Saker and Mahmoud *Advances in Difference Equations* (2019) 2019:129
https://doi.org/10.1186/s13662-019-2072-x

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

A connection between weighted Hardy's inequality and half-linear dynamic equations

S.H. Saker[1] and R.R. Mahmoud[2,3*]

*Correspondence:
ramy.ramadan.rus@cas.edu.om;
rrm00@fayoum.edu.eg
[2]Department of Mathematics, Rustaq College of Education, Rustaq, Sultanate of Oman
[3]Department of Mathematics, Faculty of Science, Fayoum University, Fayoum, Egypt
Full list of author information is available at the end of the article

Abstract

In this paper, we give an affirmative answer to the following question: Is the solvability of some nonlinear dynamic equations on a time scale $\mathbb{T}$ not only sufficient but in a certain sense also necessary for the validity of some dynamic Hardy-type inequalities with two different weights? In fact, this answer will give a new characterization of the weights in a weighted Hardy-type inequality on time scales. The results contain the results when $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{N}$, and when $\mathbb{T} = q^{\mathbb{N}_0}$ as special cases. Some applications are given for illustrations.

MSC: 26D10; 26D15; 39A13; 34A40; 34N05

Keywords: Hardy's inequality; Dynamic equations; Time scales

1 Introduction

In 1920 Hardy [13] proved the discrete inequality

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}a_i\right)^p \le \left(\frac{p}{p-1}\right)^p\sum_{n=1}^{\infty}a_n^p, \quad p>1, \tag{1.1}$$

where $a_n \ge 0$ for $n \ge 1$. This inequality has been discovered in his attempt to give an elementary proof of Hilbert's inequality for double series that was known at that time. In 1925 Hardy [14] proved the continuous inequality using the calculus of variations, which states that for $f \ge 0$ integrable over any finite interval $(0,x)$ and f^p integrable and convergent over $(0,\infty)$ and $p>1$, then

$$\int_0^{\infty}\left(\frac{1}{x}\int_0^x f(t)\,dt\right)^p dx \le \left(\frac{p}{p-1}\right)^p\int_0^{\infty}f^p(x)\,dx. \tag{1.2}$$

The constant $(p/(p-1))^p$ in (1.1) and (1.2) is the best possible. For generalizations, extensions, and applications of these inequalities, we refer the reader to the papers [10–12, 14, 15] and the books [16, 18, 19, 23]. A systematic investigation of the (generalized) Hardy inequality with weights that started in the late fifties and early sixties was connected with the name of Beesack [4, Theorem 3.1.1, p. 47]. In particular, Beesack proved that the inequality

$$\int_a^b w(x)\left(\int_a^x f(s)\,ds\right)^p dx \le \int_a^b v(x)f^p(x)\,dx \tag{1.3}$$

with two weighted functions holds if there exists a positive solution y (with a positive derivative y' on (a, b)) of the differential equation

$$\frac{d}{dx}\left[v(x)\left(\frac{dy}{dx}\right)^{p-1}\right] + w(x)y^{p-1}(x) = 0.$$

It should be mentioned that Beesack dealt not only with the case $p > 1$, but also with $p < 0$ and even with $0 < p < 1$. Beesack's approach was extended to a class of inequalities containing the Hardy inequality (1.3) as a special case; see, e.g., Beesack [5, Theorem 3.1, p. 711] or Shum [26]. Some of the restrictions on the solution y and on the weights v, w were removed by Tomaselli [32]. He followed up the earlier paper of Talenti [28], who considered a little more special weight functions. As usual several authors have been interested in finding some discrete results analogous to continuous results, and accordingly this subject has become a topic of ongoing research. For example, Chen [9] and [8], Liao [20, Proposition 2.2, p. 812] investigated similar results for discrete Hardy's inequality and its relation with difference equations.

In recent years the study of dynamic inequalities on time scales has received a lot of attention and has become a major field in pure and applied mathematics. All of these disciplines are concerned with the properties of these inequalities of various types; for more details, we refer to the books [2, 3] and the papers [1, 21, 22, 27]. The general idea is to prove a result for an inequality where the domain of the unknown function is a so-called time scale $\mathbb{T}$, which is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. The study of dynamic inequalities on time scales helps avoid proving results twice—once for a differential inequality and once again for a difference inequality. For more details, we refer the reader to [3] for recent results of Hardy-type inequalities on time scales.

Motivated by the above results, we naturally raise the question now: *Is the solvability of some nonlinear dynamic equations on time scales not only sufficient but in a certain sense also necessary for the validity of some Hardy-type inequality*?

In this paper, we try to give an affirmative answer to this question and give the new characterizations of the weights in Hardy-type inequalities on time scales and their relevance with nonlinear dynamic equations. The main results will be proved in the next section by employing Hölder's inequality, Minkowski's inequality, and a chain rule on time scales for delta-integral inequalities. Since the dynamic inequalities for nabla-integral on a time scale $\mathbb{T}$ have received a lot of attention, it is worth here to mention that the results in Theorem 3.1 can be reformulated via the nabla-integral (∇-integral) calculus. This also gives us the possibility to predict the shape of our results for diamond $\Diamond_\alpha$-integral functions (see [29–31]).

This paper is organized as follows. In Sect. 2, we present some preliminaries about the theory of time scales and state the basic formulae that will be needed in the sequel. In Sect. 3, we shall state and prove the main results of this paper. In particular, Theorem 3.1 gives us a clear explanation of the possibility of linking dynamic Hardy-type inequality containing weights with half-linear dynamic equations. As a special case of Theorem 3.1, when $\mathbb{T} = \mathbb{N}$, we will obtain the discrete result obtained by Liao [20, Proposition 2.2, p. 812]. Finally, when $\mathbb{T} = q^{\mathbb{N}_0}$, we will obtain the q-difference analogue for our results. For illustrations, we will give some applications of our results and get the sharp constants of well-known inequalities.

2 Preliminaries on time scales

In this section, we present preliminaries and the basic lemmas used in our subsequent discussions. For more details, we refer the reader to the paper by Hilger [17] and the monograph by Bohner and Peterson [6]. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. We assume throughout that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$. The forward jump operator and the backward jump operator are defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \quad \text{and} \quad \rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

A point $t \in \mathbb{T}$ is said to be left-dense if $\rho(t) = t$ and $t > \inf \mathbb{T}$, right-dense if $\sigma(t) = t$, left-scattered if $\rho(t) < t$, and right-scattered if $\sigma(t) > t$. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be right-dense continuous (rd-continuous) provided f is continuous at right-dense points and at left-dense points in $\mathbb{T}$, left-hand limits exist and are finite. The set of all such rd-continuous functions is denoted by $C_{\mathrm{rd}}(\mathbb{T})$.

The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$, and for any function $f : \mathbb{T} \to \mathbb{R}$, the notation $f^{\sigma}(t)$ denotes $f(\sigma(t))$. Without loss of generality, we assume that $\sup \mathbb{T} = \infty$ and define the time scale interval $[a, b]_{\mathbb{T}}$ by $[a, b]_{\mathbb{T}} := [a, b] \cap \mathbb{T}$.

Recall the following product and quotient rules for the derivative of the product fg and the quotient f/g (where $gg^{\sigma} \neq 0$, here $g^{\sigma} = g \circ \sigma$) of two (delta) differentiable functions f and g

$$(fg)^{\Delta} = f^{\Delta}g + f^{\sigma}g^{\Delta} = fg^{\Delta} + f^{\Delta}g^{\sigma}, \quad \text{and} \quad \left(\frac{f}{g}\right)^{\Delta} = \frac{f^{\Delta}g - fg^{\Delta}}{gg^{\sigma}}. \tag{2.1}$$

The chain rule formula on time scales [6, Theorem 1.90, p. 32] is given by (here $x : \mathbb{T} \to (0, \infty)$ is assumed to be (delta) differentiable)

$$\left(x^{\gamma}(t)\right)^{\Delta} = \gamma \int_0^1 \left[hx^{\sigma} + (1 - h)x\right]^{\gamma - 1} dhx^{\Delta}(t), \quad \gamma \in \mathbb{R}. \tag{2.2}$$

In this paper we will use the (delta) integral which we can be defined as follows. If $G^{\Delta}(t) = g(t)$, then the Cauchy (delta) integral of g is defined by

$$\int_a^t g(s)\Delta s := G(t) - G(a).$$

It was shown (see [6, Theorem 1.70, p. 26]) that if $g \in C_{\mathrm{rd}}(\mathbb{T})$, then the Cauchy integral $G(t) := \int_{t_0}^t g(s)\Delta s$ exists, $t_0 \in \mathbb{T}$, and satisfies $G^{\Delta}(t) = g(t)$, $t \in \mathbb{T}$. An infinite integral is defined as follows:

$$\int_a^{\infty} g(t)\Delta t = \lim_{b \to \infty} \int_a^b g(t)\Delta t.$$

The integration on discrete time scales is defined by

$$\int_a^b g(t)\Delta t = \sum_{t \in [a,b)} \mu(t)g(t).$$

Note that if $\mathbb{T} = \mathbb{R}$, then

$$\sigma(t) = t, \qquad \mu(t) = 0, \qquad f^{\Delta}(t) = f'(t), \qquad \int_a^b f(t)\Delta t = \int_a^b f(t)\,dt.$$

If $\mathbb{T} = \mathbb{N}$, then $\sigma(t) = t+1$, $\int_a^b f(t)\Delta t = \sum_{t=a}^{b-1} f(t)$. If $\mathbb{T} = h\mathbb{N}$, $h > 0$, then $\sigma(t) = t+h$, $\mu(t) = h$, and

$$\int_a^b f(t)\Delta t = \sum_{k=0}^{\frac{b-a-h}{h}} f(a+kh)h.$$

If $\mathbb{T} = \{t : t = q^k, k \in \mathbb{N}_0, q > 1\}$, then $\sigma(t) = qt$, $\mu(t) = (q-1)t$,

$$\int_{t_0}^{\infty} f(t)\Delta t = \sum_{k=n_0}^{\infty} f\bigl(q^k\bigr)\mu\bigl(q^k\bigr), \quad \text{where } t_0 = q^{n_0}.$$

The integration by parts formula on time scales is given by

$$\int_a^b u(t)\upsilon^{\Delta}(t)\Delta t = \bigl[u(t)\upsilon(t)\bigr]_a^b - \int_a^b u^{\Delta}(t)\upsilon^{\sigma}(t)\Delta t. \tag{2.3}$$

Hölder's inequality on time scales [6, Theorem 6.13, p. 259] is given by

$$\int_a^b \bigl|u(t)\upsilon(t)\bigr|\Delta t \le \biggl[\int_a^b \bigl|u(t)\bigr|^q \Delta t\biggr]^{\frac{1}{q}} \biggl[\int_a^b \bigl|\upsilon(t)\bigr|^p \Delta t\biggr]^{\frac{1}{p}}, \tag{2.4}$$

where $a, b \in \mathbb{T}$, $u, \upsilon \in C_{\mathrm{rd}}([a,b]_{\mathbb{T}}, \mathbb{R})$, $p > 1$, and $1/p + 1/q = 1$.

3 Main results

Throughout the paper, we will assume that the functions in the statements of the theorems are nonnegative and rd-continuous functions and (without mentioning) the integrals in the statements of the theorems are assumed to exist. Now we state and prove the basic lemmas that will be used in the proofs of our main results. The first lemma is adapted from [25, Lemma 2.6, p. 593].

Lemma 3.1 *Let $\mathbb{T}$ be a time scale with $a, b \in \mathbb{T}$ and $f, g \in \mathrm{C}_{\mathrm{rd}}([a,b]_{\mathbb{T}}, \mathbb{R})$. If $m \ge 1$, then*

$$\biggl(\int_a^b f(t)\biggl(\int_a^{\sigma(t)} g(s)\Delta s\biggr)^m \Delta t\biggr)^{\frac{1}{m}} \le \int_a^b g(s)\biggl(\int_s^b f(t)\Delta t\biggr)^{\frac{1}{m}} \Delta s. \tag{3.1}$$

From now on, we will deal with the following half-linear dynamic equation:

$$\lambda\bigl(\nu^{\frac{q}{p}}(x)\bigl(y^{\Delta}(x)\bigr)^{\frac{q}{p^*}}\bigr)^{\Delta} + w(x)y^{\frac{q}{p^*}}\bigl(\sigma(x)\bigr) = 0, \tag{3.2}$$

where p^* is the conjugate of p, and the weighted dynamic Hardy-type inequality

$$\biggl(\int_a^b w(x)\biggl(\int_a^x u(s)\,ds\biggr)^q \Delta x\biggr)^{\frac{1}{q}} \le C_L\biggl(\int_a^b \nu(x)u^p(x)\Delta x\biggr)^{\frac{1}{p}} \tag{3.3}$$

for $1 < p \le q < \infty$.

Actually, the main question that we will give the affirmative answer to states that the solvability of the dynamic equation (3.2) not only is necessary for the validity of the weighted dynamic Hardy-type inequality (3.3) but also is sufficient. The next result will guarantee the first direction, which emphasizes the need to achieve the equation in order to prove the legitimacy of the inequality. In the rest of the paper, we will assume that the function $v(x)$ satisfies the condition

$$\int_a^\infty v^{-\frac{1}{p-1}}(x)\Delta x = \infty. \tag{3.4}$$

Lemma 3.2 *Let $\mathbb{T}$ be a time scale with $a, b \in \mathbb{T}$, $1 < p \leq q < \infty$, $u \in C_{\mathrm{rd}}([a,b]_{\mathbb{T}}, \mathbb{R})$ be a nonnegative function, w, v be positive rd-continuous functions on $(a,b)_{\mathbb{T}}$,*

$$\int_a^x v^{1-p^*}(t)\Delta t < \infty \quad \text{for } x \in [a,b]_{\mathbb{T}}, \tag{3.5}$$

and there exists a number $\lambda > 0$ such that the dynamic equation (3.2) has a positive solution $y(x)$. Then the following inequality

$$\left(\int_a^b w(x)\left(\int_a^{\sigma(x)} f(t)\Delta t\right)^q \Delta x\right)^{\frac{1}{q}} \leq C\left(\int_a^b v(x)f^p(x)\Delta x\right)^{\frac{1}{p}} \tag{3.6}$$

holds for every positive function $f(x)$ on $[a,b]_{\mathbb{T}}$, with the constant

$$C = \lambda^{\frac{1}{q}}. \tag{3.7}$$

Proof Suppose that $y(x)$ is a positive solution of (3.2). By utilizing Lemma 1.2.1 in [24, Lemma 1.2.1, p. 17] and condition (3.4), we see that y satisfies

$$y(x), y^{\triangle}(x) > 0 \quad \text{and} \quad y^{\triangle\triangle}(x) < 0 \quad \text{for } x \in [a,b]_{\mathbb{T}}. \tag{3.8}$$

For, x, $t \in (a,b)_{\mathbb{T}}$ denote

$$\varphi(x) := -\lambda\left(v^{\frac{q}{p}}(x)\left(y^{\triangle}(x)\right)^{\frac{q}{p^*}}\right)^{\triangle}, \tag{3.9}$$

$$\psi(t) := f^p(t)\left(y^{\triangle}(t)\right)^{\frac{-p}{p^*}}. \tag{3.10}$$

Then (3.2) yields that $\varphi(x) = w(x)y^{\frac{q}{p^*}}(\sigma(x))$ and the time scales Hölder's inequality together with (3.8) imply that

$$\begin{aligned}\left(\int_a^{\sigma(x)} f(t)\Delta t\right)^q w(x) &= \left(\int_a^{\sigma(x)} f(t)\left(y^{\triangle}(t)\right)^{\frac{-1}{p^*}}\left(y^{\triangle}(t)\right)^{\frac{1}{p^*}}\Delta t\right)^q w(x) \\ &\leq w(x)\left(\int_a^{\sigma(x)} \psi(t)\Delta t\right)^{\frac{q}{p}}\left(\int_a^{\sigma(x)} y^{\triangle}(t)\Delta t\right)^{\frac{q}{p^*}} \\ &= w(x)\left(\int_a^{\sigma(x)} \psi(t)\Delta t\right)^{\frac{q}{p}}\left(y(\sigma(x)) - y(a)\right)^{\frac{q}{p^*}}\end{aligned}$$

$$\begin{aligned}&\le w(x)y^{\frac{q}{p^*}}\big(\sigma(x)\big)\bigg(\int_a^{\sigma(x)}\psi(t)\Delta t\bigg)^{\frac{q}{p}}\\&=\varphi(x)\bigg(\int_a^{\sigma(x)}\psi(t)\Delta t\bigg)^{\frac{q}{p}}.\end{aligned}$$

Integrating from a to b with respect to x and denoting that $r = q/p$, we get that

$$\bigg(\int_a^b\bigg(\int_a^{\sigma(x)}f(t)\Delta t\bigg)^q w(x)\Delta x\bigg)^{\frac{1}{r}}\le\bigg(\int_a^b\varphi(x)\bigg(\int_a^{\sigma(x)}\psi(t)\Delta t\bigg)^r\Delta x\bigg)^{\frac{1}{r}}.$$

Applying the time scales Minkowski's inequality (3.1), we have that

$$\bigg(\int_a^b\bigg(\int_a^{\sigma(x)}f(t)\Delta t\bigg)^q w(x)\Delta x\bigg)^{\frac{1}{r}}\le\int_a^b\psi(t)\bigg(\int_t^b\varphi(x)\Delta x\bigg)^{\frac{1}{r}}\Delta t. \tag{3.11}$$

Using (3.9) to estimate the inner integral on the right-hand side yields that

$$\begin{aligned}\int_t^b\varphi(x)\Delta x&=-\lambda\int_t^b\Big(\nu^{\frac{q}{p}}(x)\big(y^{\triangle}(x)\big)^{\frac{q}{p^*}}\Big)^{\triangle}\Delta x\\&=\lambda\nu^{\frac{q}{p}}(x)\big(y^{\triangle}(x)\big)^{\frac{q}{p^*}}\Big|_b^t\\&=\lambda\Big(\nu^{\frac{q}{p}}(t)\big(y^{\triangle}(t)\big)^{\frac{q}{p^*}}-\nu^{\frac{q}{p}}(b)\big(y^{\triangle}(b)\big)^{\frac{q}{p^*}}\Big)\\&\le\lambda\nu^{\frac{q}{p}}(t)\big(y^{\triangle}(t)\big)^{\frac{q}{p^*}},\end{aligned}$$

which leads directly to

$$\bigg(\int_t^b\varphi(x)\Delta x\bigg)^{\frac{1}{r}}\le\lambda^{\frac{p}{q}}\nu(t)\big(y^{\triangle}(t)\big)^{\frac{p}{p^*}}.$$

Substituting this estimate in (3.11) and using (3.10), we have that

$$\begin{aligned}\bigg(\int_a^b\bigg(\int_a^{\sigma(x)}f(t)\Delta t\bigg)^q w(x)\Delta x\bigg)^{\frac{p}{q}}&\le\int_a^b\psi(t)\lambda^{\frac{p}{q}}\nu(t)\big(y^{\triangle}(t)\big)^{\frac{p}{p^*}}\Delta t\\&=\lambda^{\frac{p}{q}}\int_a^b\nu(t)f^p(t)\big(y^{\triangle}(t)\big)^{\frac{-p}{p^*}}\big(y^{\triangle}(t)\big)^{\frac{p}{p^*}}\Delta t\\&=\lambda^{\frac{p}{q}}\int_a^b\nu(t)f^p(t)\Delta t.\end{aligned}$$

Finally, taking $1/p$ power to both sides, we get the required inequality (3.6) with constant C as in (3.7). The proof is complete. □

Now the remaining part, which ensures that our answer to the main question is fully covered, is to prove the other direction, i.e., the sufficient condition, which is the main job of the next Lemmas 3.4–3.5. To prove these lemmas, we need the following auxiliary result, in which we will make use of Riccati-like inequality to get a useful integral inequality in the sequel.

Lemma 3.3 *Suppose that $y(x)$ is a positive solution of (3.2) and set*

$$z(x)=\frac{y(x)}{y^{\triangle}(x)}v^{1-p^*}(x)\quad \text{for } x\in[a,b]_{\mathbb{T}}.$$

Then $z(x)>0$ and satisfies the dynamic inequality

$$z^{\triangle}(x)>\frac{p^*}{\lambda q}w(x)z^{\frac{q}{p^*}+1}(x)+v^{1-p^*}(x). \tag{3.12}$$

Proof For convenience, we sometimes skip the argument x in the computations. By using the quotient rule to differentiate

$$z(x)=\frac{y(x)}{y^{\triangle}(x)}v^{1-p^*}(x),$$

we get that

$$\begin{aligned} z^{\triangle}&=\frac{y^{\triangle}[y(\sigma(x))(v^{1-p^*})^{\triangle}+y^{\triangle}v^{1-p^*}]-y^{\triangle\triangle}[yv^{1-p^*}]}{y^{\triangle}(x)y^{\triangle}(\sigma(x))}\\ &=\frac{y^{\triangle}[y(\sigma(x))(1-p^*)v^{-p^*}+y^{\triangle}v^{1-p^*}]-y^{\triangle\triangle}[yv^{1-p^*}]}{y^{\triangle}(x)y^{\triangle}(\sigma(x))}\\ &=\frac{y(\sigma(x))y^{\triangle}(1-p^*)v^{-p^*}+y^{\triangle}y^{\triangle}v^{1-p^*}-y^{\triangle\triangle}yv^{1-p^*}}{y^{\triangle}(x)y^{\triangle}(\sigma(x))}\\ &=\frac{y^{\triangle}v^{1-p^*}}{y^{\triangle}(\sigma(x))}+\left(1-p^*\right)\frac{y(\sigma(x))v^{-p^*}}{y^{\triangle}(\sigma(x))}-\frac{yy^{\triangle\triangle}v^{1-p^*}}{y^{\triangle}y^{\triangle}(\sigma(x))}. \end{aligned}$$

From (3.8) it follows that $y^{\triangle}(x)>y^{\triangle}(\sigma(x))$, and then we get that

$$\begin{aligned} z^{\triangle}&>\frac{y^{\triangle}(\sigma(x))v^{1-p^*}}{y^{\triangle}(\sigma(x))}+\left(1-p^*\right)\frac{y(\sigma(x))v^{-p^*}}{y^{\triangle}(\sigma(x))}-\frac{yy^{\triangle\triangle}v^{1-p^*}}{y^{\triangle}y^{\triangle}}\\ &=v^{1-p^*}+\left(1-p^*\right)\frac{y(\sigma(x))v^{-p^*}}{y^{\triangle}(\sigma(x))}-\frac{yy^{\triangle\triangle}v^{1-p^*}}{(y^{\triangle})^2}\\ &>v^{1-p^*}+\left(1-p^*\right)v^{-p^*}\frac{y(\sigma(x))}{y^{\triangle}(\sigma(x))}-v^{1-p^*}\frac{y(\sigma(x))y^{\triangle\triangle}}{(y^{\triangle})^2}. \end{aligned} \tag{3.13}$$

For the last inequality, we have used the fact that $y(x)<y(\sigma(x))$ since $y^{\triangle}(x)>0$. But, since

$$wy^{\frac{q}{p^*}}(\sigma(x))=-\lambda\left(v^{\frac{q}{p}}\left(y^{\triangle}\right)^{\frac{q}{p^*}}\right)^{\triangle},$$

it follows by using the chain rule (noting that $y^{\triangle\triangle}(x)<0$) that

$$\begin{aligned} wy^{\frac{q}{p^*}}(\sigma(x))&=-\lambda\left(v^{\frac{q}{p}}\left(\left(y^{\triangle}\right)^{\frac{q}{p^*}}\right)^{\triangle}+\left(v^{\frac{q}{p}}\right)^{\triangle}\left(y^{\triangle}(\sigma(x))\right)^{\frac{q}{p^*}}\right)\\ &<-\lambda\left(v^{\frac{q}{p}}\frac{q}{p^*}\left(y^{\triangle}\right)^{\frac{q}{p^*}-1}y^{\triangle\triangle}(x)+\frac{q}{p}v^{\frac{q}{p}-1}\left(y^{\triangle}(\sigma(x))\right)^{\frac{q}{p^*}}\right)\\ &=-\lambda\frac{q}{p^*}\left(v^{\frac{q}{p}}\left(y^{\triangle}\right)^{\frac{q}{p^*}-1}y^{\triangle\triangle}+\frac{p^*}{p}v^{\frac{q}{p}-1}\left(y^{\triangle}(\sigma(x))\right)^{\frac{q}{p^*}}\right), \end{aligned}$$

which leads to

$$
\begin{aligned}
&w\frac{y^{\frac{q}{p^*}+1}(\sigma(x))}{(y^{\triangle})^{\frac{q}{p^*}+1}} \\
&\quad< -\lambda\frac{q}{p^*}\left(v^{\frac{q}{p}}\frac{y(\sigma(x))}{(y^{\triangle})^2}y^{\triangle\triangle}+\frac{p^*}{p}v^{\frac{q}{p}-1}y(\sigma(x))\frac{(y^{\triangle}(\sigma(x)))^{\frac{q}{p^*}}}{(y^{\triangle})^{\frac{q}{p^*}+1}}\right) \\
&\quad< -\lambda\frac{q}{p^*}v^{\frac{q}{p}+p^*-1}\left(v^{1-p^*}\frac{y(\sigma(x))}{(y^{\triangle})^2}y^{\triangle\triangle}+\frac{p^*}{p}v^{-p^*}y(\sigma(x))\frac{(y^{\triangle}(\sigma(x)))^{\frac{q}{p^*}}}{(y^{\triangle})^{\frac{q}{p^*}+1}}\right),
\end{aligned}
$$

and hence,

$$
\begin{aligned}
&w\left(\frac{y(\sigma(x))v^{1-p^*}}{y^{\triangle}}\right)^{\frac{q}{p^*}+1} \\
&\quad< \lambda\frac{q}{p^*}\left(-v^{1-p^*}\frac{y(\sigma(x))}{(y^{\triangle})^2}y^{\triangle\triangle}+(1-p^*)v^{-p^*}y(\sigma(x))\frac{(y^{\triangle}(\sigma(x)))^{\frac{q}{p^*}}}{(y^{\triangle})^{\frac{q}{p^*}+1}}\right),
\end{aligned}
$$

and then we get that

$$
\begin{aligned}
&\frac{p^*}{\lambda q}w\left(\frac{y(\sigma(x))v^{1-p^*}}{y^{\triangle}}\right)^{\frac{q}{p^*}+1} \\
&\quad< (1-p^*)v^{-p^*}y(\sigma(x))\frac{(y^{\triangle}(\sigma(x)))^{\frac{q}{p^*}}}{(y^{\triangle})^{\frac{q}{p^*}+1}}-v^{1-p^*}\frac{y(\sigma(x))}{(y^{\triangle})^2}y^{\triangle\triangle}.
\end{aligned}
$$

Since $y^{\triangle}(x) > y^{\triangle}(\sigma(x))$ and $(1-p^*)$ is always negative, we obtain that

$$
\begin{aligned}
&\frac{p^*}{\lambda q}w\left(\frac{y(\sigma(x))v^{1-p^*}}{y^{\triangle}}\right)^{\frac{q}{p^*}+1} \\
&\quad< (1-p^*)v^{-p^*}y(\sigma(x))\frac{(y^{\triangle}(\sigma(x)))^{\frac{q}{p^*}}}{(y^{\triangle}(\sigma(x)))^{\frac{q}{p^*}+1}}-v^{1-p^*}\frac{y(\sigma(x))}{(y^{\triangle})^2}y^{\triangle\triangle} \\
&\quad= (1-p^*)v^{-p^*}\frac{y(\sigma(x))}{y^{\triangle}(\sigma(x))}-v^{1-p^*}\frac{y(\sigma(x))}{(y^{\triangle})^2}y^{\triangle\triangle}.
\end{aligned}
\tag{3.14}
$$

Since $y^{\triangle}(x) > 0$, it follows that $y(x) < y(\sigma(x))$, and hence

$$
\begin{aligned}
\frac{p^*}{\lambda q}w\left(\frac{y(\sigma(x))v^{1-p^*}}{y^{\triangle}}\right)^{\frac{q}{p^*}+1} &> \frac{p^*}{\lambda q}w\left(\frac{y(x)v^{1-p^*}}{y^{\triangle}}\right)^{\frac{q}{p^*}+1} \\
&= \frac{p^*}{\lambda q}wz^{\frac{q}{p^*}+1}.
\end{aligned}
\tag{3.15}
$$

Finally, assembling (3.13), (3.14), and (3.15), we get that

$$
z^{\triangle}(x) > \frac{p^*}{\lambda q}w(x)z^{\frac{q}{p^*}+1}(x)+v^{1-p^*}(x),
\tag{3.16}
$$

which is the desired inequality (3.12). The proof is complete. □

Lemma 3.4 *Let $\mathbb{T}$ be a time scale with $a,b \in \mathbb{T}$, $1 < p \leq q < \infty$, $u \in C_{\mathrm{rd}}([a,b]_{\mathbb{T}},\mathbb{R})$ is a nonnegative function, and let w, v be positive rd-continuous functions on $[a,b]_{\mathbb{T}}$. Denote*

$$K = \frac{p^*}{q} \inf_f \sup_{a<x<b} \frac{1}{f(x)} \int_a^x w(t)\left(f(t) + \int_a^t v^{1-p^*}(s)\Delta s\right)^{\frac{q}{p^*+1}} \Delta t, \tag{3.17}$$

where the infimum is taken for every positive function $f(t)$ defined on $[a,b]_{\mathbb{T}}$.

(i) *If there exists a positive constant λ such that the dynamic equation (3.2) has a positive solution $y(x)$, then*

$$K \leq \lambda < \infty. \tag{3.18}$$

(ii) *If $K < \infty$, then the dynamic equation (3.2) has a positive solution $y(x)$ satisfying (3.8) for every $\lambda > K$.*

Proof (i) Suppose that $y(x)$ is a positive solution of (3.2) which satisfies (3.8) and set

$$z(x) = \frac{y(x)}{y^{\triangle}(x)} v^{1-p^*}(x),$$

which leads directly to that $z(x)$ is a positive solution on $[a,b]_{\mathbb{T}}$ for the following dynamic inequality:

$$z^{\triangle}(x) \geq \frac{p^*}{\lambda q} w(x) z^{\frac{q}{p^*}+1}(x) + v^{1-p^*}(x). \tag{3.19}$$

Since

$$z(x) \geq \int_a^x z^{\triangle}(x)\Delta x,$$

then we have that

$$z(x) \geq \frac{p^*}{\lambda q} \int_a^x w(t) z^{\frac{q}{p^*}+1}(t)\Delta t + \int_a^x v^{1-p^*}(t)\Delta t.$$

Now, assume that

$$f(x) = z(x) - \int_a^x v^{1-p^*}(t)\Delta t,$$

then we get that $f(x) > 0$ for $x \in [a,b]_{\mathbb{T}}$, and

$$\lambda \geq \frac{p^*}{q} \frac{1}{f(x)} \int_a^x w(t)\left(f(t) + \int_a^t v^{1-p^*}(s)\Delta s\right)^{\frac{q}{p^*}+1} \Delta t,$$

which gives the validity of (3.18) according to the definition of K (3.17).

(ii) Assume that $\lambda > K$. In view of definition (3.17), there is a positive function $f(x)$ such that

$$f(x) \geq \frac{p^*}{\lambda q} \int_a^x w(t)\left(f(t) + \int_a^t v^{1-p^*}(s)\Delta s\right)^{\frac{q}{p^*}+1} \Delta t. \tag{3.20}$$

We will formulate a solution for problem (3.2)–(3.8) as follows. First, define for $n \in \mathbb{N}$ the following sequence $\{z_n(x)\}$ of functions:

$$\begin{aligned} z_0(x) &= f(x) + \int_a^x v^{1-p^*}(t)\Delta t, \\ z_{n+1}(x) &= \frac{p^*}{\lambda q}\int_a^x w(t) z_n^{\frac{q}{p^*}+1}(t)\Delta t + \int_a^x v^{1-p^*}(t)\Delta t. \end{aligned} \tag{3.21}$$

It is obvious that $z_n(x) > 0$ for $x \in [a,b]_{\mathbb{T}}$, and using (3.20) we get that

$$\int_a^x w(t) z_0^{\frac{q}{p^*}+1}(t)\Delta t < \infty \tag{3.22}$$

and

$$z_0(x) - z_1(x) = f(x) - \frac{p^*}{\lambda q}\int_a^x w(t) z_0^{\frac{q}{p^*}+1}(t)\Delta t > 0,$$

which leads us to

$$z_n(x) - z_{n+1}(x) = \frac{p^*}{\lambda q}\int_a^x w(t)\bigl(z_{n-1}^{\frac{q}{p^*}+1}(t) - z_n^{\frac{q}{p^*}+1}(t)\bigr)\Delta t > 0.$$

Thus the sequence $\{z_n(x)\}$ is decreasing on $[a,b]_{\mathbb{T}}$ and asserts with the positivity of $z_n(x)$ to the existence of a nonnegative function $z(x)$ on $[a,b]_{\mathbb{T}}$ such that

$$z(x) = \lim_{n\to\infty} z_n(x).$$

Now, we obtain from (3.21) that

$$z(x) = \frac{p^*}{\lambda q}\int_a^x w(t) z^{\frac{q}{p^*}+1}(t)\Delta t + \int_a^x v^{1-p^*}(t)\Delta t.$$

Actually, this formula asserts that $z(x) > 0$ belong to $C_{\mathrm{rd}}([a,b]_{\mathbb{T}},\mathbb{R})$ and satisfies the dynamic inequality (3.19). The proof is complete. □

Remark 3.1 According to Lemma 3.4, we have shown that the number K from (3.17) is finite if and only if there is $\lambda \in (0,\infty)$ such that problem (3.2), (3.8) is solvable. Consequently, using in addition Lemma 3.2, Theorem 3.1 will be proved if we show that the validity of Hardy's inequality implies the finiteness of the number K. This will follow from the next assertion.

Lemma 3.5 *Let $\mathbb{T}$ be a time scale with $a, b \in \mathbb{T}$, $1 < p \le q < \infty$, $u \in C_{\mathrm{rd}}([a,b]_{\mathbb{T}},\mathbb{R})$ is a nonnegative function, and let w, v be positive on $[a,b]_{\mathbb{T}}$. Suppose that K is defined by* (3.17)

$$B_L = \sup_{a<x<b}\left(\int_x^b w(t)\Delta t\right)^{\frac{1}{q}}\left(\int_a^{\sigma(x)} v^{1-p^*}(t)\Delta t\right)^{\frac{1}{p^*}},$$

and let C_L be the best possible constant in (3.3). *Then*

$$C_L \le K^{\frac{1}{q}} \le k(p,q) B_L, \tag{3.23}$$

where

$$k(p,q) = \left(1 + \frac{q}{p^*}\right)^{\frac{1}{q}} \left(1 + \frac{p^*}{q}\right)^{\frac{1}{p^*}}. \tag{3.24}$$

Proof First, we prove the left inequality on (3.23) by contradiction. For this purpose, suppose that $K^{\frac{1}{q}} < C_L$ and assume that there exists a constant λ_0 such that

$$K^{\frac{1}{q}} < \lambda_0 < C_L, \tag{3.25}$$

which gives that $K < \lambda_0^q$ and problem (3.2)–(3.8) is solvable for $\lambda = \lambda_0^q$ (due to Lemma 3.4). Now, Lemma 3.2 implies that

$$C_L \leq \lambda^{\frac{1}{q}} = \lambda_0,$$

which contradicts (3.25). Next, we prove the right inequality on (3.23). Suppose that $B_L < \infty$, then we have that

$$0 < \int_t^b w(y)\Delta y < \infty \quad \text{for } t \in (a,b)_{\mathbb{T}},$$

and the function

$$f(t) = sB_L^{p^*}\left(\int_t^b w(y)\Delta y\right)^{-\frac{p^*}{q}} - \int_a^t v^{1-p^*}(y)\Delta y$$

is finite for every $s \in (1,\infty)$. Moreover, we get that

$$sB_L^{p^*} > B_L^{p^*} \geq \left(\int_t^b w(y)\Delta y\right)^{\frac{p^*}{q}} \left(\int_a^t v^{1-p^*}(y)\Delta y\right),$$

which gives directly that $f(t) > 0$ for $t \in (a,b)_{\mathbb{T}}$. From (3.17), we can write that

$$\begin{aligned}
K &\leq \frac{p^*}{q} \sup_{a<x<b} \frac{\int_a^{\sigma(x)} w(t)[sB_L^{p^*}(\int_t^b w(y)\Delta y)^{-\frac{p^*}{q}}]^{\frac{q}{p^*}+1}\Delta t}{sB_L^{p^*}(\int_t^b w(y)\Delta y)^{-\frac{p^*}{q}} - \int_a^t v^{1-p^*}(y)\Delta y} \\
&= \frac{p^*}{q}\left(sB_L^{p^*}\right)^{\frac{q}{p^*}+1} \sup_{a<x<b} \frac{h(x)}{sB_L^{p^*} - (\int_x^b w(y)\Delta y)^{\frac{p^*}{q}}(\int_a^x v^{1-p^*}(y)\Delta y)} \\
&\leq \frac{p^*}{q}\left(sB_L^{p^*}\right)^{\frac{q}{p^*}+1} \sup_{a<x<b} \frac{h(x)}{sB_L^{p^*} - (\int_x^b w(y)\Delta y)^{\frac{p^*}{q}}(\int_a^{\sigma(x)} v^{1-p^*}(y)\Delta y)} \\
&\leq \frac{p^*}{q}\frac{(sB_L^{p^*})^{\frac{q}{p^*}+1}}{(s-1)B_L^{p^*}} \sup_{a<x<b} h(x),
\end{aligned} \tag{3.26}$$

where

$$\begin{aligned} h(x) &= \int_a^{\sigma(x)} w(t)\left(\int_t^b w(y)\Delta y\right)^{-\frac{p^*}{q}(\frac{q}{p^*}+1)}\left(\int_x^b w(y)\Delta y\right)^{\frac{p^*}{q}}\Delta t \\ &\le \frac{q}{p^*}\left[1-\left(\int_a^b w(y)\Delta y\right)^{-\frac{p^*}{q}}\left(\int_x^b w(y)\Delta y\right)^{\frac{p^*}{q}}\right] \\ &\le \frac{q}{p^*}. \end{aligned} \tag{3.27}$$

If we set

$$g(s) = \left(\frac{s}{s-1}\right)^{\frac{1}{q}} s^{\frac{1}{p^*}} \quad \text{for } s \in (1,\infty),$$

then, using (3.26) and (3.27), we obtain that

$$K^{\frac{1}{q}} \le g(s)B_L.$$

But, we know that (see [25, Theorem 3.1, p. 594])

$$k(p,q) = \inf_{1<s<\infty} g(s),$$

which claims the second inequality in (3.23). This completes the proof. □

By combining the above results together (necessary and sufficient conditions), we are ready to state our main result in this paper.

Theorem 3.1 *Let $\mathbb{T}$ be a time scale with $a,b \in \mathbb{T}$, $1 < p \le q < \infty$, $u \in C_{\mathrm{rd}}([a,b]_{\mathbb{T}},\mathbb{R})$ is a nonnegative function, and let w, v be positive rd-continuous functions on $(a,b)_{\mathbb{T}}$. If*

$$\int_a^x v^{1-p^*}(t)\Delta t < \infty \quad \textit{for } x \in [a,b]_{\mathbb{T}}, \tag{3.28}$$

then inequality (3.3) holds with a finite constant C_L if and only if there is a number $\lambda > 0$ such that the half-linear dynamic equation (3.2) has a solution $y(x)$ satisfying (3.8).

Remark 3.2 As a special case of Theorem 3.1 (when $\mathbb{T} = \mathbb{N}$), we get the following result which connects the discrete Hardy-type inequality with the half-linear difference equation. It is worth to mention here that the next result coincides with the one obtained by Liao [20, Proposition 2.2, p. 812], while there are some parts of Liao's proof that were essentially based on the idea of the variational principle presented in [7] and [16, p. 181] which we did not rely on in our proof.

Corollary 3.1 *Suppose that $1 < p \le q < \infty$, w and v are positive sequences on the discrete interval $I = \{1,2,\dots,N\}$ with $N \le \infty$. If*

$$\sum_{n=1}^{\infty} v_n^{\frac{-1}{p-1}} = \infty \quad \textit{and} \quad \sum_{n=1}^{N} v_n^{1-p^*} < \infty \quad \textit{for } n \in I,$$

then the following discrete weighted Hardy-type inequality

$$\left(\sum_{n=1}^{N} w_n\left(\sum_{n=1}^{k} u_n\right)^{q}\right)^{\frac{1}{q}} \leq C_1\left(\sum_{n=1}^{N} v_n u_n^p\right)^{\frac{1}{p}}$$

holds for an arbitrary non-negative sequence u on I, with a finite constant C_1, if and only if there is a number $\lambda > 0$ such that the difference equation

$$\lambda\Delta\left(v_n^{\frac{q}{p}}(\Delta y_n)^{\frac{q}{p^*}}\right) + w_n y_{n+1}^{\frac{q}{p^*}} = 0$$

has a positive solution y_n for $n \in I$.

Remark 3.3 As a special case of Theorem 3.1 (when $\mathbb{T} = q^{\mathbb{N}_0}$), we get the following result which connects the discrete Hardy-type inequality with the half-linear difference equation. It is worth to mention here that the next result is entirely new and has not been dealt with before to the knowledge of the authors. Assume that

$$Hx\left(q^k\right) = \begin{cases} \sum_{k=1}^{n} q^k x(q^k); & n = 1, 2, \ldots, N, \\ 0; & n = 0. \end{cases}$$

Corollary 3.2 *Suppose that $1 < p \leq q < \infty$, w and v are positive sequences defined on $\mathbb{T} = q^{\mathbb{N}_0}$. If*

$$\sum_{k=1}^{N} q^k v^{-1/(p-1)}\left(q^k\right) = \infty \quad \text{and} \quad \sum_{k=1}^{N} q^k v^{1-p^*}\left(q^k\right) < \infty \quad \text{for } n \in q^{\mathbb{N}_0},$$

then the following discrete weighted Hardy-type inequality

$$\left(\sum_{k=1}^{N} q^k w\left(q^k\right)\left(\sum_{k=1}^{n} q^k u\left(q^k\right)\right)^{q}\right)^{\frac{1}{q}} \leq C_2\left(\sum_{n=1}^{N} q^k v\left(q^k\right) u^p\left(q^k\right)\right)^{\frac{1}{p}}$$

holds for an arbitrary non-negative sequence u on $q^{\mathbb{N}_0}$, with a finite constant C_1, if and only if there is a number $\lambda > 0$ such that the following second-order q-difference equation

$$w\left(q^k\right) H y^{\frac{q}{p^*}}\left(q^k\right) + \lambda\left[v^{\frac{q}{p}}\left(q^{k+1}\right) y^{\frac{q}{p^*}}\left(q^{k+1}\right) - v^{\frac{q}{p}}\left(q^k\right) y^{\frac{q}{p^*}}\left(q^k\right)\right] = 0$$

has a positive solution y_n for $n \in I$.

Now, let us conclude this section with some applications that illustrate and clarify the main ideas of the paper. Specifically, we consider the special case $\mathbb{T} = \mathbb{R}$.

Example 1 By setting $a = 0$, $b = \infty$, $p = q$, $v(x) = x^{p-k}$, $w(x) = (\frac{|k-1|}{p})^p x^{-k}$ with $k > 1$ and $F(0) = 0$, then the general weighted inequality (3.6) reduces to the following inequality:

$$\left(\int_0^\infty x^{-k}\left(\int_0^x f(t)\,dt\right)^p dx\right) < \left(\frac{p}{|k-1|}\right)^p \int_0^\infty x^{-k}\left(xf(x)\right)^p dx$$

due to Hardy [16, Theorem 330, p. 245]. In this case, for the corresponding differential equation of (3.2), we have $y(x) = x^{\frac{k-1}{p}}$ which satisfies the corresponding conditions (3.8).

Example 2 By setting $a = 0$, $b = \infty$, $p = q$, $v(x) = 1$, $w(x) = (\frac{p-1}{p})^p x^{-p}$ with $p > 1$ and $F(0) = 0$, then the general weighted inequality (3.6) reduces to the following inequality:

$$\left(\int_0^\infty \left(\frac{F(x)}{x}\right)^p dx\right) < \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)\,dx,$$

where $\int_0^x f(t)\,dt$, due to Hardy [16, Theorem 327, p. 240]. In this case, for the corresponding differential equation of (3.2), we have $y(x) = x^{\frac{p-1}{p}}$ which satisfies the corresponding conditions (3.8).

As another application for our main results, we could get the following inequality (see [16, Theorem 256, p. 182]).

Example 3 If $p > 1$, $y' > 0$, and $y(x) = \int_0^x y'(t)\,dt$, then

$$\left(\int_0^{\frac{\pi}{2}} y^p\,dx\right) \leq \frac{1}{p-1}\left(\frac{p}{2}\sin\frac{p}{2}\right)^p \int_0^{\frac{\pi}{2}} (y')^p\,dx,$$

where $y(x)$ is the solution of the equation

$$x = \frac{p}{2}\sin\frac{p}{2}\int_0^y \left(1-t^p\right)^{\frac{-1}{p}} dt, \quad 0 \leq y \leq 1.$$

Acknowledgements
The authors would like to thank the referees for their valuable comments which improved the final version of the manuscript. We would like also to thank Rustaq College of Education, Rustaq, Sultanate of Oman for supporting this research.

Funding
This research was funded by Rustaq College of Education, Ministry of Higher Education, Sultanate of Oman.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally to the writing of this paper. Both authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Faculty of Science, Mansoura University, Mansoura, Egypt. [2]Department of Mathematics, Rustaq College of Education, Rustaq, Sultanate of Oman. [3]Department of Mathematics, Faculty of Science, Fayoum University, Fayoum, Egypt.

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Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 9 December 2018 Accepted: 26 March 2019 Published online: 01 April 2019

References

1. Agarwal, P., Dragomir, S.S., Jleli, M., Samet, B.: Advances in Mathematical Inequalities and Applications. Springer, Singapore (2018)
2. Agarwal, R.P., O'Regan, D., Saker, S.H.: Dynamic Inequalities on Time Scales. Springer, Cham (2014)
3. Agarwal, R.P., O'Regan, D., Saker, S.H.: Hardy Type Inequalities on Time Scales. Springer, Cham (2016)
4. Beesack, P.R.: Hardy's inequality and its extensions. Pac. J. Math. **11**, 39–61 (1961)
5. Beesack, P.R.: Integral inequalities involving a function and its derivatives. Am. Math. Mon. **78**, 705–741 (1971)

6. Bohner, M., Peterson, A.: Dynamic Equations on Time Scales: An Introduction with Applications. Birkhäuser, Boston (2001)
7. Buttazzo, G., Giaquinta, M., Hildebrandt, S.: One-Dimensional Variational Problems: An Introduction. Oxford Lecture Series in Mathematics and Its Applications, vol. 15. Oxford University Press, New York (1998)
8. Chen, F.: The optimal constant in Hardy-type inequalities. Acta Math. Sin. **31**, 731–754 (2015)
9. Chen, M.F.: Bilateral Hardy-type inequalities. Acta Math. Sin. **29**, 1–32 (2013)
10. Copson, E.T.: Note on series of positive terms. J. Lond. Math. Soc. **2**, 9–12 (1927)
11. Copson, E.T.: Note on series of positive terms. J. Lond. Math. Soc. **3**, 49–51 (1928)
12. Copson, E.T.: Some integral inequalities. Proc. R. Soc. Edinb. A **75**(13), 157–163 (1976)
13. Hardy, G.H.: Note on a theorem of Hilbert. Math. Z. **6**, 314–317 (1920)
14. Hardy, G.H.: Notes on some points in the integral calculus (LX): an inequality between integrals. Messenger Math. **54**, 150–156 (1925)
15. Hardy, G.H.: Notes on some points in the integral calculus (LXIV). Messenger Math. **57**, 12–16 (1928)
16. Hardy, G.H., Littlewood, J.E., Polya, G.: Inequalities, 2nd edn. Cambridge University Press, Cambridge (1952)
17. Hilger, S.: Analysis on measure chain—a unified approach to continuous and discrete calculus. Results Math. **18**, 18–56 (1990)
18. Kufner, A., Maligranda, L., Persson, L.E.: The Hardy Inequalities: About Its History and Some Related Results. Vydavatelski Servis Publishing House, Pilsen (2007)
19. Kufner, A., Persson, L.-E.: Weighted Inequalities of Hardy Type. World Scientific, Singapore (2003)
20. Liao, Z.W.: Discrete Hardy-type inequalities. Adv. Nonlinear Stud. **15**, 805–834 (2015)
21. Liu, X., Zhang, L., Agarwal, P., Wang, G.: On some new integral inequalities of Gronwall–Bellman–Bihari type with delay for discontinuous functions and their applications. Indag. Math. **27**(1), 1–10 (2016)
22. Mehrez, K., Agarwal, P.: New Hermite–Hadamard type integral inequalities for convex functions and their applications. J. Comput. Appl. Math. **350**, 274–285 (2019)
23. Opic, B., Kufner, A.: Hardy-Type Inequalities. Pitman Research Notes in Mathematics, vol. 219. Longman, Harlow (1990)
24. Saker, S.H.: Oscillation Theory of Dynamic Equations on Time Scales: Second and Third Orders. Lambert Academic Publishing, Berlin (2010)
25. Saker, S.H., Mahmoud, R.R., Peterson, A.: Weighted Hardy-type dynamic inequalities on time scales. Mediterr. J. Math. **13**(2), 585–606 (2016)
26. Shum, D.T.: On a class of new inequalities. Trans. Am. Math. Soc. **204**, 299–341 (1975)
27. Surang, S., Ntouyas, S.K., Agarwal, P., Tariboon, J.: Noninstantaneous impulsive inequalities via conformable fractional calculus. J. Inequal. Appl. **2018**, 261 (2018)
28. Talenti, G.: Sopra una disuguaglianza integrale. Ann. Sc. Norm. Super. Pisa **21**, 167–188 (1967)
29. Tian, J., Zhu, Y.-R., Cheung, W.-S.: *N*-Tuple diamond-alpha integral and inequalities on time scales. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. (2018). https://doi.org/10.1007/s13398-018-0609-6
30. Tian, J.-F.: Triple diamond-alpha integral and Hölder-type inequalities. J. Inequal. Appl. **2018**, 111 (2018)
31. Tian, J.-F., Ha, M.-H.: Extensions of Hölder-type inequalities on time scales and their applications. J. Nonlinear Sci. Appl. **11**, 937–953 (2017)
32. Tomaselli, G.: A class of inequalities. Boll. Unione Mat. Ital. **21**(1), 622–631 (1969)