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The coupling integrable couplings of the generalized coupled Burgers equation hierarchy and its Hamiltonian structure

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Abstract

In this paper, we mainly give the Lie algebras E, F and H of three kinds and their commutator, respectively. Next, we establish three isospectral problems with the help of their corresponding loop algebras $\widetilde{E}$, $\widetilde{F}$, and $\widetilde{H}$. Based on on the Tu scheme, coupling integrable couplings of three kinds for the generalized coupled Burgers equation hierarchy are obtained. Finally, we obtain the Hamiltonian structure of one of the coupling integrable couplings of the generalized coupled Burgers equation hierarchy by using the quadratic-form identity.

MSC: 35B06

Keywords: Isospectral problem; Burgers equation hierarchy; Quadratic-form identity; Hamiltonian structure

1 Introduction

The development of soliton theory has undergone a rapid development in the 1960s. Integrable couplings are a new important and interesting topic in the field of soliton theory [1–3]. A lot of integrable nonlinear evolution equations, such as the Schroedinger equation and the KdV equation, were discussed. The notion of integrable couplings was first introduced by Virasoro [4, 5]. Integrable couplings are coupled systems of integrable equations which contain given integrable equations as their sub-systems [6]. Integrable couplings have much richer mathematical structures than scalar integrable equations [7–31]. Recently, Inc, Yusuf, Aliyu and Baleanu discussed a Lie symmetry analysis and conservation laws for the time fractional simplified modified Kawahara equation, the generalized shallow water wave equation, the time fractional dispersive long-wave equation and the time fractional generalized Burgers–Huxley equation. They also studied a time fractional third-order variant Boussinesq system and gave a symmetry analysis, explicit solutions, conservation laws and numerical approximations [32–37]. So it is important to study integrable couplings in soliton theory. Zhang even proposed an efficient method for constructing nonlinear evolution equations and their resulting Hamiltonian structure. Ma called it the Tu scheme. Hence, Hu developed the trace identity, and got an efficient method to work out the soliton equations and the Hamiltonian structure. In [38] Zhang gave three kinds of coupling integrable couplings of the KdV hierarchy of evolution equations. The three Lie algebras E, F and H in [38] can be used to obtain the other coupling integrable couplings of the soliton equations. In this paper, we introduce three higher-dimensional Lie algebras and their corresponding loop algebras $\widetilde{E}$, $\widetilde{F}$, and $\widetilde{H}$, and consider three Lax pairs for the zero curvature equation

$$V_x = [U, V]. \tag{1.1}$$

With the help of the loop algebras, we obtain three kinds of coupling integrable couplings of the generalized coupled Burgers equation hierarchy. And the one coupling integrable coupling of the generalized coupled Burgers equation hierarchy has a Hamiltonian structure obtained by employing the quadratic-form identity [39].

2 Three higher-dimensional Lie algebras

Let

$$E = \mathrm{span}\{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}.$$

For $\forall a = \sum_{i=1}^{7} a_i e_i$, $b = \sum_{i=1}^{7} b_i e_i \in E$, we have

$$[a,b] = \begin{pmatrix} 2a_3b_2 - 2a_2b_3 \\ 2a_1b_3 - 2a_3b_1 \\ 2a_1b_2 - 2a_2b_1 \\ a_1b_4 - a_4b_1 + a_2b_5 - a_5b_2 + a_3b_5 - a_5b_3 \\ a_5b_1 - a_1b_5 + a_2b_4 - a_4b_2 + a_4b_3 - a_3b_4 \\ a_1b_6 - a_6b_1 + a_2b_7 - a_7b_2 + a_3b_7 - a_7b_3 \\ a_7b_1 - a_7b_1 + a_2b_6 - a_6b_2 + a_6b_3 - a_3b_6 \end{pmatrix}. \tag{2.1}$$

It is easy to see that E is a Lie algebra equipped with the commutator (2.1).

Denote

$$F = \mathrm{span}\{f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9\}.$$

For $\forall a = \sum_{i=1}^{7} a_i f_i$, $b = \sum_{i=1}^{7} b_i f_i \in L_2$, we easily obtain

$$[a,b] = \begin{pmatrix} 2a_3b_2 - 2a_2b_3 \\ 2a_1b_3 - 2a_3b_1 \\ 2a_1b_2 - 2a_2b_1 \\ 2a_7b_2 - 2a_2b_7 + 2a_3b_6 - 2a_6b_3 \\ 2a_3b_8 - 2a_8b_3 + 2a_9b_2 - 2a_2b_9 \\ 2a_1b_7 - 2a_7b_1 + 2a_4b_3 - 2a_3b_4 \\ 2a_1b_6 - 2a_6b_1 + 2a_4b_2 - 2a_2b_4 \\ 2a_1b_9 - 2a_9b_1 + 2a_5b_3 - 2a_3b_5 \\ 2a_1b_8 - 2a_8b_1 + 2a_5b_2 - 2a_2b_5 \end{pmatrix} \tag{2.2}$$

and F is a Lie algebra equipped with the commutator (2.2).

The Lie algebras E and F are all expanding Lie algebra of the well-known Lie algebra

$$R^3 = \left\{r = (r_1, r_2, r_3)^T, r_i \in R\right\},$$

which along with the commutator leads to

$$[a,b] = \begin{pmatrix} 2a_3b_2 - 2a_2b_3 \\ 2a_1b_3 - 2a_3b_1 \\ 2a_1b_2 - 2a_2b_1 \end{pmatrix} \quad \forall a = (a_1,a_2,a_3)^T, b = (b_1,b_2,b_3)^T \in R^3. \tag{2.3}$$

Assume that L_s is a s-dim Lie algebra,

$$K(L_s) = \left\{ A = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}, A_i \in L_s, i = 1,2,3 \right\}.$$

For

$$B = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} \in K(L_s)$$

and

$$[A,B] = \bigl([A_1,B_1],[A_1,B_2] + [A_2,B_1],[A_1,B_3] + [A_3,B_1] + [A_3 + B_3]\bigr)^T \in K(L_s),$$

$K(L_s)$ is a Lie algebra. By taking $s = 3$ and $[A_1,B_1]$ is the same as (2.3), then we can get another Lie algebra $K(L_s)$, which can be denoted H. The Lie algebra H is isomorphic to the Lie algebra F,

$$H = \operatorname{span}\{f_1,f_2,f_3,f_4,f_5,f_6,f_7,f_8,f_9\},$$

along with the commutator

$$[a,b] = \begin{pmatrix} 2a_3b_2 - 2a_2b_3 \\ 2a_1b_3 - 2a_3b_1 \\ 2a_1b_2 - 2a_2b_1 \\ 2a_3b_5 - 2a_5b_3 + 2a_6b_2 - 2a_2b_6 \\ 2a_1b_6 - 2a_6b_1 + 2a_4b_3 - 2a_3b_4 \\ 2a_1b_5 - 2a_5b_1 + 2a_4b_2 - 2a_2b_4 \\ 2a_3b_8 - 2a_8b_3 + 2a_9b_2 - 2a_2b_9 + 2a_9b_8 - 2a_8b_9 \\ 2a_1b_9 - 2a_9b_1 + 2a_7b_3 - 2a_3b_7 + 2a_7b_9 - 2a_9b_7 \\ 2a_1b_8 - 2a_8b_1 + 2a_7b_2 - 2a_2b_7 + 2a_7b_8 - 2a_8b_7 \end{pmatrix}, \tag{2.4}$$

where $a = \sum_{i=1}^{9} a_ie_i$, $b = \sum_{i=1}^{9} b_ie_i$.

By applying Lie algebra E, F and H, we can obtain three kinds of coupling integrable couplings of the generalized coupled Burgers equation hierarchy.

3 Three kinds of coupling integrable couplings of the generalized coupled Burgers equation hierarchy

In this section, we shall show how to get three kinds of coupling integrable couplings of the generalized coupled Burgers equation hierarchy by making use of the Lie algebra E, F and H.

(I) *The first kind of coupling integrable couplings of the generalized coupled Burgers equation hierarchy*.

The loop algebra of Lie algebra E is given by $\widetilde{E} = \mathrm{span}\{e_i(n) = e_i\lambda^n, i = 1,2,3,4,5,6,7\}$, $[e_i(m), e_j(n)] = [e_i, e_j]\lambda^{m+n}$, $m,n \in Z$, $i,j = 1,2,3,4,5,6,7$, equipped with the commutator (2.1). Consider a Lax pair for zero curvature equation as follows:

$$\begin{cases} U = e_1(1) + qh_2(0) + h_2(1) + rh_3(0) + u_1h_4(0) + u_2h_5(0) + s_1h_6(0) + s_2h_7(0), \\ V = \sum_{m\geq 0}(V_{1m}h_1(-m) + V_{2m}h_2(-m) + V_{3m}h_3(-m) + V_{4m}h_4(-m) \\ \qquad + V_{5m}h_5(-m) + V_{6m}h_6(-m) + V_{7m}h_7(-m)). \end{cases} \tag{3.1}$$

The stationary zero equation

$$V_x = [U, V] \tag{3.2}$$

changes into

$$\begin{cases} V_{1,mx} = -2V_{3,m+1} - 2qV_{3m} + 2rV_{2m}, \\ V_{2,mx} = 2V_{3m} - 2rV_{1m}, \\ V_{3,mx} = -2V_{1,m+1} - 2qV_{1m} + 2V_{2m}, \\ V_{4,mx} = V_{5,m+1} + qV_{5m} + V_{4m} - u_1V_{1m} - u_2V_{2m} + rV_{5m} - u_2V_{3m}, \\ V_{5,mx} = V_{4,m+1} + qV_{4m} + u_2V_{1m} - V_{5m} - u_1V_{2m} + u_1V_{3m} - rV_{4m}, \\ V_{6,mx} = V_{7,m+1} + qV_{7m} + V_{6m} - s_1V_{1m} - s_2V_{2m} + rV_{7m} - s_2V_{3m}, \\ V_{7,mx} = V_{6,m+1} + qV_{6m} + s_2V_{1m} - V_{7m} - s_1V_{2m} + s_1V_{3m} - rV_{6m}. \end{cases} \tag{3.3}$$

Denoting

$$\begin{aligned} V_+^{(n)} = {} & \sum_{m=0}^{n}\bigl(V_{1m}h_1(n-m) + V_{2m}h_2(n-m) + V_{3m}h_3(n-m) + V_{4m}h_4(n-m) \\ & + V_{5m}h_5(n-m) + V_{6m}h_6(n-m) + V_{7m}h_7(n-m)\bigr), \end{aligned}$$

the stationary zero curvature equation (3.2) can be decomposed into

$$-V_{+x}^{(n)} + \bigl[U, V_+^{(n)}\bigr] = V_{-x}^{(n)} - \bigl[U, V_-^{(n)}\bigr].$$

Simple computation results in

$$\begin{aligned} -V_{+x}^{(n)} + \bigl[U, V_+^{(n)}\bigr] = {} & 2V_{3,n+1}h_1(0) + 2V_{1,n+1}h_3(0) - V_{5,n+1}h_4(0) - V_{4,n+1}h_5(0) \\ & - V_{7,n+1}h_6(0) - V_{6,n+1}h_7(0). \end{aligned}$$

Taking $V^{(n)} = V_+^{(n)} - \frac{1}{r}V_{3,n+1}h_2(0)$, we have

$$\begin{aligned} -V_x^{(n)} + \bigl[U, V^{(n)}\bigr] = {} & \left(\frac{1}{r}V_{3,n+1}\right)_x h_2(0) - \frac{1}{r}V_{2,n+1x}h_3(0) \\ & + \left(-V_{5,n+1} + \frac{u_2}{r}V_{3,n+1}\right)h_4(0) \end{aligned}$$

$$+\left(-V_{4,n+1}+\frac{u_1}{r}V_{3,n+1}\right)h_5(0)+\left(-V_{7,n+1}+\frac{s_2}{r}V_{3,n+1}\right)h_6(0)$$
$$+\left(-V_{6,n+1}+\frac{s_1}{r}V_{3,n+1}\right)h_7(0).$$

Hence the zero curvature equation

$$U_t-V_x^{(n)}+\left[U,V^{(n)}\right]=0 \tag{3.4}$$

gives rise to the Lax integrable hierarchy

$$u_t=\begin{pmatrix} q\\ r\\ u_1\\ u_2\\ s_1\\ s_2\end{pmatrix}_t=\begin{pmatrix} -\partial(\frac{1}{r}V_{3,n+1})\\ \frac{1}{r}\partial(V_{2,n+1})\\ V_{5,n+1}-\frac{u_2}{r}V_{3,n+1}\\ V_{4,n+1}-\frac{u_1}{r}V_{3,n+1}\\ V_{7,n+1}-\frac{s_2}{r}V_{3,n+1}\\ V_{6,n+1}-\frac{s_1}{r}V_{3,n+1}\end{pmatrix}. \tag{3.5}$$

When we take $s_1=s_2=0$, (3.5) can be reduced to an integrable coupling of the generalized coupled Burgers equation hierarchy,

$$u_t=\begin{pmatrix} q\\ r\\ u_1\\ u_2\end{pmatrix}_t=\begin{pmatrix} -\partial(\frac{1}{r}V_{3,n+1})\\ \frac{1}{r}\partial(V_{2,n+1})\\ V_{5,n+1}-\frac{u_2}{r}V_{3,n+1}\\ V_{4,n+1}-\frac{u_1}{r}V_{3,n+1}\end{pmatrix}. \tag{3.6}$$

And (3.4) can be reduced to another generalized coupled Burgers equation hierarchy by taking $u_1=u_2=0$,

$$u_t=\begin{pmatrix} q\\ r\\ s_1\\ s_2\end{pmatrix}_t=\begin{pmatrix} -\partial(\frac{1}{r}V_{3,n+1})\\ \frac{1}{r}\partial(V_{2,n+1})\\ V_{7,n+1}-\frac{s_2}{r}V_{3,n+1}\\ V_{6,n+1}-\frac{s_1}{r}V_{3,n+1}\end{pmatrix}. \tag{3.7}$$

So we call (3.5) the first kind of coupling integrable couplings of the generalized coupled Burgers equation hierarchy.

When we set $V_{1,0}=0$, $V_{2,0}=\beta$, $V_{3,0}=V_{4,0}=V_{5,0}=V_{6,0}=V_{7,0}=0$, we can obtain $V_{1,1}=\beta$, $V_{2,1}=0$, $V_{3,1}=\beta r$, $V_{4,1}=\beta u_1$, $V_{5,1}=\beta u_2$, $V_{6,1}=\beta s_1$, $V_{7,1}=\beta s_2,\ldots$.

Hence, when we take $n=2$ in Eq. (3.5), we obtain the first kind of the coupling integrable couplings of the generalized coupled Burgers equation, that is,

$$\begin{cases} q_t=-\beta(\frac{r_{xx}}{4r}+\frac{q_x}{r}+q^2+\frac{r^2}{2})_x,\\ r_t=\beta(\frac{r_{xx}}{2r}+\frac{q_x}{r}-q_xr-2qr_x),\\ u_{1t}=\beta u_{2xx}-2\beta u_{1x}q-\beta u_1q_x-\beta u_{2x}-\frac{\beta}{2}u_1r_x-\beta u_{1x}r-\beta\frac{u_2r_{xx}}{4r}-\beta\frac{u_2q_x}{2r},\\ u_{2t}=\beta u_{1xx}-2\beta u_{2x}q-\beta u_2q_x+\beta u_{1x}+\frac{\beta}{2}u_2r_x+\beta u_{2x}r-\beta\frac{u_1r_{xx}}{4r}-\beta\frac{u_1q_x}{2r},\\ s_{1t}=\beta s_{2xx}-2\beta s_{1x}q-\beta s_1q_x-\beta s_{2x}-\frac{\beta}{2}s_1r_x-\beta s_{1x}r-\beta\frac{s_2r_{xx}}{4r}-\beta\frac{s_2q_x}{2r},\\ s_{2t}=\beta s_{1xx}-2\beta s_{2x}q-\beta s_2q_x+\beta s_{1x}+\frac{\beta}{2}s_2r_x+\beta s_{2x}r-\beta\frac{s_1r_{xx}}{4r}-\beta\frac{s_1q_x}{2r}.\end{cases} \tag{3.8}$$

When we take $u_1 = u_2 = s_1 = s_2 = 0$, $\beta = -2$ in (3.9), we can get equations as follows:

$$\begin{cases} q_t = (\frac{r_{xx}}{2r} + \frac{q_x}{r} + 2q^2 + r^2)_x, \\ r_t = -\frac{r_{xx}}{r} q_{xx} - \frac{2q_x}{r} + 2q_x r + 4qr_x. \end{cases} \tag{3.9}$$

And when we take $r = -1$ in (3.8), we can get the famous generalized Burgers equation

$$q_t = 4qq_x - q_{xx}. \tag{3.10}$$

(II) *The second kind of coupling integrable couplings of the generalized coupled Burgers equation hierarchy.*

We have the loop algebra of the Lie algebra F:

$$\begin{aligned} &\widetilde{F} = \operatorname{span}\{f_i(n) = f_i\lambda^n, i = 1,2,3,4,5,6,7,8,9\}, \\ &[f_i(m), f_j(n)] = [f_i, f_j]\lambda^{m+n}, \quad m, n \in Z, i, j = 1,2,3,4,5,6,7,8,9, \end{aligned}$$

of which the resulting commutators are defined the same as (2.2).

By using the loop algebra $\widetilde{F}$, we introduce an isospectral Lax pair for zero curvature equation as follows:

$$\begin{cases} U = f_1(0) + qf_2(0) + f_2(1) + rf_3(0) + u_1 f_4(0) + u_2 f_5(0) + s_1 f_7(0) + s_2 f_9(0), \\ V = \sum_{m\geq 0}(V_{1m} f_1(-m) + V_{2m} f_2(-m) + V_{3m} f_3(-m) + V_{4m} f_4(-m) \\ \qquad + V_{5m} f_5(-m) + V_{6m} f_6(-m) + V_{7m} f_7(-m) + V_{8m} f_8(-m) + V_{9m} f_9(-m)). \end{cases} \tag{3.11}$$

The stationary zero curvature

$$V_x = [U, V]$$

is presented by the recurrence relations as follows:

$$\begin{cases} V_{1,mx} = -2V_{3,m+1} - 2qV_{3m} + 2rV_{2m}, \\ V_{2,mx} = 2V_{3m} - 2rV_{1m}, \\ V_{3,mx} = -2V_{1,m+1} - 2qV_{1m} + 2V_{2m}, \\ V_{4,mx} = -2V_{7,m+1} - 2qV_{7m} + 2s_1 V_{2m} + 2rV_{6m}, \\ V_{5,mx} = -2V_{9,m+1} - 2qV_{9m} + 2s_2 V_{2m} + 2rV_{8m}, \\ V_{6,mx} = 2V_{7m} - 2s_1 V_{1m} + 2u_1 V_{3m} - 2rV_{4m}, \\ V_{7,mx} = -2V_{4,m+1} - 2qV_{4m} + +2u_1 V_{2m} + 2V_{6m}, \\ V_{8,mx} = 2V_{9m} - 2s_2 V_{1m} + 2u_2 V_{3m} - 2rV_{5m}, \\ V_{9,mx} = -2V_{5,m+1} - 2qV_{5m} + 2u_2 V_{2m} + 2V_{8m}. \end{cases} \tag{3.12}$$

Noting that

$$V_+^{(n)} = \sum_{m=0}^{n} \big(V_{1m} f_1(n-m) + V_{2m} f_2(n-m) + V_{3m} f_3(n-m) + V_{4m} f_4(n-m)$$

$$+ V_{5m}f_5(n-m) + V_{6m}f_6(n-m) + V_{7m}f_7(n-m) + V_{8m}f_8(n-m) + V_{9m}f_9(n-m)\big),$$

we obtain by a direct calculation

$$\begin{aligned}-V_{+x}^{(n)} + \big[U, V_+^{(n)}\big] &= 2V_{3,n+1}f_1(0) + 2V_{1,n+1}f_3(0) + 2V_{7,n+1}f_4(0) + 2V_{9,n+1}f_5(0) \\ &\quad + 2V_{4,n+1}f_7(0) + 2V_{5,n+1}f_9(0).\end{aligned}$$

Taking $V^{(n)} = V_+^{(n)} - \frac{1}{r}V_{3,n+1}f_2(0)$, after a calculation, we get

$$\begin{aligned}-V_x^{(n)} + \big[U, V^{(n)}\big] &= \left(\frac{1}{r}V_{3,n+1}\right)_x f_2(0) - \frac{1}{r}V_{2,n+1x}f_3(0) + \left(2V_{7,n+1} - \frac{2s_1}{r}V_{3,n+1}\right)f_4(0) \\ &\quad + \left(2V_{9,n+1} - \frac{2s_2}{r}V_{3,n+1}\right)f_5(0) + \left(2V_{4,n+1} - \frac{2u_1}{r}V_{3,n+1}\right)f_7(0) \\ &\quad + \left(2V_{5,n+1} - \frac{2u_2}{r}V_{3,n+1}\right)f_9(0).\end{aligned}$$

The zero curvature equation

$$U_t - V_x^{(n)} + \big[U, V^{(n)}\big] = 0 \tag{3.13}$$

admits

$$u_t = \begin{pmatrix} q \\ r \\ u_1 \\ u_2 \\ s_1 \\ s_2 \end{pmatrix}_t = \begin{pmatrix} -(\frac{1}{r}V_{3,n+1})_x \\ \frac{1}{r}V_{2,n+1x} \\ -2V_{7,n+1} + \frac{2s_1}{r}V_{3,n+1} \\ -2V_{9,n+1} + \frac{2s_2}{r}V_{3,n+1} \\ -2V_{4,n+1} + \frac{2u_1}{r}V_{3,n+1} \\ -2V_{5,n+1} + \frac{2u_2}{r}V_{3,n+1} \end{pmatrix}, \tag{3.14}$$

which can be regarded as a composition of two integrable coupling of the generalized coupled Burgers equation hierarchy as follows:

$$u_t = \begin{pmatrix} q \\ r \\ u_1 \\ u_2 \end{pmatrix}_t = \begin{pmatrix} -(\frac{1}{r}V_{3,n+1})_x \\ \frac{1}{r}V_{2,n+1x} \\ -2V_{7,n+1} + \frac{2s_1}{r}V_{3,n+1} \\ -2V_{9,n+1} + \frac{2s_2}{r}V_{3,n+1} \end{pmatrix}, \tag{3.15}$$

$$u_t = \begin{pmatrix} q \\ r \\ s_1 \\ s_2 \end{pmatrix}_t = \begin{pmatrix} -(\frac{1}{r}V_{3,n+1})_x \\ \frac{1}{r}V_{2,n+1x} \\ -2V_{4,n+1} + \frac{2u_1}{r}V_{3,n+1} \\ -2V_{5,n+1} + \frac{2u_2}{r}V_{3,n+1} \end{pmatrix}. \tag{3.16}$$

When we set $V_{1,0} = 0$, $V_{2,0} = \beta$, $V_{3,0} = V_{4,0} = V_{5,0} = V_{6,0} = V_{7,0} = V_{8,0} = V_{9,0} = 0$, we can obtain $V_{1,1} = \beta$, $V_{2,1} = 0$, $V_{3,1} = \beta r$, $V_{4,1} = \beta u_1$, $V_{5,1} = \beta u_2$, $V_{6,1} = 0$, $V_{7,1} = \beta s_1$, $V_{8,1} = 0$, $V_{9,1} = \beta s_2, \ldots$.

Hence, when we take $n = 2$ in (3.14), we get the second kind of the coupling integrable couplings of the generalized coupled Burgers equation as follows:

$$\begin{cases} q_t = -\beta(\frac{r_{xx}}{4r} + \frac{q_x}{r} + q^2 + \frac{r^2}{2})_x, \\ r_t = \beta(\frac{r_{xx}}{2r} + \frac{q_x}{r} - q_x r - 2qr_x), \\ u_{1t} = -\frac{\beta}{2}s_{1xx} - 2\beta u_{1x}q - \beta u_1 q_x + 2\beta u_1 r - 2\beta s_1 r^2 + \frac{\beta}{2}\frac{s_1 r_{xx}}{r} + \beta\frac{s_1 q_x}{r}, \\ u_{2t} = -\frac{\beta}{2}s_{2xx} - 2\beta u_{2x}q - \beta u_2 q_x + 2\beta u_2 r - 2\beta s_2 r^2 + \frac{\beta}{2}\frac{s_2 r_{xx}}{r} + \beta\frac{s_2 q_x}{r}, \\ s_{1t} = -\frac{\beta}{2}u_{1xx} - 2\beta s_{1x}q - \beta s_1 q_x + 2\beta u_1 - 2\beta s_1 r + \frac{\beta}{2}\frac{u_1 r_{xx}}{r} + \beta\frac{u_1 q_x}{r}, \\ s_{2t} = -\frac{\beta}{2}u_{2xx} - 2\beta s_{2x}q - \beta s_2 q_x + 2\beta u_2 - 2\beta s_2 r + \frac{\beta}{2}\frac{u_2 r_{xx}}{r} + \beta\frac{u_2 q_x}{r}. \end{cases} \tag{3.17}$$

When take $u_1 = u_2 = s_1 = s_2 = 0$, $r = -1$, $\beta = -2$ in (3.17), we can also get the same as (3.10).

(III) *The third kind of coupling integrable couplings of the generalized coupled Burgers equation hierarchy.*

By using the loop algebra of Lie algebra H as follows:

$$\widetilde{H} = \operatorname{span}\{f_i(n) = f_i\lambda^n, i = 1,2,3,4,5,6,7,8,9\},$$

along with the commutator

$$[f_i(m), f_j(n)] = [f_i, f_j]\lambda^{m+n}, \quad m,n \in Z, i,j = 1,2,3,4,5,6,7,8,9,$$

we get the third type of coupling integrable couplings of the generalized coupled Burgers equation hierarchy.

A Lax pair for zero curvature equation is given as follows:

$$\begin{cases} U = f_1(0) + qf_2(0) + f_2(1) + rf_3(0) + u_1 f_4(0) + u_2 f_6(0) + s_1 f_7(0) + s_2 f_9(0), \\ V = \sum_{m\geq 0}(V_{1m}f_1(-m) + V_{2m}f_2(-m) + V_{3m}f_3(-m) + V_{4m}f_4(-m) \\ \qquad + V_{5m}f_5(-m) + V_{6m}f_6(-m) + V_{7m}f_7(-m) + V_{8m}f_8(-m) + V_{9m}f_9(-m)). \end{cases} \tag{3.18}$$

A solution to the stationary zero curvature equation

$$V_x = [U, V]$$

is presented now:

$$\begin{cases} V_{1,mx} = -2V_{3,m+1} - 2qV_{3m} + 2rV_{2m}, \\ V_{2,mx} = 2V_{3m} - 2rV_{1m}, \\ V_{3,mx} = -2V_{1,m+1} - 2qV_{1m} + 2V_{2m}, \\ V_{4,mx} = -2V_{6,m+1} - 2qV_{6m} + 2rV_{5m} + 2u_2V_{2m}, \\ V_{5,mx} = 2V_{6m} - 2u_2V_{1m} + 2u_1V_{3m} - 2rV_{4m}, \\ V_{6,mx} = -2V_{4,m+1} - 2qV_{4m} + 2V_{5m} + 2u_1V_{2m}, \\ V_{7,mx} = -2V_{9,m+1} - 2qV_{9m} + 2rV_{8m} + 2s_2V_{2m} + 2s_2V_{8m}, \\ V_{8,mx} = 2V_{9m} - 2s_2V_{1m} + 2s_1V_{3m} - 2rV_{7m} + 2s_1V_{9m} - 2s_2V_{7m}, \\ V_{9,mx} = -2V_{7,m+1} - 2qV_{7m} + 2V_{8m} + 2s_1V_{2m} + 2s_1V_{8m}. \end{cases} \tag{3.19}$$

Setting

$$\begin{aligned} V_{+}^{(n)} = \sum_{m=0}^{n} \bigl(& V_{1m} f_1(n-m) + V_{2m} f_2(n-m) + V_{3m} f_3(n-m) + V_{4m} f_4(n-m) \\ & + V_{5m} f_5(n-m) + V_{6m} f_6(n-m) + V_{7m} f_7(n-m) + V_{8m} f_8(n-m) + V_{9m} f_9(n-m) \bigr), \end{aligned}$$

we have

$$\begin{aligned} & -V_{+x}^{(n)} + \bigl[U, V_{+}^{(n)}\bigr] \\ & \quad = 2V_{3,n+1} f_1(0) + 2V_{1,n+1} f_3(0) + 2V_{6,n+1} f_4(0) \\ & \qquad + 2V_{4,n+1} f_6(0) + 2V_{9,n+1} f_7(0) + 2V_{7,n+1} f_9(0). \end{aligned}$$

Taking $V^{(n)} = V_{+}^{(n)} - \frac{1}{r} V_{3,n+1} f_2(0)$, we obtain

$$\begin{aligned} -V_{x}^{(n)} + \bigl[U, V^{(n)}\bigr] = {} & \left(\frac{1}{r} V_{3,n+1}\right)_x f_2(0) - \frac{1}{r} V_{2,n+1x} f_3(0) + \left(2V_{6,n+1} - 2\frac{u_2}{r} V_{3,n+1}\right) f_4(0) \\ & + \left(2V_{4,n+1} - 2\frac{u_1}{r} V_{3,n+1}\right) f_6(0) + \left(2V_{9,n+1} - 2\frac{s_2}{r} V_{3,n+1}\right) f_7(0) \\ & + \left(2V_{7,n+1} - 2\frac{s_1}{r} V_{3,n+1}\right) f_9(0). \end{aligned}$$

Thus, the zero curvature equation

$$U_t - V_{x}^{(n)} + \bigl[U, V^{(n)}\bigr] = 0 \tag{3.20}$$

admits the Lax integrable hierarchy

$$u_t = \begin{pmatrix} q \\ r \\ u_1 \\ u_2 \\ s_1 \\ s_2 \end{pmatrix}_t = \begin{pmatrix} -(\frac{1}{r} V_{3,n+1})_x \\ \frac{1}{r} V_{2,n+1x} \\ -2V_{6,n+1} + 2\frac{u_2}{r} V_{3,n+1} \\ -2V_{4,n+1} + 2\frac{u_1}{r} V_{3,n+1} \\ -2V_{9,n+1} + 2\frac{s_2}{r} V_{3,n+1} \\ -2V_{7,n+1} + 2\frac{s_1}{r} V_{3,n+1} \end{pmatrix}. \tag{3.21}$$

When we take $s_1 = s_2 = 0$ and $u_1 = u_2 = 0$ in (3.21), respectively, we get two integrable coupling of the generalized coupled Burgers equation hierarchy

$$u_t = \begin{pmatrix} q \\ r \\ u_1 \\ u_2 \end{pmatrix}_t = \begin{pmatrix} -(\frac{1}{r} V_{3,n+1})_x \\ \frac{1}{r} V_{2,n+1x} \\ -2V_{6,n+1} + 2\frac{u_2}{r} V_{3,n+1} \\ -2V_{4,n+1} + 2\frac{u_1}{r} V_{3,n+1} \end{pmatrix}, \tag{3.22}$$

$$u_t = \begin{pmatrix} q \\ r \\ s_1 \\ s_2 \end{pmatrix}_t = \begin{pmatrix} -(\frac{1}{r} V_{3,n+1})_x \\ \frac{1}{r} V_{2,n+1x} \\ -2V_{9,n+1} + 2\frac{s_2}{r} V_{3,n+1} \\ -2V_{7,n+1} + 2\frac{s_1}{r} V_{3,n+1} \end{pmatrix}. \tag{3.23}$$

So we call (3.21) the third type of coupling integrable couplings of the generalized coupled Burgers equation hierarchy.

Set $V_{1,0} = 0$, $V_{2,0} = \beta$, $V_{3,0} = V_{4,0} = V_{5,0} = V_{6,0} = V_{7,0} = V_{8,0} = V_{9,0} = 0$, we can obtain $V_{1,1} = \beta$, $V_{2,1} = 0$, $V_{3,1} = \beta r$, $V_{4,1} = \beta u_1$, $V_{5,1} = 0$, $V_{6,1} = \beta u_2$, $V_{7,1} = \beta s_1$, $V_{8,1} = 0$, $V_{9,1} = \beta s_2, \ldots$.

And when we take $n = 2$ in (3.21), we get the third hind of the coupling integrable couplings of the generalized coupled Burgers equation, that is,

$$\begin{cases} q_t = -\beta(\frac{r_{xx}}{4r} + \frac{q_x}{r} + q^2 + \frac{r^2}{2})_x, \\ r_t = \beta(\frac{r_{xx}}{2r} + \frac{q_x}{r} - q_x r - 2qr_x), \\ u_{1t} = -\frac{\beta}{2}u_{2xx} - 2\beta u_{1x}q - \beta u_1 q_x + 2\beta u_1 r - 2\beta u_2 r^2 + \frac{\beta}{2}\frac{u_2 r_{xx}}{r} + \beta\frac{u_2 q_x}{r}, \\ u_{2t} = -\frac{\beta}{2}u_{1xx} - 2\beta u_{2x}q - \beta u_2 q_x + 2\beta u_1 - 2\beta u_2 r + \frac{\beta}{2}\frac{u_1 r_{xx}}{r} + \beta\frac{u_1 q_x}{r}, \\ s_{1t} = -\frac{\beta}{2}s_{2xx} - 2\beta s_{1x}q - \beta s_1 q_x + 2\beta s_1 r - 2\beta s_2 r^2 - 3\beta s_2^2 r + \beta s_1^2 r + 2\beta s_1 s_2 \\ \qquad - \beta s_2^3 + \beta s_1^2 s_2 + \frac{\beta}{2}\frac{s_2 r_{xx}}{r} + \beta\frac{s_2 q_x}{r}, \\ s_{2t} = -\frac{\beta}{2}s_{1xx} - 2\beta s_{2x}q - \beta s_2 q_x + 2\beta s_1 - 2\beta s_2 r + 3\beta s_1^2 r - \beta s_2^2 - 2\beta s_1 s_2 r \\ \qquad + \beta s_1^3 - \beta s_1 s_2^2 + \frac{\beta}{2}\frac{s_1 r_{xx}}{r} + \beta\frac{s_1 q_x}{r}. \end{cases} \tag{3.24}$$

Similarly when take $u_1 = u_2 = s_1 = s_2 = 0$, $n = 2$, $\alpha = -2$ in (3.24), we can obtain (3.10).

4 Hamiltonian structure of coupling integrable couplings of the generalized coupled Burgers equation hierarchy

In this section we shall deduce the Hamiltonian forms of the coupling integrable couplings (3.21) by using the quadratic-form [39]. The commutator (2.4) can be written as

$$[a,b]^T = a^T R(b), \tag{4.1}$$

where

$$R(b) = \begin{pmatrix} 0 & 2b_3 & 2b_2 & 0 & 2b_6 & 2b_5 & 0 & 2b_9 & 2b_8 \\ -2b_3 & 0 & -2b_1 & -2b_6 & 0 & -2b_4 & -2b_9 & 0 & -2b_7 \\ 2b_2 & -2b_1 & 0 & 2b_5 & -2b_4 & 0 & 2b_8 & -2b_7 & 0 \\ 0 & 0 & 0 & 0 & 2b_3 & 2b_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2b_3 & 0 & -2b_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2b_2 & -2b_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2b_3+2b_9 & 2b_2+2b_8 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2b_3-2b_9 & 0 & -2b_1-2b_7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2b_2+2b_8 & -2b_1-2b_7 & 0 \end{pmatrix}. \tag{4.2}$$

The constant symmetric matrix F satisfying the matrix equation

$$R(b)F = -\big(R(b)F\big)^T, \qquad F = F^T, \tag{4.3}$$

shows that

$$F=\begin{pmatrix}1&0&0&1&0&0&1&0&0\\0&1&0&0&1&0&0&1&0\\0&0&-1&0&0&-1&0&0&-1\\1&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0\\0&0&-1&0&0&0&0&0&0\\1&0&0&0&0&0&1&0&0\\0&1&0&0&0&0&0&1&0\\0&0&-1&0&0&0&0&0&-1\end{pmatrix}. \tag{4.4}$$

In the linear space R^9, define a functional,

$$\begin{aligned}\{a,b\}&=a^TFb\\&=a_1b_1+a_2b_2-a_3b_3+a_7b_7\\&\quad+a_8b_8-a_9b_9+(a_1b_4+a_4b_1)+(a_1b_7+a_7b_1)+(a_2b_5+a_5b_2)\\&\quad+(a_2b_8+a_8b_2)-(a_3b_6+a_6b_3)-(a_3b_9+a_9b_3).\end{aligned} \tag{4.5}$$

The Lax pair U and V in (3.18) can be written as

$$\begin{cases}U=(1,q+\lambda,r,u_1,0,u_2,s_1,0,s_2)^T,\\V=(V_1,V_2,V_3,V_4,V_5,V_6,V_7,V_8,V_9),\end{cases} \tag{4.6}$$

where $V_i=\sum_{m\geq 0}V_{im}\lambda^{-m}$, $i=1,2,\ldots,9$. With the help of (4.2), we have

$$\begin{aligned}&\left\{V,\frac{\partial U}{\partial\lambda}\right\}=V_2+V_5+V_8,\\&\left\{V,\frac{\partial U}{\partial q}\right\}=V_2+V_5+V_8,\\&\left\{V,\frac{\partial U}{\partial r}\right\}=-V_3-V_6-V_9,\\&\left\{V,\frac{\partial U}{\partial u_1}\right\}=V_1,\qquad\left\{V,\frac{\partial U}{\partial u_2}\right\}=-V_3,\\&\left\{V,\frac{\partial U}{\partial s_1}\right\}=V_1+V_7,\qquad\left\{V,\frac{\partial U}{\partial s_2}\right\}=-V_3-V_9,\end{aligned}$$

substituting the above into the quadratic-form identity yields

$$\frac{\delta}{\delta\overline{u}}(V_2+V_5+V_8)=\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\begin{pmatrix}V_2+V_5+V_8\\-V_3-V_6-V_9\\V_1\\-V_3\\V_1+V_7\\-V_3-V_9\end{pmatrix}, \tag{4.7}$$

where

$$\frac{\delta}{\delta u}=\begin{pmatrix}\frac{\delta}{\delta q}\\ \frac{\delta}{\delta r}\\ \frac{\delta}{\delta u_1}\\ \frac{\delta}{\delta u_2}\\ \frac{\delta}{\delta s_1}\\ \frac{\delta}{\delta s_2}\end{pmatrix}.$$

Comparing the coefficients of λ^{-n-2} on both sides of (4.7), we get

$$\frac{\delta}{\delta \overline{u}}(V_{2,n+2}+V_{5,n+2}+V_{8,n+2})=(\gamma-n-1)\begin{pmatrix}V_{2,n+1}+V_{5,n+1}+V_{8,n+1}\\ -V_{3,n+1}-V_{6,n+1}-V_{9,n+1}\\ V_{1,n+1}\\ -V_{3,n+1}\\ V_{1,n+1}+V_{7,n+1}\\ -V_{3,n+1}-V_{9,n+1}\end{pmatrix}. \tag{4.8}$$

Taking $n=0$ in (4.8), we obtain $\gamma=0$.

So we get

$$\frac{\delta H_{n+2}}{\delta u}=\begin{pmatrix}V_{2,n+1}+V_{5,n+1}+V_{8,n+1}\\ -V_{3,n+1}-V_{6,n+1}-V_{9,n+1}\\ V_{1,n+1}\\ -V_{3,n+1}\\ V_{1,n+1}+V_{7,n+1}\\ -V_{3,n+1}-V_{9,n+1}\end{pmatrix}, \tag{4.9}$$

where

$$H_{n+2}=-\frac{V_{2,n+2}+V_{5,n+2}+V_{8,n+2}}{n+1}.$$

So (3.21) can be written as a Hamiltonian form

$$\begin{aligned}u_t=\begin{pmatrix}q\\ r\\ u_1\\ u_2\\ s_1\\ s_2\end{pmatrix}_t&=\begin{pmatrix}-(\frac{1}{r}V_{3,n+1})_x\\ \frac{1}{r}V_{2,n+1x}\\ -2V_{6,n+1}+2\frac{u_2}{r}V_{3,n+1}\\ -2V_{4,n+1}+2\frac{u_1}{r}V_{3,n+1}\\ -2V_{9,n+1}+2\frac{s_2}{r}V_{3,n+1}\\ -2V_{7,n+1}+2\frac{s_1}{r}V_{3,n+1}\end{pmatrix}\\ &=\begin{pmatrix}0&0&0&\partial\frac{1}{r}&0&0\\ 0&0&-2&-\frac{2}{r}&0&0\\ 0&2&0&-2\frac{u_2}{r}&0&-2\\ \frac{1}{r}\partial&\frac{2}{r}&2\frac{u_2}{r}&0&2+2\frac{s_2}{r}&2\frac{s_1}{r}\\ 0&0&0&-2-2\frac{s_2}{r}&0&2\\ 0&0&2&-2\frac{s_1}{r}&-2&0\end{pmatrix}\end{aligned}$$

$$\times \begin{pmatrix} V_{2,n+1}+V_{5,n+1}+V_{8,n+1} \\ -V_{3,n+1}-V_{6,n+1}-V_{9,n+1} \\ V_{1,n+1} \\ -V_{3,n+1} \\ V_{1,n+1}+V_{7,n+1} \\ -V_{3,n+1}-V_{9,n+1} \end{pmatrix}$$

$$= J\frac{\delta H_{n+2}}{\delta u}, \tag{4.10}$$

where $\partial = \frac{\partial}{\partial x}$ and J is a Hamiltonian operator. We can obtain a recursive operator from (3.19),

$$L = \begin{pmatrix} l_{11} & l_{12} & l_{13} & l_{14} & l_{15} & l_{16} \\ l_{21} & l_{22} & l_{23} & l_{24} & l_{25} & l_{26} \\ l_{31} & l_{32} & l_{33} & l_{34} & l_{35} & l_{36} \\ l_{41} & l_{42} & l_{43} & l_{44} & l_{45} & l_{46} \\ l_{51} & l_{52} & l_{53} & l_{54} & l_{55} & l_{56} \\ l_{61} & l_{62} & l_{63} & l_{64} & l_{65} & l_{66} \end{pmatrix}, \tag{4.11}$$

where

$$\begin{aligned}
&l_{11} = \frac{1}{2r}\partial - \partial^{-1}q\partial, \qquad l_{12} = \frac{1}{r}\partial - \partial^{-1}r\partial, \qquad l_{13} = \frac{u_2}{r}\partial - \partial^{-1}u_1\partial, \\
&l_{14} = \frac{u_1}{r}\partial - \partial^{-1}u_2\partial, \qquad l_{15} = \frac{s_2}{r}\partial - \partial^{-1}s_1\partial, \qquad l_{16} = \frac{s_1}{r}\partial - \partial^{-1}s_2\partial, \\
&l_{21} = -r - \frac{1}{4}\partial\frac{1}{r}\partial, \qquad l_{22} = -q - \frac{1}{2}\partial\frac{1}{r}, \qquad l_{23} = 2u_2\partial^{-1}r - \frac{1}{2}\partial\frac{u_2}{r}, \qquad l_{24} = -\frac{1}{2}\partial\frac{u_1}{r}, \\
&l_{25} = 2s_2\partial^{-1}r + 2s_2\partial^{-1}s_2 - \frac{1}{2}\partial\frac{s_2}{r}, \qquad l_{26} = -\frac{1}{2}\partial\frac{s_1}{r} + 2s_2\partial^{-1} + 2s_2\partial^{-1}s_1, \\
&l_{31} = l_{32} = l_{35} = l_{36} = 0, \qquad l_{33} = -q - 2\partial^{-1}r, \qquad l_{34} = \frac{\partial}{2} - 2\partial^{-1}, \\
&l_{41} = l_{42} = l_{45} = l_{46} = 0, \qquad l_{43} = \frac{\partial}{2} + 2r\partial^{-1}r, \qquad l_{34} = 2r\partial^{-1} - q, \\
&l_{51} = l_{52} = l_{53} = l_{54} = 0, \qquad l_{55} = -q - 2\partial^{-1}r - 2\partial^{-1}s_2 - 2s_1\partial^{-1}r - 2s_1\partial^{-1}s_2, \\
&l_{56} = \frac{1}{2}\partial - 2\partial^{-1} - 2\partial^{-1}s_1 - 2s_1\partial^{-1} - 2s_1\partial^{-1}s_1, \\
&l_{61} = l_{62} = l_{63} = l_{64} = 0, \qquad l_{65} = \frac{1}{2}\partial + 2r\partial^{-1}r + 2r\partial^{-1}s_2 + 2s_2\partial^{-1}r + 2s_2\partial^{-1}s_2, \\
&l_{66} = -q + 2r\partial^{-1} + 2r\partial^{-1}s_1 + 2s_2\partial^{-1} + 2s_2\partial^{-1}s_1.
\end{aligned}$$

L satisfies

$$\begin{pmatrix} V_{2,n+1}+V_{5,n+1}+V_{8,n+1} \\ -V_{3,n+1}-V_{6,n+1}-V_{9,n+1} \\ V_{1,n+1} \\ -V_{3,n+1} \\ V_{1,n+1}+V_{7,n+1} \\ -V_{3,n+1}-V_{9,n+1} \end{pmatrix} = L \begin{pmatrix} V_{2n}+V_{5n}+V_{8n} \\ -V_{3n}-V_{6n}-V_{9n} \\ V_{1n} \\ -V_{3n} \\ V_{1n}+V_{7n} \\ -V_{3n}-V_{9n} \end{pmatrix}.$$

The Hamiltonian form of the coupling integrable couplings (3.21) can be written as

$$u_t = J\frac{\delta H_{n+2}}{\delta u} = JL^n \begin{pmatrix} 0 \\ -\beta r - \beta u_2 - \beta s_2 \\ \beta \\ -\beta r \\ \beta + \beta s_1 \\ -\beta r - \beta s_2 \end{pmatrix}. \tag{4.12}$$

5 Conclusion

In this paper we get three kinds of coupling integrable couplings of the generalized coupled Burgers equation hierarchy, which are new results. And we obtain the Hamiltonian structure of one coupling integrable couplings of the generalized coupled Burgers equation hierarchy by using the quadratic-form identity.

6 Results and discussion

Firstly, we introduce three Lie algebras E, F and H. With the help of their corresponding loop algebras $\widetilde{E}$, $\widetilde{F}$, and $\widetilde{H}$, we establish three isospectral problems, respectively. Then, by taking advantage of the Tu scheme, we get three kinds of coupling integrable couplings of the generalized coupled Burgers equation hierarchy as follows:

$$\begin{cases} q_t = -\beta(\frac{r_{xx}}{4r} + \frac{q_x}{r} + q^2 + \frac{r^2}{2})_x, \\ r_t = \beta(\frac{r_{xx}}{2r} + \frac{q_x}{r} - q_x r - 2qr_x), \\ u_{1t} = -\frac{\beta}{2}u_{2xx} - 2\beta u_{1x}q - \beta u_1 q_x + 2\beta u_1 r - 2\beta u_2 r^2 + \frac{\beta}{2}\frac{u_2 r_{xx}}{r} + \beta\frac{u_2 q_x}{r}, \\ u_{2t} = -\frac{\beta}{2}u_{1xx} - 2\beta u_{2x}q - \beta u_2 q_x + 2\beta u_1 - 2\beta u_2 r + \frac{\beta}{2}\frac{u_1 r_{xx}}{r} + \beta\frac{u_1 q_x}{r}, \\ s_{1t} = -\frac{\beta}{2}s_{2xx} - 2\beta s_{1x}q - \beta s_1 q_x + 2\beta s_1 r - 2\beta s_2 r^2 - 3\beta s_2^2 r + \beta s_1^2 r + 2\beta s_1 s_2 \\ \qquad - \beta s_2^3 + \beta s_1^2 s_2 + \frac{\beta}{2}\frac{s_2 r_{xx}}{r} + \beta\frac{s_2 q_x}{r}, \\ s_{2t} = -\frac{\beta}{2}s_{1xx} - 2\beta s_{2x}q - \beta s_2 q_x + 2\beta s_1 - 2\beta s_2 r + 3\beta s_1^2 r - \beta s_2^2 - 2\beta s_1 s_2 r + \beta s_1^3 \\ \qquad - \beta s_1 s_2^2 + \frac{\beta}{2}\frac{s_1 r_{xx}}{r} + \beta\frac{s_1 q_x}{r}, \end{cases} \tag{6.1}$$

$$\begin{cases} q_t = -\beta(\frac{r_{xx}}{4r} + \frac{q_x}{r} + q^2 + \frac{r^2}{2})_x, \\ r_t = \beta(\frac{r_{xx}}{2r} + \frac{q_x}{r} - q_x r - 2qr_x), \\ u_{1t} = \beta u_{2xx} - 2\beta u_{1x}q - \beta u_1 q_x - \beta u_{2x} - \frac{\beta}{2}u_1 r_x - \beta u_{1x} r - \beta\frac{u_2 r_{xx}}{4r} - \beta\frac{u_2 q_x}{2r}, \\ u_{2t} = \beta u_{1xx} - 2\beta u_{2x}q - \beta u_2 q_x + \beta u_{1x} + \frac{\beta}{2}u_2 r_x + \beta u_{2x} r - \beta\frac{u_1 r_{xx}}{4r} - \beta\frac{u_1 q_x}{2r}, \\ s_{1t} = \beta s_{2xx} - 2\beta s_{1x}q - \beta s_1 q_x - \beta s_{2x} - \frac{\beta}{2}s_1 r_x - \beta s_{1x} r - \beta\frac{s_2 r_{xx}}{4r} - \beta\frac{s_2 q_x}{2r}, \\ s_{2t} = \beta s_{1xx} - 2\beta s_{2x}q - \beta s_2 q_x + \beta s_{1x} + \frac{\beta}{2}s_2 r_x + \beta s_{2x} r - \beta\frac{s_1 r_{xx}}{4r} - \beta\frac{s_1 q_x}{2r}, \end{cases} \tag{6.2}$$

$$\begin{cases} q_t = -\beta(\frac{r_{xx}}{4r} + \frac{q_x}{r} + q^2 + \frac{r^2}{2})_x, \\ r_t = \beta(\frac{r_{xx}}{2r} + \frac{q_x}{r} - q_x r - 2qr_x), \\ u_{1t} = -\frac{\beta}{2}s_{1xx} - 2\beta u_{1x}q - \beta u_1 q_x + 2\beta u_1 r - 2\beta s_1 r^2 + \frac{\beta}{2}\frac{s_1 r_{xx}}{r} + \beta\frac{s_1 q_x}{r}, \\ u_{2t} = -\frac{\beta}{2}s_{2xx} - 2\beta u_{2x}q - \beta u_2 q_x + 2\beta u_2 r - 2\beta s_2 r^2 + \frac{\beta}{2}\frac{s_2 r_{xx}}{r} + \beta\frac{s_2 q_x}{r}, \\ s_{1t} = -\frac{\beta}{2}u_{1xx} - 2\beta s_{1x}q - \beta s_1 q_x + 2\beta u_1 - 2\beta s_1 r + \frac{\beta}{2}\frac{u_1 r_{xx}}{r} + \beta\frac{u_1 q_x}{r}, \\ s_{2t} = -\frac{\beta}{2}u_{2xx} - 2\beta s_{2x}q - \beta s_2 q_x + 2\beta u_2 - 2\beta s_2 r + \frac{\beta}{2}\frac{u_2 r_{xx}}{r} + \beta\frac{u_2 q_x}{r}, \end{cases} \tag{6.3}$$

$$\begin{cases} q_t = -\beta(\frac{r_{xx}}{4r} + \frac{q_x}{r} + q^2 + \frac{r^2}{2})_x, \\ r_t = \beta(\frac{r_{xx}}{2r} + \frac{q_x}{r} - q_x r - 2qr_x), \\ u_{1t} = -\frac{\beta}{2}u_{2xx} - 2\beta u_{1x}q - \beta u_1 q_x + 2\beta u_1 r - 2\beta u_2 r^2 + \frac{\beta}{2}\frac{u_2 r_{xx}}{r} + \beta\frac{u_2 q_x}{r}, \\ u_{2t} = -\frac{\beta}{2}u_{1xx} - 2\beta u_{2x}q - \beta u_2 q_x + 2\beta u_1 - 2\beta u_2 r + \frac{\beta}{2}\frac{u_1 r_{xx}}{r} + \beta\frac{u_1 q_x}{r}, \\ s_{1t} = -\frac{\beta}{2}s_{2xx} - 2\beta s_{1x}q - \beta s_1 q_x + 2\beta s_1 r - 2\beta s_2 r^2 - 3\beta s_2^2 r + \beta s_1^2 r + 2\beta s_1 s_2 \\ \qquad - \beta s_2^3 + \beta s_1^2 s_2 + \frac{\beta}{2}\frac{s_2 r_{xx}}{r} + \beta\frac{s_2 q_x}{r}, \\ s_{2t} = -\frac{\beta}{2}s_{1xx} - 2\beta s_{2x}q - \beta s_2 q_x + 2\beta s_1 - 2\beta s_2 r + 3\beta s_1^2 r - \beta s_2^2 - 2\beta s_1 s_2 r + \beta s_1^3 \\ \qquad - \beta s_1 s_2^2 + \frac{\beta}{2}\frac{s_1 r_{xx}}{r} + \beta\frac{s_1 q_x}{r}. \end{cases} \tag{6.4}$$

Finally, we obtain the Hamiltonian structure of one of coupling integrable couplings of the generalized coupled Burgers equation hierarchy by using the quadratic-form identity as follows:

$$u_t = J\frac{\delta H_{n+2}}{\delta u} = JL^n \begin{pmatrix} 0 \\ -\beta r - \beta u_2 - \beta s_2 \\ \beta \\ -\beta r \\ \beta + \beta s_1 \\ -\beta r - \beta s_2 \end{pmatrix}. \tag{6.5}$$

In this paper we can only get the Hamiltonian structure of one of the coupling integrable couplings of the generalized coupled Burgers equation hierarchy by using the quadratic-form identity. How to get all the Hamiltonian structure of one coupling integrable couplings of the generalized coupled Burgers equation hierarchy is worthy of further studying.

7 Methods and experiment
Not applicable.

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