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On a coupled system of fractional differential equations with nonlocal non-separated boundary conditions

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Abstract

We solve a coupled system of nonlinear fractional differential equations equipped with coupled fractional nonlocal non-separated boundary conditions by using the Banach contraction principle and the Leray–Schauder fixed point theorem. Finally, we give examples to demonstrate our results.

MSC: 34A08; 34B15

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1 Introduction

In last few years, some physical phenomena were described through fractional differential equations and compared with integer order differential equations which have better results, which is why researchers of different areas have paid great attention to study fractional differential equation. Fractional differential equations arise in the mathematical modeling of systems and processes occurring in many engineering and scientific disciplines such as physics, chemistry, polymer rheology, control theory, diffusive transport akin to diffusion, electrical networks, probability, etc. For details, see [14–16, 22, 23, 27, 29, 37]. In the last few decades, fractional-order differential equations equipped with a variety of boundary conditions have been studied. The literature on the topic includes the existence and uniqueness results related to classical, periodic/anti-periodic, nonlocal, multi-point, and integral boundary conditions; for instance, see [1, 3, 6, 7, 9, 11, 17, 19, 24, 26, 28, 30, 34, 36] and the references therein.

The existence and uniqueness of positive solutions for such problems have become an important area of investigation in recent years. Ahmad and Nieto [6] investigated the existence and uniqueness of solutions for an anti-periodic fractional boundary value problem

$$\begin{aligned} &{}^cD^q x(t) = f\big(t, x(t)\big), \quad t \in [0,T], 1 < q \le 2, T > 0, \\ &x(0) = -x(T), \qquad {}^cD^p x(0) = -{}^cD^p x(T), \quad 0 < p < 1, \end{aligned}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order q, f is a given continuous function.

Liu and Liu [24] investigated the existence and uniqueness of solutions for fractional differential equations with fractional non-separated boundary conditions in the form of

$$\begin{aligned}&{}^cD^{\alpha}x(t)=f\big(t,x(t)\big),\quad t\in[0,T],1<\alpha\leq 2,T>0,\\&a_1x(0)+b_1x(T)=c_1,\qquad a_2\big({}^cD^{\gamma}x(0)\big)+b_2\big({}^cD^{\gamma}x(T)\big)=c_2,\quad 0<\gamma<1,\end{aligned}$$

where ${}^cD^{\alpha}$ denotes the Caputo fractional derivative of order α, f is a continuous function on $[0,T]\times\mathbb{R}$ and a_i, b_i, c_i, $i=1,2$ are real constants such that $a_1+b_1\neq 0$ and $b_2\neq 0$.

The system of fractional differential equations boundary value problems has also received much attention and its research has developed very rapidly; see [2, 4, 5, 8, 10, 12, 13, 20, 21, 25, 31–33, 35]. Recently, Alsulalt et al. [13] established the existence and uniqueness results for a nonlinear coupled system of Caputo type fractional differential equations supplemented with non-separated coupled boundary conditions.

In this paper, motivated by the aforementioned work, we consider the existence and uniqueness of solutions for a coupled system of fractional differential equation

$$\begin{cases}{}^cD^{\alpha}_{0^+}u(t)=f(t,u(t),v(t)),\quad 0<t<1,\\{}^cD^{\beta}_{0^+}v(t)=g(t,u(t),v(t)),\quad 0<t<1,\end{cases}\tag{1}$$

subject to the fractional non-separated coupled boundary conditions

$$\begin{cases}u(0)=\lambda_1v(1),\qquad {}^cD^{\gamma}_{0^+}u(1)=\lambda_2{}^cD^{\gamma}_{0^+}v(\xi),\quad 0<\gamma<1,\\v(0)=\mu_1u(1),\qquad {}^cD^{\gamma}_{0^+}v(1)=\mu_2{}^cD^{\gamma}_{0^+}u(\xi),\quad 0<\gamma<1,\end{cases}\tag{2}$$

where $\alpha,\beta\in(1,2]$, $\xi\in(0,1)$, ${}^cD^{\alpha}_{0^+}$ and ${}^cD^{\beta}_{0^+}$ are the Caputo fractional derivatives of order α and β, respectively, $f,g\in C([0,1]\times[0,+\infty)\times[0,+\infty),[0,+\infty))$ and λ_i, μ_i, $i=1,2$ real constants with $\mu_1\lambda_1\neq 1$ and $\mu_2\lambda_2\xi^{2(1-\gamma)}\neq 1$.

This paper is organized as follows. In Sect. 2, we recall some basic definitions of fractional calculus and present an auxiliary lemma, which plays a major role in obtaining the main result. In Sect. 3, we established the existence and uniqueness results for a nonlinear coupled system of fractional differential equation (1)–(2). Finally, as an application, we give two examples to illustrate our results.

2 Preliminaries

Let us now recall some basic definitions of fractional derivative [37] and prove a lemma before stating our main results.

Definition 2.1 The fractional integral of order q with the lower limit zero for a function $f:[0,\infty)\to\mathbb{R}$ is defined as

$$I^qf(t)=\frac{1}{\Gamma(q)}\int_0^t\frac{f(s)}{(t-s)^{1-q}}\,ds,\quad t>0,q>0,$$

provided the right hand side is point-wise defined on $[0,\infty)$, where $\Gamma(\cdot)$ is the gamma function, which is defined by $\Gamma(q)=\int_0^{\infty}t^{q-1}e^{-t}\,dt$.

Definition 2.2 The Riemann–Liouville fractional derivative of order $q > 0$, $n - 1 < q < n$, $n \in \mathbb{N}$, is defined as

$$D_{0^+}^q f(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-q-1} f(s)\,ds,$$

where the function $f(t)$ has an absolutely continuous derivative up to order $(n - 1)$.

Definition 2.3 The Caputo derivative of order q for a function $f : [0,\infty) \to \mathbb{R}$ with $f(t) \in C^n[0,\infty)$ is defined by

$${}^cD^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{q+1-n}}\,ds = I^{n-q} f^{(n)}(t), \quad t > 0, n-1 < q < n.$$

Furthermore, we noted that the Riemann–Liouville fractional derivative of a constant is usually nonzero which can cause serious problems in real world applications. We have

$$\begin{aligned} {}^cD^q f(t) &= \frac{1}{\Gamma(n-q)} \int_0^t \frac{f^n(s)}{(t-s)^{q+1-n}}\,ds \\ &= D^q f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{\Gamma(k-q+1)} t^{k-q} \\ &= D^q \left[f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} t^k \right], \quad t > 0, n-1 < q < n. \end{aligned}$$

So, we preferred to use Caputo's definition which gives better results than those of Riemann–Liouville.

Lemma 2.1 *Let $\Delta = 1 - \lambda_2\mu_2\xi^{2(1-\gamma)} \neq 0$ and $\lambda_1\mu_1 \neq 1$. Let $\phi, \psi \in C([0,1],\mathbb{R})$. Then the solution of the linear fractional differential equations*:

$$\begin{cases} {}^cD_{0^+}^\alpha u(t) = \phi(t), & t \in [0,1], 1 < \alpha \leq 2, \\ {}^cD_{0^+}^\beta v(t) = \psi(t), & t \in [0,1], 1 < \beta \leq 2, \end{cases} \tag{3}$$

supplemented with the boundary conditions (2) *is given by*

$$\begin{aligned} u(t) = {}& \frac{\mu_2\Gamma(2-\gamma)}{\Delta} \left[\frac{\lambda_1(\mu_1\lambda_2\xi^{1-\gamma}+1)}{1-\lambda_1\mu_1} + \lambda_2\xi^{1-\gamma} t \right] A_3 \\ & - \frac{\Gamma(2-\gamma)}{\Delta} \left[\frac{\lambda_1(\mu_1\lambda_2\xi^{1-\gamma}+1)}{1-\lambda_1\mu_1} + \lambda_2\xi^{1-\gamma} t \right] B_3 \\ & + \frac{\lambda_2\Gamma(2-\gamma)}{\Delta} \left[\frac{\lambda_1(\mu_1+\mu_2\xi^{1-\gamma})}{1-\lambda_1\mu_1} + t \right] B_2 \\ & - \frac{\Gamma(2-\gamma)}{\Delta} \left[\frac{\lambda_1(\mu_1+\mu_2\xi^{1-\gamma})}{1-\lambda_1\mu_1} + t \right] A_2 \\ & + \frac{\lambda_1}{1-\mu_1\lambda_1}(\mu_1A_1 + B_1) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi(s)\,ds \end{aligned} \tag{4}$$

and

$$\begin{aligned} v(t) &= \frac{\mu_2 \Gamma(2-\gamma)}{\Delta}\left[\frac{\mu_1(\lambda_1+\lambda_2\xi^{1-\gamma})}{1-\lambda_1\mu_1}+t\right]A_3 \\ &\quad - \frac{\Gamma(2-\gamma)}{\Delta}\left[\frac{\mu_1(\lambda_1+\lambda_2\xi^{1-\gamma})}{1-\lambda_1\mu_1}+t\right]B_3 \\ &\quad + \frac{\lambda_2 \Gamma(2-\gamma)}{\Delta}\left[\frac{\mu_1(\lambda_1\mu_2\xi^{1-\gamma}+1)}{1-\lambda_1\mu_1}+\mu_2\xi^{1-\gamma}t\right]B_2 \\ &\quad - \frac{\Gamma(2-\gamma)}{\Delta}\left[\frac{\mu_1(\lambda_1\mu_2\xi^{1-\gamma}+1)}{1-\lambda_1\mu_1}+\mu_2\xi^{1-\gamma}t\right]A_2 \\ &\quad + \frac{\mu_1}{1-\mu_1\lambda_1}(A_1+\lambda_1 B_1) + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\psi(s)\,ds, \end{aligned} \tag{5}$$

where

$$\begin{aligned} A_1 &= \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\phi(s)\,ds, & B_1 &= \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\psi(s)\,ds, \\ A_2 &= \int_0^1 \frac{(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}\phi(s)\,ds, & B_2 &= \int_0^\xi \frac{(\xi-s)^{\beta-\gamma-1}}{\Gamma(\beta-\gamma)}\psi(s)\,ds, \\ A_3 &= \int_0^\xi \frac{(\xi-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}\phi(s)\,ds, & B_3 &= \int_0^1 \frac{(1-s)^{\beta-\gamma-1}}{\Gamma(\beta-\gamma)}\psi(s)\,ds. \end{aligned}$$

Proof It is well know [22] that the general solution of the fractional differential equations in (3) can be written by

$$u(t) = c_0 + c_1 t + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\phi(s)\,ds, \tag{6}$$

$$v(t) = d_0 + d_1 t + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\psi(s)\,ds, \tag{7}$$

where $c_i, d_i, i = 0, 1$ are arbitrary constants. Since

$$^cD^\gamma k = 0 \quad (k \text{ is a constant}), \qquad ^cD^\gamma t = \frac{t^{1-\gamma}}{\Gamma(2-\gamma)}, \qquad ^cD^\gamma I^\alpha y(t) = I^{\alpha-\gamma} y(t),$$

from (6) and (7), we have

$$\begin{aligned} ^cD^\gamma u(t) &= c_1 \frac{t^{1-\gamma}}{\Gamma(2-\gamma)} + \int_0^t \frac{(t-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}\phi(s)\,ds, \\ ^cD^\gamma v(t) &= d_1 \frac{t^{1-\gamma}}{\Gamma(2-\gamma)} + \int_0^t \frac{(t-s)^{\beta-\gamma-1}}{\Gamma(\beta-\gamma)}\psi(s)\,ds. \end{aligned}$$

Using the boundary conditions in (6), we have

$$\begin{aligned} u(0) = \lambda_1 v(1) \quad &\Rightarrow \quad c_0 = \lambda_1(d_0 + d_1 + B_1), \\ v(0) = \mu_1 u(1) \quad &\Rightarrow \quad d_0 = \mu_1(c_0 + c_1 + A_1). \end{aligned}$$

Using the boundary conditions in (7), we have

$$
\begin{aligned}
{}^cD^\gamma u(1) = \lambda_2 {}^cD^\gamma v(\xi) \quad &\Rightarrow \quad c_1 = \lambda_2 \xi^{1-\gamma} d_1 + \Gamma(2-\gamma)[\lambda_2 B_2 - A_2], \\
{}^cD^\gamma v(1) = \mu_2 {}^cD^\gamma u(\xi) \quad &\Rightarrow \quad d_1 = \mu_2 \xi^{1-\gamma} c_1 - \Gamma(2-\gamma)[\mu_2 A_3 - B_3].
\end{aligned}
$$

From the last two relations we find

$$
\begin{aligned}
c_1 &= \frac{\Gamma(2-\gamma)}{\Delta}\big[\lambda_2\mu_2\xi^{1-\gamma}A_3 - \lambda_2\xi^{1-\gamma}B_3 + \lambda_2 B_2 - A_2\big], \\
d_1 &= \frac{\Gamma(2-\gamma)}{\Delta}\big[\mu_2 A_3 - B_3 - \mu_2\xi^{1-\gamma}A_2 + \mu_2\lambda_2\xi^{1-\gamma}B_2\big].
\end{aligned}
$$

Substituting c_1 and d_1 in the first two relations, we find

$$
\begin{aligned}
c_0 &= \frac{\lambda_1}{1-\lambda_1\mu_1}[\mu_1 c_1 + \mu_1 A_1 + d_1 + B_1] \\
&= \frac{\lambda_1}{1-\lambda_1\mu_1}\bigg[\frac{\Gamma(2-\gamma)\mu_2(\mu_1\lambda_2\xi^{1-\gamma}+1)}{\Delta}A_3 - \frac{\Gamma(2-\gamma)(\mu_1\lambda_2\xi^{1-\gamma}+1)}{\Delta}B_3 \\
&\quad - \frac{\Gamma(2-\gamma)(\mu_1+\mu_2\xi^{1-\gamma})}{\Delta}A_2 + \frac{\Gamma(2-\gamma)\lambda_2(\mu_1+\mu_2\xi^{1-\gamma})}{\Delta}B_2 + \mu_1 A_1 + B_1\bigg]
\end{aligned}
$$

and

$$
\begin{aligned}
d_0 &= \frac{\mu_1}{1-\lambda_1\mu_1}[\lambda_1 d_1 + \lambda_1 B_1 + c_1 + A_1] \\
&= \frac{\mu_1}{1-\lambda_1\mu_1}\bigg[\frac{\Gamma(2-\gamma)\mu_2(\lambda_1+\lambda_2\xi^{1-\gamma})}{\Delta}A_3 - \frac{\Gamma(2-\gamma)(\lambda_1+\lambda_2\xi^{1-\gamma})}{\Delta}B_3 \\
&\quad - \frac{\Gamma(2-\gamma)(\lambda_1\mu_2\xi^{1-\gamma}+1)}{\Delta}A_2 + \frac{\Gamma(2-\gamma)\lambda_2(\mu_2\lambda_1\xi^{1-\gamma}+1)}{\Delta}B_2 + \lambda_1 B_1 + A_1\bigg].
\end{aligned}
$$

Inserting the values of $c_i, d_i, i = 0, 1$ in (6) and (7), we get solutions (4) and (5). The converse of the above proof is as follows.

For any $t \in [0, 1]$, taking the γ-fractional derivative for (4) and (5) yields

$$
\begin{aligned}
{}^cD^\gamma u(t) &= \frac{t^{1-\gamma}}{\Delta}\big[\mu_2\lambda_2\xi^{1-\gamma}A_3 - \lambda_2\xi^{1-\gamma}B_3 + \lambda_2 B_2 - A_2\big] \\
&\quad + \int_0^t \frac{(t-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}\phi(s)\,ds
\end{aligned}
$$

and

$$
\begin{aligned}
{}^cD^\gamma v(t) &= \frac{t^{1-\gamma}}{\Delta}\big[\mu_2 A_3 - B_3 + \lambda_2\mu_2\xi^{1-\gamma}B_2 - \mu_2\xi^{1-\gamma}A_2\big] \\
&\quad + \int_0^t \frac{(t-s)^{\beta-\gamma-1}}{\Gamma(\beta-\gamma)}\psi(s)\,ds.
\end{aligned}
$$

Checking the first boundary condition, we see that

$$
u(0) = \lambda_1 v(1), \qquad {}^cD^\gamma_{0^+} u(1) = \lambda_2 {}^cD^\gamma_{0^+} v(\xi).
$$

Moreover, in checking the second boundary condition we get

$$v(0) = \mu_1 u(1), \qquad {}^cD^{\gamma}_{0^+} v(1) = \mu_2 {}^cD^{\gamma}_{0^+} u(\xi).$$

Taking the α-fractional derivative and β-fractional derivative yields

$${}^cD^{\alpha}_{0^+} u(t) = \phi(t); \qquad {}^cD^{\beta}_{0^+} v(t) = \psi(t),$$

which is what we set out to prove. □

3 Main results

Let $X = \{u(t)|u(t) \in C([0,1],\mathbb{R})\}$ denote the Banach space of all continuous functions from $[0,1]$ into $\mathbb{R}$ equipped with the norm $\|u\| = \sup\{|u(t)|, t \in [0,1]\}$. Obviously, $(X, \|\cdot\|)$ is a Banach space. Then the product space $(X \times X, \|(u,v)\|)$ is also a Banach space equipped with the norm $\|(u,v)\| = \|u\| + \|v\|$.

In view of Lemma 2.1, we define the operator $Q : X \times X \to X \times X$ by

$$Q(u,v) = \bigl(Q_1(u,v), Q_2(u,v)\bigr).$$

Here

$$\begin{aligned} Q_1(u,v)(t) = {} & \frac{\mu_2 \Gamma(2-\gamma)}{\Delta}\left[\frac{\lambda_1(\mu_1\lambda_2\xi^{1-\gamma}+1)}{1-\lambda_1\mu_1} + \lambda_2\xi^{1-\gamma} t\right] A_{3f} \\ & - \frac{\Gamma(2-\gamma)}{\Delta}\left[\frac{\lambda_1(\mu_1\lambda_2\xi^{1-\gamma}+1)}{1-\lambda_1\mu_1} + \lambda_2\xi^{1-\gamma} t\right] B_{3g} \\ & - \frac{\Gamma(2-\gamma)}{\Delta}\left[\frac{\lambda_1(\mu_1+\mu_2\xi^{1-\gamma})}{1-\lambda_1\mu_1} + t\right] A_{2f} \\ & + \frac{\lambda_2\Gamma(2-\gamma)}{\Delta}\left[\frac{\lambda_1(\mu_1+\mu_2\xi^{1-\gamma})}{1-\lambda_1\mu_1} + t\right] B_{2g} \\ & + \frac{\lambda_1}{1-\mu_1\lambda_1}(\mu_1 A_{1f} + B_{1g}) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s,u(s),v(s)\bigr)\,ds \end{aligned}$$

and

$$\begin{aligned} Q_2(u,v)(t) = {} & \frac{\mu_2 \Gamma(2-\gamma)}{\Delta}\left[\frac{\mu_1(\lambda_1+\lambda_2\xi^{1-\gamma})}{1-\lambda_1\mu_1} + t\right] A_{3f} \\ & - \frac{\Gamma(2-\gamma)}{\Delta}\left[\frac{\mu_1(\lambda_1+\lambda_2\xi^{1-\gamma})}{1-\lambda_1\mu_1} + t\right] B_{3g} \\ & - \frac{\Gamma(2-\gamma)}{\Delta}\left[\frac{\mu_1(\lambda_1\mu_2\xi^{1-\gamma}+1)}{1-\lambda_1\mu_1} + \mu_2\xi^{1-\gamma} t\right] A_{2f} \\ & + \frac{\lambda_2\Gamma(2-\gamma)}{\Delta}\left[\frac{\mu_1(\lambda_1\mu_2\xi^{1-\gamma}+1)}{1-\lambda_1\mu_1} + \mu_2\xi^{1-\gamma} t\right] B_{2g} \\ & + \frac{\mu_1}{1-\mu_1\lambda_1}(\lambda_1 B_{1g} + A_{1f}) + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g\bigl(s,u(s),v(s)\bigr)\,ds. \end{aligned}$$

Here

$$A_{1f} = \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s,u(s),v(s)\bigr)\,ds, \qquad B_{1g} = \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} g\bigl(s,u(s),v(s)\bigr)\,ds,$$

$$A_{2f} = \int_0^1 \frac{(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} f\bigl(s,u(s),v(s)\bigr)\,ds, \qquad B_{2g} = \int_0^\xi \frac{(\xi-s)^{\beta-\gamma-1}}{\Gamma(\beta-\gamma)} g\bigl(s,u(s),v(s)\bigr)\,ds,$$
$$A_{3f} = \int_0^\xi \frac{(\xi-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} f\bigl(s,u(s),v(s)\bigr)\,ds, \qquad B_{3g} = \int_0^1 \frac{(1-s)^{\beta-\gamma-1}}{\Gamma(\beta-\gamma)} g\bigl(s,u(s),v(s)\bigr)\,ds.$$

We use the following notations for convenience:

$$\begin{aligned}
\sigma_1 ={}& \frac{|\mu_2|\Gamma(2-\gamma)}{|1-\lambda_2\mu_2\xi^{2(1-\gamma)}|}\left[\frac{|\lambda_1|(|\mu_1||\lambda_2|\xi^{1-\gamma}+1)}{|1-\lambda_1\mu_1|}+|\lambda_2|\xi^{1-\gamma}\right]\frac{\xi^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} \\
&+\frac{\Gamma(2-\gamma)}{|1-\lambda_2\mu_2\xi^{2(1-\gamma)}|}\left[\frac{|\lambda_1|(|\mu_1|+|\mu_2|\xi^{1-\gamma})}{|1-\lambda_1\mu_1|}+1\right]\frac{1}{\Gamma(\alpha-\gamma+1)} \\
&+\left[\frac{|\lambda_1||\mu_1|}{|1-\mu_1\lambda_1|}+1\right]\frac{1}{\Gamma(\alpha+1)}, \\
\sigma_2 ={}& \frac{|\lambda_2|\Gamma(2-\gamma)}{|1-\lambda_2\mu_2\xi^{2(1-\gamma)}|}\left[\frac{|\lambda_1|(|\mu_1|+|\mu_2|\xi^{1-\gamma})}{|1-\lambda_1\mu_1|}+1\right]\frac{\xi^{\beta-\gamma}}{\Gamma(\beta-\gamma+1)} \\
&+\frac{\Gamma(2-\gamma)}{|1-\lambda_2\mu_2\xi^{2(1-\gamma)}|}\left[\frac{|\lambda_1|(|\mu_1||\lambda_2|\xi^{1-\gamma}+1)}{|1-\lambda_1\mu_1|}+|\lambda_2|\xi^{1-\gamma}\right]\frac{1}{\Gamma(\beta-\gamma+1)} \\
&+\frac{|\lambda_1|}{|1-\mu_1\lambda_1|}\frac{1}{\Gamma(\beta+1)}, \\
\sigma_3 ={}& \frac{|\mu_2|\Gamma(2-\gamma)}{|1-\lambda_2\mu_2\xi^{2(1-\gamma)}|}\left[\frac{|\mu_1|(|\lambda_1|+|\lambda_2|)\xi^{1-\gamma})}{|1-\lambda_1\mu_1|}+1\right]\frac{\xi^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} \\
&+\frac{\Gamma(2-\gamma)}{|1-\lambda_2\mu_2\xi^{2(1-\gamma)}|}\left[\frac{|\mu_1|(|\lambda_1||\mu_2|\xi^{1-\gamma}+1)}{|1-\lambda_1\mu_1|}+|\mu_2|\xi^{1-\gamma}\right]\frac{1}{\Gamma(\alpha-\gamma+1)} \\
&+\frac{|\mu_1|}{|1-\mu_1\lambda_1|}\frac{1}{\Gamma(\alpha+1)}, \\
\sigma_4 ={}& \frac{|\lambda_2|\Gamma(2-\gamma)}{|1-\lambda_2\mu_2\xi^{2(1-\gamma)}|}\left[\frac{|\mu_1|(|\lambda_1||\mu_2|\xi^{1-\gamma}+1)}{|1-\lambda_1\mu_1|}+|\mu_2|\xi^{1-\gamma}\right]\frac{\xi^{\beta-\gamma}}{\Gamma(\beta-\gamma+1)} \\
&+\frac{\Gamma(2-\gamma)}{1-\lambda_2\mu_2\xi^{2(1-\gamma)}}\left[\frac{|\mu_1|(|\lambda_1|+|\lambda_2|\xi^{1-\gamma})}{|1-\lambda_1\mu_1}|+1\right]\frac{1}{\Gamma(\beta-\gamma+1)} \\
&+\left[\frac{|\lambda_1||\mu_1|}{|1-\mu_1\lambda_1|}+1\right]\frac{1}{\Gamma(\beta+1)}.
\end{aligned}$$

Now we are in a position to present our main results. The methods used to prove the existence and uniqueness solutions of boundary value problem (1)–(2) go via Banach's contraction principle.

Theorem 3.1 *Assume that*:

($H1$) $f,g:[0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ *are functions and there exist positive constants* l_1 *and* l_2 *such that*, *for all* $t\in[0,1]$ *and* $x_i,y_i\in\mathbb{R}$, $i=1,2$, *we have*

$$\begin{aligned}
\bigl|f(t,x_1,x_2)-f(t,y_1,y_2)\bigr| &\le l_1\bigl(|x_1-y_1|+|x_2-y_2|\bigr), \\
\bigl|g(t,x_1,x_2)-g(t,y_1,y_2)\bigr| &\le l_2\bigl(|x_1-y_1|+|x_2-y_2|\bigr).
\end{aligned}$$

If $(\sigma_1+\sigma_3)l_1+(\sigma_2+\sigma_4)l_2<1$ *then system* (1)–(2) *has a unique solution on* $[0,1]$.

Proof Define $\sup_{t\in[0,1]} f(t,0,0) = \rho_1 < \infty$ and $\sup_{t\in[0,1]} g(t,0,0) = \rho_2 < \infty$ and $r > 0$ such that

$$r > \frac{(\sigma_1+\sigma_3)\rho_1 + (\sigma_2+\sigma_4)\rho_2}{1-(\sigma_1+\sigma_3)l_1-(\sigma_2+\sigma_4)l_2}.$$

We show that $Q(B_r) \subset B_r$, where $B_r = \{(u,v) \in X \times X : \|(u,v)\| \le r\}$.

By assumption $(H1)$, for $(u,v) \in B_r$, $t \in [0,1]$, we have

$$\begin{aligned} \bigl|f\bigl(t,u(t),v(t)\bigr)\bigr| &\le \bigl|f\bigl(t,u(t),v(t)\bigr) - f(t,0,0)\bigr| + \bigl|f(t,0,0)\bigr| \\ &\le l_1\bigl(\bigl|u(t)\bigr| + \bigl|v(t)\bigr|\bigr) + \rho_1 \\ &\le l_1\bigl(\|u\| + \|v\|\bigr) + \rho_1 \le l_1 r + \rho_1, \end{aligned}$$

and $|g(t,u(t),v(t))| \le l_2(\|u\|+\|v\|) + \rho_2 \le l_2 r + \rho_2$, which leads to

$$\begin{aligned} &\bigl|Q_1(u,v)(t)\bigr| \\ &\quad\le \frac{|\mu_2|\Gamma(2-\gamma)}{|1-\lambda_2\mu_2\xi^{2(1-\gamma)}|}\biggl[\frac{|\lambda_1|(|\mu_1||\lambda_2|\xi^{1-\gamma}+1)}{|1-\lambda_1\mu_1|} + |\lambda_2|\xi^{1-\gamma}\biggr]\frac{\xi^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}(l_1 r + \rho_1) \\ &\qquad + \frac{\Gamma(2-\gamma)}{|1-\lambda_2\mu_2\xi^{2(1-\gamma)}|}\biggl[\frac{|\lambda_1|(|\mu_1||\lambda_2|\xi^{1-\gamma}+1)}{|1-\lambda_1\mu_1|} + |\lambda_2|\xi^{1-\gamma}\biggr]\frac{1}{\Gamma(\beta-\gamma+1)}(l_2 r + \rho_2) \\ &\qquad + \frac{\Gamma(2-\gamma)}{|1-\lambda_2\mu_2\xi^{2(1-\gamma)}|}\biggl[\frac{|\lambda_1|(|\mu_1|+|\mu_2|\xi^{1-\gamma})}{|1-\lambda_1\mu_1|} + 1\biggr]\frac{1}{\Gamma(\alpha-\gamma+1)}(l_1 r + \rho_1) \\ &\qquad + \frac{|\lambda_2|\Gamma(2-\gamma)}{|1-\lambda_2\mu_2\xi^{2(1-\gamma)}|}\biggl[\frac{|\lambda_1|(|\mu_1|+|\mu_2|\xi^{1-\gamma})}{|1-\lambda_1\mu_1|} + 1\biggr]\frac{\xi^{\beta-\gamma}}{\Gamma(\beta-\gamma+1)}(l_2 r + \rho_2) \\ &\qquad + \frac{|\lambda_1|}{|1-\mu_1\lambda_1|}\biggl[\frac{|\mu_1|}{\Gamma(\alpha+1)}(l_1 r + \rho_1) + \frac{1}{\Gamma(\beta+1)}(l_2 r + \rho_2)\biggr] + \frac{1}{\Gamma(\alpha+1)}(l_1 r + \rho_1). \end{aligned}$$

Hence,

$$\bigl\|Q_1(u,v)\bigr\| \le (\sigma_1 l_1 + \sigma_2 l_2)r + \sigma_1\rho_1 + \sigma_2\rho_2.$$

In the same way, we obtain

$$\bigl\|Q_2(u,v)\bigr\| \le (\sigma_3 l_1 + \sigma_4 l_2)r + \sigma_3\rho_1 + \sigma_4\rho_2.$$

Consequently,

$$\bigl\|Q(u,v)\bigr\| \le \bigl[(\sigma_1+\sigma_3)l_1 + (\sigma_2+\sigma_4)l_2\bigr]r + (\sigma_1+\sigma_3)\rho_1 + (\sigma_2+\sigma_4)\rho_2 \le r.$$

Now, for $(u_2,v_2),(u_1,v_1) \in X \times X$ and for any $t \in [0,1]$, we get

$$\begin{aligned} &\bigl|Q_1(u_2,v_2)(t) - Q_1(u_1,v_1)(t)\bigr| \\ &\quad\le \frac{|\mu_2|\Gamma(2-\gamma)}{|1-\lambda_2\mu_2\xi^{2(1-\gamma)}|}\biggl[\frac{|\lambda_1|(|\mu_1||\lambda_2|\xi^{1-\gamma}+1)}{|1-\lambda_1\mu_1|} + |\lambda_2|\xi^{1-\gamma}\biggr] \\ &\qquad \times \frac{\xi^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} l_1\bigl(\|u_2-u_1\| + \|v_2-v_1\|\bigr) \end{aligned}$$

$$
\begin{aligned}
&+\frac{\Gamma(2-\gamma)}{|1-\lambda_2\mu_2\xi^{2(1-\gamma)}|}\left[\frac{|\lambda_1|(|\mu_1||\lambda_2|\xi^{1-\gamma}+1)}{|1-\lambda_1\mu_1|}+|\lambda_2|\xi^{1-\gamma}\right]\\
&\times\frac{1}{\Gamma(\beta-\gamma+1)}l_2\bigl(\|u_2-u_1\|+\|v_2-v_1\|\bigr)\\
&+\frac{\Gamma(2-\gamma)}{|1-\lambda_2\mu_2\xi^{2(1-\gamma)}|}\left[\frac{|\lambda_1|(|\mu_1|+|\mu_2|\xi^{1-\gamma})}{|1-\lambda_1\mu_1|}+1\right]\\
&\times\frac{1}{\Gamma(\alpha-\gamma+1)}l_1\bigl(\|u_2-u_1\|+\|v_2-v_1\|\bigr)\\
&+\frac{|\lambda_2|\Gamma(2-\gamma)}{|1-\lambda_2\mu_2\xi^{2(1-\gamma)}|}\left[\frac{|\lambda_1|(|\mu_1|+|\mu_2|\xi^{1-\gamma})}{|1-\lambda_1\mu_1|}+1\right]\\
&\times\frac{\xi^{\beta-\gamma}}{\Gamma(\beta-\gamma+1)}l_2\bigl(\|u_2-u_1\|+\|v_2-v_1\|\bigr)\\
&+\frac{|\lambda_1|}{|1-\mu_1\lambda_1|}\frac{1}{\Gamma(\beta+1)}l_2\bigl(\|u_2-u_1\|+\|v_2-v_1\|\bigr)\\
&+\frac{|\lambda_1||\mu_1|}{|1-\mu_1\lambda_1|}\frac{1}{\Gamma(\alpha+1)}l_1\bigl(\|u_2-u_1\|+\|v_2-v_1\|\bigr)\\
&+\frac{1}{\Gamma(\alpha+1)}l_1\bigl(\|u_2-u_1\|+\|v_2-v_1\|\bigr)\\
=&(\sigma_1 l_1+\sigma_2 l_2)\bigl(\|u_2-u_1\|+\|v_2-v_1\|\bigr),
\end{aligned}
$$

and consequently we obtain

$$
\bigl\|Q_1(u_2,v_2)(t)-Q_1(u_1,v_1)(t)\bigr\|\le(\sigma_1 l_1+\sigma_1 l_2)\bigl(\|u_2-u_1\|+\|v_2-v_1\|\bigr). \tag{8}
$$

Similarly,

$$
\bigl\|Q_2(u_2,v_2)(t)-Q_2(u_1,v_1)(t)\bigr\|\le(\sigma_3 l_1+\sigma_4 l_2)\bigl(\|u_2-u_1\|+\|v_2-v_1\|\bigr). \tag{9}
$$

It follows from (8) and (9) that

$$
\bigl\|Q(u_2,v_2)(t)-Q(u_1,v_1)\bigr\|\le\bigl[(\sigma_1+\sigma_3)l_1+(\sigma_2+\sigma_4)l_2\bigr]\bigl(\|u_2-u_1\|+\|v_2-v_1\|\bigr).
$$

Since $(\sigma_1+\sigma_3)l_1+(\sigma_2+\sigma_4)l_2<1$, Q is a contraction operator. So, by Banach's fixed point theorem, the operator Q has a unique fixed point, which is the unique solution of problem (1)–(2). □

The second result is based on the Leray–Schauder alternative.

Lemma 3.2 (Leray–Schauder alternative [18]) *Let $F: E\to E$ be a completely continuous operator (i.e., a map restricted to any bounded set in E is compact). Let*

$$
\varepsilon(F)=\bigl\{x\in E: x=\lambda F(x)\ \textit{for some}\ 0<\lambda<1\bigr\}.
$$

Then either the set $\varepsilon(F)$ is unbounded or F has at least one fixed point.

Theorem 3.3 *Assume that*:

($H2$) $f, g : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *are continuous functions and there exist real constants* $k_i, \gamma_i \geq 0$ $(i = 0,1,2)$ *and* $k_0 > 0$, $\gamma_0 > 0$ *such that* $\forall x_i \in \mathbb{R}$ $(i = 1,2)$, *we have*

$$\bigl|f(t,x_1,x_2)\bigr| \leq k_0 + k_1|x_1| + k_2|x_2|,$$

$$\bigl|g(t,x_1,x_2)\bigr| \leq \gamma_0 + \gamma_1|x_1| + \gamma_2|x_2|.$$

If $(\sigma_1 + \sigma_3)\kappa_1 + (\sigma_2 + \sigma_4)\gamma_1 < 1$ *and* $(\sigma_1 + \sigma_3)\kappa_2 + (\sigma_2 + \sigma_4)\gamma_2 < 1$ *then system* (1)–(2) *has at least one solution on* $[0,1]$.

Proof First we show that the operator $Q : X \times X \to X \times X$ is completely continuous. By the continuity of functions f and g, the operator Q is continuous.

Let $\Omega \subset X \times X$ be bounded. Then there exist positive constants K_1 and K_2 such that $|f(t,u(t).v(t))| \leq K_1$, $|g(t,u(t).v(t))| \leq K_2$, $\forall (u,v) \in \Omega$. Then, for any $(u,v) \in \Omega$, we have

$$\begin{aligned}
&\bigl|Q_1(u,v)(t)\bigr| \\
&\quad\leq \frac{|\mu_2|\Gamma(2-\gamma)}{|1-\lambda_2\mu_2\xi^{2(1-\gamma)}|}\left[\frac{|\lambda_1|(|\mu_1||\lambda_2|\xi^{1-\gamma}+1)}{|1-\lambda_1\mu_1|} + |\lambda_2|\xi^{1-\gamma}\right]\frac{\xi^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}K_1 \\
&\qquad+ \frac{\Gamma(2-\gamma)}{|1-\lambda_2\mu_2\xi^{2(1-\gamma)}|}\left[\frac{|\lambda_1|(|\mu_1||\lambda_2|\xi^{1-\gamma}+1)}{|1-\lambda_1\mu_1|} + |\lambda_2|\xi^{1-\gamma}\right]\frac{1}{\Gamma(\beta-\gamma+1)}K_2 \\
&\qquad+ \frac{\Gamma(2-\gamma)}{|1-\lambda_2\mu_2\xi^{2(1-\gamma)}|}\left[\frac{|\lambda_1|(|\mu_1|+|\mu_2|\xi^{1-\gamma})}{|1-\lambda_1\mu_1|} + 1\right]\frac{1}{\Gamma(\alpha-\gamma+1)}K_1 \\
&\qquad+ \frac{|\lambda_2|\Gamma(2-\gamma)}{|1-\lambda_2\mu_2\xi^{2(1-\gamma)}|}\left[\frac{|\lambda_1|(|\mu_1|+|\mu_2|\xi^{1-\gamma})}{|1-\lambda_1\mu_1|} + 1\right]\frac{\xi^{\beta-\gamma}}{\Gamma(\beta-\gamma+1)}K_2 \\
&\qquad+ \frac{|\lambda_1|}{|1-\mu_1\lambda_1|}\left[\frac{|\mu_1|}{\Gamma(\alpha+1)}K_1 + \frac{1}{\Gamma(\beta+1)}K_2\right] + \frac{1}{\Gamma(\alpha+1)}K_1,
\end{aligned}$$

which implies that $\|Q_1(u,v)\| \leq \sigma_1 K_1 + \sigma_2 K_2$. Similarly, we get $\|Q_2(u,v)\| \leq \sigma_3 K_1 + \sigma_4 K_2$. Thus, it follows from the above inequalities that the operator Q is uniformly bounded, since $\|Q(u,v)\| \leq (\sigma_1 + \sigma_3)K_1 + (\sigma_2 + \sigma_4)K_2$.

Next, we show that Q is equicontinuous. Let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$. Then we have

$$\begin{aligned}
&\bigl|Q_1\bigl(u(t_2),v(t_2)\bigr) - Q_1\bigl(u(t_1),v(t_1)\bigr)\bigr| \\
&\quad\leq K_1\frac{\xi^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}\frac{|\mu_2||\lambda_2|\xi^{1-\gamma}\Gamma(2-\gamma)}{|1-\lambda_2\mu_2\xi^{2(1-\gamma)}|}(t_2-t_1) \\
&\qquad+ K_2\frac{1}{\Gamma(\beta-\gamma+1)}\frac{|\lambda_2|\xi^{1-\gamma}\Gamma(2-\gamma)}{|1-\lambda_2\mu_2\xi^{2(1-\gamma)}|}(t_2-t_1) \\
&\qquad+ K_1\frac{1}{\Gamma(\alpha-\gamma+1)}\frac{\Gamma(2-\gamma)}{|1-\lambda_2\mu_2\xi^{2(1-\gamma)}|}(t_2-t_1) \\
&\qquad+ K_2\frac{\xi^{\beta-\gamma}}{\Gamma(\beta-\gamma+1)}\frac{|\lambda_2|\Gamma(2-\gamma)}{|1-\lambda_2\mu_2\xi^{2(1-\gamma)}|}(t_2-t_1) \\
&\qquad+ K_1\left|\frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}\,ds - \frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}\,ds\right| \\
&\quad\leq K_1\frac{\xi^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}\frac{|\mu_2||\lambda_2|\xi^{1-\gamma}\Gamma(2-\gamma)}{|1-\lambda_2\mu_2\xi^{2(1-\gamma)}|}(t_2-t_1)
\end{aligned}$$

$$
\begin{aligned}
&+ K_2 \frac{1}{\Gamma(\beta-\gamma+1)} \frac{|\lambda_2|\xi^{1-\gamma}\Gamma(2-\gamma)}{|1-\lambda_2\mu_2\xi^{2(1-\gamma)}|}(t_2-t_1) \\
&+ K_1 \frac{1}{\Gamma(\alpha-\gamma+1)} \frac{\Gamma(2-\gamma)}{|1-\lambda_2\mu_2\xi^{2(1-\gamma)}|}(t_2-t_1) \\
&+ K_2 \frac{\xi^{\beta-\gamma}}{\Gamma(\beta-\gamma+1)} \frac{|\lambda_2|\Gamma(2-\gamma)}{|1-\lambda_2\mu_2\xi^{2(1-\gamma)}|}(t_2-t_1) \\
&+ K_1 \left| \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\,ds \right| + K_1 \left| \int_0^{t_1} \frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}\,ds \right| \\
\le{}& K_1 \frac{\xi^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} \frac{|\mu_2||\lambda_2|\xi^{1-\gamma}\Gamma(2-\gamma)}{|1-\lambda_2\mu_2\xi^{2(1-\gamma)}|}(t_2-t_1) \\
&+ K_2 \frac{1}{\Gamma(\beta-\gamma+1)} \frac{|\lambda_2|\xi^{1-\gamma}\Gamma(2-\gamma)}{|1-\lambda_2\mu_2\xi^{2(1-\gamma)}|}(t_2-t_1) \\
&+ K_1 \frac{1}{\Gamma(\alpha-\gamma+1)} \frac{\Gamma(2-\gamma)}{|1-\lambda_2\mu_2\xi^{2(1-\gamma)}|}(t_2-t_1) \\
&+ K_2 \frac{\xi^{\beta-\gamma}}{\Gamma(\beta-\gamma+1)} \frac{|\lambda_2|\Gamma(2-\gamma)}{|1-\lambda_2\mu_2\xi^{2(1-\gamma)}|}(t_2-t_1) + K_1 \frac{(t_2^\alpha - t_1^\alpha)}{\Gamma(\alpha+1)}.
\end{aligned}
$$

Analogously, we can obtain

$$
\begin{aligned}
&\bigl|Q_2\bigl(u(t_2),v(t_2)\bigr) - Q_2\bigl(u(t_1),v(t_1)\bigr)\bigr| \\
&\quad\le K_1 \frac{\xi^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} \frac{|\mu_2|\Gamma(2-\gamma)}{|1-\lambda_2\mu_2\xi^{2(1-\gamma)}|}(t_2-t_1) \\
&\qquad+ K_2 \frac{1}{\Gamma(\beta-\gamma+1)} \frac{\Gamma(2-\gamma)}{|1-\lambda_2\mu_2\xi^{2(1-\gamma)}|}(t_2-t_1) \\
&\qquad+ K_1 \frac{1}{\Gamma(\alpha-\gamma+1)} \frac{|\mu_2|\xi^{1-\gamma}\Gamma(2-\gamma)}{|1-\lambda_2\mu_2\xi^{2(1-\gamma)}|}(t_2-t_1) \\
&\qquad+ K_2 \frac{\xi^{\beta-\gamma}}{\Gamma(\beta-\gamma+1)} \frac{|\mu_2||\lambda_2|\xi^{1-\gamma}\Gamma(2-\gamma)}{|1-\lambda_2\mu_2\xi^{2(1-\gamma)}|}(t_2-t_1) + K_2 \frac{(t_2^\beta - t_1^\beta)}{\Gamma(\beta+1)}.
\end{aligned}
$$

Then we can easily show that the operator $Q(u,v)$ is equicontinuous. As a consequence of steps together with the Arzela–Ascoli theorem, we find that the operator $Q(u,v)$ is completely continuous.

Finally, it will be verified that the set $\varepsilon = \{(u,v) \in X \times X | (u,v) = \lambda Q(u,v), 0 \le \lambda \le 1\}$ is bounded. Let $(u,v) \in \varepsilon$, with $(u,v) = \lambda Q(u,v)$. For any $t \in [0,1]$, we have

$$
u(t) = \lambda Q_1(u,v)(t), \qquad v(t) = \lambda Q_2(u,v)(t).
$$

Then

$$
\begin{aligned}
\bigl|u(t)\bigr| &\le \sigma_1\bigl(k_0 + k_1|u| + k_2|v|\bigr) + \sigma_2\bigl(\gamma_0 + \gamma_1|u| + \gamma_2|v|\bigr) \\
&= \sigma_1 k_0 + \sigma_2\gamma_0 + (\sigma_1 k_1 + \sigma_2\gamma_1)|u| + (\sigma_1 k_2 + \sigma_2\gamma_2)|v|
\end{aligned}
$$

and

$$
\begin{aligned}
\bigl|v(t)\bigr| &\le \sigma_3\bigl(k_0 + k_1|u| + k_2|v|\bigr) + \sigma_4\bigl(\gamma_0 + \gamma_1|u| + \gamma_2|v|\bigr) \\
&= \sigma_3 k_0 + \sigma_4\gamma_0 + (\sigma_3 k_1 + \sigma_4\gamma_1)|u| + (\sigma_3 k_2 + \sigma_4\gamma_2)|v|,
\end{aligned}
$$

which implies

$$\|u\| + \|v\| \le (\sigma_1 + \sigma_3)k_0 + (\sigma_2 + \sigma_4)\gamma_0 + \big[(\sigma_1 + \sigma_3)k_1 + (\sigma_2 + \sigma_4)\gamma_1\big]\|u\| + \big[(\sigma_1 + \sigma_3)k_2 + (\sigma_2 + \sigma_4)\gamma_2\big]\|v\|.$$

Consequently,

$$\big\|(u,v)\big\| \le \frac{(\sigma_1 + \sigma_3)k_0 + (\sigma_2 + \sigma_4)\gamma_0}{\sigma_0}$$

where

$$\sigma_0 = \min\big\{1 - \big[(\sigma_1 + \sigma_3)k_1 + (\sigma_2 + \sigma_4)\gamma_1\big], \qquad 1 - \big[(\sigma_1 + \sigma_3)k_2 + (\sigma_2 + \sigma_4)\gamma_2\big]\big\},$$

which proves that ε is bounded. Thus, by Lemma 3.2, the operator Q has at least one fixed point. Hence, the boundary value problem (1)–(2) has at least one solution. □

4 Some examples

In this section, we will present some examples to illustrate the main results.

Example 1 Consider the following system of fractional boundary value problem:

$$\begin{cases} {}^cD_{0^+}^{3/2}u(t) = \frac{e^t}{18} + \frac{\cos^2 u(t)}{9} + \frac{25(|v(t)|)}{10+|v(t)|}, & t \in (0,1), \\ {}^cD_{0^+}^{3/2}v(t) = \frac{1+\sin^2(u(t))}{\sqrt[3]{t(1-t)^2}} + \frac{|v(t)|}{18(1+|v(t)|)} + \frac{1}{4}, & t \in (0,1), \\ u(0) = \frac{1}{2}v(1), & {}^cD_{0^+}^{1/2}u(1) = \frac{1}{3}{}^cD_{0^+}^{1/2}v(1/4), \\ v(0) = \frac{3}{7}u(1), & {}^cD_{0^+}^{1/2}v(1) = \frac{2}{3}{}^cD_{0^+}^{1/2}u(1/4). \end{cases}$$

Here $\alpha = \beta = \frac{3}{2}$, $\gamma = \frac{1}{2}$, $\lambda_1 = \frac{1}{2}$, $\lambda_2 = \frac{1}{3}$, $\mu_1 = \frac{3}{7}$, $\mu_2 = \frac{2}{3}$, $\xi = \frac{1}{4}$. By simple calculation,, we found that $\sigma_1 = 2.755358$, $\sigma_2 = 2.847316$, $\sigma_3 = 2.10908$, $\sigma_4 = 2.49716$. Note that

$$\big|f(t,u_1,u_2) - f(t,v_1,v_2)\big| \le \frac{1}{18}|u_1 - u_2| + \frac{1}{18}|v_1 - v_2|,$$
$$\big|g(t,u_1,u_2) - g(t,v_1,v_2)\big| \le \frac{1}{18}|u_1 - u_2| + \frac{1}{18}|v_1 - v_2|,$$

and $(\sigma_1 + \sigma_3)l_1 + (\sigma_2 + \sigma_4)l_2 \approx 0.56716 < 1$. Thus all the conditions of Theorem 3.1 are satisfied.

Example 2 Consider the following fractional boundary value problem:

$$\begin{cases} {}^cD_{0^+}^{3/2}u(t) = \frac{1}{6(t+1)^2}u(t) + \frac{1}{32}\sin v(t) + 1, & t \in (0,1), \\ {}^cD_{0^+}^{3/2}v(t) = \frac{1}{64\pi}\sin(2\pi u(t)) + \frac{1}{32}v(t) + \frac{1}{2}, & t \in (0,1), \\ u(0) = \frac{1}{3}v(1), & {}^cD_{0^+}^{1/3}u(1) = \frac{1}{2}{}^cD_{0^+}^{1/3}v(1/5), \\ v(0) = \frac{3}{5}u(1), & {}^cD_{0^+}^{1/3}v(1) = \frac{2}{5}{}^cD_{0^+}^{1/3}u(1/5). \end{cases}$$

Here $\alpha = \beta = \frac{3}{2}$, $\gamma = \frac{1}{3}$, $\lambda_1 = \frac{1}{3}$, $\lambda_2 = \frac{1}{2}$, $\mu_1 = \frac{3}{5}$, $\mu_2 = \frac{2}{5}$, $\xi = \frac{1}{5}$. Note that

$$\left|f(t,x_1,x_2)\right| \le 1 + \frac{1}{32}|x_1| + \frac{1}{32}|x_2|,$$
$$\left|g(t,x_1,x_2)\right| \le \frac{1}{2} + \frac{1}{32}|x_1| + \frac{1}{32}|x_2|.$$

We get $k_1 = \frac{1}{32}$, $k_2 = \frac{1}{32}$, $\gamma_1 = \frac{1}{32}$, $\gamma_2 = \frac{1}{32}$. By simple calculation, we found that $\sigma_1 = 2.14850$, $\sigma_2 = 2.1845$, $\sigma_3 = 2.10890$, $\sigma_4 = 2.32131$, then we have $(\sigma_1+\sigma_3)k_1 + (\sigma_2+\sigma_4)\gamma_1 \approx 0.27386 < 1$ and $(\sigma_1+\sigma_3)k_2 + (\sigma_2+\sigma_4)\gamma_2 < 1$. By Theorem 3.3, the coupled boundary value problem has at least one positive solution.

5 Conclusions

In this work, we have established the existence and uniqueness results for a nonlinear coupled system of Caputo type fractional differential equations supplemented with coupled fractional nonlocal non-separated boundary conditions by using the Banach contraction principle and the Leray–Schauder fixed point theorem. Finally, we give examples to demonstrate our results.

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