Zi and Wang *Advances in Difference Equations* (2019) 2019:80
https://doi.org/10.1186/s13662-019-2016-5

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Positive solutions for Caputo fractional differential system with coupled boundary conditions

Yumei Zi[1] and Ying Wang[1*]

*Correspondence: lywy1981@163.com
[1]School of Mathematics and Statistics, Linyi University, Linyi, People's Republic of China

Abstract

This paper focuses on the Caputo fractional differential system involving coupled integral boundary conditions and parameters. Using the properties of the Green function, the Leray–Schauder's alternative and the Banach contraction principle, the existence and uniqueness results of the system are established. An example is then given to demonstrate the validity of the result.

MSC: 34A08; 34B18

Keywords: Existence; Uniqueness; Caputo fractional differential system; Coupled integral boundary conditions

1 Introduction

Fractional differential systems have been of great interest recently. This paper mainly presents the existence and uniqueness solution of the following general fractional differential system involving coupled integral boundary conditions and parameters:

$$\begin{cases} {}^{c}D^{\alpha_1}u(t) + \lambda_1 f_1(t,u(t),v(t)) = 0, \\ {}^{c}D^{\alpha_2}v(t) + \lambda_2 f_2(t,u(t),v(t)) = 0, \quad 0 < t < 1, \\ u'(0) = u''(0) = \cdots = u^{(n-1)}(0) = 0, \qquad u(1) = \mu_1 \int_0^1 a(s)v(s)\,dA_1(s), \\ v'(0) = v''(0) = \cdots = v^{(m-1)}(0) = 0, \qquad v(1) = \mu_2 \int_0^1 b(s)u(s)\,dA_2(s), \end{cases} \tag{1}$$

where $\lambda_i > 0$ is a parameter, $n-1 < \alpha_1 \le n$, $m-1 < \alpha_2 \le m$, $n,m \ge 2$, $D_{0+}^{\alpha_i}$ is the standard Caputo derivative; $\mu_i > 0$ is a constant, $\int_0^1 a(s)v(s)\,dA_1(s)$, $\int_0^1 b(s)u(s)\,dA_2(s)$ denote the Riemann–Stieltjes integral with a signed measure, that is, $A_i : [0,1] \to [0,+\infty)$ is the function of bounded variation; $a,b : [0,1] \to [0,+\infty)$ are continuous, $f_i : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ is a continuous function, $i = 1,2$.

In the mathematical context, fractional differential equations involving different boundary value conditions have aroused the interest of many scholars, see references [1–21] to name a few. Bai and Qiu [22] discussed the following nonlinear fractional differential equation with two-point boundary value conditions by using the Krasnoselskii's fixed point theorem:

$$\begin{cases} D_{0^+}^{\alpha} u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(1) = u''(0) = 0, \end{cases}$$

where $2 < \alpha \leq 3$, $D_{0^+}^{\alpha}$ is Caputo derivative.

Wang et al. [23] gave the existence and uniqueness results for the coupled fractional differential system

$$\begin{cases} D_{0^+}^{\alpha} u(t) + f(t, v(t)) = 0, \\ D_{0^+}^{\beta} v(t) + g(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = v(0) = 0, \qquad u(1) = au(\xi), \qquad v(1) = bv(\xi), \end{cases}$$

where $1 < \alpha, \beta < 2$, $0 \leq a, b < 1$, $0 < \xi < 1$, $D_{0^+}^{\alpha}$, $D_{0^+}^{\beta}$ are two standard Riemann–Liouville fractional derivatives, $f, g : [0,1] \times [0, +\infty) \to [0, +\infty)$ are continuous. The whole discussion was based on the Banach fixed point theorem and the nonlinear alternative of Leray–Schauder type.

Recently, Henderson and Luca in [24] considered the system of fractional differential equations

$$\begin{cases} D_{0^+}^{\alpha_1} u(t) + \lambda_1 f_1(t, u(t), v(t)) = 0, \\ D_{0^+}^{\alpha_2} v(t) + \lambda_2 f_2(t, u(t), v(t)) = 0, \quad 0 < t < 1, \end{cases} \tag{2}$$

with the multi-point boundary conditions

$$\begin{cases} u(0) = u'(0) = \cdots = u^{(n-2)} = 0, \qquad u(1) = \sum_{i=1}^{p} a_i u(\xi_i), \\ v(0) = v'(0) = \cdots = v^{(m-2)} = 0, \qquad v(1) = \sum_{i=1}^{q} b_i v(\eta_i), \end{cases}$$

where $n-1 < \alpha_1 \leq n$, $m-1 < \alpha_2 \leq m$, $n, m \geq 2$, $\lambda_i > 0$ is a parameter, $D_{0^+}^{\alpha_i}$, $D_{0^+}^{\beta_i}$ are Riemann–Liouville derivatives; $a_i > 0$, $b_i > 0$ are constants, $f_i : [0,1] \times [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is a continuous function. By the use of Krasnoselskii's fixed point theorem, the authors in [24] got the existence of positive solutions for the above system. System (2) with coupled boundary value conditions

$$\begin{cases} u(0) = u'(0) = \cdots = u^{(n-2)} = 0, \qquad u(1) = \mu_1 \int_0^1 v(s)\, dA_1(s), \\ v(0) = v'(0) = \cdots = v^{(m-2)} = 0, \qquad v(1) = \mu_2 \int_0^1 u(s)\, dA_2(s) \end{cases}$$

has also been discussed in [25, 26], where $\mu_i > 0$ is a constant.

Fractional differential systems involving derivatives with coupled boundary conditions have witnessed significant development, as shown by [27–30], but most of the authors considered the fractional equations with Riemann–Liouville derivatives. The equation discussed in this paper is exactly the Caputo fractional equation. The purpose of this paper is to investigate the existence and uniqueness of positive solutions for Caputo fractional differential systems with coupled integral boundary conditions. In this paper, the Caputo derivatives of orders α_1 and α_2 can be different, and in case $dA_1(s) = dA_2(s) = ds$ or $g(s)\,ds$, system (1) reduces to a multi-point boundary value problem as well.

2 Preliminaries and lemmas

Definition 2.1 ([31, 32]) The Caputo fractional order derivative of order $\alpha > 0$, $n-1 < \alpha < n$, $n \in \mathbb{N}$ is defined as

$$^{\mathrm{c}}D^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}u^{(n)}(s)\,ds,$$

where $u \in C^n(J,\mathbb{R})$, $\mathbb{R} = (-\infty,+\infty)$, $\mathbb{N}$ denotes the natural number set, $n = [\alpha] + 1$, and $[\alpha]$ denotes the integer part of α.

Definition 2.2 ([31, 32]) Let $\alpha > 0$ and let u be piecewise continuous on $(0,+\infty)$ and integrable on any finite subinterval of J. Then for $t > 0$, we call

$$I^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)\,ds,$$

the Riemann–Liouville fractional integral of u of order α.

Lemma 2.1 ([31, 32]) *Let $n-1 < \alpha \le n$, $u \in C^n[0,1]$. Then*

$$I^{\alpha}\bigl(^{\mathrm{c}}D^{\alpha}u\bigr)(t) = u(t) + c_0 + c_1t + c_2t^2 + \cdots + c_{n-1}t^{n-1},$$

where $c_i \in \mathbb{R}$ $(i = 1,2,\ldots,n-1)$, n is the smallest integer greater than or equal to α.

Lemma 2.2 *Assume the following condition* (H_0) *holds*:

(H_0)

$$k_1 = \int_0^1 a(t)\,dA_1(t) > 0, \qquad k_2 = \int_0^1 b(t)\,dA_2(t) > 0, \qquad 1-\mu_1\mu_2k_1k_2 > 0.$$

Let $h_i \in C(0,1)\cap L(0,1)$ $(i = 1,2)$. Then the system with the coupled boundary conditions

$$\begin{cases} ^{\mathrm{c}}D^{\alpha_1}u(t) + h_1(t) = 0, \qquad ^{\mathrm{c}}D^{\alpha_2}v(t) + h_2(t) = 0, \quad 0 < t < 1, \\ u'(0) = u''(0) = \cdots = u^{(n-1)}(0) = 0, \qquad u(1) = \mu_1\int_0^1 a(t)v(s)\,dA_1(s), \\ v'(0) = v''(0) = \cdots = v^{(m-1)}(0) = 0, \qquad v(1) = \mu_2\int_0^1 b(t)u(s)\,dA_2(s) \end{cases} \tag{3}$$

has a unique integral representation

$$\begin{cases} u(t) = \int_0^1 K_1(t,s)h_1(s)\,ds + \int_0^1 H_1(t,s)h_2(s)\,ds, \\ v(t) = \int_0^1 K_2(t,s)h_2(s)\,ds + \int_0^1 H_2(t,s)h_1(s)\,ds, \end{cases} \tag{4}$$

where

$$\begin{aligned} K_1(t,s) &= \frac{\mu_1\mu_2 k_1}{1-\mu_1\mu_2 k_1 k_2}\int_0^1 G_1(t,s)b(t)\,dA_2(t) + G_1(t,s), \\ H_1(t,s) &= \frac{\mu_1}{1-\mu_1\mu_2 k_1 k_2}\int_0^1 G_2(t,s)a(t)\,dA_1(t), \\ K_2(t,s) &= \frac{\mu_2\mu_1 k_2}{1-\mu_1\mu_2 k_1 k_2}\int_0^1 G_2(t,s)a(t)\,dA_1(t) + G_2(t,s), \\ H_2(t,s) &= \frac{\mu_2}{1-\mu_1\mu_2 k_1 k_2}\int_0^1 G_1(t,s)b(t)\,dA_2(t), \end{aligned} \tag{5}$$

and

$$G_i(t,s) = \begin{cases} \frac{(1-s)^{\alpha_i-1}-(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)}, & 0 \le s \le t \le 1, \\ \frac{(1-s)^{\alpha_i-1}}{\Gamma(\alpha_i)}, & 0 \le t \le s \le 1, \end{cases} \quad i = 1,2, \tag{6}$$

Proof By Lemma 2.1, system (3) is equivalent to the following integral equations:

$$u(t) = -\int_0^t \frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)} h_1(s)\,ds + c_1 + c_2 t + c_3 t^2 + \cdots + c_n t^{n-1}, \tag{7}$$

$$v(t) = -\int_0^t \frac{(t-s)^{\alpha_2-1}}{\Gamma(\alpha_2)} h_2(s)\,ds + \overline{c}_1 + \overline{c}_2 t + \overline{c}_3 t^2 + \cdots + \overline{c}_m t^{m-1}. \tag{8}$$

Conditions $u'(0) = u''(0) = \cdots = u^{(n-1)}(0) = 0$, $v'(0) = v''(0) = \cdots = v^{(m-1)}(0) = 0$ imply that

$$c_2 = c_3 = \cdots = c_n = 0, \qquad \overline{c}_2 = \overline{c}_3 = \cdots = \overline{c}_m = 0.$$

That is,

$$\begin{aligned} u(t) &= -\int_0^t \frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)} h_1(s)\,ds + c_1, \\ v(t) &= -\int_0^t \frac{(t-s)^{\alpha_2-1}}{\Gamma(\alpha_2)} h_2(s)\,ds + \overline{c}_1. \end{aligned}$$

So, we get

$$\begin{aligned} c_1 &= u(1) + \int_0^1 \frac{(1-s)^{\alpha_1-1}}{\Gamma(\alpha_1)} h_1(s)\,ds, \\ \overline{c}_1 &= v(1) + \int_0^1 \frac{(1-s)^{\alpha_2-1}}{\Gamma(\alpha_2)} h_2(s)\,ds. \end{aligned}$$

Together with (6), we have

$$\begin{aligned} u(t) &= u(1) + \int_0^1 \frac{(1-s)^{\alpha_1-1}}{\Gamma(\alpha_1)} h_1(s)\,ds - \int_0^t \frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)} h_1(s)\,ds \\ &= u(1) + \int_0^1 G_1(t,s) h_1(s)\,ds, \end{aligned} \tag{9}$$

$$\begin{aligned} v(t) &= v(1) + \int_0^1 \frac{(1-s)^{\alpha_2-1}}{\Gamma(\alpha_2)} h_2(s)\,ds - \int_0^t \frac{(t-s)^{\alpha_2-1}}{\Gamma(\alpha_2)} h_2(s)\,ds \\ &= v(1) + \int_0^1 G_2(t,s) h_2(s)\,ds. \end{aligned} \tag{10}$$

Multiplying (9) and (10) by $b(t)$, $a(t)$, and integrating with respect to $dA_2(t)$, $dA_1(t)$, respectively, we have

$$\begin{aligned} &\int_0^1 b(t)u(t)\,dA_2(t) = u(1)\int_0^1 b(t)\,dA_2(t) + \int_0^1 b(t)\int_0^1 G_1(t,s)h_1(s)\,ds\,dA_2(t), \\ &\int_0^1 a(t)v(t)\,dA_1(t) = v(1)\int_0^1 a(t)\,dA_1(t) + \int_0^1 a(t)\int_0^1 G_2(t,s)h_2(s)\,ds\,dA_1(t). \end{aligned} \tag{11}$$

Therefore, we obtain

$$\begin{aligned} &\frac{1}{\mu_2}v(1) - k_2 u(1) = \int_0^1 b(t)\int_0^1 G_1(t,s)h_1(s)\,ds\,dA_2(t), \\ &-k_1 v(1) + \frac{1}{\mu_1}u(1) = \int_0^1 a(t)\int_0^1 G_2(t,s)h_2(s)\,ds\,dA_1(t). \end{aligned}$$

Note that

$$\begin{vmatrix} \frac{1}{\mu_1} & -k_1 \\ -k_2 & \frac{1}{\mu_2} \end{vmatrix} = \frac{1-\mu_1\mu_2 k_1 k_2}{\mu_1\mu_2} \neq 0.$$

Then, system (11) has a unique solution for $u(1)$ and $v(1)$. By Cramer's rule, we get

$$\begin{aligned} u(1) = \frac{\mu_1}{1-\mu_1\mu_2 k_1 k_2}\biggl(&\int_0^1 a(t)\int_0^1 G_2(t,s)h_2(s)\,ds\,dA_1(t) \\ &+ \mu_2 k_1 \int_0^1 b(t)\int_0^1 G_1(t,s)h_1(s)\,ds\,dA_2(t)\biggr), \end{aligned} \tag{12}$$

$$\begin{aligned} v(1) = \frac{\mu_2}{1-\mu_1\mu_2 k_1 k_2}\biggl(&\int_0^1 b(t)\int_0^1 G_1(t,s)h_1(s)\,ds\,dA_2(t) \\ &+ \mu_1 k_2 \int_0^1 a(t)\int_0^1 G_2(t,s)h_2(s)\,ds\,dA_1(t)\biggr). \end{aligned} \tag{13}$$

Substituting (12) and (13) into (9) and (10), respectively, we can obtain (4). The proof is completed. □

Lemma 2.3 *The Green function* $G_i(t,s)$ $(i=1,2)$ *defined by* (6) *has the following properties*:

$$\frac{(1-s)^{\alpha_i-1}(1-t^{\alpha_i-1})}{\Gamma(\alpha_i)} \le G_i(t,s) \le \frac{(1-s)^{\alpha_i-1}}{\Gamma(\alpha_i)}, \quad t,s\in[0,1], i=1,2. \tag{14}$$

Proof From the definition of $G_i(t,s)$ $(i=1,2)$, for $0\le t\le s\le 1$, it is obvious that (14) holds.

For $0 \le s \le t \le 1$, we have $t - ts \ge t - s$, and then

$$\begin{aligned}(1-s)^{\alpha_i-1} - (t-s)^{\alpha_i-1} &\ge (1-s)^{\alpha_i-1} - (t-ts)^{\alpha_i-1}\\ &\ge (1-s)^{\alpha_i-1} - t^{\alpha_i-1}(1-s)^{\alpha_i-1}\\ &= (1-s)^{\alpha_i-1}\bigl(1-t^{\alpha_i-1}\bigr),\end{aligned}$$

so, we know $\frac{(1-s)^{\alpha_i-1}(1-t^{\alpha_i-1})}{\Gamma(\alpha_i)} \le G_i(t,s)$. From the definition of $G_i(t,s)$, we also obtain $G_i(t,s) \le \frac{(1-s)^{\alpha_i}}{\Gamma(\alpha_i)}$. Thus, (14) holds. The proof is completed. □

Lemma 2.4 *For $t,s \in [0,1]$, the functions $K_i(t,s)$ and $H_i(t,s)$ $(i = 1,2)$ defined by (5) satisfy*

$$K_1(t,s), H_2(t,s) \le \rho(1-s)^{\alpha_1-1}, \qquad K_2(t,s), H_1(t,s) \le \rho(1-s)^{\alpha_2-1}, \tag{15}$$

$$K_1(t,s), H_2(t,s) \ge \varrho(1-s)^{\alpha_1-1}, \qquad K_2(t,s), H_1(t,s) \ge \varrho(1-s)^{\alpha_2-1}, \tag{16}$$

where

$$\begin{aligned}\rho = \max\biggl\{&\frac{\mu_1\mu_2k_1}{\Gamma(\alpha_1)(1-\mu_1\mu_2k_1k_2)}\int_0^1 b(t)\,dA_2(t) + \frac{1}{\Gamma(\alpha_1)},\\ &\frac{\mu_2}{\Gamma(\alpha_1)(1-\mu_1\mu_2k_1k_2)}\int_0^1 b(t)\,dA_2(t),\\ &\frac{\mu_1\mu_2k_2}{\Gamma(\alpha_2)(1-\mu_1\mu_2k_1k_2)}\int_0^1 a(t)\,dA_1(t) + \frac{1}{\Gamma(\alpha_2)},\\ &\frac{\mu_1}{\Gamma(\alpha_2)(1-\mu_1\mu_2k_1k_2)}\int_0^1 a(t)\,dA_1(t)\biggr\},\\ \varrho = \max\biggl\{&\frac{\mu_1\mu_2k_1}{\Gamma(\alpha_1)(1-\mu_1\mu_2k_1k_2)}\int_0^1 b(t)\bigl(1-t^{\alpha_1-1}\bigr)\,dA_2(t),\\ &\frac{\mu_2}{\Gamma(\alpha_1)(1-\mu_1\mu_2k_1k_2)}\int_0^1 b(t)\bigl(1-t^{\alpha_1-1}\bigr)\,dA_2(t),\\ &\frac{\mu_1\mu_2k_2}{\Gamma(\alpha_2)(1-\mu_1\mu_2k_1k_2)}\int_0^1 a(t)\bigl(1-t^{\alpha_2-1}\bigr)\,dA_1(t),\\ &\frac{\mu_1}{\Gamma(\alpha_2)(1-\mu_1\mu_2k_1k_2)}\int_0^1 a(t)\bigl(1-t^{\alpha_2-1}\bigr)\,dA_1(t)\biggr\}.\end{aligned}$$

Proof By Lemma 2.3, together with the definitions of $K_i(t,s)$ and $H_i(t,s)$ in (5), for any $t,s \in [0,1]$, we have

$$\begin{aligned}K_1(t,s) &= \frac{\mu_1\mu_2k_1}{1-\mu_1\mu_2k_1k_2}\int_0^1 G_1(t,s)b(t)\,dA_2(t) + G_1(t,s)\\ &\le \frac{\mu_1\mu_2k_1}{1-\mu_1\mu_2k_1k_2}\int_0^1 \frac{(1-s)^{\alpha_1-1}b(t)}{\Gamma(\alpha_1)}\,dA_2(t) + \frac{(1-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}\\ &= \biggl(\frac{\mu_1\mu_2k_1}{\Gamma(\alpha_1)(1-\mu_1\mu_2k_1k_2)}\int_0^1 b(t)\,dA_2(t) + \frac{1}{\Gamma(\alpha_1)}\biggr)(1-s)^{\alpha_1-1}\\ &\le \rho(1-s)^{\alpha_1-1},\end{aligned} \tag{17}$$

$$
\begin{aligned}
H_2(t,s) &= \frac{\mu_2}{1-\mu_1\mu_2k_1k_2}\int_0^1 G_1(t,s)b(t)\,dA_2(t) \\
&\le \frac{\mu_2}{1-\mu_1\mu_2k_1k_2}\int_0^1 \frac{(1-s)^{\alpha_1-1}b(t)}{\Gamma(\alpha_1)}\,dA_2(t) \\
&= \left(\frac{\mu_2}{\Gamma(\alpha_1)(1-\mu_1\mu_2k_1k_2)}\int_0^1 b(t)\,dA_2(t)\right)(1-s)^{\alpha_1-1} \\
&= \rho(1-s)^{\alpha_1-1}. \qquad (18)
\end{aligned}
$$

Similarly as in (17)–(18), we have $K_2(t,s), H_1(t,s) \le \rho(1-s)^{\alpha_2-1}$, so the second inequality of (15) holds.

By Lemma 2.3, for any $t,s \in [0,1]$, we also have

$$
\begin{aligned}
K_1(t,s) &= \frac{\mu_1\mu_2k_1}{1-\mu_1\mu_2k_1k_2}\int_0^1 G_1(t,s)b(t)\,dA_2(t) + G_1(t,s) \\
&\ge \frac{\mu_1\mu_2k_1}{1-\mu_1\mu_2k_1k_2}\int_0^1 \frac{b(t)(1-s)^{\alpha_1-1}(1-t^{\alpha_1-1})}{\Gamma(\alpha_1)}\,dA_2(t) \\
&= \left(\frac{\mu_1\mu_2k_1}{\Gamma(\alpha_1)(1-\mu_1\mu_2k_1k_2)}\int_0^1 b(t)\bigl(1-t^{\alpha_1-1}\bigr)\,dA_2(t)\right)(1-s)^{\alpha_1-1} \\
&\ge \varrho(1-s)^{\alpha_1-1}, \qquad (19)
\end{aligned}
$$

$$
\begin{aligned}
H_2(t,s) &= \frac{\mu_2}{1-\mu_1\mu_2k_1k_2}\int_0^1 G_1(t,s)b(t)\,dA_2(t) \\
&\ge \frac{\mu_2}{1-\mu_1\mu_2k_1k_2}\int_0^1 \frac{b(t)(1-s)^{\alpha_1-1}(1-t^{\alpha_1-1})}{\Gamma(\alpha_1)}\,dA_2(t) \\
&= \left(\frac{\mu_2}{\Gamma(\alpha_1)(1-\mu_1\mu_2k_1k_2)}\int_0^1 b(t)\bigl(1-t^{\alpha_1-1}\bigr)\,dA_2(t)\right)(1-s)^{\alpha_1-1} \\
&= \varrho(1-s)^{\alpha_1-1}. \qquad (20)
\end{aligned}
$$

Similarly as in (19)–(20), we have $K_2(t,s), H_1(t,s) \ge \varrho(1-s)^{\alpha_2-1}$, so the second inequality of (16) holds. The proof is completed. □

Let $X = C[0,1] \times C[0,1]$, then X is a Banach space with the norm

$$
\bigl\|(u,v)\bigr\| = \|u\| + \|v\|, \qquad \|u\| = \max_{t\in[0,1]}\bigl|u(t)\bigr|, \qquad \|v\| = \max_{t\in[0,1]}\bigl|v(t)\bigr|.
$$

For any $(u,v) \in X$, we can define an integral operator $T : X \to X$ by

$$
T(u,v)(t) = \bigl(T_1(u,v)(t), T_2(u,v)(t)\bigr), \quad 0 \le t \le 1, \qquad (21)
$$

$$
\begin{aligned}
T_1(u,v)(t) = \lambda_1 &\int_0^1 K_1(t,s)f_1\bigl(s,u(s),v(s)\bigr)\,ds \\
&+ \lambda_2\int_0^1 H_1(t,s)f_2\bigl(s,u(s),v(s)\bigr)\,ds, \quad 0 \le t \le 1,
\end{aligned}
$$

$$T_2(u,v)(t) = \lambda_2 \int_0^1 K_2(t,s) f_2\big(s,u(s),v(s)\big)\,ds + \lambda_1 \int_0^1 H_2(t,s) f_1\big(s,u(s),v(s)\big)\,ds, \quad 0 \le t \le 1.$$

Then (u,v) is a positive solutions of system (1) if and only if (u,v) is a fixed point of T. It can be proved that the following Lemma 2.5 is correct.

Lemma 2.5 *$T : X \to X$ is a completely continuous operator.*

Lemma 2.6 ([33]) *Let E be a Banach space. Assume that $T : E \to E$ is a completely continuous operator. Let $V = \{x \in E | x = \mu Tx, 0 < \mu < 1\}$. Then either the set V is unbounded, or T has at least one fixed point.*

3 Main results

Theorem 3.1 *Assume that there exist real constants $m_i > 0$, and $n_i, l_i \ge 0$, such that $\forall t \in [0,1]$, $x, y \in [0,+\infty)$,*

$$f_i(t,x,y) \le m_i + n_i|x| + l_i|y|, \quad i = 1,2. \tag{22}$$

In addition, assume that

$$2M_1 n_1 + 2M_2 n_2 < 1, \qquad 2M_1 l_1 + 2M_2 l_2 < 1,$$

where

$$M_1 = \lambda_1 \int_0^1 \rho(1-s)^{\alpha_1 - 1}\,ds, \qquad M_2 = \lambda_2 \int_0^1 \rho(1-s)^{\alpha_2 - 1}\,ds. \tag{23}$$

Then system (1) has at least one solution.

Proof Let us confirm that the set $V = \{(u,v) \in X : (u,v) = \varsigma T(u,v), 0 \le \varsigma \le 1\}$ is bounded. Let $(u,v) \in V$, then $(u,v) = \varsigma T(u,v)$. For any $t \in [0,1]$, we have $u = \varsigma T_1(u,v)$, $v = \varsigma T_2(u,v)$. Then, by Lemma 2.4, we obtain

$$\begin{aligned} \big|u(t)\big| &\le \left|\lambda_1 \int_0^1 K_1(t,s) f_1\big(s,u(s),v(s)\big)\,ds + \lambda_2 \int_0^1 H_1(t,s) f_2\big(s,u(s),v(s)\big)\,ds\right| \\ &\le \lambda_1 \int_0^1 \rho(1-s)^{\alpha_1-1} f_1\big(s,u(s),v(s)\big)\,ds + \lambda_2 \int_0^1 \rho(1-s)^{\alpha_2-1} f_2\big(s,u(s),v(s)\big)\,ds \\ &\le M_1\big(m_1 + n_1\|u\| + l_1\|v\|\big) + M_2\big(m_2 + n_2\|u\| + l_2\|v\|\big), \end{aligned} \tag{24}$$

$$\begin{aligned} \big|v(t)\big| &\le \left|\lambda_2 \int_0^1 K_2(t,s) f_2\big(s,u(s),v(s)\big)\,ds + \lambda_1 \int_0^1 H_2(t,s) f_1\big(s,u(s),v(s)\big)\,ds\right| \\ &\le \lambda_2 \int_0^1 \rho(1-s)^{\alpha_2-1} f_1\big(s,u(s),v(s)\big)\,ds + \lambda_1 \int_0^1 \rho(1-s)^{\alpha_1-1} f_1\big(s,u(s),v(s)\big)\,ds \\ &\le M_2\big(m_2 + n_2\|u\| + l_2\|v\|\big) + M_1\big(m_1 + n_1\|u\| + l_1\|v\|\big). \end{aligned} \tag{25}$$

Combined with (24) and (25), we know

$$\begin{aligned}\|u\| + \|v\| &\le 2M_1\big(m_1 + n_1\|u\| + l_1\|v\|\big) + 2M_2\big(m_2 + n_2\|u\| + l_2\|v\|\big)\\ &\le 2M_1m_1 + 2M_2m_2 + (2M_1n_1 + 2M_2n_2)\|u\| + (2M_1l_1 + 2M_2l_2)\|v\|.\end{aligned}$$

Therefore

$$\begin{aligned}\big\|(u,v)\big\| &= \|u\| + \|v\|\\ &\le \frac{2M_1m_1 + 2M_2m_2}{\min\{1-(2M_1n_1+2M_2n_2), 1-(2M_1l_1+2M_2l_2)\}}.\end{aligned}$$

So we have proved that the set V is bounded. Thus, by Lemma 2.6, operator T has at least one fixed point. Hence system (1) has at least one solution. The proof is complete. □

Theorem 3.2 *Assume that* $f_i : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ *is continuous and there exist real constants* $\gamma_i, \delta_i \ge 0$ *such that* $\forall t \in [0,1]$, $x_i, y_i \in [0,+\infty)$,

$$\big|f_i(t,x_1,y_1) - f_i(t,x_2,y_2)\big| \le \gamma_i|x_1 - x_2| + \delta_i|y_1 - y_2|, \quad i = 1,2. \tag{26}$$

In addition, assume that $2M_1(\gamma_1 + \delta_1) + 2M_2(\gamma_2 + \delta_2) < 1$, *where* M_1, M_2 *are defined as* (23). *Then system* (1) *has a unique solution.*

Proof Denoting $\sup|f_i(t,0,0)| = \Theta_i < +\infty$, by (26), we have

$$\big|f_i(t,x,y)\big| \le \Theta_i + \gamma_i|x| + \delta_i|y|, \quad i = 1,2.$$

Let $r = \frac{2M_1\Theta_1+2M_2\Theta_2}{1-2M_1(\gamma_1+\delta_1)-2M_2(\gamma_2+\delta_2)}$, $K_r = \{(u,v) \in X : \|(u,v)\| < r\}$, we show that $TK_r \subset K_r$. For any $(u,v) \in K_r$, we have

$$\begin{aligned}\big|T_1(u,v)(t)\big| &\le \max_{t\in[0,1]}\left|\lambda_1\int_0^1 K_1(t,s)f_1\big(s,u(s),v(s)\big)\,ds + \lambda_2\int_0^1 H_1(t,s)f_2\big(s,u(s),v(s)\big)\,ds\right|\\ &\le \lambda_1\int_0^1 \rho(1-s)^{\alpha_1-1}f_1\big(s,u(s),v(s)\big)\,ds\\ &\quad + \lambda_2\int_0^1 \rho(1-s)^{\alpha_2-1}f_2\big(s,u(s),v(s)\big)\,ds\\ &\le \lambda_1\int_0^1 \rho(1-s)^{\alpha_1-1}\big(\Theta_1 + \gamma_1\|u\| + \delta_1\|v\|\big)\,ds\\ &\quad + \lambda_2\int_0^1 \rho(1-s)^{\alpha_2-1}\big(\Theta_2 + \gamma_2\|u\| + \delta_2\|v\|\big)\,ds\\ &\le M_1\big(\Theta_1 + \gamma_1\|u\| + \delta_1\|v\|\big) + M_2\big(\Theta_2 + \gamma_2\|u\| + \delta_2\|v\|\big).\end{aligned}$$

Hence

$$\big\|T_1(u,v)\big\| \le M_1\big(\Theta_1 + (\gamma_1 + \delta_1)r\big) + M_2\big(\Theta_2 + (\gamma_2 + \delta_2)r\big). \tag{27}$$

Similarly as in (27), for any $(u,v) \in K_r$, we can get

$$\big\|T_2(u,v)\big\| \le M_1\big(\Theta_1 + (\gamma_1 + \delta_1)r\big) + M_2\big(\Theta_2 + (\gamma_2 + \delta_2)r\big). \tag{28}$$

By (27) and (28),

$$\begin{aligned}\|T(u,v)\| &= \|T_1(u,v)\| + \|T_2(u,v)\| \\ &\le 2\big(M_1\big(\Theta_1 + (\gamma_1+\delta_1)r\big) + M_2\big(\Theta_2 + (\gamma_2+\delta_2)r\big)\big) \\ &\le r.\end{aligned}$$

Now for $(u_1,v_1),(u_2,v_2)\in X$, and for any $t\in[0,1]$, we have

$$\begin{aligned}&\big|T_1(u_2,v_2)(t) - T_1(u_1,v_1)(t)\big| \\ &\quad\le \lambda_1\int_0^1 K_1(t,s)\big|f_1\big(s,u_2(s),v_2(s)\big) - f_1\big(s,u_1(s),v_1(s)\big)\big|\,ds \\ &\qquad + \lambda_2\int_0^1 H_1(t,s)\big|f_2\big(s,u_2(s),v_2(s)\big) - f_2\big(s,u_1(s),v_1(s)\big)\big|\,ds \\ &\quad\le \lambda_1\int_0^1 \rho(1-s)^{\alpha_1-1}\big|f_1\big(s,u_2(s),v_2(s)\big) - f_1\big(s,u_1(s),v_1(s)\big)\big|\,ds \\ &\qquad + \lambda_2\int_0^1 \rho(1-s)^{\alpha_2-1}\big|f_2\big(s,u_2(s),v_2(s)\big) - f_2\big(s,u_1(s),v_1(s)\big)\big|\,ds \\ &\quad\le M_1\big(\gamma_1\|u_2-u_1\| + \delta_1\|v_2-v_1\|\big) + M_2\big(\gamma_2\|u_2-u_1\| + \delta_2\|v_2-v_1\|\big) \\ &\quad\le \big(M_1(\gamma_1+\delta_1) + M_2(\gamma_2+\delta_2)\big)\big(\|u_2-u_1\| + \|v_2-v_1\|\big).\end{aligned}$$

Consequently, for $(u_1,v_1),(u_2,v_2)\in X$, we obtain

$$\big\|T_1(u_2,v_2) - T_1(u_1,v_1)\big\| \le \big(M_1(\gamma_1+\delta_1) + M_2(\gamma_2+\delta_2)\big)\big(\|u_2-u_1\| + \|v_2-v_1\|\big). \tag{29}$$

By a similar proof as for (29), for $(u_1,v_1),(u_2,v_2)\in X$, we get

$$\big\|T_2(u_2,v_2) - T_2(u_1,v_1)\big\| \le \big(M_1(\gamma_1+\delta_1) + M_2(\gamma_2+\delta_2)\big)\big(\|u_2-u_1\| + \|v_2-v_1\|\big). \tag{30}$$

It follows from (29) and (30) that

$$\big\|T(u_2,v_2) - T(u_1,v_1)\big\| \le \big(2M_1(\gamma_1+\delta_1) + 2M_2(\gamma_2+\delta_2)\big)\big(\|u_2-u_1\| + \|v_2-v_1\|\big).$$

Since $(2M_1(\gamma_1+\delta_1) + 2M_2(\gamma_2+\delta_2)) < 1$, T is a contraction operator. By the contraction mapping principle, operator T has a unique fixed point, so system (1) has a unique solution. The proof is complete. □

4 Examples

An example is given to illustrate our main results in this paper. Consider the following problem:

$$\begin{cases} {}^{c}D^{\frac{5}{2}}u(t) + f_1(t,u(t),v(t)) = 0, \\ {}^{c}D^{\frac{7}{3}}v(t) + 2f_2(t,u(t),v(t)) = 0, \quad 0<t<1, \\ u'(0) = u''(0) = 0, \qquad u(1) = v(\frac{1}{3}) + v(\frac{1}{2}), \\ v'(0) = v''(0) = 0, \qquad v(1) = \frac{1}{2}\int_0^1 u(s)\,ds^2. \end{cases} \tag{31}$$

Let $\alpha_1 = \frac{5}{2}$, $\alpha_2 = \frac{7}{3}$, $\lambda_1 = 1$, $\lambda_2 = 2$, $\mu_1 = 1$, $\mu_2 = \frac{1}{2}$, $a(t) = b(t) = 1$,

$$A(t) = \begin{cases} 0, & 0 \le t < \frac{1}{3}, \\ 1, & \frac{1}{3} \le t < \frac{1}{2}, \\ 2, & \frac{1}{2} \le t \le 1, \end{cases} \qquad B(t) = t^2.$$

For $t \in [0,1]$, $x, y \in [0,+\infty)$, take

$$f_1(t,x,y) = \frac{t}{1+e^t}\left(1 + \frac{1}{5}\sin^2 x + \frac{1}{10}\cos y\right),$$
$$f_2(t,x,y) = \frac{t}{(1+t)^3}\left(1 + 3\cos x + \frac{1}{4}y\right).$$

Notice that

$$\bigl|f_1(t,x,y)\bigr| = \left|\frac{t}{1+e^t}\left(1 + \frac{1}{5}\sin^2 x + \frac{1}{10}\cos y\right)\right| \le 1 + \frac{1}{5}|x| + \frac{1}{10}|y|,$$
$$\bigl|f_2(t,x,y)\bigr| = \left|\frac{t}{(2+t)^3}\left(\frac{2}{3} + 3\cos x + 2y\right)\right| \le \frac{1}{12} + \frac{3}{8}|x| + \frac{1}{2}|y|,$$
$$2M_1 n_1 + 2M_2 n_2 \doteq 0.63467 < 1, \qquad 2M_1 l_1 + 2M_2 l_2 \doteq 0.83802 < 1.$$

Therefore, all conditions of Theorem 3.1 are satisfied, and hence system (31) has at least one solution.

Acknowledgements
The author thanks the anonymous reviewers for carefully reading this paper and constructive comments.

Funding
This work is supported by the National Natural Science Foundation of China (11701252, 11671185, 61703194), a Project of Shandong Province Higher Educational Science and Technology Program (J16LI03), the Natural Science Foundation of Shandong Province of China (ZR2018MA016), the Science Research Foundation for Doctoral Authorities of Linyi University (LYDX2016BS080), and the Applied Mathematics Enhancement Program of Linyi University.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Publisher's Note

Received: 22 November 2018 Accepted: 11 February 2019 Published online: 28 February 2019

References
1. Wang, Y., Liu, L., Zhang, X., Wu, Y.: Positive solutions of an abstract fractional semipositone differential system model for bioprocesses of HIV infection. Appl. Math. Comput. **258**, 312–324 (2015)
2. Zhang, X., Mao, C., Liu, L., Wu, Y.: Exact iterative solution for an abstract fractional dynamic system model for bioprocess. Qual. Theory Dyn. Syst. **16**, 205–222 (2017)
3. Jiang, J., Liu, L., Wu, Y.: Multiple positive solutions of singular fractional differential system involving Stieltjes integral conditions. Electron. J. Qual. Theory Differ. Equ. **2012**, 43 (2012)
4. Liu, L., Zhang, X., Jiang, J., Wu, Y.: The unique solution of a class of sum mixed monotone operator equations and its application to fractional boundary value problems. J. Nonlinear Sci. Appl. **9**, 2943–2958 (2016)

5. Wang, Y.: Positive solutions for fractional differential equation involving the Riemann–Stieltjes integral conditions with two parameters. J. Nonlinear Sci. Appl. **9**, 5733–5740 (2016)
6. Sun, F., Liu, L., Zhang, X., Wu, Y.: Spectral analysis for a singular differential system with integral boundary conditions. Mediterr. J. Math. **13**, 4763–4782 (2016)
7. Hao, X.: Positive solution for singular fractional differential equations involving derivatives. Adv. Differ. Equ. **2016**, 139 (2016)
8. Zou, Y., He, G.: On the uniqueness of solutions for a class of fractional differential equations. Appl. Math. Lett. **74**, 68–73 (2017)
9. Liu, L., Sun, F., Zhang, X., Wu, Y.: Bifurcation analysis for a singular differential system with two parameters via to degree theory. Nonlinear Anal., Model. Control **22**(1), 31–50 (2017)
10. Zhang, X., Liu, L., Wu, Y., Cui, Y.: New result on the critical exponent for solution of an ordinary fractional differential problem. J. Funct. Spaces **2017**, Article ID 3976469 (2017)
11. Nyamoradi, N., Baleanu, D., Agarwal, R.P.: Existence and uniqueness of positive solutions to fractional boundary value problems with nonlinear boundary conditions. Bound. Value Probl. **2013**, 266 (2013)
12. Min, D., Liu, L., Wu, Y.: Uniqueness of positive solutions for the singular fractional differential equations involving integral boundary value conditions. Bound. Value Probl. **2018**, 23 (2018)
13. Zhang, X., Zhong, Q.: Uniqueness of solution for higher-order fractional differential equations with conjugate type integral conditions. Fract. Calc. Appl. Anal. **20**(6), 1471–1484 (2017)
14. Cui, Y., Ma, W., Wang, X., Su, X.: Uniqueness theorem of differential system with coupled integral boundary conditions. Electron. J. Qual. Theory Differ. Equ. **2018**, 9 (2018)
15. Yan, F., Zuo, M., Hao, X.: Positive solution for a fractional singular boundary value problem with p-Laplacian operator. Bound. Value Probl. **2018**, 51 (2018)
16. Hao, X., Zuo, M., Liu, L.: Multiple positive solutions for a system of impulsive integral boundary value problems with sign-changing nonlinearities. Appl. Math. Lett. **82**, 24–31 (2018)
17. Zhang, X., Liu, L., Zou, Y.: Fixed-point theorems for systems of operator equations and their applications to the fractional differential equations. J. Funct. Spaces **2017**, Article ID 7469868 (2017)
18. Hao, X., Wang, H.: Positive solutions of semipositone singular fractional differential systems with a parameter and integral boundary conditions. Open Math. **16**, 581–596 (2018)
19. Hao, X., Sun, H., Liu, L.: Existence results for fractional integral boundary value problem involving fractional derivatives on an infinite interval. Math. Methods Appl. Sci. **41**(16), 6984–6996 (2018)
20. Jiang, J., Liu, W., Wang, H.: Positive solutions to singular Dirichlet-type boundary value problems of nonlinear fractional differential equations. Adv. Differ. Equ. **2018**, 169 (2018)
21. Zhang, X., Jiang, J., Wu, Y., Cui, Y.: Existence and asymptotic properties of solutions for a nonlinear Schrödinger elliptic equation from geophysical fluid flows. Appl. Math. Lett. **90**, 229–237 (2019)
22. Bai, Z., Qiu, T.: Existence of positive solution for singular fractional differential equation. Appl. Math. Comput. **215**(7), 2761–2767 (2009)
23. Wang, J., Xiang, H., Liu, Z.: Positive solution to nonzero boundary values problem for a coupled system of nonlinear fractional differential equations. Int. J. Differ. Equ. **2010**, Article ID 186928 (2010)
24. Henderson, J., Luca, R.: Positive solutions for a system of nonlocal fractional boundary value problems. Fract. Calc. Appl. Anal. **16**(4), 985–1008 (2013)
25. Wang, Y., Liu, L., Wu, Y.: Positive solutions for a class of higher-order singular semipositone fractional differential systems with coupled integral boundary conditions and parameters. Adv. Differ. Equ. **2014**, 268 (2014)
26. Wang, Y., Liu, L., Zhang, X., Wu, Y.: Positive solutions for $(n-1,1)$-type singular fractional differential system with coupled integral boundary conditions. Abstr. Appl. Anal. **2014**, Article ID 142391 (2014)
27. ur Rehman, M., Ali Khan, R.: A note on boundary value problems for a coupled system of fractional differential equations. Comput. Math. Appl. **61**, 2630–2637 (2011)
28. Jiang, J., Liu, L., Wu, Y.: Positive solutions to singular fractional differential system with coupled boundary conditions. Commun. Nonlinear Sci. Numer. Simul. **18**, 3061–3074 (2013)
29. Liu, L., Li, H., Wu, Y.: Existence and uniqueness of positive solutions for singular fractional differential systems with coupled integral boundary conditions. J. Nonlinear Sci. Appl. **10**, 243–262 (2017)
30. Wang, Y., Jiang, J.: Existence and nonexistence of positive solutions for the fractional coupled system involving generalized p-Laplacian. Adv. Differ. Equ. **2017**, 337 (2017)
31. Miller, K.S., Ross, B.: An Introduction to the Fractional Calculus and Fractional Differential Equations. Wiley, New York (1993)
32. Podlubny, I.: Fractional Differential Equations. Mathematics in Science and Engineering, vol. 198. Academic Press, New York (1999)
33. Grans, A., Dugundji, J.: Fixed Point Theorems. Springer, New York (2005)