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Stability and Hopf bifurcation for a stage-structured predator–prey model incorporating refuge for prey and additional food for predator

Yuzhen Bai[1*] and Yunyun Li[1,2]

*Correspondence:
baiyu99@126.com
[1]School of Mathematical Sciences, Qufu Normal University, Qufu, P.R. China
Full list of author information is available at the end of the article

Abstract

In this paper, we study a stage-structured predator–prey model incorporating refuge for prey and additional food for predator. By analyzing the corresponding characteristic equations, we investigate the local stability of equilibria and the existence of Hopf bifurcation at the positive equilibrium taking the time delay as a bifurcation parameter. Furthermore, we obtain the direction of the Hopf bifurcation and the stability of bifurcating periodic solutions applying the center manifold theorem and normal form theory. Numerical simulations are illustrated to verify our main results.

MSC: 34B27; 34B05; 34B15

Keywords: Stage-structured predator–prey model; Refuge; Additional food; Hopf bifurcation

1 Introduction

Since the first mathematical model for predator–prey was developed independently by Lotka [1] and Volterra [2], the predator–prey models in ecology have received great attention [3–7]. Researchers studied the predator–prey models by analyzing their life history. In the natural world, species can be divided into two stages: immaturity and maturity. Therefore, the predator–prey models with stage structure are more reasonable than the ones without stage structure. With the prey species as immature individual organisms, we suppose that they are not attacked by predators, but as mature individuals, in order to reduce their rate of encounter with predators, prey refuges play an important role in affording the prey some degree of protection from predation. Kuang [4] showed that a time delay could destroy the stability of the positive equilibrium and cause a Hopf bifurcation. The delayed predator–prey models with stage structure or refuge have been studied by many authors, see [8–12]. Especially, Wei and Fu [13] investigated Hopf bifurcation and stability of a delayed predator–prey model with stage structure for prey incorporating prey refuge,

$$\begin{cases} \dot{x}_1(t) = ax_2(t) - bx_1(t) - \alpha x_1(t), \\ \dot{x}_2(t) = \alpha x_1(t) - cx_2(t) - dx_2^2(t) - \frac{\beta(1-m)x_2(t)y(t)}{a_1+b_1(1-m)x_2(t)+c_1y(t)}, \\ \dot{y}(t) = \frac{d\beta(1-m)x_2(t-\tau)y(t-\tau)}{a_1+b_1(1-m)x_2(t-\tau)+c_1y(t-\tau)} - ry(t), \end{cases} \tag{1.1}$$

where $x_1(t)$, $x_2(t)$ and $y(t)$ denote the densities of immature prey, mature prey and predator at time t, respectively. m is a refuge parameter with $m \in [0,1)$, $\tau \geq 0$ is the time delay due to the gestation of the predator.

Prey refuge can protect the prey from the attack of predators in some degree. What will happen if the predators cannot eat the prey? Now, additional food is very important for the predators. In fact, additional food is an important component of most predators. Recently, the effects of the additional food to predator in prey–predator models were investigated [14–19]. Srinivasu et al. [14] reported the dynamics of prey–predator system in the presence of additional food for predator and discussed the effect of quality and quantity of the additional food. Ghosh et al. [18] considered a predator–prey model with logistic growth rate and prey refuge in presence of additional food for predator

$$\begin{cases} \dot{N}(t) = r_1N(1-\frac{N}{K}) - \frac{c_1(1-c')e_1NP}{a+h_2e_2A'+h_1e_1N}, \\ \dot{P}(t) = \frac{b_1[(1-c')e_1N+e_2A']P}{a+h_2e_2A'+h_1e_1N} - rP, \end{cases} \tag{1.2}$$

where $N(t)$ and $P(t)$ represent the densities of the prey and predator at time t, respectively. The parameters c' is a refuge parameter with $c' \in [0,1)$.

Motivated by the above work, we propose a delayed predator–prey model with stage structure for prey incorporating refuge and providing additional food to the predator,

$$\begin{cases} \dot{x}_1(t) = ax_2(t) - bx_1(t) - \alpha x_1(t), \\ \dot{x}_2(t) = \alpha x_1(t) - cx_2(t) - dx_2^2(t) - \frac{k_1(1-m)e_1x_2(t)y(t)}{a_1+h_2e_2A'+h_1e_1x_2(t)}, \\ \dot{y}(t) = \frac{k_2[(1-m)e_1x_2(t-\tau)+e_2A']y(t-\tau)}{a_1+h_2e_2A'+h_1e_1x_2(t-\tau)} - ry(t), \end{cases} \tag{1.3}$$

where $x_1(t)$, $x_2(t)$ and $y(t)$ denote the densities of immature prey species, mature prey species and predator species at time t, respectively. a is the intrinsic growth rate of the immature prey species. b, c and r denote the death rates of immature prey, mature prey and predator, respectively. α is the transformation rate from immature prey to mature prey. d is intra species competition rate of mature prey. m is a refuge parameter with $m \in [0,1)$, $k_1(1-m)$ is the capturing rate of the predator. k_2 is the conversion rate of nutrients into the production of predator species. $\tau \geq 0$ is the time delay due to the gestation of the predator. h_1 and e_1, respectively represent the handling time of the predator per unit quantity of mature prey, ability of the predator to detect the mature prey. h_2 and e_2, respectively, represent the handling time of the predator per unit quantity of additional food, the ability of the predator to identify the additional food. A' represents the biomass of the additional food. All the parameters are nonnegative constants.

Define $k_1 := \frac{k_1}{h_1}$, $k_2 := \frac{k_2}{h_1}$, $a_1 := \frac{a_1}{e_1 h_1}$, $\beta = \frac{h_2}{h_1}$, $\eta = \frac{e_2}{e_1}$. The model (1.3) can be written as

$$\begin{cases} \dot{x}_1(t) = ax_2(t) - bx_1(t) - \alpha x_1(t), \\ \dot{x}_2(t) = \alpha x_1(t) - cx_2(t) - dx_2^2(t) - \frac{k_1(1-m)x_2(t)y(t)}{a_1+\beta\eta A'+x_2(t)}, \\ \dot{y}(t) = \frac{k_2[(1-m)x_2(t-\tau)+\eta A']y(t-\tau)}{a_1+\beta\eta A'+x_2(t-\tau)} - ry(t). \end{cases} \tag{1.4}$$

By denoting $u_1(t) = \frac{x_1(t)}{a_1}$, $u_2(t) = \frac{x_2(t)}{a_1}$, $v(t) = \frac{k_1 y(t)}{a_1}$, $d_1 = a_1 d$, $\xi = \frac{\eta A'}{a_1}$, the model (1.4) reduces to the following form:

$$\begin{cases} \dot{u}_1(t) = au_2(t) - bu_1(t) - \alpha u_1(t), \\ \dot{u}_2(t) = \alpha u_1(t) - cu_2(t) - d_1 u_2^2(t) - \frac{(1-m)u_2(t)v(t)}{1+\beta\xi+u_2(t)}, \\ \dot{v}(t) = \frac{k_2[(1-m)u_2(t-\tau)+\xi]v(t-\tau)}{1+\beta\xi+u_2(t-\tau)} - rv(t), \end{cases} \tag{1.5}$$

where the term β and ξ are the parameters which characterize the "quality" and "quantity" of additional food, respectively. The initial conditions for model (1.5) take the form

$$\begin{cases} u_1(\theta) = \varphi_1(\theta) \geq 0, \quad u_2(\theta) = \varphi_2(\theta) \geq 0, \quad v(\theta) = \varphi_3(\theta) \geq 0, \\ \theta \in [-\tau, 0), \quad \varphi_1(0) > 0, \quad \varphi_2(0) > 0, \quad \varphi_3(0) > 0, \end{cases} \tag{1.6}$$

where $(\varphi_1(\theta), \varphi_2(\theta), \varphi_3(\theta)) \in C\{[-\tau, 0], R_+^3\}$, $R_+^3 = \{(u_1, u_2, v) : u_1 \geq 0, u_2 \geq 0, v \geq 0\}$.

From the fundamental theory of functional differential equations [20], the model (1.5) has a unique solution $(u_1(t), u_2(t), v(t))$ satisfying the initial conditions (1.6). It is easy to show that all solutions of (1.5) with initial conditions (1.6) are defined on $[0, +\infty)$ and remain positive for all $t \geq 0$.

The main contributions of the present paper are: (1) A stage-structured predator–prey model incorporating refuge for prey and additional food for predator is formulated. (2) The existence and local stability of equilibria and the existence of Hopf bifurcation of the model are given. (3) The direction of the Hopf bifurcation and the stability of bifurcating periodic solutions are obtained by applying the center manifold theorem and the normal form theory. (4) Numerical simulations are illustrated to show our main results.

In this paper, we assume the following conditions hold.

(H_1) $a\alpha - (b+\alpha)c > 0$; ($H_2$) $a\alpha - (b+\alpha)c < 0$;

(H_3) $r + (r\beta - k_2)\xi > 0$; ($H_4$) $r + (r\beta - k_2)\xi < 0$;

(H_5) $k_2(1-m) - r > 0$; (H_6) $k_2(1-m) - r < 0$;

(H_7) $0 < u_2^* < \dfrac{a\alpha - (b+\alpha)c}{d_1(b+\alpha)}$; ($H_8$) $(b+\alpha)c < a\alpha < (b+\alpha)(c + 2d_1u_2^* + A)$;

(H_9) $(b+\alpha)c < a\alpha < (b+\alpha)(c + 2d_1u_2^*)$; ($H_{10}$) $(b+\alpha)(c + 2d_1u_2^* + A) < a\alpha$;

where u_2^*, A can be found in Sect. 2 and Sect. 3, respectively.

Now we give the biological interpretation of the conditions (H_1)–(H_{10}).

From (H_1), we have $a\alpha > (b+\alpha)c$, which means that the prey species keeps a linear net growth without the predator species. The condition (H_3) can be rewritten in the form

$r > \frac{k_2\xi}{1+\beta\xi}$. It means that additional food cannot ensure the survival of the predator species without the prey species. The condition (H_5) is explained that prey species can ensure the survival of predator species without additional food even if the prey species have refuge. It is clear that the conditions (H_2), (H_4), (H_6) have opposite interpretation with (H_1), (H_3), (H_5), respectively. The term $\frac{a\alpha-(b+\alpha)c}{d_1(b+\alpha)}$ in condition (H_7) is the ratio of net growth with their own retarded growth of prey species. This ratio is a critical value for u_2^*. The condition (H_8) can be simplified as $0 < a\alpha - (b+\alpha)c < (b+\alpha)(2d_1u_2^* + A)$. It implies that on the one hand the prey species keep linear net growth and on the other hand this growth is limited by some value. Obviously, if (H_9) holds, then (H_8) holds. The condition (H_{10}) shows that the linear net growth of the prey species is higher than the limit value.

2 Equilibria of the model (1.5)

In order to obtain the equilibria of the model (1.5), we consider the prey nullcline and predator nullcline of this model, which are given by

$$\begin{cases} au_2 - bu_1 - \alpha u_1 = 0, \\ \alpha u_1 - cu_2 - d_1u_2^2 - \frac{(1-m)u_2v}{1+\beta\xi+u_2} = 0, \\ \frac{k_2[(1-m)u_2+\xi]v}{1+\beta\xi+u_2} - rv = 0. \end{cases}$$

Obviously, the model (1.5) always has a trivial equilibrium $E_0(0,0,0)$.

If the condition (H_1) holds, then the model (1.5) has a predator-extinction equilibrium $E_1(\bar{u}_1, \bar{u}_2, 0)$, where $\bar{u}_1 = \frac{a[a\alpha-(b+\alpha)c]}{(b+\alpha)^2d_1}$, $\bar{u}_2 = \frac{a\alpha-(b+\alpha)c}{(b+\alpha)d_1}$.

If the conditions (H_1), (H_3), (H_5) and (H_7) hold, which imply

$$\beta > \frac{k_2}{r} - \frac{1}{\xi}, \quad \text{and} \quad 0 < m < \min\left\{1 - \frac{r}{k_2}, 1 - \frac{r}{k_2} - \frac{d_1[r + (r\beta - k_2)\xi](b+\alpha)}{\alpha k_2(a-c) - bc}\right\},$$

then there exists a unique coexisting equilibrium $E_2(u_1^*, u_2^*, v^*)$ of the model (1.5), where

$$u_1^* = \frac{a}{b+\alpha}u_2^*, \qquad u_2^* = \frac{r + (r\beta - k_2)\xi}{k_2(1-m) - r},$$
$$v^* = \frac{[a\alpha - (b+\alpha)c - d_1(b+\alpha)u_2^*](1 + \beta\xi + u_2^*)}{(1-m)(b+\alpha)}.$$

3 Local stability of the equilibria

Let $E(u_1, u_2, v)$ be any arbitrary equilibrium, then Jacobian matrix at E is given by

$$J_{(u_1,u_2,v)} = \begin{pmatrix} -b-\alpha & a & 0 \\ \alpha & -c - 2d_1u_2 - \frac{(1-m)(1+\beta\xi)v}{(1+\beta\xi+u_2)^2} & -\frac{(1-m)u_2}{1+\beta\xi+u_2} \\ 0 & \frac{k_2[(1-m)(1+\beta\xi)-\xi]ve^{-\lambda\tau}}{(1+\beta\xi+u_2)^2} & \frac{k_2[(1-m)u_2+\xi]e^{-\lambda\tau}}{1+\beta\xi+u_2} - r \end{pmatrix}.$$

(a) Trivial equilibrium point: At the trivial equilibrium point $E_0(0,0,0)$, the Jacobian matrix is given by

$$J_{(0,0,0)} = \begin{pmatrix} -b-\alpha & a & 0 \\ \alpha & -c & 0 \\ 0 & 0 & \frac{k_2\xi e^{-\lambda\tau}}{1+\beta\xi} - r \end{pmatrix},$$

and the characteristic equation at E_0 becomes

$$\left(\lambda + r - \frac{k_2\xi e^{-\lambda\tau}}{1+\beta\xi}\right)\left[\lambda^2 + (b+\alpha+c)\lambda + c(b+\alpha) - a\alpha\right] = 0, \tag{3.1}$$

then the equation

$$\lambda^2 + (b+\alpha+c)\lambda + c(b+\alpha) - a\alpha = 0$$

has two roots, and we have $\lambda_1 + \lambda_2 = -(b+\alpha+c) < 0$, $\lambda_1\lambda_2 = c(b+\alpha) - a\alpha$.

If (H_1) holds, then $\lambda_1\lambda_2 < 0$, that is, E_0 is an unstable saddle; If (H_2) holds, then $\lambda_1\lambda_2 > 0$, that is, $\mathrm{Re}(\lambda_i) < 0$, $i = 1,2$. Another root of (3.1) is determined by the equation

$$\lambda + r - \frac{k_2\xi e^{-\lambda\tau}}{1+\beta\xi} = 0. \tag{3.2}$$

Denote

$$f_1(\lambda) = \lambda + r - \frac{k_2\xi e^{-\lambda\tau}}{1+\beta\xi}.$$

If (H_2) and (H_3) hold, we claim that E_0 is locally asymptotically stable. Otherwise, there is a root λ satisfying $\mathrm{Re}(\lambda) \geq 0$, it follows from (3.2) that

$$\mathrm{Re}(\lambda) = \frac{k_2\xi}{1+\beta\xi}e^{-\tau\,\mathrm{Re}\,\lambda}\cos(\tau\,\mathrm{Im}\,\lambda) - r \leq \frac{k_2\xi}{1+\beta\xi} - r < 0,$$

which is contradiction. Hence the equilibrium E_0 is locally asymptotically stable.

If (H_4) holds, it is easy to show that, for real λ, $f_1(0) = r - \frac{k_2\xi}{1+\beta\xi} < 0$, and

$$\lim_{\lambda\to+\infty} f_1(\lambda) = +\infty.$$

Hence, $f_1(\lambda) = 0$ has a positive real root.

From the above discussions, we can get the following theorem.

Theorem 3.1 *For the model* (1.5):

(i) *If* (H_1) *or* (H_4) *holds, then the trivial equilibrium* $E_0(0,0,0)$ *is unstable.*

(ii) *If* (H_2) *and* (H_3) *hold, then the trivial equilibrium* $E_0(0,0,0)$ *is locally asymptotically stable.*

Remark 3.1 It is easy to understand Theorem 3.1 from the biological meaning of (H_1)–(H_4).

(b) Predator-extinction equilibrium point: At equilibrium point $E_1(\bar{u}_1,\bar{u}_2,0)$, the Jacobian matrix is given by

$$J_{(\bar{u}_1,\bar{u}_2,0)} = \begin{pmatrix} -b-\alpha & a & 0 \\ \alpha & -c-2d_1\bar{u}_2 & -\frac{(1-m)\bar{u}_2}{1+\beta\xi+\bar{u}_2} \\ 0 & 0 & \frac{k_2[(1-m)\bar{u}_2+\xi]e^{-\lambda\tau}}{1+\beta\xi+\bar{u}_2} - r \end{pmatrix},$$

and the characteristic equation at E_1 becomes

$$\left(\lambda + r - \frac{k_2[(1-m)\bar{u}_2 + \xi]e^{-\lambda\tau}}{1+\beta\xi+\bar{u}_2}\right)\left[\lambda^2 + (b+\alpha+c+2d_1\bar{u}_2)\lambda + (b+\alpha)(c+2d_1\bar{u}_2) - a\alpha\right] = 0, \tag{3.3}$$

then the equation

$$\lambda^2 + (b+\alpha+c+2d_1\bar{u}_2)\lambda + (b+\alpha)(c+2d_1\bar{u}_2) - a\alpha = 0$$

has two roots, and

$$\begin{aligned}\lambda_1+\lambda_2 &= -(b+\alpha+c+2d_1\bar{u}_2) < 0,\\ \lambda_1\lambda_2 &= (b+\alpha)(c+2d_1\bar{u}_2) - a\alpha\\ &= a\alpha - c(b+\alpha).\end{aligned}$$

If (H_1) holds, then $\lambda_1\lambda_2 > 0$, that is $\operatorname{Re}\lambda_i < 0$, $i = 1,2$. Another root of (3.3) is determined by

$$\lambda + r - \frac{k_2[(1-m)\bar{u}_2+\xi]}{1+\beta\xi+\bar{u}_2}e^{-\lambda\tau} = 0. \tag{3.4}$$

Denote

$$f_2(\lambda) = \lambda + r - \frac{k_2[(1-m)\bar{u}_2+\xi]}{1+\beta\xi+\bar{u}_2}e^{-\lambda\tau}.$$

If (H_4) and (H_5) hold, it is easy to show that, for real λ,

$$\begin{aligned}f_2(0) &= r - \frac{k_2[(1-m)\bar{u}_2+\xi]}{1+\beta\xi+\bar{u}_2}\\ &= \frac{1}{1+\beta\xi+\bar{u}_2}\left[r(1+\beta\xi) - k_2\xi + \left[r - k_2(1-m)\right]\bar{u}_2\right]\\ &< 0,\end{aligned}$$

and $\lim_{\lambda\to+\infty} f_2(\lambda) = +\infty$. Hence, $f_2(\lambda) = 0$ has a positive real root.

If (H_3) and (H_6) hold, we have $f_2(0) > 0$. We claim that E_1 is locally asymptotically stable. Otherwise, there is a root λ satisfying $\operatorname{Re}\lambda \geq 0$. It follows from (3.4) that

$$\begin{aligned}\operatorname{Re}\lambda &= \frac{k_2[(1-m)\bar{u}_2+\xi]}{1+\beta\xi+\bar{u}_2}e^{-\tau\operatorname{Re}\lambda}\cos(\tau\operatorname{Im}\lambda) - r\\ &\leq \frac{k_2[(1-m)\bar{u}_2+\xi]}{1+\beta\xi+\bar{u}_2} - r\\ &= -f_2(0) < 0,\end{aligned}$$

which is a contradiction. Hence, when (H_3) and (H_6) hold, then $\operatorname{Re}\lambda < 0$.

Based on the above discussions, the following theorem can be obtained.

Theorem 3.2 *Suppose that* (H_1) *holds. For the model* (1.5), *we have*:

(i) *If* (H_4) *and* (H_5) *hold, then the predator-extinction equilibrium* $E_1(\bar{u}_1, \bar{u}_2, 0)$ *is unstable.*

(ii) *If* (H_3) *and* (H_6) *hold, then the predator-extinction equilibrium* $E_1(\bar{u}_1, \bar{u}_2, 0)$ *is locally asymptotically stable.*

Remark 3.2 It is easy to understand Theorem 3.2 from the biological meaning of (H_3)–(H_6).

(c) Co-existing equilibrium point: At the coexisting equilibrium point $E_2(u_1^*, u_2^*, v^*)$, the Jacobian matrix is given by

$$J_{(u_1^*,u_2^*,v^*)} = \begin{pmatrix} -b-\alpha & a & 0 \\ \alpha & -c-2d_1u_2^* - \frac{(1-m)(1+\beta\xi)v^*}{(1+\beta\xi+u_2^*)^2} & -\frac{(1-m)u_2^*}{1+\beta\xi+u_2^*} \\ 0 & \frac{k_2[(1-m)(1+\beta\xi)-\xi]v^*}{(1+\beta\xi+u_2^*)^2}e^{-\lambda\tau} & \frac{k_2[(1-m)u_2^*+\xi]e^{-\lambda\tau}}{1+\beta\xi+u_2^*} - r \end{pmatrix},$$

and the characteristic equation at E_2 becomes

$$\begin{aligned} &\lambda^3 + \left(b+\alpha+c+2d_1u_2^*+A+r-Be^{-\lambda\tau}\right)\lambda^2 \\ &\quad + \Bigg[(b+\alpha)\left(c+2d_1u_2^*+A\right) - a\alpha \\ &\quad + \left(r-Be^{-\lambda\tau}\right)\left(b+\alpha+c+2d_1u_2^*+A\right) + \frac{k_2C(1-m)u_2^*}{1+\beta\xi+u_2^*}e^{-\lambda\tau}\Bigg]\lambda \\ &\quad + \left(r-Be^{-\lambda\tau}\right)\left[(b+\alpha)\left(c+2d_1u_2^*+A\right)-a\alpha\right] + \frac{k_2C(1-m)(b+\alpha)u_2^*}{1+\beta\xi+u_2^*}e^{-\lambda\tau} = 0, \end{aligned} \tag{3.5}$$

where $A = \frac{(1-m)(1+\beta\xi)}{(1+\beta\xi+u_2^*)^2}v^* > 0$, $B = \frac{k_2[(1-m)u_2^*+\xi]}{1+\beta\xi+u_2^*} = r > 0$, and $C = A - \frac{\xi v^*}{(1+\beta\xi+u_2^*)^2} > 0$, when $m < 1 - \frac{\xi}{1+\beta\xi}$.

One can rewrite (3.5) so that it has the following form:

$$\begin{aligned} &\lambda^3 + \left(b+\alpha+c+2d_1u_2^*+A+r\right)\lambda^2 \\ &\quad + \left[(b+\alpha)\left(c+2d_1u_2^*+A\right) - a\alpha + r\left(b+\alpha+c+2d_1u_2^*+A\right)\right]\lambda \\ &\quad + r\left[(b+\alpha)\left(c+2d_1u_2^*+A\right)-a\alpha\right] \\ &\quad + e^{-\lambda\tau}\Bigg[-r\lambda^2 + \left[\frac{k_2C(1-m)u_2^*}{1+\beta\xi+u_2^*} - r\left(b+\alpha+c+2d_1u_2^*+A\right)\right]\lambda \\ &\quad + \frac{k_2C(1-m)(b+\alpha)u_2^*}{1+\beta\xi+u_2^*} - r\left[(b+\alpha)\left(c+2d_1u_2^*+A\right)-a\alpha\right]\Bigg] = 0. \end{aligned} \tag{3.6}$$

Let

$$\begin{aligned} P_1 &= b+\alpha+c+2d_1u_2^*+A+r, \\ P_2 &= (b+\alpha)\left(c+2d_1u_2^*+A\right) - a\alpha + r\left(b+\alpha+c+2d_1u_2^*+A\right), \\ P_3 &= r\left[(b+\alpha)\left(c+2d_1u_2^*+A\right)-a\alpha\right], \end{aligned}$$

$$P_4 = \frac{k_2C(1-m)u_2^*}{1+\beta\xi+u_2^*} - r\bigl(b+\alpha+c+2d_1u_2^*+A\bigr),$$
$$P_5 = \frac{k_2C(1-m)(b+\alpha)u_2^*}{1+\beta\xi+u_2^*} - r\bigl[(b+\alpha)\bigl(c+2d_1u_2^*+A\bigr)-a\alpha\bigr].$$

Equation (3.6) can be written as

$$\lambda^3 + P_1\lambda^2 + P_2\lambda + P_3 + e^{-\lambda\tau}\bigl(-r\lambda^2 + P_4\lambda + P_5\bigr) = 0. \tag{3.7}$$

Case 3.1. $\tau = 0$.
Equation (3.7) turns to

$$\lambda^3 + (P_1 - r)\lambda^2 + (P_2 + P_4)\lambda + (P_3 + P_5) = 0, \tag{3.8}$$

then

$$\begin{aligned}
P_1 - r &= b+\alpha+c+2d_1u_2^*+A+r-r = b+\alpha+c+2d_1u_2^*+A > 0,\\
P_3 + P_5 &= r\bigl[(b+\alpha)\bigl(c+2d_1u_2^*+A\bigr)-a\alpha\bigr] + \frac{k_2C(1-m)(b+\alpha)u_2^*}{1+\beta\xi+u_2^*}\\
&\quad - r\bigl[(b+\alpha)\bigl(c+2d_1u_2^*+A\bigr)-a\alpha\bigr]\\
&= \frac{k_2C(1-m)(b+\alpha)u_2^*}{1+\beta\xi+u_2^*} > 0,\\
&(P_1 - r)(P_2 + P_4) - (P_3 + P_5)\\
&\quad = \bigl(b+\alpha+c+2d_1u_2^*+A\bigr)\Bigl[(b+\alpha)\bigl(c+2d_1u_2^*+A\bigr)-a\alpha\\
&\qquad + r\bigl(b+\alpha+c+2d_1u_2^*+A\bigr) + \frac{k_2C(1-m)u_2^*}{1+\beta\xi+u_2^*} - r\bigl(b+\alpha+c+2d_1u_2^*+A\bigr)\Bigr]\\
&\qquad - \frac{k_2C(1-m)(b+\alpha)u_2^*}{1+\beta\xi+u_2^*}\\
&\quad = \bigl(c+2d_1u_2^*+A\bigr)\frac{k_2C(1-m)u_2^*}{1+\beta\xi+u_2^*}\\
&\qquad + \bigl(b+\alpha+c+2d_1u_2^*+A\bigr)\bigl[(b+\alpha)\bigl(c+2d_1u_2^*+A\bigr)-a\alpha\bigr].
\end{aligned}$$

If the condition (H_8) holds, then $(P_1 - r)(P_2 + P_4) - (P_3 + P_5) > 0$. By the Routh–Hurwitz criterion, we see that the coexisting equilibrium point E_2 is locally asymptotically stable.
Case 3.2. $\tau > 0$.
Let $\lambda = i\omega$ ($\omega > 0$) be a root of (3.7), then

$$\bigl(-i\omega^3 + i\omega P_2 + P_3 - P_1\omega^2\bigr) + (\cos\omega\tau - i\sin\omega\tau)\bigl(iP_4\omega + r\omega^2 + P_5\bigr) = 0. \tag{3.9}$$

Separating real part and imaginary part of (3.9), we have

$$\begin{aligned}
P_4\omega\cos\omega\tau - \bigl(r\omega^2 + P_5\bigr)\sin\omega\tau &= \omega^3 - P_2\omega,\\
P_4\omega\sin\omega\tau + \bigl(r\omega^2 + P_5\bigr)\cos\omega\tau &= P_1\omega^2 - P_3,
\end{aligned}$$

that is,

$$\begin{cases} \cos\omega\tau = \frac{(rP_1+P_4)\omega^4-(rP_3+P_2P_4-P_1P_5)\omega^2-P_3P_5}{(P_4\omega)^2+(P_5+r\omega^2)^2}, \\ \sin\omega\tau = \frac{-r\omega^5+(P_1P_5-P_5+rP_2)\omega^2+(P_2P_5-P_3P_4)\omega}{(P_4\omega)^2+(P_5+r\omega^2)^2}. \end{cases} \tag{3.10}$$

Taking the square on both sides of (3.10) implies that

$$\omega^6 + \left(P_1^2 - 2P_2 - r^2\right)\omega^4 + \left(P_2^2 - 2P_1P_3 - P_4^2 - 2rP_5\right)\omega^2 + P_3^2 - P_5^2 = 0. \tag{3.11}$$

Suppose $\nu = \omega^2$. Then (3.11) becomes

$$\nu^3 + \left(P_1^2 - 2P_2 - r^2\right)\nu^2 + \left(P_2^2 - 2P_1P_3 - P_4^2 - 2rP_5\right)\nu + P_3^2 - P_5^2 = 0, \tag{3.12}$$

where

$$\begin{aligned} P_1^2 - 2P_2 - r^2 &= \left(b + \alpha + c + 2d_1u_2^* + A + r\right)^2 - 2\left[(b+\alpha)\left(c + 2d_1u_2^* + A\right)\right. \\ &\quad \left. - a\alpha + r\left(b + \alpha + c + 2d_1u_2^* + A\right)\right] - r^2 \\ &= (b+\alpha)^2 + \left(c + 2d_1u_2^* + A\right)^2 + 2a\alpha > 0, \end{aligned}$$

$$\begin{aligned} &P_2^2 - 2P_1P_3 - P_4^2 - 2rP_5 \\ &\quad = \left[(b+\alpha)\left(c + 2d_1u_2^* + A\right) - a\alpha + r\left(b + \alpha + c + 2d_1u_2^* + A\right)\right]^2 \\ &\quad\quad - 2r\left(b + \alpha + c + 2d_1u_2^* + A + r\right)\left[(b+\alpha)\left(c + 2d_1u_2^* + A\right) - a\alpha\right] \\ &\quad\quad - 2r\left[\frac{k_2C(1-m)(b+\alpha)u_2^*}{1 + \beta\xi + u_2^*} - r\left[(b+\alpha)\left(c + 2d_1u_2^* + A\right) - a\alpha\right]\right] \\ &\quad\quad - \left[\frac{k_2C(1-m)u_2^*}{1 + \beta\xi + u_2^*} - r\left(b + \alpha + c + 2d_1u_2^* + A\right)\right]^2. \end{aligned}$$

Denoting $m_1 = b + \alpha$, $m_2 = c + 2d_1u_2^* + A$, $m_3 = \frac{k_2C(1-m)u_2^*}{1+\beta\xi+u_2^*}$, we have

$$\begin{aligned} &P_2^2 - 2P_1P_3 - P_4^2 - 2rP_5 \\ &\quad = \left[m_1m_2 - a\alpha + r(m_1 + m_2)\right]^2 - 2r(m_1 + m_2 + r)(m_1m_2 - a\alpha) \\ &\quad\quad - 2r\left[m_3(b+\alpha) - r(m_1m_2 - a\alpha)\right] - \left[m_3 - r(m_1 + m_2)\right]^2 \\ &\quad = (m_1m_2 - a\alpha)^2 + 2rm_3\left[m_1 + m_2 - (b+\alpha)\right] - m_3^2 \\ &\quad = \left[(b+\alpha)\left(c + 2d_1u_2^* + A\right) - a\alpha\right]^2 \\ &\quad\quad + \frac{k_2C(1-m)u_2^*}{1 + \beta\xi + u_2^*}\left[2r\left(c + 2d_1u_2^* + A\right) - \frac{k_2C(1-m)u_2^*}{1 + \beta\xi + u_2^*}\right] \\ &\quad = \left[(b+\alpha)\left(c + 2d_1u_2^* + A\right) - a\alpha\right]^2 \\ &\quad\quad + \frac{k_2C(1-m)u_2^*}{1 + \beta\xi + u_2^*}\left[2\frac{k_2[(1-m)u_2^* + \xi]}{1 + \beta\xi + u_2^*}\left(c + 2d_1u_2^* + A\right) - \frac{k_2C(1-m)u_2^*}{1 + \beta\xi + u_2^*}\right] \\ &\quad = \left[(b+\alpha)\left(c + 2d_1u_2^* + A\right) - a\alpha\right]^2 \end{aligned}$$

$$
\begin{aligned}
&+\frac{k_2^2C(1-m)u_2^*}{1+\beta\xi+u_2^*}\times\left[\left(c+2d_1u_2^*\right)\left[2(1-m)u_2^*+\xi\right]+\xi v^*+\frac{\xi(1-m)u_2^*v^*}{(1+\beta\xi+u_2^*)^2}\right]\\
&>0,\\
P_3^2-P_5^2&=(P_3+P_5)(P_3-P_5)\\
&=\frac{k_2C(1-m)(b+\alpha)u_2^*}{1+\beta\xi+u_2^*}\Bigg[r\left[(b+\alpha)\left(c+2d_1u_2^*+A\right)-a\alpha\right]\\
&\quad-\frac{k_2C(1-m)(b+\alpha)u_2^*}{1+\beta\xi+u_2^*}+r\left[(b+\alpha)\left(c+2d_1u_2^*+A\right)-a\alpha\right]\Bigg]\\
&=\frac{k_2C(1-m)(b+\alpha)u_2^*}{1+\beta\xi+u_2^*}\Bigg[\frac{k_2C(1-m)(b+\alpha)u_2^*}{1+\beta\xi+u_2^*}\\
&\quad+\frac{2k_2(1-m)u_2^*[(b+\alpha)(c+2d_1u_2^*)-a\alpha]}{1+\beta\xi+u_2^*}\\
&\quad+\frac{2k_2\xi[(b+\alpha)(c+2d_1u_2^*+A)-a\alpha]}{1+\beta\xi+u_2^*}+\frac{k_2\xi(1-m)(b+\alpha)u_2^*v^*}{(1+\beta\xi+u_2^*)^3}\Bigg].
\end{aligned}
$$

If (H_9) holds, then we have $a\alpha<(b+\alpha)(c+2d_1u_2^*+A)$. Obviously, if ($H_9$) holds, it implies that $P_3^2-P_5^2>0$, and $(P_1-r)(P_2+P_4)-(P_3+P_5)>0$, then (3.11) has no positive real roots. Therefore, by Theorem 3.4.1 in [10], all roots of (3.11) have negative real parts for all $\tau\ge 0$, which implies that the positive equilibrium $E_2(u_1^*,u_2^*,v^*)$ is locally asymptotically stable for all $\tau\ge 0$.

If (H_{10}) $a\alpha>(b+\alpha)(c+2d_1u_2^*+A)$ holds, which implies that

$$P_3-P_5=2r\left[(b+\alpha)\left(c+2d_1u_2^*+A\right)-a\alpha\right]-\frac{k_2C(1-m)(b+\alpha)u_2^*}{1+\beta\xi+u_2^*}<0,$$

then $P_3^2-P_5^2<0$. Hence, there exists a unique positive root ω_0 satisfying (3.11). From (3.10), we get

$$
\begin{cases}
\cos\omega_0\tau=\frac{(rP_1+P_4)\omega_0^4-(rP_3+P_2P_4-P_1P_5)\omega_0^2-P_3P_5}{(P_4\omega_0)^2+(P_5+r\omega_0^2)^2},\\
\sin\omega_0\tau=\frac{-r\omega_0^5+(P_1P_5-P_5+rP_2)\omega_0^2+(P_2P_5-P_3P_4)\omega_0}{(P_4\omega_0)^2+(P_5+r\omega_0^2)^2}.
\end{cases}
$$

Denote

$$\tau_n=\frac{1}{\omega_0}\arccos\frac{(rP_1+P_4)\omega_0^4-(rP_3+P_2P_4-P_1P_5)\omega_0^2-P_3P_5}{(P_4\omega_0)^2+(P_5+r\omega_0^2)^2}+\frac{2n\pi}{\omega_0},\quad n=0,1,2,\ldots.$$

Taking $\tau_0=\min\{\tau_n:n=0,1,2,\ldots\}$, we see that $\pm i\omega_0$ is a pair of purely imaginary roots of (3.7) with $\tau=\tau_n$. Differentiating the two sides of (3.7) with respect to τ, it follows that

$$\left(3\lambda^2+2P_1\lambda+P_2\right)\frac{d\lambda}{d\tau}+(-2r\lambda+P_4)e^{-\lambda\tau}\frac{d\lambda}{d\tau}+\left(-r\lambda^2+P_4\lambda+P_5\right)\left(-\tau e^{-\lambda\tau}\frac{d\lambda}{d\tau}-\lambda e^{-\lambda\tau}\right)=0,$$

then

$$\left(\frac{d\lambda}{d\tau}\right)^{-1}=-\frac{3\lambda^2+2P_1\lambda+P_2}{\lambda(\lambda^3+P_1\lambda^2+P_2\lambda+P_3)}+\frac{-2r\lambda+P_4}{\lambda(-r\lambda^2+P_4\lambda+P_5)}-\frac{\tau}{\lambda},$$

$$\begin{aligned}
&\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\lambda=i\omega_0}\\
&= \frac{3\omega_0^2 - P_2 - 2iP_1\omega_0}{i\omega_0(-i\omega_0^3 + iP_2\omega_0 + P_3 - P_1\omega_0^2)} + \frac{P_4 - 2ir\omega_0}{i\omega_0(r\omega_0^2 + P_5 + iP_4\omega_0)} - \frac{\tau}{i\omega_0}\\
&= \frac{[(3\omega_0^2 - P_2)(\omega_0^3 - P_2\omega_0) - 2P_1\omega_0(P_3 - P_1\omega_0^2)] + i[-2P_1\omega_0(\omega_0^3 - P_2\omega_0) + (3\omega_0^2 - P_2)(P_1\omega_0^2 - P_3)]}{\omega_0[(\omega_0^3 - P_2\omega_0)^2 + (P_3 - P_1\omega_0^2)^2]}\\
&\quad + \frac{[-P_4^2\omega_0 - 2r\omega_0(P_5 + r\omega_0^2)] - i[P_4(P_5 + r\omega_0^2) - 2rP_4\omega_0^2]}{\omega_0[(P_4\omega_0)^2 + (P_5 + r\omega_0^2)^2]} - \frac{\tau}{i\omega_0}\\
&= \frac{E + iF}{\omega_0[(\omega_0^3 - P_2\omega_0)^2 + (P_3 - P_1\omega_0^2)^2]} + \frac{E' - iF'}{\omega_0[(P_4\omega_0)^2 + (P_5 + r\omega_0^2)^2]} - \frac{\tau}{i\omega_0},
\end{aligned}$$

where

$$\begin{aligned}
&E = (3\omega_0^2 - P_2)(\omega_0^3 - P_2\omega_0) - 2P_1\omega_0(P_3 - P_1\omega_0^2),\\
&F = -2P_1\omega_0(\omega_0^3 - P_2\omega_0) + (3\omega_0^2 - P_2)(P_1\omega_0^2 - P_3),\\
&E' = -P_4^2\omega_0 - 2r\omega_0(P_5 + r\omega_0^2), \qquad F' = P_4P_5 - rP_4\omega_0^2.
\end{aligned}$$

Since

$$(\omega_0^3 - P_2\omega_0)^2 + (P_3 - P_1\omega_0^2)^2 = (P_4\omega_0)^2 + (P_5 + r\omega_0^2)^2,$$

we have

$$\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\lambda=i\omega_0} = \frac{1}{\omega_0}\left[\frac{E + E' + i(F - F')}{(P_4\omega_0)^2 + (P_5 + r\omega_0^2)^2} - \frac{\tau}{i\omega_0}\right].$$

By simple computation, we derive that

$$\begin{aligned}
\operatorname{sgn}\left\{\frac{d\operatorname{Re}\lambda}{d\tau}\right\}_{\lambda=i\omega_0} &= \operatorname{sgn}\left\{\operatorname{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\right\}_{\lambda=i\omega_0}\\
&= \operatorname{sgn}\left\{\frac{1}{\omega_0}\frac{E + E'}{(P_4\omega_0)^2 + (P_5 + r\omega_0^2)^2}\right\}\\
&= \operatorname{sgn}\left\{\frac{3\omega_0^4 + 2\omega_0^2(P_1^2 - 2P_2 - r^2) + (P_2^2 - 2P_1P_3 - P_4^2 - 2rP_5)}{(P_4\omega_0)^2 + (P_5 + r\omega_0^2)^2}\right\}\\
&> 0.
\end{aligned}$$

Therefore, the transversal condition holds and a Hopf bifurcation occurs at $\omega = \omega_0$, $\tau = \tau_0$. In conclusion, we have the following results.

Theorem 3.3 *Assume that* (H_1), (H_3), (H_5), (H_7) *hold and* $m < 1 - \frac{\xi}{1+\beta\xi}$. *For the model* (1.5), *we have*:

(i) *If* (H_9) *holds, then the coexisting equilibrium* $E_2(u_1^*, u_2^*, v^*)$ *is locally asymptotically stable for all* $\tau \geq 0$.

(ii) *If* (H_{10}) *holds, then there exists a positive number* τ_0, *such that* $E_2(u_1^*, u_2^*, v^*)$ *is locally asymptotically stable for* $0 \leq \tau < \tau_0$ *and unstable for* $\tau > \tau_0$. *Furthermore, the model* (1.5) *undergoes a Hopf bifurcation at* E_2 *when* $\tau = \tau_0$.

4 Stability of bifurcated periodic solutions

In this section, we will establish the direction and stability of periodic solutions bifurcating from the positive equilibrium E_2, and we shall derive explicit formulae for determining the properties of the Hopf bifurcation at τ_0 by using the normal form theory and the center manifold theorem introduced by Hassard et al. [21].

For the model (1.5), expanding the nonlinear part by Taylor expansion, we rewrite (1.5) in the following form:

$$\begin{cases} \dot{u}_1(t) = a_{11}u_1(t) + a_{12}u_2(t) + f_1, \\ \dot{u}_2(t) = a_{21}u_1(t) + a_{22}u_2(t) + a_{23}v(t) + f_2, \\ \dot{v}(t) = a_{31}v(t) + b_{31}u_2(t-\tau) + b_{32}v(t-\tau) + f_3, \end{cases} \tag{4.1}$$

where

$$a_{11} = -(b+\alpha), \qquad a_{12} = a, \qquad a_{21} = \alpha, \qquad a_{22} = -c - 2d_1u_2^* - \frac{v^*(1-m)(1+\beta\xi)}{(1+\beta\xi+u_2^*)^2},$$

$$a_{23} = -\frac{(1-m)u_2^*}{1+\beta\xi+u_2^*}, \qquad a_{31} = -r, \qquad b_{31} = \frac{k_2(1-m)v^*(1+\beta\xi) - k_2\xi v^*}{(1+\beta\xi+u_2^*)^2},$$

$$b_{32} = \frac{k_2[(1-m)u_2^* + \xi]}{1+\beta\xi+u_2^*}, \qquad f_1 = 0,$$

$$f_2 = a_{24}u_2^2(t) + a_{25}u_2(t)v(t) + a_{26}u_2^3(t) + a_{27}u_2^2(t)v(t),$$

$$f_3 = a_{32}u_2^2(t-\tau) + a_{33}u_2(t-\tau)v(t-\tau) + a_{34}u_2^3(t-\tau),$$

$$a_{24} = -2d_1 + \frac{2v^*(1-m)(1+\beta\xi)}{(1+\beta\xi+u_2^*)^2}, \qquad a_{25} = -\frac{(1-m)(1+\beta\xi)}{(1+\beta\xi+u_2^*)^2},$$

$$a_{26} = -\frac{4v^*(1-m)(1+\beta\xi)}{(1+\beta\xi+u_2^*)^3}, \qquad a_{27} = -\frac{4(1-m)(1+\beta\xi)}{(1+\beta\xi+u_2^*)^2},$$

$$a_{32} = \frac{2k_2[(1-m)(1+\beta\xi) - \xi]v^*}{(1+\beta\xi+u_2^*)^3}, \qquad a_{33} = \frac{k_2[(1-m)(1+\beta\xi) - \xi]}{(1+\beta\xi+u_2^*)^2},$$

$$a_{34} = -\frac{6k_2[(1-m)(1+\beta\xi) - \xi]v^*}{(1+\beta\xi+u_2^*)^4}.$$

The linearized model (4.1) is

$$\begin{cases} \dot{u}_1(t) = a_{11}u_1(t) + a_{12}u_2(t), \\ \dot{u}_2(t) = a_{21}u_1(t) + a_{22}u_2(t) + a_{23}v(t), \\ \dot{v}(t) = a_{31}v(t) + b_{31}u_2(t-\tau) + b_{32}v(t-\tau). \end{cases} \tag{4.2}$$

Let $\tau = \tau_0 + \mu$, $\mu \in \mathbb{R}$, $t = s\tau$, $u_1(s\tau) = \hat{u}_1(s)$, $u_2(s\tau) = \hat{u}_2(s)$, $v(s\tau) = \hat{v}(s)$, denote $u_1 = \hat{u}_1$, $u_2 = \hat{u}_2$, $v = \hat{v}$, then (4.1) is transformed into the model

$$\begin{cases} \dot{u}_1(t) = (\tau_0+\mu)[a_{11}u_1(t) + a_{12}u_2(t)], \\ \dot{u}_2(t) = (\tau_0+\mu)[a_{21}u_1(t) + a_{22}u_2(t) + a_{23}v(t) + f_{22}(t)], \\ \dot{v}(t) = (\tau_0+\mu)[a_{31}v(t) + b_{31}u_2(t-1) + b_{32}v(t-1) + f_{33}(t)], \end{cases} \tag{4.3}$$

where

$$f_{22}(t) = a_{24}u_2^2(t) + a_{25}u_2(t)v(t) + a_{26}u_2^3(t) + a_{27}u_2^2(t)v(t),$$
$$f_{33}(t) = a_{32}u_2^2(t-1) + a_{33}u_2(t-1)v(t-1) + a_{34}u_2^3(t-1).$$

Denote $C^k[-1,0] = \{\varphi|\varphi : [-1,0] \to \mathbb{R}^3\}$, each component of φ has a Kth-order continuous derivative. Let $\phi(\theta) = (\phi_1(\theta), \phi_2(\theta), \phi_3(\theta))^T \in C[-1,0]$ be the initial data of model (1.5).

Define the operators

$$L_\mu\phi = (\tau_0 + \mu)\big[A'\phi(0) + B'\phi(-1)\big], \qquad f(\mu,\phi) = (\tau_0 + \mu)(0, f_{22}, f_{33}),$$

with

$$A' = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{31} \end{pmatrix},$$
$$B' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & b_{31} & b_{32} \end{pmatrix}$$

and $L_\mu : C[-1,0] \to \mathbb{R}^3, f : R \times C[-1,0] \to \mathbb{R}^3$. Then (4.3) can be rewritten as $u_t' = L_\mu(u_t) + f(\mu, u_t)$.

By the Riesz representation theorem there exists a function $\eta(\theta,\mu)$ of bounded variation for $\theta \in [-1,0]$ such that $L_\mu\phi = \int_{-1}^0 d\eta(\theta,\mu)\phi(\theta)$, for $\theta \in [-1,0]$. In fact, we can choose

$$\eta(\theta,\mu) = (\tau_0 + \mu)A'\delta(\theta) + (\tau_0 + \mu)B'\delta(\theta + 1),$$

where $\delta(\theta)$ is the Dirac function.

For $\phi \in C^1[-1,0]$, define

$$(A_\mu\phi)(\theta) = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & \theta \in [-1,0), \\ \int_{-1}^0 d\eta(\theta,\mu)\phi(\theta), & \theta = 0, \end{cases}$$

and

$$(R_\mu\phi)(\theta) = \begin{cases} 0, & \theta \in [-1,0), \\ f(\mu,\theta), & \theta = 0. \end{cases}$$

The model (4.3) is equivalent to $u_t' = A_\mu u_t + R_\mu u_t$, where $u_t = u(t+\theta)$, $\theta \in [-1,0]$.

For $\varphi \in C^1[-1,0]$, define

$$\big(A^*\psi\big)(s) = \begin{cases} -\frac{d\psi(s)}{ds}, & s \in (0,1], \\ \int_{-1}^0 d\eta^T(s,0)\psi(-s), & s = 0, \end{cases}$$

and the bilinear inner product

$$\langle\psi(s),\phi(\theta)\rangle = \bar{\psi}(0)\phi(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta} \bar{\psi}(\xi-\theta)\,d\eta(\theta)\phi(\xi)\,d\xi, \tag{4.4}$$

where $\psi(\theta) \in C^1[-1,0]$, $\eta(\theta) = \eta(\theta,0)$, and A_0 and A^* are adjoint operators. From the discussion in Sect. 3, we know that $\pm i\omega_0\tau_0$ are the eigenvalues of A_0. Hence, they are also eigenvalues of A^*.

Suppose that $q(\theta) = (1,q_1,q_2)^T e^{i\omega_0\tau_0\theta}$ is the eigenvector of A_0, corresponding to $i\omega_0\tau_0$, then $q(0) = (1,q_1,q_2)^T$, and $q(-1) = q(0)e^{-i\omega_0\tau_0}$. By a direct calculation, we get

$$\tau_0 \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & b_{31}e^{-i\omega_0\tau_0} & a_{31}+b_{32}e^{-i\omega_0\tau_0} \end{pmatrix} \begin{pmatrix} 1 \\ q_1 \\ q_2 \end{pmatrix} = i\omega_0\tau_0 \begin{pmatrix} 1 \\ q_1 \\ q_2 \end{pmatrix},$$

then

$$q_1 = \frac{i\omega_0 - a_{11}}{a_{12}}, \qquad q_2 = \frac{(i\omega_0 - a_{22})(i\omega_0 - a_{11}) - a_{21}a_{12}}{a_{12}a_{23}}.$$

Similarly, we can calculate the eigenvector $q^*(s) = D(1,q_1^*,q_2^*)e^{i\omega_0\tau_0 s}$ of A^* belong to the eigenvector $-i\omega_0\tau_0$, then we get

$$\tau_0 D \begin{pmatrix} 1 & q_1^* & q_2^* \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & b_{31}e^{i\omega_0\tau_0} & a_{31}+b_{32}e^{i\omega_0\tau_0} \end{pmatrix} = -i\omega_0\tau_0 D \begin{pmatrix} 1 & q_1^* & q_2^* \end{pmatrix},$$

then

$$q_1^* = \frac{-a_{12}(a_{31}+b_{32}e^{i\omega_0\tau_0}+i\omega_0)}{(a_{22}+i\omega_0)(a_{31}+b_{32}e^{i\omega_0\tau_0}+i\omega_0) - a_{23}b_{31}e^{i\omega_0\tau_0}},$$
$$q_2^* = \frac{a_{12}a_{23}}{(a_{22}+i\omega_0)(a_{31}+b_{32}e^{i\omega_0\tau_0}+i\omega_0) - a_{23}b_{31}e^{i\omega_0\tau_0}}.$$

We normalize q and q^* by the condition $\langle q^*(s),q(\theta)\rangle = 1$. Clearly $\langle q^*(s),q(\theta)\rangle = 0$. In order to ensure that $\langle q^*(s),q(\theta)\rangle = 1$, we need to determine the value of D. By (4.4), we have

$$\begin{aligned}
\langle q^*(s),q(\theta)\rangle &= \bar{D}(1,\bar{q}_1^*,\bar{q}_2^*)(1,q_1,q_2)^T \\
&\quad - \int_{-1}^{0}\int_{\xi=0}^{\theta} \bar{D}(1,\bar{q}_1^*,\bar{q}_2^*)e^{-i\omega_0\tau_0(\xi-\theta)}\,d\eta(\theta)(1,q_1,q_2)^T e^{i\omega_0\tau_0\xi}\,d\xi \\
&= \bar{D}(1+\bar{q}_1^*q_1+\bar{q}_2^*q_2) - \bar{D}\int_{-1}^{0}(1,\bar{q}_1^*,\bar{q}_2^*)\theta e^{i\omega_0\tau_0\theta}\,d\eta(\theta)(1,q_1,q_2)^T \\
&= \bar{D}\bigl[1+\bar{q}_1^*q_1+\bar{q}_2^*q_2+\tau_0\bar{q}_2^*(b_{31}q_1+b_{32}q_2)e^{-i\omega_0\tau_0}\bigr],
\end{aligned}$$

therefore $\bar{D} = \frac{1}{1+\bar{q}_1^*q_1+\bar{q}_2^*q_2+\tau_0\bar{q}_2^*(b_{31}q_1+b_{32}q_2)e^{-i\omega_0\tau_0}}$.

In the remainder of this section, following the algorithms given in [21] and using a similar computation process as in [22], we get the coefficients that will be used to determine several important qualities

$$
\begin{aligned}
g_{20} &= 2\tau_0\bar{D}\big(k_{11}\bar{q}_1^* + k_{21}\bar{q}_2^*\big), \qquad g_{11} = \tau_0\bar{D}\big(k_{12}\bar{q}_1^* + k_{22}\bar{q}_2^*\big),\\
g_{02} &= 2\tau_0\bar{D}\big(k_{13}\bar{q}_1^* + k_{23}\bar{q}_2^*\big), \qquad g_{21} = 2\tau_0\bar{D}\big(k_{14}\bar{q}_1^* + k_{24}\bar{q}_2^*\big),
\end{aligned}
$$

where

$$
\begin{aligned}
k_{11} &= a_{24}q_1^2 + a_{25}q_1q_2, \qquad k_{12} = 2a_{24}q_1\bar{q}_1 + a_{25}(q_1\bar{q}_2 + q_2\bar{q}_1),\\
k_{13} &= a_{24}\bar{q}_1^2 + a_{25}\bar{q}_1\bar{q}_2,\\
k_{14} &= a_{24}\big[\bar{q}_1 w_{20}^{(2)}(0) + 2q_1 w_{11}^{(2)}(0)\big] + 3a_{26}q_1^2\bar{q}_1 + a_{27}\big(q_1^2\bar{q}_2 + 2q_1q_2\bar{q}_1\big)\\
&\quad + a_{25}\left[\frac{1}{2}\bar{q}_2 w_{20}^{(2)}(0) + q_2 w_{11}^{(2)}(0) + \frac{1}{2}\bar{q}_1 w_{20}^{(3)}(0) + q_1 w_{11}^{(3)}(0)\right],\\
k_{21} &= a_{32}q_1^2 + a_{33}q_1q_2 e^{-2i\omega_0\tau_0}, \qquad k_{22} = \big[2a_{32}q_1\bar{q}_1 + a_{33}(q_1\bar{q}_2 + q_2\bar{q}_1)\big]e^{-2i\omega_0\tau_0},\\
k_{23} &= \big(a_{32}\bar{q}_1^2 + a_{33}\bar{q}_1\bar{q}_2\big)e^{-2i\omega_0\tau_0},\\
k_{24} &= a_{32}\big[\bar{q}_1 w_{20}^{(2)}(-1) + 2q_1 w_{11}^{(2)}(-1)\big]e^{-i\omega_0\tau_0} + 3a_{34}q_1^2\bar{q}_1 e^{-3i\omega_0\tau_0}\\
&\quad + a_{33}\left[\frac{1}{2}\bar{q}_2 w_{20}^{(2)}(-1) + q_2 w_{11}^{(2)}(-1) + \frac{1}{2}\bar{q}_1 w_{20}^{(3)}(-1) + q_1 w_{11}^{(3)}(-1)\right]e^{-i\omega_0\tau_0},
\end{aligned}
$$

and

$$
\begin{aligned}
w_{20}(\theta) &= \frac{ig_{20}}{\omega_0\tau_0}q(0)e^{i\omega_0\tau_0\theta} + \frac{i\bar{g}_{20}}{3\omega_0\tau_0}\bar{q}(0)e^{-i\omega_0\tau_0\theta} + E_1 e^{2i\omega_0\tau_0\theta}\\
&= \frac{ig_{20}}{\omega_0\tau_0}q(\theta) + \frac{i\bar{g}_{20}}{3\omega_0\tau_0}\bar{q}(\theta) + E_1 e^{2i\omega_0\tau_0\theta},\\
w_{11}(\theta) &= -\frac{ig_{11}}{\omega_0\tau_0}q(0)e^{i\omega_0\tau_0\theta} + \frac{i\bar{g}_{11}}{\omega_0\tau_0}\bar{q}(0)e^{-i\omega_0\tau_0\theta} + E_2\\
&= -\frac{ig_{11}}{\omega_0\tau_0}q(\theta) + \frac{i\bar{g}_{11}}{\omega_0\tau_0}\bar{q}(\theta) + E_2.
\end{aligned}
$$

Moreover, E_1 and E_2 satisfy the following equations:

$$
\begin{aligned}
&\begin{pmatrix} 2i\omega_0 - a_{11} & -a_{12} & 0\\ -a_{21} & 2i\omega_0 - a_{22} & -a_{23}\\ 0 & -b_{31}e^{-2i\omega_0\tau_0} & 2i\omega_0 - a_{31} - b_{32}e^{-2i\omega_0\tau_0} \end{pmatrix} E_1 = 2\begin{pmatrix} 0\\ k_{11}\\ k_{21} \end{pmatrix},\\
&\begin{pmatrix} -a_{11} & -a_{12} & 0\\ -a_{21} & -a_{22} & -a_{23}\\ 0 & -b_{31} & -a_{31} - b_{32} \end{pmatrix} E_2 = \begin{pmatrix} 0\\ k_{12}\\ k_{22} \end{pmatrix}.
\end{aligned}
$$

Furthermore, g_{ij} is expressed by the parameters and delay in (1.5). Thus, we can compute the following values:

$$
C_1(0) = \frac{i}{2\omega_0\tau_0}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2},
$$

$$\mu_2 = -\frac{\operatorname{Re} C_1(0)}{\operatorname{Re} \frac{d\lambda(\tau_0)}{d\tau}},$$

$$\beta_2 = 2 \operatorname{Re} C_1(0),$$

$$T_2 = -\frac{\operatorname{Im} C_1(0) + \mu_2 \operatorname{Im} \frac{d\lambda(\tau_0)}{d\tau}}{\omega_0 \tau_0},$$

which determine the properties of bifurcation period solutions at $\tau = \tau_0$ on the center manifold. From the above discussions, we have the following result.

Theorem 4.1 *For model* (1.5), *the following results hold*:

(i) *The sign of* μ_2 *determines the directions of the Hopf bifurcation*: *if* $\mu_2 > 0$, *then the Hopf bifurcation is supercritical and the bifurcating periodic solutions exist for* $\tau > \tau_0$; *if* $\mu_2 < 0$, *then the Hopf bifurcation is subcritical and the bifurcating periodic solutions exist for* $\tau < \tau_0$.

(ii) *The sign of* β_2 *determines the stability of the bifurcating periodic solutions*: *the bifurcating periodic solutions are stable if* $\beta_2 < 0$; *the bifurcating periodic solutions are unstable if* $\beta_2 > 0$.

(iii) *The sign of* T_2 *determines the period of the bifurcating periodic solutions*: *the period increases if* $T_2 > 0$ *and decreases* $T_2 < 0$.

5 Numerical simulations

We perform the numerical simulations of the model (1.5) to verify our theoretical results.

Taking $a = 3$; $b = \frac{1}{4}$; $\alpha = \frac{1}{4}$; $c = \frac{1}{8}$; $d_1 = \frac{1}{8}$; $m = \frac{1}{2}$; $k_2 = \frac{3}{2}$; $\xi = \frac{1}{3}$; $\beta = \frac{1}{2}$; $r = \frac{1}{4}$; we see that the conditions (H_1) or (H_4) hold. Theorem 3.1(i) is verified numerically in Fig. 1(a).

Taking $a = 3$; $b = \frac{1}{4}$; $\alpha = \frac{1}{4}$; $c = 2$; $d_1 = \frac{1}{8}$; $m = \frac{1}{2}$; $k_2 = \frac{3}{2}$; $\xi = \frac{1}{3}$; $\beta = \frac{1}{2}$; $r = 2$; it is clear that the conditions (H_2) and (H_3) hold. Theorem 3.1(ii) is verified numerically in Fig. 1(b).

If we choose $a = 4.3$; $b = \frac{1}{4}$; $\alpha = \frac{2}{9}$; $c = 2$; $d_1 = \frac{1}{8}$; $m = \frac{1}{4}$; $k_2 = 3$; $\xi = 5$; $\beta = \frac{1}{2}$; $r = 2$; then the conditions (H_4) and (H_5) hold. Theorem 3.2(i) is verified numerically by Fig. 2(a).

If we choose $a = \frac{7}{2}$; $b = \frac{1}{4}$; $\alpha = \frac{1}{2}$; $c = 2$; $d_1 = \frac{1}{8}$; $m = \frac{1}{2}$; $k_2 = \frac{3}{2}$; $\xi = \frac{1}{3}$; $\beta = \frac{1}{2}$; $r = 2$; then the conditions (H_3) and (H_6) hold. Theorem 3.2(ii) is verified numerically by Fig. 2(b).

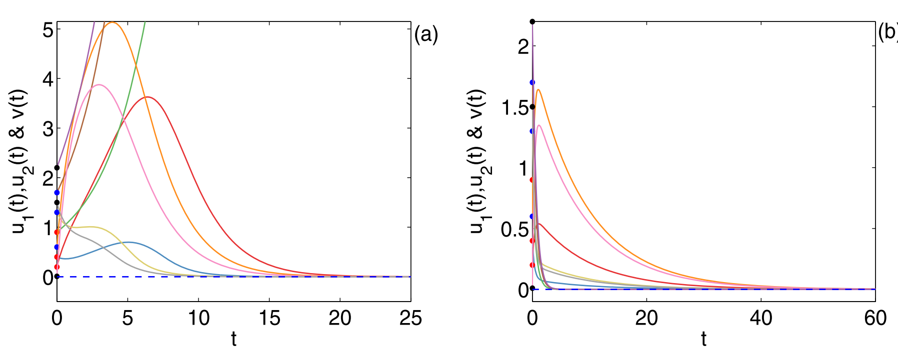


Figure 1 Time series of $u_1(t)$, $u_2(t)$ and $v(t)$ generated by the model (1.5). Three initial values are chosen as $(u_1(0), u_2(0), v(0)) = (0.2, 0.4, 0.9), (0.6, 1.3, 1.7), (0.01, 1.5, 2.2)$ marked red points, blue points and black points, respectively. The dashed line stands for $u_1(t) = 0$, $u_2(t) = 0$ and $v(t) = 0$, respectively. (**a**) The trivial equilibrium $E_0(0, 0, 0)$ is unstable. (**b**) The trivial equilibrium $E_0(0, 0, 0)$ is locally asymptotically stable

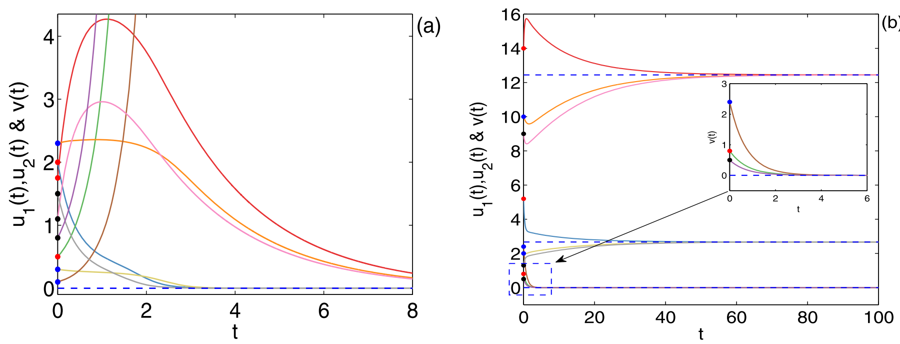
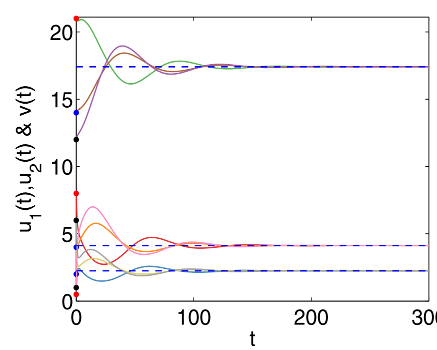


Figure 2 Time series of $u_1(t)$, $u_2(t)$ and $v(t)$ generated by the model (1.5): (**a**) Three initial values are chosen as $(u_1(0), u_2(0), v(0)) = (1.75, 2, 0.5), (2.3, 0.3, 0.1), (1.1, 1.5, 0.8)$ marked red points, blue points and black points, respectively. The dashed line stands for $u_2(t) = 0$. (**b**) Three initial values are chosen as $(u_1(0), u_2(0), v(0)) = (14, 5.2, 0.8), (10, 2, 2.4), (9, 1.3, 0.5)$ marked red points, blue points and black points, respectively. The dashed line stands for $u_1(t) = 12.444444444446397$, $u_2(t) = 2.666666666667100$ and $v(t) = 0$, respectively

Figure 3 Time series of $u_1(t)$, $u_2(t)$ and $v(t)$ generated by the model (1.5): Three initial values are chosen as $(u_1(0), u_2(0), v(0)) = (8, 0.5, 21), (2, 4, 14), (1, 6, 12)$ marked red points, blue points and black points, respectively. The dashed line stands for $u_1(t) = 4.125000892829194$, $u_2(t) = 2.250000487010802$ and $v(t) = 17.406948324710228$, respectively



Taking $a = 1.1$; $b = \frac{1}{10}$; $\alpha = \frac{1}{2}$; $c = \frac{1}{10}$; $d_1 = \frac{1}{12}$; $m = \frac{1}{4}$; $k_2 = 0.5$; $\xi = 3.5$; $\beta = 5$; $r = \frac{1}{8}$; we see that the conditions $m < 1 - \frac{\xi}{1+\beta\xi}$, (H_8) and $(P_1 - r)(P_2 + P_4) - (P_3 + P_5) > 0$ hold. The numerical result of Case 3.1 can be seen in Fig. 3.

When $\tau \geq 0$, taking $a = 1.1$; $b = 0.1$; $\alpha = \frac{1}{2}$; $c = \frac{1}{10}$; $d_1 = \frac{1}{4}$; $m = \frac{1}{3}$; $k_2 = 0.5$; $\xi = 3.5$; $\beta = 4.5$; $r = \frac{1}{8}$; we see that the conditions $\beta = 4.5 > \frac{k_2}{r} - \frac{1}{\xi} = 3.714285714285714$, $0 < m = \frac{1}{3} < \min\{1 - \frac{r}{k_2}, 1 - \frac{r}{k_2} - \frac{d_1[r+(r\beta-k_2)\xi](b+\alpha)}{\alpha k_2(a-c)-bc}\} = 0.5351562500000000$, $m = \frac{1}{3} < 1 - \frac{\xi}{1+\beta\xi} = 0.7910447761194030$ and (H_9) hold. The numerical result of Theorem 3.3(i) is presented by Fig. 4(a) for $\tau = 0$ and Fig. 4(b) for $\tau = 10$.

Taking $a = 6$; $b = 0.2$; $\alpha = \frac{5}{3}$; $c = \frac{1}{10}$; $d_1 = \frac{1}{4}$; $m = \frac{1}{3}$; $k_2 = 0.5$; $\xi = 3.5$; $\beta = 4.5$; $r = \frac{1}{8}$; we see that (H_1), (H_3), (H_5), (H_7), (H_{10}) hold and $\tau_0 = 0.1509514710143546$. The numerical result of Theorem 3.3(ii) is presented by Fig. 5 for $\tau = 0.06$ and Fig. 6 for $\tau = 0.5$.

6 Conclusions

In this paper, we study a delayed predator–prey model with stage structure for prey incorporating refuge and provide additional food to the predator. By analyzing the corresponding characteristic equations, we investigate the local stability of the equilibria of the model. We discuss the existence of Hopf bifurcation by choosing time delay as a parameter. We find that time delay can causes a stable equilibrium to become unstable one, even occur Hopf bifurcation, when time delay passes through some critical values. Furthermore, by

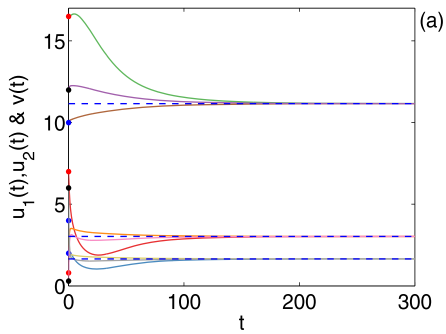


Figure 4 Time series of $u_1(t)$, $u_2(t)$ and $v(t)$ generated by the model (1.5): Three initial values are chosen as $(u_1(0), u_2(0), v(0)) = (7, 0.8, 16.5), (2, 4, 10), (0.3, 6, 12)$ marked red points, blue points and black points, respectively. The dashed line stands for $u_1(t) = 3.025001035342410$, $u_2(t) = 1.650000564732131$ and $v(t) = 11.155001498913562$, respectively. (**a**) $\tau = 0$. (**b**) $\tau = 10$

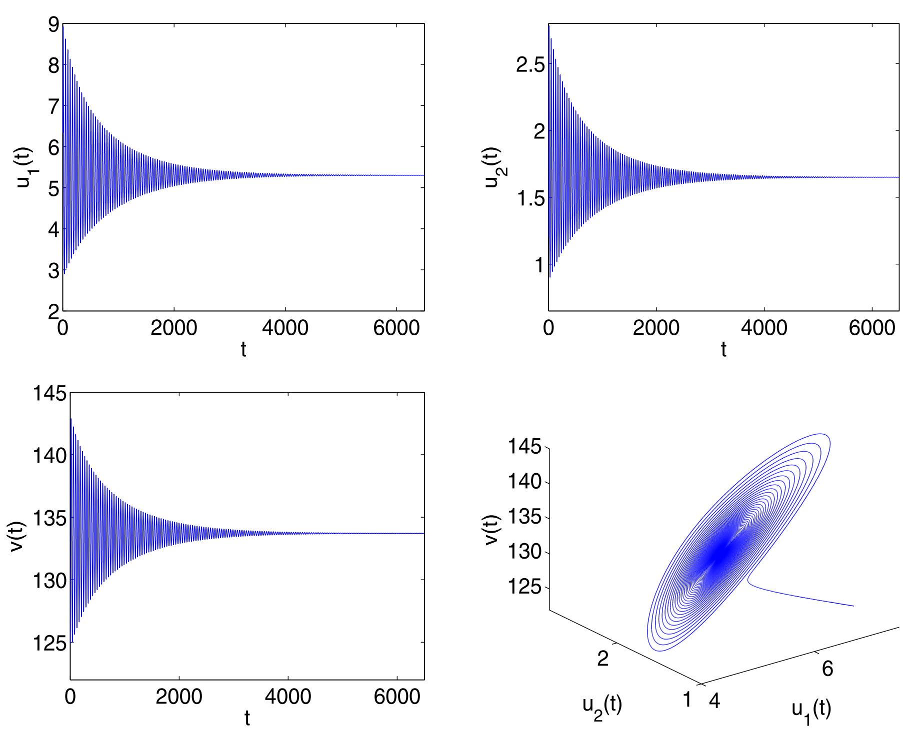


Figure 5 Time series of $u_1(t)$, $u_2(t)$, $v(t)$ and phase portrait of the model (1.5) with $\tau = 0.06 < \tau_0 = 0.1509514710143546$. An orbit from the initial condition $(u_1(0), u_2(0), v(0)) = (7, 1.2, 125)$ located in a sufficiently small neighborhood of the equilibrium $E_2(5.303571428572600, 1.650000000000400, 133.712142857161200)$ converges to the equilibrium E_2. The simulation results indicate that the equilibrium E_2 is locally asymptotically stable

applying the normal form method and center manifold theorem, we investigate the direction of Hopf bifurcation and the stability of the bifurcated periodic solutions. We give numerical simulations to show our main results.

From Theorem 3.3, we see that, for the stability of a coexisting equilibrium point, the refuge has to be bounded by a value which depends on the quantity and the quality of additional food. Results obtained in this paper provide a useful platform to understand

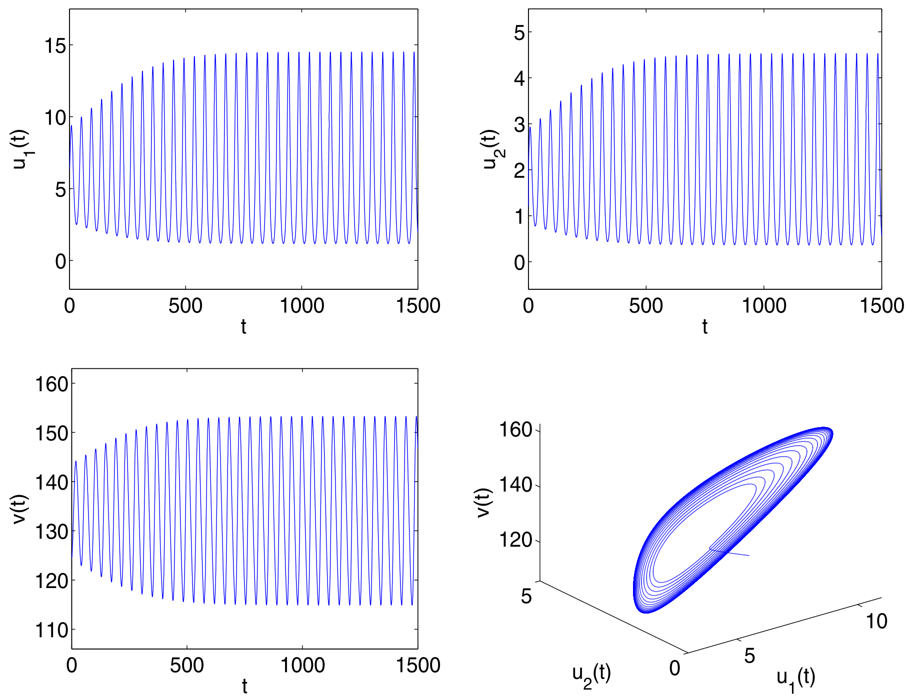


Figure 6 Time series of $u_1(t)$, $u_2(t)$, $v(t)$ and phase portrait of the model (1.5) with $\tau = 0.5 > \tau_0 = 0.1509514710143546$. An orbit from the initial condition $(u_1(0), u_2(0), v(0)) = (7, 1.2, 125)$ located in a sufficiently small neighborhood of the equilibrium $E_2(5.303571428572600, 1.650000000000400, 133.712142857161200)$ converges to a periodic solution. The simulation results indicate that the equilibrium E_2 is unstable and periodic solution is stable. The periodic attractor bifurcates from the equilibrium and surrounds the equilibrium E_2 as the delay τ crosses the critical value $\tau_0 = 0.1509514710143546$

the roles of refuge and additional food. Therefore, refuge and additional food can be taken as population controllers to study the prey–predator models.

Our results can be compared with the ones in Sahoo [17] which considered the role of additional food in eco-epidemiological system with disease in the prey. So, we can extend our predator–prey model to an eco-epidemiological system based on [17, 23].

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors have equally contributed to obtaining new results in this paper and also read and approved the final manuscript.

Author details
[1]School of Mathematical Sciences, Qufu Normal University, Qufu, P.R. China. [2]No. 1 Middle School of Qufu, Qufu, P.R. China.

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