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Representation by several orthogonal polynomials for sums of finite products of Chebyshev polynomials of the first, third and fourth kinds

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Abstract

The classical linearization problem concerns with determining the coefficients in the expansion of the product of two polynomials in terms of any given sequence of polynomials. As a generalization of this, we consider here sums of finite products of Chebyshev polynomials of the first, third, and fourth kinds, which are different from the ones previously studied. We represent each of them as linear combinations of Hermite, extended Laguerre, Legendre, Gegenbauer, and Jacobi polynomials. Here, the coefficients involve some terminating hypergeometric functions ${}_2F_1$, ${}_2F_2$, and ${}_1F_1$. These representations are obtained by explicit computations.

Keywords: Sums of finite products; Chebyshev polynomials; Hermite polynomial; Extended Laguerre polynomial; Legendre polynomial; Gegenbauer polynomial; Jacobi polynomial

1 Introduction and preliminaries

Here in this section, before stating the necessary basic facts about orthogonal polynomials, we will first fix some notations that will be used throughout this paper. We will limit those facts as minimum as possible. So the interested reader may want to refer to some general books on orthogonal polynomials; for instance [2, 3, 26].

For any nonnegative integer n, the falling factorial polynomials $(x)_n$ and the rising factorial polynomials $\langle x\rangle_n$ are, respectively, given by

$$(x)_0 = 1, \qquad (x)_n = x(x-1)\cdots(x-n+1) \quad (n \geq 1), \tag{1}$$

$$\langle x\rangle_0 = 1, \qquad \langle x\rangle_n = x(x+1)\cdots(x+n-1) \quad (n \geq 1). \tag{2}$$

The two factorial polynomials are related by

$$(-1)^n(x)_n = \langle -x\rangle_n, \qquad (-1)^n\langle x\rangle_n = (-x)_n, \tag{3}$$

$$\frac{(2n-2j)!}{(n-j)!} = \frac{2^{2n-2j}(-1)^j\langle\frac{1}{2}\rangle_n}{\langle\frac{1}{2}-n\rangle_j} \quad (n \geq j \geq 0), \tag{4}$$

$$\frac{(2n+2j)!}{(n+j)!} = 2^{2n+2j}\left\langle\frac{1}{2}\right\rangle_n\left\langle n+\frac{1}{2}\right\rangle_j \quad (n,j \geq 0), \tag{5}$$

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n)!\sqrt{\pi}}{2^{2n}n!} \quad (n \geq 0), \tag{6}$$

$$\frac{\Gamma(x+1)}{\Gamma(x+1-n)} = (x)_n, \qquad \frac{\Gamma(x+n)}{\Gamma(x)} = \langle x\rangle_n \quad (n \geq 0), \tag{7}$$

where $\Gamma(x)$ is the gamma function. The hypergeometric function is defined by

$${}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;x) = \sum_{n=0}^{\infty}\frac{\langle a_1\rangle_n\cdots\langle a_p\rangle_n}{\langle b_1\rangle_n\cdots\langle b_q\rangle_n}\frac{x^n}{n!}. \tag{8}$$

Next, we would like to state some basic facts about Chebyshev polynomials of the first kind $T_n(x)$, second kind $U_n(x)$, third kind $V_n(x)$, and fourth kind $W_n(x)$. Also, we will mention those facts about Hermite polynomials $H_n(x)$, extended Laguerre polynomials $L_n^{\alpha}(x)$, Legendre polynomials $P_n(x)$, Gegenbauer polynomials $C_n^{(\lambda)}(x)$, and Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$. All of these facts can be found also in [5–9, 12, 14].

The above-mentioned polynomials are given, in terms of generating functions, in the following:

$$F_1(t,x) = \frac{1-xt}{1-2xt+t^2} = \sum_{n=0}^{\infty}T_n(x)t^n, \tag{9}$$

$$F_2(t,x) = \frac{1}{1-2xt+t^2} = \sum_{n=0}^{\infty}U_n(x)t^n, \tag{10}$$

$$F_3(t,x) = \frac{1-t}{1-2xt+t^2} = \sum_{n=0}^{\infty}V_n(x)t^n, \tag{11}$$

$$F_4(t,x) = \frac{1+t}{1-2xt+t^2} = \sum_{n=0}^{\infty}W_n(x)t^n, \tag{12}$$

$$e^{2xt-t^2} = \sum_{n=0}^{\infty}H_n(x)\frac{t^n}{n!}, \tag{13}$$

$$(1-t)^{-\alpha-1}\exp\left(-\frac{xt}{1-t}\right) = \sum_{n=0}^{\infty}L_n^{\alpha}(x)t^n \quad (\alpha > -1), \tag{14}$$

$$\left(1-2xt+t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}P_n(x)t^n, \tag{15}$$

$$\frac{1}{(1-2xt+t^2)^{\lambda}} = \sum_{n=0}^{\infty}C_n^{(\lambda)}(x)t^n \quad \left(\lambda > -\frac{1}{2}, \lambda \neq 0, |t| < 1, |x| \leq 1\right), \tag{16}$$

$$\frac{\alpha+\beta}{R(1-t+R)^{\alpha}(1+t+R)^{\beta}} = \sum_{n=0}^{\infty}P_n^{(\alpha,\beta)}(x)t^n \quad \left(R = \sqrt{1-2xt+t^2}, \alpha, \beta > -1\right). \tag{17}$$

They are also given, in terms of explicit expressions, as follows:

$$T_n(x) = {}_2F_1\left(-n,n;\frac{1}{2};\frac{1-x}{2}\right) = \frac{n}{2}\sum_{l=0}^{[\frac{n}{2}]}(-1)^l\frac{1}{n-l}\binom{n-l}{l}(2x)^{n-2l} \quad (n \geq 1), \tag{18}$$

$$
\begin{aligned}
U_n(x) &= (n+1)_2F_1\left(-n, n+2; \frac{3}{2}; \frac{1-x}{2}\right) \\
&= \sum_{l=0}^{[\frac{n}{2}]} (-1)^l \binom{n-l}{l} (2x)^{n-2l} \quad (n \geq 0), \qquad (19)\\
V_n(x) &= {}_2F_1\left(-n, n+1; \frac{1}{2}; \frac{1-x}{2}\right) \\
&= \sum_{l=0}^{n} \binom{n+l}{2l} 2^l (x-1)^l \quad (n \geq 0), \qquad (20)\\
W_n(x) &= (2n+1)_2F_1\left(-n, n+1; \frac{3}{2}; \frac{1-x}{2}\right) \\
&= (2n+1) \sum_{l=0}^{n} \frac{2^l}{2l+1} \binom{n+l}{2l} (x-1)^l \quad (n \geq 0), \qquad (21)\\
H_n(x) &= n! \sum_{l=0}^{[\frac{n}{2}]} \frac{(-1)^l}{l!(n-2l)!} (2x)^{n-2l} \quad (n \geq 0), \qquad (22)\\
L_n^{\alpha}(x) &= \frac{\langle \alpha+1 \rangle_n}{n!} {}_1F_1(-n, \alpha+1; x) \\
&= \sum_{l=0}^{n} \frac{(-1)^l \binom{n+\alpha}{n-l}}{l!} x^l \quad (n \geq 0), \qquad (23)\\
P_n(x) &= {}_2F_1\left(-n, n+1; 1; \frac{1-x}{2}\right) \\
&= \frac{1}{2^n} \sum_{l=0}^{[\frac{n}{2}]} (-1)^l \binom{n}{l} \binom{2n-2l}{n} x^{n-2l} \quad (n \geq 0), \qquad (24)\\
C_n^{(\lambda)}(x) &= \binom{n+2\lambda-1}{n} {}_2F_1\left(-n, n+2\lambda; \lambda+\frac{1}{2}; \frac{1-x}{2}\right) \\
&= \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \frac{\Gamma(n-k+\lambda)}{\Gamma(\lambda) k!(n-2k)!} (2x)^{n-2k} \quad (n \geq 0), \qquad (25)\\
P_n^{(\alpha,\beta)}(x) &= \frac{\langle \alpha+1 \rangle_n}{n!} {}_2F_1\left(-n, 1+\alpha+\beta+n; \alpha+1; \frac{1-x}{2}\right) \\
&= \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \binom{n+\beta}{k} \left(\frac{x-1}{2}\right)^k \left(\frac{x+1}{2}\right)^{n-k} \quad (n \geq 0). \qquad (26)
\end{aligned}
$$

Next, we state Rodrigues-type formulas for Hermite and extended Laguerre polynomials, and Rodrigues' formulas for Legendre, Gegenbauer and Jacobi polynomials:

$$
H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}, \qquad (27)
$$

$$
L_n^{\alpha}(x) = \frac{1}{n!} x^{-\alpha} e^x \frac{d^n}{dx^n} \left(e^{-x} x^{n+\alpha}\right), \qquad (28)
$$

$$
P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left(x^2-1\right)^n, \qquad (29)
$$

$$\left(1-x^2\right)^{\lambda-\frac{1}{2}}C_n^{(\lambda)}(x)=\frac{(-2)^n}{n!}\frac{\langle\lambda\rangle_n}{\langle n+2\lambda\rangle_n}\frac{d^n}{dx^n}\left(1-x^2\right)^{n+\lambda-\frac{1}{2}}, \tag{30}$$

$$(1-x)^{\alpha}(1+x)^{\beta}P_n^{(\alpha,\beta)}(x)=\frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}(1-x)^{n+\alpha}(1+x)^{n+\beta}. \tag{31}$$

The special polynomials in (27)–(31) satisfy the following orthogonality properties with respect to various weight functions:

$$\int_{-\infty}^{\infty}e^{-x^2}H_n(x)H_m(x)\,dx=2^n n!\sqrt{\pi}\delta_{m,n}, \tag{32}$$

$$\int_0^{\infty}x^{\alpha}e^{-x}L_n^{\alpha}(x)L_m^{\alpha}(x)\,dx=\frac{1}{n!}\Gamma(\alpha+n+1)\delta_{m,n}, \tag{33}$$

$$\int_{-1}^{1}P_n(x)P_m(x)\,dx=\frac{2}{2n+1}\delta_{m,n}, \tag{34}$$

$$\int_{-1}^{1}\left(1-x^2\right)^{\lambda-\frac{1}{2}}C_n^{(\lambda)}(x)C_m^{(\lambda)}(x)\,dx=\frac{\pi 2^{1-2\lambda}\Gamma(n+2\lambda)}{n!(n+\lambda)\Gamma(\lambda)^2}\delta_{m,n}, \tag{35}$$

$$\begin{aligned}&\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}P_n^{(\alpha,\beta)}(x)P_m^{(\alpha,\beta)}(x)\,dx\\&\quad=\frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)\Gamma(n+1)}\delta_{m,n}.\end{aligned} \tag{36}$$

In this paper, we will consider the following sums of finite products of Chebyshev polynomials of the first, third and fourth kinds:

$$\alpha_{m,r}(x)=\sum_{i_1+\cdots+i_{r+1}=m}T_{i_1}(x)\cdots T_{i_{r+1}}(x)\quad(m,r\geq 0), \tag{37}$$

$$\beta_{m,r}(x)=\sum_{i_1+\cdots+i_{r+1}=m}V_{i_1}(x)\cdots V_{i_{r+1}}(x)\quad(m,r\geq 0), \tag{38}$$

$$\gamma_{m,r}(x)=\sum_{i_1+\cdots+i_{r+1}=m}W_{i_1}(x)\cdots W_{i_{r+1}}(x)\quad(m,r\geq 0), \tag{39}$$

where all the sums in (37)–(39) run over all nonnegative integers $i_1,\ldots,i_{r+1}$, with $i_1+\cdots+i_{r+1}=m$. Here, we observe that $\alpha_{m,r}(x)$, $\beta_{m,r}(x)$, and $\gamma_{m,r}(x)$ all have degree m.

Our goal here is to express each of the sums of products in (37)–(39) as linear combinations of $H_n(x)$, $L_n^{\alpha}(x)$, $P_n(x)$, $C_n^{(\lambda)}(x)$, and $P_n^{(\alpha,\beta)}(x)$. An important observation here is that $\alpha_{m,r}(x)$, $\beta_{m,r}(x)$, and $\gamma_{m,r}(x)$ can be expressed in terms of $U_{m-j+r}^{(r)}(x)$, $(j=1,\ldots,m)$ (see Lemmas 1 and 2) by using the generating functions in (10). Then our results for $\alpha_{m,r}(x)$, $\beta_{m,r}(x)$, and $\gamma_{m,r}(x)$ will be obtained by making use of Lemmas 1 and 2, the general formulas in Propositions 1 and 2, and integration by parts. We note here that each of the sums in (37)–(39) are also expressed in terms of all four kinds of Chebyshev polynomials in (21).

Before we state the main theorems, we would like to mention some previous work directly related to the results in the present paper. For this purpose, let us put

$$\begin{aligned}&\sum_{l=0}^{m}\sum_{i_1+\cdots+i_{r+1}=m-l}\binom{r+l}{r}x^l T_{i_1}(x)\cdots T_{i_{r+1}}(x)\\&\quad-\sum_{l=0}^{m-2}\sum_{i_1+\cdots+i_{r+1}=m-l-2}\binom{r+l}{r}x^l T_{i_1}(x)\cdots T_{i_{r+1}}(x)\quad(m\geq 2,r\geq 1),\end{aligned} \tag{40}$$

$$\sum_{l=0}^{m}\sum_{i_1+\cdots+i_{r+1}=l}\binom{r-1+m-l}{r-1}V_{i_1}(x)\cdots V_{i_{r+1}}(x)\quad(m\geq 0,r\geq 1),\tag{41}$$

$$\sum_{l=0}^{m}\sum_{i_1+\cdots+i_{r+1}=l}(-1)^{m-l}\binom{r-1+m-l}{r-1}W_{i_1}(x)\cdots W_{i_{r+1}}(x)\quad(m\geq 0,r\geq 1).\tag{42}$$

We studied Eq. (40) in [13, 16] and (41) and (42) in [4, 19] and were able to express each of them in terms of the Chebyshev polynomials of all kinds, Hermite polynomials, extended Laguerre polynomials. Legendre polynomials, Gegenbauer polynomials, and Jacobi polynomials. It is worth mentioning that some terminating hypergeometric functions like ${}_1F_1$, ${}_2F_0$, ${}_2F_1$, and ${}_3F_2$ appear as coefficients in such expressions. The impetus for these studies was the observation that the sums in (40)–(42) are, respectively, equal to $\frac{1}{2^{r-1}r!}T_{m+r}^{(r)}(x)$, $\frac{1}{2^r r!}V_{m+r}^{(r)}(x)$, and $\frac{1}{2^r r!}W_{m+r}^{(r)}(x)$. In fact, these equalities can easily be seen by differentiating the generating functions in (9), (11) and (12). The next three theorems are the main results in this paper.

Theorem 1 *For any nonnegative integers m, r, the following identities hold*:

$$\begin{aligned}
&\sum_{i_1+\cdots+i_{r+1}=m}T_{i_1}(x)\cdots T_{i_{r+1}}(x)\\
&=\frac{1}{r!}\sum_{s=0}^{[\frac{m}{2}]}\frac{1}{(m-2s)!}\sum_{l=0}^{s}\frac{(-1)^l(m+r-l)!}{l!(s-l)!}\\
&\quad\times{}_2F_1\left(2l-m,-r-1;l-m-r;\frac{1}{2}\right)H_{m-2s}(x) &(43)\\
&=\frac{2^m}{r!}\sum_{k=0}^{m}\frac{(-1)^k}{\Gamma(\alpha+k+1)}\sum_{l=0}^{[\frac{m-k}{2}]}\frac{(-\frac{1}{4})^l(m+r-l)!\Gamma(m-2l+\alpha+1)}{l!(m-k-2l)!}\\
&\quad\times{}_2F_1\left(2l-m,-r-1;l-m-r;\frac{1}{2}\right)L_k^{\alpha}(x) &(44)\\
&=\frac{4^m}{r!}\sum_{s=0}^{[\frac{m}{2}]}2^{1-2s}(2m-4s+1)\sum_{l=0}^{s}\frac{(-\frac{1}{4})^l(m+r-l)!(m-s-l+1)!}{l!(s-l)!(2m-2s-2l+2)!}\\
&\quad\times{}_2F_1\left(2l-m,-r-1;l-m-r;\frac{1}{2}\right)P_{m-2s}(x) &(45)\\
&=\frac{\Gamma(\lambda)}{r!}\sum_{s=0}^{[\frac{m}{2}]}(m-2s+\lambda)\sum_{l=0}^{s}\frac{(-1)^l(m+r-l)!}{l!(s-l)!\Gamma(m+\lambda-s-l+1)}\\
&\quad\times{}_2F_1\left(2l-m,-r-1;l-m-r;\frac{1}{2}\right)C_{m-2s}^{(\lambda)}(x) &(46)\\
&=\frac{(-2)^m}{r!}\sum_{k=0}^{m}\frac{(-2)^k\Gamma(k+\alpha+\beta+1)}{\Gamma(2k+\alpha+\beta+1)}\sum_{l=0}^{[\frac{m-k}{2}]}\frac{(-\frac{1}{4})^l(m+r-l)!}{l!(m-k-2l)!}\\
&\quad\times{}_2F_1\left(2l-m,-r-1;l-m-r;\frac{1}{2}\right)\\
&\quad\times{}_2F_1(2l+k-m,k+\beta+1;2k+\alpha+\beta+2;2)P_k^{(\alpha,\beta)}(x). &(47)
\end{aligned}$$

Theorem 2 *For any nonnegative integers m, r, the following identities hold*:

$$
\begin{aligned}
&\sum_{i_1+\cdots+i_{r+1}=m} V_{i_1}(x)\cdots V_{i_{r+1}}(x) \\
&= (-1)^m(r+1)\sum_{k=0}^{m}\frac{(-1)^k}{k!}\sum_{s=0}^{[\frac{m-k}{2}]}\frac{(k+2s+r)!}{(m-k-2s)!(r+1-m+k+2s)!s!} \\
&\quad\times {}_1F_1(-s,-k-2s-r;-1)H_k(x) \qquad (48)\\
&= \frac{2^m}{r!}\sum_{k=0}^{m}\frac{(-1)^k}{\Gamma(\alpha+k+1)}\sum_{l=0}^{[\frac{m-k}{2}]}\frac{(-\frac{1}{4})^l(m+r-l)!\Gamma(m+\alpha-2l+1)}{l!(m-k-2l)!} \\
&\quad\times {}_2F_2\left(2l+k-m,-r-1;l-m-r;2l-m-\alpha;-\frac{1}{2}\right)L_k^{\alpha}(x) \qquad (49)\\
&= (-1)^m(r+1)\sum_{k=0}^{m}(2k+1) \\
&\quad\times\sum_{s=0}^{[\frac{m-k}{2}]}\frac{(k+2s+r)!}{s!(m-k-2s)!(r+1-m+k+2s)!(k+s+\frac{1}{2})_{k+s}} \\
&\quad\times {}_2F_1\left(-s,-k-s-\frac{1}{2};-k-2s-r;1\right)P_k(x) \qquad (50)\\
&= (-1)^m(r+1)\Gamma(\lambda)\sum_{k=0}^{m}(k+\lambda)(-1)^k \\
&\quad\times\sum_{s=0}^{[\frac{m-k}{2}]}\frac{(k+2s+r)!}{s!(m-k-2s)!(r+1-m+k+2s)!\Gamma(k+\lambda+s+1)} \\
&\quad\times {}_2F_1(-s,-k-\lambda-s;-k-2s-r;1)C_k^{(\lambda)}(x) \qquad (51)\\
&= (-2)^m(r+1)\sum_{k=0}^{m}\frac{(-2)^k\Gamma(k+\alpha+\beta+1)}{\Gamma(2k+\alpha+\beta+1)} \\
&\quad\times\sum_{j=0}^{m-k}\frac{2^{-j}}{j!(r+1-j)!}\sum_{l=0}^{[\frac{m-j-k}{2}]}\frac{(-\frac{1}{4})^l(m-j+r-l)!}{l!(m-j-k-2l)!} \\
&\quad\times {}_2F_1(j+k+2l-m,k+\beta+1;2k+\alpha+\beta+2;2)P_k^{(\alpha,\beta)}(x). \qquad (52)
\end{aligned}
$$

Theorem 3 *For any nonnegative integers m, r, the following identities hold*:

$$
\begin{aligned}
&\sum_{i_1+\cdots+i_{r+1}=m} W_{i_1}(x)\cdots W_{i_{r+1}}(x) \\
&= (r+1)\sum_{k=0}^{m}\frac{1}{k!}\sum_{s=0}^{[\frac{m-k}{2}]}\frac{(k+2s+r)!}{(m-k-2s)!(r+1-m+k+2s)!s!} \\
&\quad\times {}_1F_1(-s,-k-2s-r;-1)H_k(x) \qquad (53)\\
&= \frac{2^m}{r!}\sum_{k=0}^{m}\frac{(-1)^k}{\Gamma(\alpha+k+1)}\sum_{l=0}^{[\frac{m-k}{2}]}\frac{(-\frac{1}{4})^l(m+r-l)!\Gamma(m+\alpha-2l+1)}{l!(m-k-2l)!}
\end{aligned}
$$

$$\times {}_2F_2\left(2l+k-m,-r-1;l-m-r;2l-m-\alpha;\frac{1}{2}\right)L_k^{\alpha}(x) \tag{54}$$

$$= (r+1)\sum_{k=0}^{m}(-1)^k(2k+1)$$

$$\times \sum_{s=0}^{[\frac{m-k}{2}]}\frac{(k+2s+r)!}{s!(m-k-2s)!(r+1-m+k+2s)!(k+s+\frac{1}{2})_{k+s}}$$

$$\times {}_2F_1\left(-s,-k-s-\frac{1}{2};-k-2s-r;1\right)P_k(x) \tag{55}$$

$$= (r+1)\Gamma(\lambda)\sum_{k=0}^{m}(k+\lambda)$$

$$\times \sum_{s=0}^{[\frac{m-k}{2}]}\frac{(k+2s+r)!}{s!(m-k-2s)!(r+1-m+k+2s)!\Gamma(k+\lambda+s+1)}$$

$$\times {}_2F_1(-s,-k-\lambda-s;-k-2s-r;1)C_k^{(\lambda)}(x) \tag{56}$$

$$= (-2)^m(r+1)\sum_{k=0}^{m}\frac{(-2)^k\Gamma(k+\alpha+\beta+1)}{\Gamma(2k+\alpha+\beta+1)}\sum_{j=0}^{m-k}\frac{(-\frac{1}{2})^j}{j!(r+1-j)!}$$

$$\times \sum_{l=0}^{[\frac{m-j-k}{2}]}\frac{(-\frac{1}{4})^l(m-j+r-l)!}{l!(m-j-k-2l)!}$$

$$\times {}_2F_1(j+k+2l-m,k+\beta+1;2k+\alpha+\beta+2;2)P_k^{(\alpha,\beta)}(x). \tag{57}$$

Lastly, we would like to mention some of the previous results that are related to the present work. Along the same line as this paper, certain sums of finite products of Chebyshev polynomials of the first, second, third and fourth kinds, and of Legendre, Laguerre, Fibonacci and Lucas polynomials are expressed in terms of all four kinds of Chebyshev polynomials in [10, 16, 19, 23, 25] and also in terms of Hermite, extended Laguerre, Legendre, Gegenbauer and Jacobi polynomials in [4, 11, 13, 24].

Also, some sums of finite products of Appell and non-Appell polynomials are expressed as linear combinations of Bernoulli polynomials. All of these were obtained by deriving Fourier series expansions for the functions closely related to such sums of finite products of special polynomials. Indeed, as for Appell polynomials some sums of finite products of Bernoulli and Euler polynomials are expressed in terms of Bernoulli polynomials in [1, 20]. As for non-Appell polynomials in [15, 17, 18, 22] the same are done for some sums of finite products of Chebyshev polynomials of the first, second, third, and fourth kinds, and of Legendre, Laguerre, Fibonacci and Lucas polynomials.

2 Proof of Theorem 1

Here we are going to prove Theorem 1. For this purpose, we first state Propositions 1 and 2 that will be used in showing Theorems 1, 2 and 3.

We note that the facts (a), (b), (c), (d) and (e) of Proposition 1 are, respectively, from (3.7) of [8], (2.3) of [12], (2.3) of [9], (2.3) of [6] and (2.7) of [14]. Actually, all the formulas in Proposition 1 can be derived from the orthogonalities in (32)–(36). Rodrigues' and Rodrigues-type formulas in (27)–(31), and integration by parts.

Proposition 1 *Let $q(x) \in \mathbb{R}[x]$ be a polynomial of degree n. Then the following formulas hold*:

(a) $q(x) = \sum_{k=0}^{n} C_{k,1} H_k(x)$, *where*

$$C_{k,1} = \frac{(-1)^k}{2^k k! \sqrt{\pi}} \int_{-\infty}^{\infty} q(x) \frac{d^k}{dx^k} e^{-x^2}\, dx.$$

(b) $q(x) = \sum_{k=0}^{n} C_{k,2} L_k^{\alpha}(x)$, *where*

$$C_{k,2} = \frac{1}{\Gamma(\alpha + k + 1)} \int_{0}^{\infty} q(x) \frac{d^k}{dx^k} \left(e^{-x} x^{k+\alpha}\right) dx.$$

(c) $q(x) = \sum_{k=0}^{n} C_{k,3} P_k(x)$, *where*

$$C_{k,3} = \frac{2k+1}{2^{k+1} k!} \int_{-1}^{1} q(x) \frac{d^k}{dx^k} \left(x^2 - 1\right)^k dx.$$

(d) $q(x) = \sum_{k=0}^{n} C_{k,4} C_k^{(\lambda)}(x)$, *where*

$$C_{k,4} = \frac{(k+\lambda)\Gamma(\lambda)}{(-2)^k \sqrt{\pi}\, \Gamma(k + \lambda + \frac{1}{2})} \int_{-1}^{1} q(x) \frac{d^k}{dx^k} \left(1 - x^2\right)^{k+\lambda-\frac{1}{2}} dx.$$

(e) $q(x) = \sum_{k=0}^{n} C_{k,5} P_k^{(\alpha,\beta)}(x)$, *where*

$$C_{k,5} = \frac{(-1)^k (2k + \alpha + \beta + 1)\Gamma(k + \alpha + \beta + 1)}{2^{\alpha+\beta+k+1} \Gamma(\alpha + k + 1)\Gamma(\beta + k + 1)} \int_{-1}^{1} q(x) \frac{d^k}{dx^k} (1-x)^{k+\alpha} (1+x)^{k+\beta}\, dx.$$

The next proposition is stated in [24].

Proposition 2 *Let m, k be nonnegative integers. Then we have the following.*

(a) $$\int_{-\infty}^{\infty} x^m e^{-x^2}\, dx = \begin{cases} 0, & \text{if } m \equiv 1 \pmod 2, \\ \frac{m! \sqrt{\pi}}{(\frac{m}{2})! 2^m}, & \text{if } m \equiv 0 \pmod 2. \end{cases}$$

(b) $$\int_{-1}^{1} x^m \left(1 - x^2\right)^k dx = \begin{cases} 0, & \text{if } m \equiv 1 \pmod 2, \\ \frac{2^{2k+2} k! m! (k + \frac{m}{2} + 1)!}{(\frac{m}{2})! (2k + m + 2)!}, & \text{if } m \equiv 0 \pmod 2. \end{cases}$$

(c) $$\int_{-1}^{1} x^m \left(1 - x^2\right)^{k+\lambda-\frac{1}{2}} dx = \begin{cases} 0, & \text{if } m \equiv 1 \pmod 2, \\ \frac{\Gamma(k+\lambda+\frac{1}{2})\Gamma(\frac{m}{2}+\frac{1}{2})}{\Gamma(k+\lambda+\frac{m}{2}+1)}, & \text{if } m \equiv 0 \pmod 2. \end{cases}$$

(d) $$\int_{-1}^{1} x^m (1-x)^{k+\alpha} (1+x)^{k+\beta}\, dx$$
$$= 2^{2k+\alpha+\beta+1} \sum_{s=0}^{m} \binom{m}{s} (-1)^{m-s} 2^s \frac{\Gamma(k + \alpha + 1)\Gamma(k + \beta + s + 1)}{\Gamma(2k + \alpha + \beta + s + 2)}.$$

In [27], the following lemma is stated for $m \geq r + 1$. But it is valid for any nonnegative integer m, under the usual convention $\binom{r+1}{j} = 0$, for $j > r + 1$ (see [21]).

Lemma 1 *Let m, r be any nonnegative integers. Then the following identity holds*:

$$\sum_{i_1+\cdots+i_{r+1}=m} T_{i_1}(x)\cdots T_{i_{r+1}}(x) = \frac{1}{2^r r!}\sum_{j=0}^{m}(-1)^j\binom{r+1}{j}x^j U^{(r)}_{m-j+r}(x), \tag{58}$$

where $\binom{r+1}{j}=0$, *for* $j>r+1$.

For (19), we see that the rth derivative of $U_n(x)$ is given by

$$U^{(r)}_n(x) = \sum_{l=0}^{[\frac{n-r}{2}]}(-1)^l\binom{n-l}{l}(n-2l)_r 2^{n-2l}x^{n-2l-r}. \tag{59}$$

We see easily from (59) that

$$x^j U^{(r)}_{m-j+r}(x) = \sum_{l=0}^{[\frac{m-j}{2}]}(-1)^l\binom{m-j+r-l}{l}(m-j+r-2l)_r 2^{m-j+r-2l}x^{m-2l}. \tag{60}$$

In this section, we are going to show (43), (45) and (47) in Theorem 1, leaving the others (44) and (46) as exercises to the reader.

With $\alpha_{m,r}(x)$ as in (37), we let

$$\alpha_{m,r}(x) = \sum_{k=0}^{m} C_{k,1}H_k(x). \tag{61}$$

Then, by making use of (a) of Proposition 1, (58), (60), and integration by parts k times, we have

$$\begin{aligned}
C_{k,1} &= \frac{(-1)^k}{2^k k!\sqrt{\pi}}\int_{-\infty}^{\infty}\alpha_{m,r}(x)\frac{d^k}{dx^k}e^{-x^2}\,dx \\
&= \frac{(-1)^k}{2^k k!\sqrt{\pi}2^r r!}\sum_{j=0}^{m}(-1)^j\binom{r+1}{j}\int_{-\infty}^{\infty}x^j U^{(r)}_{m-j+r}(x)\frac{d^k}{dx^k}e^{-x^2}\,dx \\
&= \frac{(-1)^k}{2^k k!\sqrt{\pi}2^r r!}\sum_{j=0}^{m}(-1)^j\binom{r+1}{j}\sum_{l=0}^{[\frac{m-j}{2}]}(-1)^l\binom{m-j+r-l}{l} \\
&\quad\times(m-j+r-2l)_r 2^{m-j+r-2l}\int_{-\infty}^{\infty}x^{m-2l}\frac{d^k}{dx^k}e^{-x^2}\,dx \\
&= \frac{(-1)^k}{2^k k!\sqrt{\pi}2^r r!}\sum_{l=0}^{[\frac{m-k}{2}]}\sum_{j=0}^{m-2l}(-1)^j\binom{r+1}{j}(-1)^l\binom{m-j+r-l}{l} \\
&\quad\times(m-j+r-2l)_r 2^{m-j+r-2l}(-1)^k(m-2l)_k\int_{-\infty}^{\infty}x^{m-2l-k}e^{-x^2}\,dx,
\end{aligned} \tag{62}$$

where we note from (a) of Proposition 2 that

$$\int_{-\infty}^{\infty}x^{m-2l-k}e^{-x^2}\,dx = \begin{cases}0, & \text{if } k\not\equiv m \pmod 2,\\ \frac{(m-2l-k)!\sqrt{\pi}}{(\frac{m-k}{2}-l)!2^{m-2l-k}}, & \text{if } k\equiv m \pmod 2.\end{cases} \tag{63}$$

Now, from (61)–(63) and after some simplifications, we obtain

$$\begin{aligned}\alpha_{m,r}(x) &= \frac{1}{r!}\sum_{\substack{0\le k\le m\\ k\equiv m(\mathrm{mod}\,2)}}\frac{1}{k!}\sum_{l=0}^{[\frac{m-k}{2}]}\frac{(-1)^l(m-2l)!}{(\frac{m-k}{2}-l)!l!}\sum_{j=0}^{m-2l}\frac{(\frac{1}{2})^j(-1)^j(r+1)_j(m+r-l-j)!}{j!(m-2l-j)!}H_k(x)\\ &= \frac{1}{r!}\sum_{\substack{0\le k\le m\\ k\equiv m(\mathrm{mod}\,2)}}\frac{1}{k!}\sum_{l=0}^{[\frac{m-k}{2}]}\frac{(-1)^l(m+r-l)!}{l!(\frac{m-k}{2}-l)!}\sum_{j=0}^{m-2l}\frac{(\frac{1}{2})^j\langle 2l-m\rangle_j\langle -r-1\rangle_j}{j!\langle l-m-r\rangle_j}H_k(x)\\ &= \frac{1}{r!}\sum_{s=0}^{[\frac{m}{2}]}\frac{1}{(m-2s)!}\sum_{l=0}^{s}\frac{(-1)^l(m+r-l)!}{l!(s-l)!}{}_2F_1\left(2l-m,-r-1;l-m-r;\frac{1}{2}\right)\\ &\quad\times H_{m-2s}(x).\end{aligned} \tag{64}$$

This shows (35) of Theorem 1.

Next, let us put

$$\alpha_{m,r}(x) = \sum_{k=0}^{m} C_{k,3}P_k(x). \tag{65}$$

Then, from (c) of Proposition 1, (58), (60) and integration by parts k times, we get

$$\begin{aligned}C_{k,3} &= \frac{(2k+1)}{2^{k+1}k!2^r r!}\sum_{l=0}^{[\frac{m-k}{2}]}\sum_{j=0}^{m-2l}(-1)^j\binom{r+1}{j}(-1)^l\\ &\quad\times\binom{m-j+r-l}{l}(m-j+r-2l)_r 2^{m-j+r-2l}(m-2l)_k\\ &\quad\times\int_{-1}^{1}x^{m-2l-k}\left(1-x^2\right)^k dx,\end{aligned} \tag{66}$$

where we observe from (b) of Proposition 2 that

$$\int_{-1}^{1}x^{m-2l-k}\left(1-x^2\right)^k dx = \begin{cases}0, & \text{if } k\not\equiv m(\mathrm{mod}\,2),\\ \frac{2^{2k+2}k!(m-2l-k)!(k+\frac{m-k}{2}-l+1)!}{(\frac{m-k}{2}-l)!(m+k-2l+2)!}, & \text{if } k\equiv m(\mathrm{mod}\,2).\end{cases} \tag{67}$$

Now, from (65)–(67) and after some simplifications, we have

$$\begin{aligned}\alpha_{m,r}(x) &= \frac{2^m}{r!}\sum_{\substack{0\le k\le m\\ k\equiv m(\mathrm{mod}\,2)}}(2k+1)2^{k+1}\sum_{l=0}^{[\frac{m-k}{2}]}\frac{(-\frac{1}{4})^l(m-2l)!(k+\frac{m-k}{2}-l+1)!}{l!(\frac{m-k}{2}-l)!(m+k-2l+2)!}\\ &\quad\times\sum_{j=0}^{m-2l}\frac{(-\frac{1}{2})^j(r+1)_j(m+r-l-j)!}{j!(m-2l-j)!}P_k(x)\\ &= \frac{2^m}{r!}\sum_{\substack{0\le k\le m\\ k\equiv m(\mathrm{mod}\,2)}}(2k+1)2^{k+1}\sum_{l=0}^{[\frac{m-k}{2}]}\frac{(-\frac{1}{4})^l(m+r-l)!(k+\frac{m-k}{2}-l+1)!}{l!(\frac{m-k}{2}-l)!(m+k-2l+2)!}\end{aligned}$$

$$\times \sum_{j=0}^{m-2l} \frac{(\frac{1}{2})^j \langle 2l-m\rangle_j \langle -r-1\rangle_j}{j!\langle l-m-r\rangle_j} P_k(x) \tag{68}$$

$$= \frac{4^m}{r!} \sum_{s=0}^{[\frac{m}{2}]} 2^{1-2s}(2m-4s+1) \sum_{l=0}^{s} \frac{(-\frac{1}{4})^l (m+r-l)!(m-s-l+1)!}{l!(s-l)!(2m-2s-2l+2)!}$$

$$\times {}_2F_1\left(2l-m,-r-1;l-m-r;\frac{1}{2}\right) P_{m-2s}(x).$$

This completes the proof for (37) in Theorem 1.

Finally, we put

$$\alpha_{m,r}(x) = \sum_{k=0}^{m} C_{k,5} P_k^{(\alpha,\beta)}(x). \tag{69}$$

Then, from (e) of Proposition 1, (58), (60) and integration by parts k times, we get

$$C_{k,5} = \frac{(-1)^k (2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)}{2^{\alpha+\beta+k+1}\Gamma(\alpha+k+1)\Gamma(\beta+k+1)2^r r!}$$

$$\times \sum_{l=0}^{[\frac{m-k}{2}]} \sum_{j=0}^{m-2l} (-1)^j \binom{r+1}{j} (-1)^l \binom{m-j+r-l}{l} (m-j+r-2l)_r$$

$$\times 2^{m-j+r-2l}(-1)^k (m-2l)_k \int_{-1}^{1} x^{m-2l-k}(1-x)^{k+\alpha}(1+x)^{k+\beta}\,dx, \tag{70}$$

where we note from (d) of Proposition 2 that

$$\int_{-1}^{1} x^{m-2l-k}(1-x)^{k+\alpha}(1+x)^{k+\beta}\,dx = 2^{2k+\alpha+\beta+1} \sum_{s=0}^{m-2l-k} \binom{m-2l-k}{s} (-1)^{m-k-s} 2^s$$

$$\times \frac{\Gamma(k+\alpha+1)\Gamma(k+\beta+s+1)}{\Gamma(2k+\alpha+\beta+s+2)}. \tag{71}$$

Now, from (69)–(71) and after some simplifications, we have

$$\alpha_{m,r}(x) = \frac{(-2)^m}{r!} \sum_{k=0}^{m} \frac{(-2)^k \Gamma(k+\alpha+\beta+1)}{\Gamma(2k+\alpha+\beta+1)} \sum_{l=0}^{[\frac{m-k}{2}]} \frac{(-\frac{1}{4})^l (m-2l)k}{l!}$$

$$\times \sum_{j=0}^{m-2l} \frac{(\frac{1}{2})^j (-1)^j (r+1)_j (m+r-l-j)!}{j!(m-2l-j)!}$$

$$\times \sum_{s=0}^{m-2l-k} \frac{2^s \langle 2l+k-m\rangle_s \langle k+\beta+1\rangle_s}{s!\langle 2k+\alpha+\beta+2\rangle_s} P_k^{(\alpha,\beta)}(x) \tag{72}$$

$$= \frac{(-2)^m}{r!} \sum_{k=0}^{m} \frac{(-2)^k \Gamma(k+\alpha+\beta+1)}{\Gamma(2k+\alpha+\beta+1)} \sum_{l=0}^{[\frac{m-k}{2}]} \frac{(-\frac{1}{4})^l (m+r-l)!}{(m-k-2l)!}$$

$$\times \sum_{j=0}^{m-2l} \frac{(\frac{1}{2})^j \langle 2l-m\rangle_j \langle -r-1\rangle_j}{j!\langle l-m-r\rangle_j}$$

$$\times {}_2F_1(2l+k-m,k+\beta+1;2k+\alpha+\beta+2;2)P_k^{(\alpha,\beta)}(x)$$
$$= \frac{(-2)^m}{r!}\sum_{k=0}^{m}\frac{(-2)^k\Gamma(k+\alpha+\beta+1)}{\Gamma(2k+\alpha+\beta+1)}\sum_{l=0}^{[\frac{m-k}{2}]}\frac{(-\frac{1}{4})^l(m+r-l)!}{(m-k-2l)!}$$
$$\times {}_2F_1\left(2l-m,-r-1;l-m-r;\frac{1}{2}\right)$$
$$\times {}_2F_1(2l+k-m,k+\beta+1;2k+\alpha+\beta+2;2)P_k^{(\alpha,\beta)}(x).$$

This finishes the proof for (39) in Theorem 1.

3 Proofs of Theorems 2 and 3

Here we will show (49) and (51) in Theorem 2 and leave the others (48), (50) and (52) as exercises to the reader. Also, we remark that Theorem 3 follows from Theorem 2 by simple observation. We start with the next lemma, which can be shown analogously to Lemma 1.

Lemma 2 *Let m, r be nonnegative integers. Then the following identities hold true*:

$$\sum_{i_1+\cdots+i_{r+1}=m} V_{i_1}(x)\cdots V_{i_{r+1}}(x) = \frac{1}{2^r r!}\sum_{j=0}^{m}(-1)^j\binom{r+1}{j}U_{m-j+r}^{(r)}(x), \tag{73}$$

$$\sum_{i_1+\cdots+i_{r+1}=m} W_{i_1}(x)\cdots W_{i_{r+1}}(x) = \frac{1}{2^r r!}\sum_{j=0}^{m}\binom{r+1}{j}U_{m-j+r}^{(r)}(x), \tag{74}$$

where $\binom{r+1}{j}=0$ *for* $j>r+1$.

With $\beta_{m,r}(x)$ as in (38), let us put

$$\beta_{m,r}(x) = \sum_{k=0}^{m} C_{k,2}L_k^{\alpha}(x). \tag{75}$$

First, we note from (59) that

$$U_{m-j+r}^{(r+k)}(x) = \sum_{l=0}^{[\frac{m-j-k}{2}]}(-1)^l\binom{m-j+r-l}{l}(m-j+r-2l)_{r+k}2^{m-j+r-2l}x^{m-j-k-2l}. \tag{76}$$

Then, from (b) of Proposition 1, (73), (76), and integration by parts k times, we get

$$\begin{aligned}
C_{k,2}(x) &= \frac{1}{\Gamma(\alpha+k+1)}\int_0^\infty \beta_{m,r}(x)\frac{d^k}{dx^k}\left(e^{-x}x^{k+\alpha}\right)dx \\
&= \frac{1}{\Gamma(\alpha+k+1)2^r r!}\sum_{j=0}^{m}(-1)^j\binom{r+1}{j}\int_0^\infty U_{m-j+r}^{(r)}(x)\frac{d^k}{dx^k}\left(e^{-x}x^{k+\alpha}\right)dx \qquad (77)\\
&= \frac{1}{\Gamma(\alpha+k+1)2^r r!}\sum_{j=0}^{m}(-1)^j\binom{r+1}{j}(-1)^k\int_0^\infty U_{m-j+r}^{(r+k)}(x)e^{-x}x^{k+\alpha}\,dx \\
&= \frac{1}{\Gamma(\alpha+k+1)2^r r!}\sum_{j=0}^{m}(-1)^j\binom{r+1}{j}(-1)^k
\end{aligned}$$

$$\times \sum_{l=0}^{[\frac{m-j-k}{2}]} (-1)^l \binom{m-j+r-l}{l} (m-j+r-2l)_{r+k}$$
$$\times 2^{m-j+r-2l} \Gamma(m+\alpha-j-2l+1).$$

Now, from (75) and (77) and using (7), we have

$$\begin{aligned}
\beta_{m,r}(x) &= \frac{2^m}{r!} \sum_{k=0}^{m} \frac{(-1)^k}{\Gamma(\alpha+k+1)} \sum_{l=0}^{[\frac{m-k}{2}]} \frac{(-\frac{1}{4})^l \Gamma(m+\alpha-2l+1)}{l!} \\
&\quad \times \sum_{j=0}^{m-k-2l} \frac{(-\frac{1}{2})^j (m+r-l-j)!(r+1)_j}{j!(m-k-2l-j)!(m+\alpha-2l)_j} L_k^{\alpha}(x) \\
&= \frac{2^m}{r!} \sum_{k=0}^{m} \frac{(-1)^k}{\Gamma(\alpha+k+1)} \sum_{l=0}^{[\frac{m-k}{2}]} \frac{(-\frac{1}{4})^l (m+r-l)!\Gamma(m+\alpha-2l+1)}{l!(m-k-2l)!} \\
&\quad \times \sum_{j=0}^{m-k-2l} \frac{(-\frac{1}{2})^j \langle 2l+k-m\rangle_j \langle -r-1\rangle_j}{j!\langle l-m-r\rangle_j \langle 2l-m-\alpha\rangle_j} L_k^{\alpha}(x) \\
&= \frac{2^m}{r!} \sum_{k=0}^{m} \frac{(-1)^k}{\Gamma(\alpha+k+1)} \sum_{l=0}^{[\frac{m-k}{2}]} \frac{(-\frac{1}{4})^l (m+r-l)!\Gamma(m+\alpha-2l+1)}{l!(m-k-2l)!} \\
&\quad \times {}_2F_2\left(2l+k-m, -r-1; l-m-r, 2l-m-\alpha; -\frac{1}{2}\right) L_k^{\alpha}(x).
\end{aligned} \tag{78}$$

This shows (41) of Theorem 2.

Next, we let

$$\beta_{m,r}(x) = \sum_{k=0}^{m} C_{k,4} C_k^{(\lambda)}(x). \tag{79}$$

Then, from (d) of Proposition 1, (73), (76), and integration by parts k times, we obtain

$$\begin{aligned}
C_{k,4} &= \frac{(k+\lambda)\Gamma(\lambda)}{(-2)^k \sqrt{\pi}\,\Gamma(k+\lambda+\frac{1}{2}) 2^r r!} \\
&\quad \times \sum_{j=0}^{m} (-1)^j \binom{r+1}{j} \int_{-1}^{1} U_{m-j+r}^{(r)}(x) \frac{d^k}{dx^k} \left(1-x^2\right)^{k+\lambda-\frac{1}{2}} dx \\
&= \frac{(k+\lambda)\Gamma(\lambda)}{2^k \sqrt{\pi}\,\Gamma(k+\lambda+\frac{1}{2}) 2^r r!} \\
&\quad \times \sum_{j=0}^{m-k} (-1)^j \binom{r+1}{j} \int_{-1}^{1} U_{m-j+r}^{(r+k)}(x) \left(1-x^2\right)^{k+\lambda-\frac{1}{2}} dx \\
&= \frac{(k+\lambda)\Gamma(\lambda)}{2^k \sqrt{\pi}\,\Gamma(k+\lambda+\frac{1}{2}) 2^r r!} \\
&\quad \times \sum_{j=0}^{m-k} (-1)^j \binom{r+1}{j} \sum_{l=0}^{[\frac{m-k-j}{2}]} (-1)^l \binom{m-j+r-l}{l} (m-j+r-2l)_{r+k} \\
&\quad \times 2^{m-j+r-2l} \int_{-1}^{1} x^{m-j-k-2l} \left(1-x^2\right)^{k+\lambda-\frac{1}{2}} dx.
\end{aligned} \tag{80}$$

From (c) of Proposition 2, we note that

$$\int_{-1}^{1} x^{m-j-k-2l}\left(1-x^2\right)^{k+\lambda-\frac{1}{2}}\,dx = \begin{cases} 0, & \text{if } j \not\equiv m-k \pmod 2, \\ \frac{\Gamma(k+\lambda+\frac{1}{2})\Gamma(\frac{m-j-k}{2}-l+\frac{1}{2})}{\Gamma(k+\lambda+\frac{m-j-k}{2}-l+1)}, & \text{if } j \equiv m-k \pmod 2. \end{cases} \tag{81}$$

By (79)–(81), and after some simplifications, we have

$$\begin{aligned} \beta_{m,r}(x) &= \frac{2^m\Gamma(\lambda)}{\sqrt{\pi}\,r!}\sum_{k=0}^{m}\frac{(k+\lambda)}{2^k}\sum_{\substack{0\le j\le m-k \\ j\equiv m-k \pmod 2}}\frac{(-\frac{1}{2})^j(r+1)!}{j!(r+1-j)!} \\ &\quad\times\sum_{l=0}^{[\frac{m-k-j}{2}]}\frac{(-\frac{1}{4})^l(m-j+r-l)!\,\Gamma(\frac{m-k-j}{2}-l+\frac{1}{2})}{l!(m-j-k-2l)!\,\Gamma(k+\lambda+\frac{m-k-j}{2}-l+1)}C_k^{(\lambda)}(x) \\ &= \frac{(-1)^m(r+1)\Gamma(\lambda)}{\sqrt{\pi}}\sum_{k=0}^{m}(-1)^k(k+\lambda) \\ &\quad\times\sum_{s=0}^{[\frac{m-k}{2}]}\frac{4^s}{(m-k-2s)!(r+1-m+k+2s)!} \\ &\quad\times\sum_{l=0}^{s}\frac{(-\frac{1}{4})^l(k+2s+r-l)!\,\Gamma(s-l+\frac{1}{2})}{l!(2s-2l)!\,\Gamma(k+\lambda+s+1-l)}C_k^{(\lambda)}(x) \\ &= (-1)^m(r+1)\Gamma(\lambda)\sum_{k=0}^{m}(-1)^k(k+\lambda) \\ &\quad\times\sum_{s=0}^{[\frac{m-k}{2}]}\frac{(k+2s+r)!}{s!(m-k-2s)!(r+1-m+k+2s)!\,\Gamma(k+\lambda+s+1)} \\ &\quad\times\sum_{l=0}^{s}\frac{\langle -s\rangle_l\langle -k-\lambda-s\rangle_l}{l!\langle -k-2s-r\rangle_l}C_k^{(\lambda)}(x) \\ &= (-1)^m(r+1)\Gamma(\lambda)\sum_{k=0}^{m}(-1)^k(k+\lambda) \\ &\quad\times\sum_{s=0}^{[\frac{m-k}{2}]}\frac{(k+2s+r)!}{s!(m-k-2s)!(r+1-m+k+2s)!\,\Gamma(k+\lambda+s+1)} \\ &\quad\times {}_2F_1(-s,-k-\lambda-s;-k-2s-r;1)C_k^{(\lambda)}(x). \end{aligned} \tag{82}$$

This finishes up the proof for (51) in Theorem 2.

Finally, we remark here that the identities (53)–(57) in Theorem 3 follow from those (48)–(52) in Theorem 2. For this purpose, we note from (73) and (74) that the only difference between $\beta_{m,r}(x)$ and $\gamma_{m,r}(x)$ (see (38), (39)) are the alternating sign $(-1)^j$ in their sums. This amounts to multiplying (48), (50), (52) by $(-1)^{m-k}$, and (51) by $(-1)^j$, and replacing ${}_2F_2(-;-;-\frac{1}{2})$ in (49) by ${}_2F_2(-;-;\frac{1}{2})$.

4 Conclusion

In this paper, we studied the classical linearization problem, determining the coefficients in the expansion of the product of two polynomials in terms of any given sequence of polynomials. Considering sums of finite products of Chebyshev polynomials of the first, third, and fourth kinds, we have represented each of them as linear combinations of Hermite, extended Laguerre, Legendre, Gegenbauer, and Jacobi polynomials by explicit computations. Also, it is shown that some terminating hypergeometric functions ${}_2F_1$, ${}_2F_2$, and ${}_1F_1$ appear in the coefficients of the combinations.

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