(t)}{u^\mu}dt$$

and

$$\int_0^T (\alpha_1)_+(t)g\left(u(t)\right)dt - \int_0^T (\alpha_1)_-(t)g\left(u(t)\right)dt \leq \int_0^T \frac{(\alpha_2)_+(t)}{u^\mu}dt.$$

In view of mean value theorem of integrals, there exists $\eta_1 \in [0, T]$ such that

$$g_L T\overline{(\alpha_1)_+} - g_M T\overline{(\alpha_1)_-} \leq T\frac{\overline{(\alpha_2)_+}}{u^\mu(\eta_1)},$$

i.e.,

$$u(\eta_1) \leq \left(\frac{\overline{(\alpha_2)_+}}{g_L\overline{(\alpha_1)_+} - g_M\overline{(\alpha_1)_-}}\right)^{\frac{1}{\mu}} := A_1.$$

Multiplying both sides of (2.2) by u^μ and integrating it over $[0, T]$, we have

$$\int_0^T \left(\phi_p\left(u'(t)\right)\right)' u^\mu(t)\,dt + \int_0^T \alpha_1(t)g\left(u(t)\right)u^\mu(t)\,dt = \int_0^T \alpha_2(t)\,dt.$$

Since $\int_0^T (\phi_p(u'(t)))' u^\mu(t)\,dt = -\mu\int_0^T |u'(t)|^{p-2}u^{\mu-1}|u'(t)|^2\,dt \leq 0$, thus

$$g_M\int_0^T (\alpha_1)_+(t)u^\mu(t)\,dt \geq T\overline{\alpha_2}.$$

From mean value theorem of integrals, there exists $\eta_2 \in [0, T]$ such that

$$u(\eta_2) \geq \left(\frac{\overline{\alpha_2}}{g_M\overline{(\alpha_1)_+}}\right)^{\frac{1}{\mu}} := A_2. \qquad \square$$

Lemma 2.3 *Let $g(u)$ satisfy the conditions of Lemma 2.2. Let*

$$F(x) = \int_1^x f(s)\,ds, \qquad K_0 := \sup_{s\in[A_2,+\infty)} F(s) < +\infty \tag{2.3}$$

and

$$\lim_{s\to 0^+}\left(F(s)-\frac{T\overline{(\alpha_2)_+}}{s^\mu}-g_M T\overline{(\alpha_1)_-}\right)>K_0, \tag{2.4}$$

where A_2 *is defined in Lemma 2.2. Then there exists a constant* $\gamma_0>0$ *such that*

$$\min_{t\in[0,T]} u(t)\geq\gamma_0 \quad \text{for } u\in\Omega.$$

Proof Let $u\in\Omega$, then u satisfies (2.2). There exist $t_1,t_2\in R$ such that $t_2-t_1\in(0,T)$ and

$$u(t_1)=\max_{t\in[0,T]}u(t), \qquad u(t_2)=\min_{t\in[0,T]}u(t).$$

From (2.3), the definitions of A_2 and $u(t_1)$, we have

$$A_2\leq u(t_1)<+\infty$$

and

$$F\bigl(u(t_1)\bigr)\leq\sup_{s\in[A_2,+\infty)}F(s):=K_0. \tag{2.5}$$

Integrating (2.2) over $[t_1,t_2]$, we have

$$\int_{t_1}^{t_2}f\bigl(u(t)\bigr)u'(t)\,dt+\int_{t_1}^{t_2}\alpha_1(t)g\bigl(u(t)\bigr)\,dt=\int_{t_1}^{t_2}\frac{\alpha_2(t)}{u^\mu(t)}\,dt. \tag{2.6}$$

From the definitions of $u(t_1)$, (2.5), and (2.6), we obtain that

$$\begin{aligned}
F\bigl(u(t_2)\bigr)&=F\bigl(u(t_1)\bigr)+\int_{t_1}^{t_2}\frac{\alpha_2(t)}{u^\mu(t)}\,dt-\int_{t_1}^{t_2}\alpha_1(t)g\bigl(u(t)\bigr)\,dt\\
&<K_0+\int_0^T\frac{(\alpha_2)_+(t)}{u^\mu(t)}\,dt+g_M\int_0^T(\alpha_1)_-(t)\,dt\\
&\leq K_0+\frac{T\overline{(\alpha_2)_+}}{u^\mu(t_2)}+g_M T\overline{(\alpha_1)_-}
\end{aligned}$$

and

$$F\bigl(u(t_2)\bigr)-\frac{T\overline{(\alpha_2)_+}}{u^\mu(t_2)}-g_M T\overline{(\alpha_1)_-}\leq K_0. \tag{2.7}$$

In view of (2.4), there exists a constant $\gamma_0>0$ such that

$$F(s)-\frac{T\overline{(\alpha_2)_+}}{s^\mu}-g_M T\overline{(\alpha_1)_-}>K_0, \quad \forall s\in(0,\gamma_0). \tag{2.8}$$

By (2.7) and (2.8), we have

$$\min_{t\in[0,T]}u(t)\geq\gamma_0 \quad \text{for } u\in\Omega. \qquad \square$$

Lemma 2.4 *Let $g(u)$ satisfy the condition of Lemma 2.2. Let*

$$G(x) = \int_1^x s^\mu f(s)\,ds, \qquad \lim_{s\to+\infty}\bigl(G(s) - g_M T\overline{\alpha_1} s^\mu\bigr) = +\infty \tag{2.9}$$

and

$$\lim_{s\to 0^+} G(s) < \rho + T\overline{\alpha_2}, \tag{2.10}$$

where

$$\rho = \inf_{s\in[A_2,+\infty)}\bigl(G(s) - g_M T\overline{\alpha_1} s^\mu\bigr),$$

A_2 is defined in Lemma 2.2. Then there exist constants $\gamma_2 > \gamma_1 > 0$ such that

$$\min_{t\in[0,T]} u(t) \geq \gamma_1 \quad \textit{for } u \in \Omega$$

and

$$\max_{t\in[0,T]} u(t) \leq \gamma_2 \quad \textit{for } u \in \Omega.$$

Proof Let $u \in \Omega$, then u satisfies (2.2). There exist $t_1, t_2 \in R$ such that $t_2 - t_1 \in (0, T)$ and

$$u(t_1) = \max_{t\in[0,T]} u(t), \qquad u(t_2) = \min_{t\in[0,T]} u(t).$$

Multiplying (2.2) by $u^\mu(t)$, and then integrating it over the interval $[t_1, t_2]$, we have

$$\begin{aligned}&\int_{t_1}^{t_2}\bigl(\phi_p\bigl(u'(t)\bigr)\bigr)' u^\mu(t)\,dt + \int_{t_1}^{t_2} f\bigl(u(t)\bigr)u'(t)u^\mu(t)\,dt + \int_{t_1}^{t_2}\alpha_1(t)g\bigl(u(t)\bigr)u^\mu(t)\,dt \\ &\quad = \int_{t_1}^{t_2}\alpha_2(t)\,dt.\end{aligned} \tag{2.11}$$

From $\int_{t_1}^{t_2}(\phi_p(u'(t)))'u^\mu(t)\,dt < 0$, (2.9), and (2.11), we have

$$\begin{aligned}G\bigl(u(t_2)\bigr) &\geq G\bigl(u(t_1)\bigr) - \int_{t_1}^{t_2}\alpha_1(t)g\bigl(u(t)\bigr)u^\mu(t)\,dt + \int_{t_1}^{t_2}\alpha_2(t)\,dt \\ &\geq G\bigl(u(t_1)\bigr) - g_M T\overline{\alpha_1}u^\mu(t_1) + T\overline{\alpha_2} \geq \rho + T\overline{\alpha_2}.\end{aligned} \tag{2.12}$$

By (2.10) there exists a constant $\gamma_1 > 0$ such that

$$G(s) < \rho + T\overline{\alpha_2}, \quad \forall s \in (0, \gamma_1). \tag{2.13}$$

By (2.12) and (2.13), we have

$$\min_{t\in[0,T]} u(t) \geq \gamma_1 \quad \text{for } u \in \Omega. \tag{2.14}$$

From Lemma 2.2 and (2.14), we have

$$\gamma_1 \le u(t_2) \le A_1. \tag{2.15}$$

By (2.12) we have

$$G\bigl(u(t_1)\bigr) - g_M T\overline{\alpha_1} u^\mu(t_1) \le G\bigl(u(t_2)\bigr) - T\overline{\alpha_2} \le \max_{\gamma_1 \le s \le A_1} G(s) - T\overline{\alpha_2}. \tag{2.16}$$

In view of (2.9), there exists a constant $\gamma_2 > \gamma_1$ such that

$$G(s) - g_M T\overline{\alpha_1} s^\mu > \max_{\gamma_1 \le s \le A_1} G(s) - T\overline{\alpha_2}, \quad s \in (\gamma_2, +\infty). \tag{2.17}$$

Thus, (2.6) and (2.17) imply

$$\max_{t\in[0,T]} u(t) = u(t_1) \le \gamma_2. \qquad \square$$

Lemma 2.5 *Let $g(u)$ satisfy the conditions of Lemma 2.2. Let*

$$F(x) = \int_0^x f(s)\,ds, \qquad B_0 := \inf_{s\in[A_2,+\infty)} F(s) > -\infty \tag{2.18}$$

and

$$\lim_{s\to 0^+} \left(F(s) + \frac{T\overline{(\alpha_2)_+}}{s^\mu} + g_M T\overline{(\alpha_1)_-} \right) < B_0, \tag{2.19}$$

where A_2 is defined in Lemma 2.2. Then there exists a constant $\gamma_3 > 0$ such that

$$\min_{t\in[0,T]} u(t) \ge \gamma_3 \quad \textit{for } u \in \Omega.$$

Proof Let $u \in \Omega$, then u satisfies (2.2). There exist $t_1, t_2 \in R$ such that $t_2 - t_1 \in (0, T)$ and

$$u(t_1) = \max_{t\in[0,T]} u(t), \qquad u(t_2) = \min_{t\in[0,T]} u(t).$$

From (2.18), the definitions of A_2 and $u(t_1)$, we have

$$A_2 \le u(t_1) < +\infty$$

and

$$F\bigl(u(t_1)\bigr) \ge \inf_{s\in[A_2,+\infty)} F(s) := B_0. \tag{2.20}$$

Integrating (2.2) over $[t_1, t_2]$ we have

$$\int_{t_1}^{t_2} f\bigl(u(t)\bigr)u'(t)\,dt + \int_{t_1}^{t_2} \alpha_1(t) g\bigl(u(t)\bigr)\,dt = \int_{t_1}^{t_2} \frac{\alpha_2(t)}{u^\mu(t)}\,dt$$

and

$$F(u(t_2)) = F(u(t_1)) + \int_{t_1}^{t_2} \frac{\alpha_2(t)}{u^\mu(t)}\,dt - \int_{t_1}^{t_2} \alpha_1(t) g(u(t))\,dt. \tag{2.21}$$

Form (2.20) and (2.21), we have

$$\begin{aligned} F(u(t_2)) &\geq B_0 - \int_{t_1}^{t_2} \frac{(\alpha_2)_-(t)}{u^\mu(t)}\,dt - \int_{t_1}^{t_2} (\alpha_1)_+(t) g(u(t))\,dt \\ &\geq B_0 - \int_0^T \frac{(\alpha_2)_-(t)}{u^\mu(t)}\,dt - \int_0^T (\alpha_1)_+(t) g(u(t))\,dt \\ &\geq B_0 - g_M T\overline{(\alpha_1)_+} - \frac{\overline{(\alpha_2)_-}}{u^\mu(t_2)}, \end{aligned}$$

i.e.,

$$F(u(t_2)) + g_M T\overline{(\alpha_1)_+} + \frac{\overline{(\alpha_2)_-}}{u^\mu(t_2)} \geq B_0. \tag{2.22}$$

In view of (2.19), there exists a constant $\gamma_3 > 0$ such that

$$F(s) + \frac{T\overline{(\alpha_2)_+}}{s^\mu} + g_M T\overline{(\alpha_1)_-} < B_0, \quad \forall s \in (0, \gamma_3). \tag{2.23}$$

By (2.22) and (2.23), we have

$$\min_{t\in[0,T]} u(t) \geq \gamma_3 \quad \text{for } u \in \Omega. \qquad \square$$

3 Main results

Theorem 3.1 *Assume that the conditions of Lemmas 2.2 and 2.3 hold. Then Eq. (1.1) has at least one positive T-periodic solution.*

Proof From the conditions of Lemma 2.3, there exists a constant $\gamma_0 > 0$ such that

$$\min_{t\in[0,T]} u(t) \geq \gamma_0 \quad \text{for } u \in \Omega. \tag{3.1}$$

Now, we prove that

$$\max_{t\in[0,T]} u(t) \leq M_1, \qquad \max_{t\in[0,T]} \left|u'(t)\right| \leq M_2, \quad \text{for } u \in \Omega, \tag{3.2}$$

where M_1, M_2 are positive constants. In fact, for $u \in \Omega$, by Lemma 2.2, there exists $\eta \in [0, T]$ such that

$$u(\eta) \leq A_1.$$

Furthermore,

$$u(t) = u(\eta) + \int_\eta^t u'(s)\,ds$$

and

$$\left|u(t)\right| \leq A_1 + \int_0^T \left|u'(s)\right| ds \leq A_1 + T^{\frac{1}{q}} \left(\int_0^T \left|u'(s)\right|^p ds \right)^{\frac{1}{p}}, \tag{3.3}$$

where $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Multiply (2.2) by $u(t)$ and integrate it over the interval $[0, T]$, then

$$\begin{aligned} \int_0^T \left|u'(t)\right|^p dt &= \int_0^T \alpha_1(t) g\big(u(t)\big) u(t)\, dt + \int_0^T \frac{\alpha_2(t)}{u^\mu} u(t)\, dt \\ &\leq \int_0^T (\alpha_1)_+(t) g_M u(t)\, dt + \int_0^T \frac{(\alpha_2)_+(t)}{\gamma_0^\mu} u(t)\, dt \\ &\leq T\overline{(\alpha_1)_+} g_M |u|_0 + \frac{T\overline{(\alpha_2)_+}}{\gamma_0^\mu} |u|_0 \\ &= \left(T\overline{(\alpha_1)_+} g_M + \frac{T\overline{(\alpha_2)_+}}{\gamma_0^\mu} \right) |u|_0. \end{aligned} \tag{3.4}$$

By (3.3) and (3.4), we have

$$|u|_0 \leq A_1 + T^{\frac{1}{q}} \left(T\overline{(\alpha_1)_+} g_M + \frac{T\overline{(\alpha_2)_+}}{\gamma_0^\mu} \right)^{\frac{1}{p}} |u|_0^{\frac{1}{p}}. \tag{3.5}$$

By (3.5), there exists a positive constant M_1 such that

$$|u|_0 \leq M_1,$$

i.e.,

$$\max_{t\in[0,T]} u(t) \leq M_1. \tag{3.6}$$

Let $u(t_1) = \max_{t\in[0,T]} u(t)$ for $u \in \Omega$, then $u'(t_1) = 0$. Integrating (2.2) over $[t_1, t]$, we have

$$\phi_p\big(u'(t)\big) = \lambda \int_{t_1}^t \left[-f\big(x(s)\big)x'(s) - \alpha_1(s) g\big(x(s)\big) + \frac{\alpha_2(s)}{x^\mu(s)} \right] ds, \quad t \in [t_1, t_1 + T].$$

Thus,

$$\begin{aligned} \left|u'(t)\right|^{p-1} &\leq \left|F\big(u(t)\big) - F\big(u(t_1)\big)\right| + g_M T\overline{(\alpha_1)_-} + \frac{T\overline{\alpha_2}}{\gamma_0^\mu} \\ &\leq 2 \max_{\gamma_0\leq s\leq M_1} \left|F(s)\right| + g_M T\overline{(\alpha_1)_-(t)} + \frac{T\overline{\alpha_2(t)}}{\gamma_0^\mu} \end{aligned}$$

and

$$\left|u'(t)\right| \leq \left(2 \max_{\gamma_0\leq s\leq M_1} \left|F(s)\right| + g_M T\overline{(\alpha_1)_-} + \frac{T\overline{\alpha_2}}{\gamma_0^\mu} \right)^{\frac{1}{p-1}} := M_2. \tag{3.7}$$

In view of (3.6) and (3.7), it follows that (3.2) holds. Let $|u|_\infty = \max\{\gamma_0, M_1, M_2\} + 1$ for $u \in \Omega$. Then condition (1) of Theorem 2.1 holds. Next, let

$$\mathcal{F}(a) = g(a)\overline{\alpha_1} - \frac{\overline{\alpha_2}}{a^\mu} = 0, \quad a \in R. \tag{3.8}$$

Clearly, Eq. (3.8) has no solution on $\partial\Omega \cap R$. Hence, condition (2) of Theorem 2.1 holds. Furthermore, since $\overline{\alpha_1}, \overline{\alpha_2}, g_L < g(u) < g_M$, $u \in \Omega$, for sufficiently large g_L, we have the following inequality:

$$g(u)\overline{\alpha_1} - \frac{\overline{\alpha_2}}{u^\mu} > 0 \quad \text{for } u \in (0, \gamma_0];$$

on the other hand, for sufficiently small g_M, we have the following inequality:

$$g(u)\overline{\alpha_1} - \frac{\overline{\alpha_2}}{u^\mu} < 0 \quad \text{for } u \in [M_1, +\infty).$$

Thus,

$$\left(g(\gamma_0)\overline{\alpha_1} - \frac{\overline{\alpha_2}}{\gamma_0{}^\mu}\right)\left(g(M_1)\overline{\alpha_1} - \frac{\overline{\alpha_2}}{M_1^\mu}\right) < 0,$$

which implies

$$d_B\big(\mathcal{F}, (\gamma_0, M_1) \cap R, 0\big) \neq 0,$$

i.e., condition (3) of Theorem 2.1 holds. By using Theorem 2.1, we see that Eq. (1.1) has at least one positive T-periodic solution. □

Remark 3.1 Inequality (3.3) can be deduced by Lemma 2.1. In fact, by Lemma 2.1, we have

$$\begin{aligned} u(t) - u(\eta_2) &\leq \max u(t) - \min u(t) \\ &\leq \frac{T^{\frac{1}{2}}}{2}\left(\int_0^T \left|u'(s)\right|^2\right)^{\frac{1}{2}} ds. \end{aligned}$$

Thus,

$$u(t) \leq A_1 + \frac{T^{\frac{p-1}{p}}}{2}\left(\int_0^T \left|u'(s)\right|^p\right)^{\frac{1}{p}} ds.$$

Theorem 3.2 *Assume that the conditions of Lemmas 2.2 and 2.4 hold. Then Eq. (1.1) has at least one positive T-periodic solution.*

Theorem 3.3 *Assume that the conditions of Lemmas 2.2 and 2.5 hold. Then Eq. (1.1) has at least one positive T-periodic solution.*

4 Examples

This section presents some examples that demonstrate the validity of our theoretical results.

Example 4.1 Consider the following equation:

$$\left(\left|u'(t)\right|^2 u'(t)\right)' - \frac{x'(t)}{x^2} + (2+\sin t)\frac{\sin^2 x+1}{\sin^2 x+2} = (2-\cos t)x^{-\frac{1}{2}}(t), \tag{4.1}$$

where

$$p=4, \qquad \phi_4(u') = \left|u'(t)\right|^2 u'(t), \qquad f(x) = -\frac{1}{x^2}, \qquad \alpha_1(t) = 2+\sin t,$$
$$g(x) = \frac{\sin^2 x+1}{\sin^2 x+2}, \qquad \alpha_2(t) = 2-\sin t, \qquad \mu = \frac{1}{2}.$$

Then we have

$$\overline{\alpha_1} = \overline{\alpha_2} = 2 > 0.$$

Clearly,

$$\overline{(\alpha_1)_+} = \overline{(\alpha_2)_+} = 2, \qquad \overline{(\alpha_1)_-} = \overline{(\alpha_2)_-} = 0, \qquad g_L = \frac{1}{3}, \qquad g_M = 1, \tag{4.2}$$

$$g_L\overline{(\alpha_1)_+} - g_M\overline{(\alpha_1)_-} = \frac{2}{3} > 0, \qquad A_2 = \left(\frac{\overline{\alpha_2}}{g_M\overline{(\alpha_1)_+}}\right)^{\frac{1}{\mu}} = 1. \tag{4.3}$$

Based on (4.2) and (4.3), it is easy to see that all the conditions of Lemma 2.2 hold.

$$F(x) = \int_1^x -\frac{1}{s^2}\,ds = \frac{1}{x} - 1, \qquad K_0 := \sup_{s\in[1,+\infty)} F(s) = 0 < +\infty, \tag{4.4}$$

$$\lim_{s\to 0^+}\left(F(s) - \frac{T\overline{(\alpha_2)_+}}{s^\mu} - g_M T\overline{(\alpha_1)_-}\right) = \lim_{s\to 0^+}\left(\frac{1}{s} - 1 - \frac{4\pi}{s^{\frac{1}{2}}}\right) = +\infty. \tag{4.5}$$

Equations (4.4) and (4.5) imply that all the conditions of Lemma 2.3 hold. Thus, based on Theorem 3.1, Eq. (4.1) has at least one positive 2π-periodic solution.

Example 4.2 Consider the following equation:

$$\left(\left|u'(t)\right|^2 u'(t)\right)' + \frac{100x'(t)}{x^{\frac{1}{2}}} + (1+\sin t)\frac{\sin^2 x+1}{\sin^2 x+2} = (100-\cos t)x^{-\frac{1}{2}}(t), \tag{4.6}$$

where

$$p=4, \qquad \phi_4(u') = \left|u'(t)\right|^2 u'(t), \qquad f(x) = \frac{100}{x^{\frac{1}{2}}}, \qquad \alpha_1(t) = 1+\sin t,$$
$$g(x) = \frac{\sin^2 x+1}{\sin^2 x+2}, \qquad \alpha_2(t) = 100-\sin t, \qquad \mu = \frac{1}{2}.$$

Then we have

$$\overline{\alpha_1} = 1 > 0, \qquad \overline{\alpha_2} = 100 > 0.$$

Clearly,

$$\overline{(\alpha_1)_+} = 1, \qquad \overline{(\alpha_1)_-} = 0, \qquad g_L = \frac{1}{3}, \qquad g_M = 1, \tag{4.7}$$

$$g_L\overline{(\alpha_1)_+} - g_M\overline{(\alpha_1)_-} = \frac{1}{3} > 0, \qquad A_2 = \left(\frac{\overline{\alpha_2}}{g_M\overline{(\alpha_1)_+}}\right)^{\frac{1}{\mu}} = 10. \tag{4.8}$$

Equation (4.7) and (4.8) imply that all the conditions of Lemma 2.2 hold.

$$G(x) = \int_1^x s^\mu f(s)\,ds = 100x - 100,$$

$$\lim_{s\to+\infty}\left(G(s) - g_M T\overline{\alpha_1}s^\mu\right) = \lim_{s\to+\infty}\left(100s - 100 - 2\pi s^{\frac{1}{2}}\right) = +\infty, \tag{4.9}$$

$$\rho = \inf_{s\in[A_2,+\infty)}\left(G(s) - g_M T\overline{\alpha_1}s^\mu\right) = 990 - 2\pi\sqrt{10} < +\infty,$$

$$\lim_{s\to 0^+} G(s) = -100 < \rho + T\overline{\alpha_2}. \tag{4.10}$$

Equation (4.9) and (4.10) imply that all the conditions of Lemma 2.4 hold. Based on Theorem 3.2, Eq. (4.6) has at least one positive 2π-periodic solution.

Finally, we give an application for Eq. (1.1) to Rayleigh–Plesset equation. In [28], Plesset and Prosperetti studied the following model:

$$\rho\left(RR'' + \frac{3}{2}(R')^2\right) = \left[P_v - P_\infty(t)\right] + P_{g0}\left(\frac{R_0}{R}\right)^{3k} - \frac{2S}{R} - \frac{4\nu R'}{R}, \tag{4.11}$$

where $R(t)$ is the ratio of the bubble at the time t, ρ is the liquid density, P_∞ is the pressure in the liquid at a large distance from the bubble. The physical meaning of the rest of the parameters in (4.11) can be seen in [4]. The transformation $R = x^{\frac{2}{5}}$ in (4.11) leads to the equation

$$x'' + \frac{4\nu}{x^{\frac{4}{5}}}x' + \frac{5[P_\infty(t) - P_v]}{2\rho}x^{\frac{1}{5}} + \frac{5S}{x^{\frac{1}{5}}} - \left(\frac{5P_{g0}R_0^{3k}}{2\rho}\right)\frac{1}{x^{\frac{6k-1}{5}}} = 0. \tag{4.12}$$

For $k \geq 1$, Hark and Torres [4] obtained the existence of positive periodic solutions investigated for (4.12) by using Schaefer's fixed point theorem. In order to study (4.12) by using Theorem 3.2, (4.12) is converted to

$$x'' + \frac{4\nu}{x^{\frac{4}{5}}}x' + \left[P_\infty(t) - P_v\right]\left(\frac{5}{2\rho}x^{\frac{1}{5}} + \frac{5S}{P_\infty(t) - P_v}x^{-\frac{1}{5}}\right) = \left(\frac{5P_{g0}R_0^{3k}}{2\rho}\right)\frac{1}{x^{\frac{6k-1}{5}}}. \tag{4.13}$$

This is a special type of Eq. (1.1). Corresponding to (1.1), we have

$$p = 2, \qquad f(x) = \frac{4\nu}{x^{\frac{4}{5}}}, \qquad g(x) = \frac{5}{2\rho}x^{\frac{1}{5}} + \frac{5S}{P_\infty(t) - P_v}x^{-\frac{1}{5}},$$

$$\alpha_1(t) = P_\infty(t) - P_v, \qquad \alpha_2(t) = \frac{5P_{g0}R_0^{3k}}{2\rho}, \qquad \mu = \frac{6k-1}{5}.$$

Assume that $P_\infty(t)$ is a T-periodic continuous function, $g(x)$ is a positive bounded function, i.e., there exist positive constants g_L and g_M such that

$$g_L \le g(u) \le g_M, \quad \forall u > 0.$$

Furthermore, assume that

$$\overline{P_\infty} > P_v, \qquad g_L\overline{(\alpha_1)_+} - g_M\overline{(\alpha_1)_-} > 0, \qquad k > \frac{1}{6}, \qquad k \ne \frac{1}{3}.$$

Let

$$G(x) = \int_{A_2}^{x} s^\mu f(s)\,ds = \int_{A_2}^{x} 4\nu s^{\mu-\frac{4}{5}}\,ds = \frac{4\nu}{\mu+\frac{1}{5}}x^{\mu+\frac{1}{5}} - \frac{4\nu}{\mu+\frac{1}{5}}A_2^{\mu+\frac{1}{5}},$$

where $A_2 = (\frac{\overline{\alpha_2}}{g_M\overline{(\alpha_1)_+}})^{\frac{1}{\mu}}$, then

$$\begin{aligned} &\lim_{s\to+\infty}\bigl(G(s) - g_M T\overline{\alpha_1}s^\mu\bigr) = \lim_{s\to+\infty}\left(\frac{4\nu}{\mu+\frac{1}{5}}s^{\mu+\frac{1}{5}} - \frac{4\nu}{\mu+\frac{1}{5}}A_2^{\mu+\frac{1}{5}} - g_M T\overline{\alpha_1}s^\mu\right) = +\infty, \\ &\rho = \inf_{s\in[A_2,+\infty)}\bigl(G(s) - g_M T\overline{\alpha_1}s^\mu\bigr) = -g_M T\overline{\alpha_1}A_2^\mu < +\infty, \\ &\lim_{s\to 0^+} G(s) = -g_M T\overline{\alpha_1}A_2^\mu < \rho + T\overline{\alpha_2}. \end{aligned}$$

Thus, based on Theorem 3.2, Eq. (4.13) has at least one positive T-periodic solution.

Remark 4.1 For $k = \frac{1}{3}$, (4.12) is changed into the following equation:

$$x'' + \frac{4\nu}{x^{\frac{4}{5}}}x' + \frac{5[P_\infty(t) - P_v]}{2\rho}x^{\frac{1}{5}} = \left(\frac{5P_{g_0}R_0}{2\rho} - 5S\right)\frac{1}{x^{\frac{1}{5}}}, \tag{4.14}$$

where $g(x) = x^{\frac{1}{5}}$. Since $g(x)$ is an unbounded function for $x > 0$, we cannot obtain existence results of periodic solutions by the results of this paper. However, Lu, Guo, and Chen [6] obtained the following theorem.

Theorem 4.1 *Assume $P_v < \overline{P_\infty}$, $S < \frac{P_{g_0}R_0}{2\rho}$. If $\nu > \frac{T\overline{[P_\infty(t)-P_v]_+}}{2\rho}$, then Rayleigh–Plesset equation (4.14) has at least one positive T-periodic solution.*

5 Conclusions

In this paper, we study a class of second-order indefinite singular equations with p-Laplacian. By employing some analytic techniques and continuation theorem due to Manásevich and Mawhin, we have presented some new sufficient criteria for the existence of positive periodic solutions for the above singular equation. These criteria possess adjustable parameters which are important in some applied fields. Finally, some examples are given to demonstrate the effectiveness of the obtained theoretical results. It is noted that there exist positive periodic solutions to a Rayleigh–Plesset equation for $k > \frac{1}{6}$. When $k \le \frac{1}{6}$, we want to obtain some existence results of periodic solutions for the Rayleigh–Plesset equation. In addition, there exist many problems for further study such as dynamic properties of indefinite singular equations.

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