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Existence and uniqueness of the global solution for a class of nonlinear fractional integro-differential equations in a Banach space

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Abstract
In this paper, by employing fixed point theory, we investigate the existence and uniqueness of solutions for a class of nonlinear fractional integro-differential equations on semi-infinite domains in a Banach space.

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Keywords: Nonlinear fractional integro-differential equations; Banach space; Infinite interval

1 Introduction

Fractional calculus and fractional differential equations describe various phenomena in diverse areas of natural science such as physics, aerodynamics, biology, control theory, chemistry, and so on, see [1–12]. In the last few decades, fractional-order models have been found to be more adequate than integer order models for some real world problems as fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes; this is the main advantage of fractional differential equations in comparison with classical integer-order models. The study of fractional calculus and fractional differential equations is gaining more and more attention. Compared with classical integer-order models [13–16], fractional-order models can describe reality more accurately.

In the past decades, results on fractional differential equations with finite domain have been extensively investigated. Some recent results on fractional differential equations with finite domain, for instance, can be found in papers [17–38] and the references cited therein. Though much of the work on fractional calculus deals with finite domain, there is a considerable development on the topic involving unbounded domain [12, 39–50].

In [40], the authors considered the existence of solutions for the following fractional order initial value problems (IVPs):

$$\begin{cases} ({}_CD^{\alpha}_{0,t}x)(t) = f(t,x(t)), \quad t \in (0,+\infty), \\ x(0) = x_0, \end{cases}$$

where $0 < \alpha < 1$, ${}_C D^{\alpha}_{0,t}$ is the Caputo derivative.

In [44], the authors studied the following fractional integro-differential equations on an infinite interval:

$$\begin{cases} (D^{\alpha}u)(t) + f(t, u(t), (Tu)(t), (Su)(t)) = \theta, & t \in (0, \infty), \\ u(0) = u'(0) = u''(0) = \cdots = u^{(n-2)} = \theta, & D^{\alpha-1}u(\infty) = u_{\infty}, \end{cases}$$

where $n - 1 < \alpha \le n$, $n \in \mathbb{N}$, $n \ge 2$, D^{α} is the Riemann–Liouville fractional derivative of order α, the existence results are obtained by using the Banach fixed point theorem.

In [26], the authors considered the fractional differential equation with the nonlinearity depending on fractional derivatives of lower order on an infinite interval:

$$\begin{cases} (D^{\alpha}_{0^+}u)(t) + f(t, u(t), (D^{\alpha-2}_{0^+}u)(t), (D^{\alpha-1}_{0^+}u)(t)) = 0, & t \in (0, \infty), \\ u(0) = u'(0) = 0, & (D^{\alpha-1}_{0^+}u)(+\infty) = \xi, \end{cases}$$

where $2 < \alpha \le 3$, D^{α}_{0+}, $D^{\alpha-1}_{0+}$ and $D^{\alpha-2}_{0+}$ denote the Riemann–Liouville fractional derivatives. The existence and uniqueness results of solutions were obtained by using the Schauder fixed point theorem and Banach contraction mapping principle.

Using the fixed point index theory, the authors [17] studied the existence and multiplicity of positive solutions of the following IVP:

$$\begin{cases} (D^{\alpha}u)(t) = f(t, u(t), (D^{\beta}u)(t)), & t \in (0, 1], \\ u^{(k)}(0) = \eta_k, & k = 0, 1, \ldots, n - 1, \end{cases}$$

where $n - 1 < \beta < \alpha < n$, $n \in \mathbb{N}$, D^{α} and D^{β} are the Caputo fractional derivatives.

Inspired by the works mentioned above, in this article we aim to investigate the existence of solutions for the following nonlinear fractional-order integro-differential equation on a semi-infinite interval:

$$\begin{cases} (D^{\alpha}_{a+}u)(t) = f(t, u(t), (D^{\beta_1}_{a+}u)(t), \ldots, (D^{\beta_k}_{a+}u)(t), \\ \qquad\qquad (T_0u)(t), (T_1 D^{\gamma_1}_{a+}u)(t), \ldots, (T_m D^{\gamma_m}_{a+}u)(t)), \\ (D^{\alpha-i}_{a+}u)(a+) = u_i, \quad i = 1, 2, \ldots, n, \end{cases} \tag{1.1}$$

where $n = -[-\alpha]$, $t \in J = [a, +\infty)$, $f \in C(J \times E^{k+m+2}, E)$, $u_1, u_2, \ldots, u_n \in E$, $(E, \|\cdot\|)$ is a real Banach space. $0 < \beta_1 < \beta_2 < \cdots < \beta_k < \alpha$, $0 < \gamma_1 < \gamma_2 < \cdots < \gamma_m < \alpha$, D^{α}_{a+}, $D^{\beta_i}_{a+}$, $D^{\gamma_j}_{a+}$ are the Riemann–Liouville fractional derivatives, and

$$(T_j u)(t) = \int_a^t k_j(t,s)u(s)\,ds, \quad j = 0, 1, \ldots, m,$$

where $k_j(t,s) \in C[D, R]$, $D = \{(t,s) | a \le s \le t < \infty\}$.

In particular, if $\alpha, \beta_1, \beta_2, \ldots, \beta_k, \gamma_1, \gamma_2, \ldots, \gamma_m$ are natural numbers, then the problem in (1.1) is reduced to the usual Cauchy problem for the ordinary differential equation:

$$\begin{cases} u^{(n)}(t) = f(t, u(t), u'(t), \ldots, u^{(n-1)}(t), (T_0u)(t), (T_1u')(t), \ldots, (T_{n-1}u^{(n-1)})(t)), \\ u^{(i)}(0) = u_i, \quad i = 1, 2, \ldots, n - 1. \end{cases} \tag{1.2}$$

Thus, fractional differential equation (1.1) is the continuation and development of integer-order differential equations (1.2).

2 Preliminaries and some lemmas

In this section, we introduce notations, definitions, and some useful lemmas, which play an important role in obtaining the main results of this paper.

Suppose that $\mu(t)$ and $f_0(t) = \|f(t,\theta,\dots,\theta)\|$ are nonnegative continuous functions on J, $k_j(t,s)$ are continuous on $D = \{(t,s)|a \le s \le t < \infty\}$. Set

$$\beta = \min\{\beta_1,\gamma_1\},\qquad p = \alpha+1,\qquad q = \frac{p}{p-1},\qquad p_0 = \beta+1,\qquad q_0 = \frac{p_0}{p_0-1},$$

$$M = \max\bigl\{\beta^{-\frac{2}{p_0}},1\bigr\},\qquad N = m+k+2,$$

$$\lambda(t) = t-a+1,\qquad \mu_*(t) = \mu(t)+1,$$

$$f_0(t) = \bigl\|f(t,\theta,\dots,\theta)\bigr\|,\qquad K(t) = \sup_{a\le s\le t,0\le j\le m}\bigl\{\bigl|k_j(t,s)\bigr|\bigr\}+1,$$

$$\varphi(t) = \lambda^{\frac{\alpha^2}{p}}(t)\bigl[\mu_*(t)K(t)+f_0(t)\bigr],\qquad \Phi(t) = \bigl(NM\Gamma(\alpha)\bigr)^{q_0}\int_a^t \varphi^{q_0}(s)\,ds,$$

$$\|u\|_\Phi = \sup_{t\in J}\bigl\{\lambda^{-\frac{\alpha^2}{p}}(t)e^{-2\Phi(t)}\bigl\|u(t)\bigr\|\bigr\},$$

$$C_\Phi = \bigl\{\|u\|_\Phi < \infty|u : J\to E \text{ is continuous}\bigr\}.$$

Then C_Φ is a Banach space with the norm $\|\cdot\|_\Phi$.

Definition 2.1 The Riemann–Liouville fractional derivative of order α for a continuous function $f : [a,\infty)\to R$ is defined by

$$D^\alpha_{a+}f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n\int_a^t (t-s)^{n-\alpha-1}f(s)\,ds,\qquad n = [\alpha]+1,$$

provided the right-hand side is defined pointwise on (a,∞).

A map $u(t)\in C(J,E)$ with its Riemann–Liouville derivative of order α existing on J is called a solution of (1.1) if it satisfies (1.1).

Lemma 2.2 (Hölder's inequality) *Suppose that* $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, $u\in L^p[a,b]$, $v\in L^q[a,b]$, *then*

$$\int_a^b u(t)v(t)\,dt \le \left(\int_a^b \bigl|u(t)\bigr|^p\,dt\right)^{\frac{1}{p}}\left(\int_a^b \bigl|v(t)\bigr|^q\,dt\right)^{\frac{1}{q}}.$$

Lemma 2.3 *Suppose that* $c > 1$, $\beta \le \varrho < \alpha$, $p_1 = \varrho+1$, $q_1 = \frac{p_1}{p_1-1}$, $W\in L^p[a,b]$, $|W(t)|\le\varphi(t)$, *then*

$$\int_a^t (t-s)^{\varrho-1}W(s)e^{c\Phi(s)}\,ds \le c^{-\frac{1}{q_0}}\bigl(N\Gamma(\alpha)\bigr)^{-1}\lambda^{\frac{\alpha^2}{p}}(t)e^{c\Phi(t)}.$$

Proof

$$\begin{aligned}
&\int_a^t (t-s)^{\varrho-1} W(s) e^{c\Phi(s)}\,ds \\
&\quad\le \left[\int_a^t (t-s)^{p_1(\varrho-1)}\,ds\right]^{\frac{1}{p_1}} \left[\int_a^t \varphi^{q_1}(s) e^{cq_1\Phi(s)}\,ds\right]^{\frac{1}{q_1}} \\
&\quad= \varrho^{-\frac{2}{p_1}} (t-a)^{\frac{\varrho^2}{p_1}} \left[\int_a^t \varphi^{q_1}(s) e^{cq_1\Phi(s)}\,ds\right]^{\frac{1}{q_1}} \\
&\quad\le \varrho^{-\frac{2}{p_1}} (t-a)^{\frac{\varrho^2}{p_1}} \left[\int_a^t \varphi^{q_0}(s) e^{cq_1\Phi(s)}\,ds\right]^{\frac{1}{q_1}} \\
&\quad\le \varrho^{-\frac{2}{p_1}} (t-a)^{\frac{\varrho^2}{p_1}} \left[\int_a^t \bigl(NM\Gamma(\alpha)\bigr)^{-q_0} e^{cq_1\Phi(s)}\,d\Phi(s)\right]^{\frac{1}{q_1}} \\
&\quad\le \varrho^{-\frac{2}{p_1}} (t-a)^{\frac{\varrho^2}{p_1}} \cdot (cq_1)^{-\frac{1}{q_1}} \bigl(NM\Gamma(\alpha)\bigr)^{-\frac{q_0}{q_1}} e^{c\Phi(t)} \\
&\quad\le c^{-\frac{1}{q_1}} \bigl(NM\Gamma(\alpha)\bigr)^{-1} \varrho^{-\frac{2}{p_1}} \lambda^{\frac{\alpha^2}{p}}(t) e^{c\Phi(t)} \\
&\quad\le c^{-\frac{1}{q_0}} \bigl(N\Gamma(\alpha)\bigr)^{-1} \lambda^{\frac{\alpha^2}{p}}(t) e^{c\Phi(t)},
\end{aligned}$$

where $M = \max\{\beta^{-\frac{2}{p_0}}, 1\}$. □

Lemma 2.4 *Suppose that* $\beta \le \varrho < \alpha$, $p_1 = \varrho + 1$, $q_1 = \frac{p_1}{p_1-1}$, $u \in C_\Phi$, *let*

$$\begin{aligned}
(\sigma_1 u)(t) &= \int_a^t (t-s)^{\alpha-1} \mu(s) \bigl\|u(s)\bigr\|\,ds, \\
(\sigma_2 u)(t) &= \int_a^t (t-s)^{\alpha-1} \mu(s) \int_a^s k_0(s,\tau) \bigl\|u(\tau)\bigr\|\,d\tau\,ds, \\
(\sigma_3 u)(t) &= \int_a^t (t-s)^{\alpha-1} \mu(s) \int_a^s (s-\tau)^{\varrho-1} \bigl\|u(\tau)\bigr\|\,d\tau\,ds, \\
(\sigma_4 u)(t) &= \int_a^t (t-s)^{\alpha-1} \mu(s) \int_a^s K(s) \int_a^\tau (\tau-\eta)^{\varrho-1} \bigl\|u(\eta)\bigr\|\,d\eta\,d\tau\,ds,
\end{aligned}$$

then

$$\begin{aligned}
(\sigma_1 u)(t) &\le 2^{-\frac{1}{q_0}} \bigl(N\Gamma(\alpha)\bigr)^{-1} \|u\|_\Phi \lambda^{\frac{\alpha^2}{p}}(t) e^{2\Phi(t)}, \\
(\sigma_2 u)(t) &\le 2^{-\frac{1}{q_0}} \bigl(N\Gamma(\alpha)\bigr)^{-1} \|u\|_\Phi \lambda^{\frac{\alpha^2}{p}}(t) e^{2\Phi(t)}, \\
(\sigma_3 u)(t) &\le 2^{-\frac{1}{q_0}} \bigl(N\Gamma(\alpha)\bigr)^{-1} \|u\|_\Phi \lambda^{\frac{\alpha^2}{p}}(t) e^{2\Phi(t)}, \\
(\sigma_4 u)(t) &\le 2^{-\frac{1}{q_0}} \bigl(N\Gamma(\alpha)\bigr)^{-1} \|u\|_\Phi \lambda^{\frac{\alpha^2}{p}}(t) e^{2\Phi(t)}.
\end{aligned}$$

Proof Notice that $(\frac{p}{\alpha})^{\frac{2}{p}} > 1$, and $\lambda(t) \geq 1$, $\mu_*(t) \geq 1$, $K(t) \geq 1$, $t \in J$, direct calculations show that, for $u \in C_\Phi$, by Lemma 2.3,

$$\begin{aligned}
(\sigma_1 u)(t) &= \int_a^t (t-s)^{\alpha-1}\mu(s)\|u(s)\|\,ds \\
&\leq \|u\|_\Phi \int_a^t (t-s)^{\alpha-1}\mu(s)\lambda^{\frac{\alpha^2}{p}}(s)e^{2\Phi(s)}\,ds \\
&\leq 2^{-\frac{1}{q_0}}\big(N\Gamma(\alpha)\big)^{-1}\|u\|_\Phi\lambda^{\frac{\alpha^2}{p}}(t)e^{2\Phi(t)}, \\
(\sigma_2 u)(t) &= \int_a^t (t-s)^{\alpha-1}\mu(s)\int_a^s k_0(s,\tau)\|u(\tau)\|\,d\tau\,ds \\
&\leq \|u\|_\Phi \int_a^t (t-s)^{\alpha-1}\mu(s)K(s)\int_a^s \lambda^{\frac{\alpha^2}{p}}(\tau)e^{2\Phi(\tau)}\,d\tau\,ds \\
&\leq \|u\|_\Phi \int_a^t (t-s)^{\alpha-1}\mu(s)K(s)e^{2\Phi(s)}\,ds \\
&\leq 2^{-\frac{1}{q_0}}\big(N\Gamma(\alpha)\big)^{-1}\|u\|_\Phi\lambda^{\frac{\alpha^2}{p}}(t)e^{2\Phi(t)}, \\
(\sigma_3 u)(t) &= \int_a^t (t-s)^{\alpha-1}\mu(s)\int_a^s (s-\tau)^{\varrho-1}\|u(\tau)\|\,d\tau\,ds \\
&\leq \|u\|_\Phi \int_a^t (t-s)^{\alpha-1}\mu(s)\int_a^s (s-\tau)^{\varrho-1}\lambda^{\frac{\alpha^2}{p}}(\tau)e^{2\Phi(\tau)}\,d\tau\,ds \\
&\leq \|u\|_\Phi \int_a^t (t-s)^{\alpha-1}\mu(s)\lambda^{\frac{\alpha^2}{p}}(s)e^{2\Phi(s)}\,ds \\
&\leq 2^{-\frac{1}{q_0}}\big(N\Gamma(\alpha)\big)^{-1}\|u\|_\Phi\lambda^{\frac{\alpha^2}{p}}(t)e^{2\Phi(t)}, \\
(\sigma_4 u)(t) &= \int_a^t (t-s)^{\alpha-1}\mu(s)\int_a^s K(s)\int_a^\tau (\tau-\eta)^{\varrho-1}\|u(\eta)\|\,d\eta\,d\tau\,ds \\
&\leq \|u\|_\Phi \int_a^t (t-s)^{\alpha-1}\mu(s)\int_a^s K(s)\int_a^\tau (\tau-\eta)^{\varrho-1}\lambda^{\frac{\alpha^2}{p}}(\eta)e^{2\Phi(\eta)}\,d\eta\,d\tau\,ds \\
&\leq \|u\|_\Phi \int_a^t (t-s)^{\alpha-1}\mu(s)K(s)\int_a^s \lambda^{\frac{\alpha^2}{p}}(\tau)e^{2\Phi(\tau)}\,d\tau\,ds \\
&\leq \|u\|_\Phi \int_a^t (t-s)^{\alpha-1}\mu(s)K(s)e^{2\Phi(s)}\,ds \\
&\leq 2^{-\frac{1}{q_0}}\big(N\Gamma(\alpha)\big)^{-1}\|u\|_\Phi\lambda^{\frac{\alpha^2}{p}}(t)e^{2\Phi(t)}.
\end{aligned}$$

□

Lemma 2.5 *$u(t) \in C_\Phi$ is a solution of problem (1.1) if and only if $u(t) \in C_\Phi$ is a solution of the integral equation*

$$\begin{aligned}
u(t) &= \sum_{j=1}^{n} \frac{u_j}{\Gamma(\alpha-j+1)}(t-a)^{\alpha-j} \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} f\big(s, u(s), \big(D_{a+}^{\beta_1}u\big)(s), \ldots, \big(D_{a+}^{\beta_k}u\big)(s), \\
&\quad (T_0 u)(s), \big(T_1 D_{a+}^{\gamma_1}u\big)(s), \ldots, \big(T_m D_{a+}^{\gamma_m}u\big)(s)\big)\,ds. \qquad (2.1)
\end{aligned}$$

Proof We only transform (1.1) to the integral equation (2.1) as the converse follows by direct computation. We know that the general solution of the fractional differential equation in (1.1) can be written as [1]

$$u(t) = \sum_{j=1}^{n} v_j(t-a)^{\alpha-j} + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} f\bigl(s, u(s), \bigl(D_{a+}^{\beta_1}u\bigr)(s), \ldots, \bigl(D_{a+}^{\beta_k}u\bigr)(s), (T_0u)(s), \bigl(T_1D_{a+}^{\gamma_1}u\bigr)(s), \ldots, \bigl(T_mD_{a+}^{\gamma_m}u\bigr)(s)\bigr)\,ds, \tag{2.2}$$

where $v_1, v_2, \ldots, v_n \in E$ are arbitrary elements. For every $i = 1, 2, \ldots, n$, by (2.2), we have

$$\bigl(D_{a+}^{\alpha-i}u\bigr)(a+) = \sum_{j=1}^{i} v_j(t-a)^{i-j} + \frac{1}{(i-1)!}\int_a^t (t-s)^{i-1} f\bigl(s, u(s), \bigl(D_{a+}^{\beta_1}u\bigr)(s), \ldots, \bigl(D_{a+}^{\beta_k}u\bigr)(s), (T_0u)(s), \bigl(T_1D_{a+}^{\gamma_1}u\bigr)(s), \ldots, \bigl(T_mD_{a+}^{\gamma_m}u\bigr)(s)\bigr)\,ds.$$

Clearly, the condition $(D_{a+}^{\alpha-i}u)(a+) = u_i$ implies that

$$v_i = \frac{u_i}{\Gamma(\alpha-i+1)}. \qquad \square$$

3 Main results

Theorem 3.1 *Suppose that there exists $\mu \in C[J, R^+]$ such that, for any $x_1, x_2, \ldots, x_{k+m+2}, y_1, y_2, \ldots, y_{k+m+2} \in E$, we have*

$$\bigl\|f(t, x_1, x_2, \ldots, x_{k+m+2}) - f(t, y_1, y_2, \ldots, y_{k+m+2})\bigr\| \le \mu(t)\sum_{j=1}^{k+m+2}\|x_j - y_j\|. \tag{3.1}$$

Then IVP (1.1) has a unique solution in C_Φ.

Proof Define an operator $A : C(J,E) \to C(J,E)$ by

$$(Au)(t) = \sum_{j=1}^{n} v_j(t-a)^{\alpha-j} + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} f\bigl(s, u(s), \bigl(D_{a+}^{\mu_1}u\bigr)(s), \ldots, \bigl(D_{a+}^{\mu_k}u\bigr)(s), (T_0u)(s), \bigl(T_1D_{a+}^{\gamma_1}u\bigr)(s), \ldots, \bigl(T_mD_{a+}^{\gamma_m}u\bigr)(s)\bigr)\,ds. \tag{3.2}$$

It follows from (3.1) that

$$\begin{aligned}&\left\|f(t,x_1,x_2,\ldots,x_{k+m+2})\right\| \le f_0(t)+\mu(t)\sum_{j=1}^{k+m+2}\|x_j\|,\\&\forall t\in J, x_1,x_2,\ldots,x_{k+m+2}\in E.\end{aligned}\tag{3.3}$$

For any $u\in C_\Phi$, by (3.1)–(3.3) and Lemma 2.4, we get

$$\begin{aligned}\left\|(Au)(t)\right\| \le{}& \left\|\sum_{j=1}^{n}v_j(t-a)^{\alpha-j}\right\|\\&+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\Bigg[f_0(s)+\mu(s)\Bigg(\left\|u(s)\right\|+\sum_{j=1}^{k}\left\|\left(D_{a+}^{\beta_j}u\right)(s)\right\|\\&+\left\|(T_0u)(s)\right\|+\sum_{j=1}^{m}\left\|\left(T_jD_{a+}^{\gamma_j}u\right)(s)\right\|\Bigg)\Bigg]\,ds\\\le{}&\left\|\sum_{j=1}^{n}v_j(t-a)^{\alpha-j}\right\|+\frac{1}{\Gamma(\alpha)}\Phi(t)\\&+(k+m+2)\cdot\frac{1}{\Gamma(\alpha)}2^{-\frac{1}{q_0}}\left(N\Gamma(\alpha)\right)^{-1}\|u\|_\Phi\lambda^{\frac{\alpha^2}{p}}(t)e^{2\Phi(t)}\\={}&\left\|\sum_{j=1}^{n}v_j(t-a)^{\alpha-j}\right\|+\frac{1}{\Gamma(\alpha)}\Phi(t)+2^{-\frac{1}{q_0}}\|u\|_\Phi\lambda^{\frac{\alpha^2}{p}}(t)e^{2\Phi(t)},\quad\forall t\in J,\end{aligned}$$

then $Au\in C_\Phi$, so $A:C_\Phi\to C_\Phi$.

On the other hand, for any $u,v\in C_\Phi$, by (3.1) and Lemma 2.4, we have

$$\begin{aligned}&\left\|(Au)(t)-(Av)(t)\right\|\\&\quad\le\int_0^t\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\Bigg[\mu(s)\Bigg(\left\|u(s)-v(s)\right\|+\sum_{j=1}^{k}\left\|\left(D_{a+}^{\beta_j}(u-v)\right)(s)\right\|\\&\qquad+\left\|\left(T_0(u-v)\right)(s)\right\|+\sum_{j=1}^{m}\left\|\left(T_jD_{a+}^{\gamma_j}(u-v)\right)(s)\right\|\Bigg)\Bigg]\,ds\\&\quad\le(k+m+2)\cdot\frac{1}{\Gamma(\alpha)}2^{-\frac{1}{q_0}}\left(N\Gamma(\alpha)\right)^{-1}\left\|(u-v)\right\|_\Phi\lambda^{\frac{\alpha^2}{p}}(t)e^{2\Phi(t)}\\&\quad=2^{-\frac{1}{q_0}}\left\|(u-v)\right\|_\Phi\lambda^{\frac{\alpha^2}{p}}(t)e^{2\Phi(t)},\quad\forall t\in J,\end{aligned}$$

then $\|Au-Av\|_\Phi\le 2^{-\frac{1}{q_0}}\|u-v\|_\Phi$. Thus the Banach contraction mapping principle implies that A has a unique fixed point in C_Φ. This completes the proof. □

4 Example

Consider the problem

$$\begin{cases} (D_{a+}^{\alpha}u)(t) = t^2 \ln(u^2(t) + u(t) + 1) + \sin(e^t(D_{a+}^{\beta}u(t)) + 2t) \\ \qquad\qquad + e^{t^2+1} \int_a^t \frac{(TD_{a+}^{\gamma}u)(s)+s^3}{s^2+1}\, ds, \quad t \in J, \\ (D_{a+}^{\alpha-i}u)(a+) = u_i, \quad i = 1, 2, \ldots, n. \end{cases} \tag{4.1}$$

Let $E = \mathbb{R}$, then (4.1) can be considered as an IVP of the form (1.1) in E, where $n = -[-\alpha]$, $t \in J = [a, +\infty)$, $u_1, u_2, \ldots, u_n \in \mathbb{R}$, $0 < \beta < \alpha$, $0 < \gamma < \alpha$, and

$$(Tu)(t) = \int_a^t k(t,s)u(s)\, ds,$$

where $k(t,s) \in C[D, R]$, $D = \{(t,s) | a \le s \le t < \infty\}$.

Let

$$f\bigl(t, x_1(t), x_2(t), x_3(t)\bigr) = t^2 \ln\bigl(x_1^2(t) + x_1(t) + 1\bigr) + \sin\bigl(e^t x_2(t) + 2t\bigr) + e^{t^2+1} \int_a^t \frac{x_3(s) + s^3}{s^2 + 1}\, ds,$$

then, for any $x_1, x_2, x_3, y_1, y_2, y_3 \in C[0, +\infty)$, we have

$$\bigl| f(t, x_1, x_2, x_3) - f(t, y_1, y_2, y_3) \bigr| \le \mu(t) \sum_{j=1}^{3} |x_j - y_j|,$$

here $\mu(t) = 2t^2 + e^t + e^{t^2+1}(t - a)$.

Obviously (3.1) holds, all the conditions of Theorem 3.1 are satisfied. Using Theorem 3.1 we can conclude that IVP (4.1) has a unique solution.

5 Concluding remarks

In this paper, we establish the conditions for the existence of a unique solution for problem (1.1), which is indeed an important and useful contribution to the existing literature on the topic. We emphasize that our method of proof is completely different from the ones used in [12–32].

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