Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

On one-soliton solutions of the Q2 equation in the ABS list

Qiang Cheng[1], Cheng Zhang[1*], Danda Zhang[2] and Da-jun Zhang[1*]

*Correspondence:
ch.zhang.maths@gmail.com;
djzhang@staff.shu.edu.cn
[1] Department of Mathematics, Shanghai University, Shanghai, P.R. China
Full list of author information is available at the end of the article

Abstract

In this paper, we derive seed and 1-soliton solutions of the Q2 equation in the Adler–Bobenko–Suris list. The seed solutions of Q2 are obtained using those of Q1(δ) and an non-auto Bäcklund transformation connecting them. Then using an auto Bäcklund transformation, two types of Q2 one-soliton solutions are obtained based on its seed solutions. These obtained solutions are new and cannot be derived as degenerations from any known soliton solutions.

PACS Codes: 02.30.Ik; 02.30.Jr; 04.20.Jb

Keywords: Soliton solutions; ABS list; The Q2 equation; Bäcklund transformation

1 Introduction

Integrable systems involve the study of physically relevant nonlinear equations, which includes many families of well-known, highly important partial and ordinary differential equations. Over the past two decades, research in discrete integrable systems has undergone a truly remarkable development (see for instance the monograph [1]). One of the key achievements was the introduction of multi-dimensional consistency [2–4] as a defining criterion for discrete integrability, which can be seen as the discrete analog of the compatible hierarchies in continuous integrable theory. It turns out that quad-equations, two-dimensional lattice equations defined on square lattices, are said to be integrable, if they are three-dimensional consistent, geometrically meaning that the equations can be consistently embedded around a cube (CAC). This led, up to few other assumptions, to the classification of integrable affine linear quad-equations [5], known as the Adler–Bobenko–Suris (ABS) list. The list contains 9 equations, named Q4, Q3(δ), Q2, Q1(δ), A2, A1(δ), H3(δ), H2 and H1. Most of these equations were known or related to known equations. For example, Q4 is a fully discretized version of the famous Krichever–Novikov equation [6, 7], Q3(δ) is related to the Nijhoff–Quispel–Capel equation [8, 9], Q1(0), H3(0) and H1 are, respectively, the lattice Schwarzian Korteweg–de Vries (KdV) equation, the lattice potential modified KdV equation and the lattice potential KdV equation [10].

Amongst the equations in the ABS list, Q2 is somewhat mysterious. It was unknown before the ABS classification; so far its solutions have only been obtained either as degeneration of Q3(δ) [9, 11–13] or from solutions of Q1(δ) using appropriate transformations [14]. Partly due to its complexity, one rarely finds solution results obtained directly from

Springer

© The Author(s) 2019. This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (http://creativecommons.org/licenses/by/4.0/), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

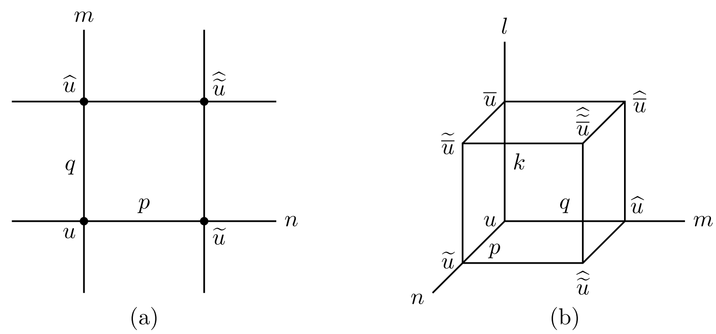

Figure 1 (**a**) Lattice on (n, m) plane. (**b**) Consistent cube

the equation itself. The Q2 equation reads

$$p(u-\widehat{u})(\widetilde{u}-\widehat{\widetilde{u}})-q(u-\widetilde{u})(\widehat{u}-\widehat{\widetilde{u}})=pq(q-p)\big(u+\widetilde{u}+\widehat{u}+\widehat{\widetilde{u}}-p^2+pq-q^2\big), \tag{1.1}$$

where $\widetilde{\ }$ and $\widehat{\ }$ serve as shifts in a $\mathbb{Z}^2$ lattice, i.e.,

$$u\equiv u_{n,m}, \qquad \widetilde{u}\equiv u_{n+1,m}, \qquad \widehat{u}\equiv u_{n,m+1}, \qquad \widehat{\widetilde{u}}\equiv u_{n+1,m+1},$$

and p, q are the lattice parameters associated with n, m respectively. See Fig. 1(a).

Due to the CAC property, the lattice equation admits the following copies:

$$p(u-\overline{u})(\widetilde{u}-\overline{\widetilde{u}})-k(u-\widetilde{u})(\overline{u}-\overline{\widetilde{u}})=pk(k-p)\big(u+\widetilde{u}+\overline{u}+\overline{\widetilde{u}}-p^2+pk-k^2\big), \tag{1.2a}$$

$$k(u-\widehat{u})(\overline{u}-\widehat{\overline{u}})-q(u-\overline{u})(\widehat{u}-\widehat{\overline{u}})=kq(q-k)\big(u+\overline{u}+\widehat{u}+\widehat{\overline{u}}-k^2+kq-q^2\big). \tag{1.2b}$$

Here $\bar{\ }$ denotes shifts in the third direction associated with the lattice parameter k (see Fig. 1(b)). The above equations also define an auto Bäcklund transformation (BT) of Q2 [5], namely, given u as a solution, $\overline{u}$ solves Q2 as well provided that u and $\overline{u}$ are connected through (1.2a)–(1.2b). In the auto BT approach, u is commonly known as a *seed* solution, while $\bar{u}$ as a new solution generated from u.

One particular aim of this paper is to derive one-soliton solutions (1SSs) of Q2 by means of the auto BT (1.2a)–(1.2b). This requires knowledge of its seed solutions, which have not been well understood either in the literature. For instance the fixed-point method [15] only provides a solution of Q2 that is a special case of a more general solution (see Sect. 2.3). Our approach to obtaining Q2's seed solutions is based on an non-auto BT between Q1(δ) and Q2 [16]. There exist two different types of (seed) solutions of Q1(δ) such as the exponential type cf. [15] and the rational type [15, 17]. These solutions allow one to derive exponential and rational types of seed solutions of Q2, and in turn, lead to different 1SSs of Q2. Note that although the idea of this paper is clear, the "integration" of the BT (1.2a)–(1.2b) to get 1SS is highly nontrivial. We also note that the solutions of Q2 we obtain here are essentially new, as they cannot be reduced as reductions of known results.

The paper is organized as follows. We will first make use of known solutions of Q1(δ) and the non-auto BT between Q1(δ) and Q2 to derive seed solutions for Q2. Fixed point idea will also be discussed. These will be done in Sect. 2. Then in Sect. 3 we derive three 1SSs for Q2 from different seed solutions. Section 4 serves for conclusions.

2 Seed solutions

We derive seed solutions of Q2 from those of Q1(δ). The Q1(δ) equation reads [5]

$$p(w-\widehat{w})(\widetilde{w}-\widehat{\widetilde{w}})-q(w-\widetilde{w})(\widehat{w}-\widehat{\widetilde{w}})=\delta^2 pq(q-p), \tag{2.1}$$

and is connected to Q2 by the following non-auto BT (Miura transformation) [16]:

$$\delta(w-\widetilde{w})(u-\widetilde{u})=-p\big(\delta^2 u+\delta^2\widetilde{u}-2w\widetilde{w}\big)+\delta p^2(w+\widetilde{w}+p), \tag{2.2a}$$

$$\delta(w-\widehat{w})(u-\widehat{u})=-q\big(\delta^2 u+\delta^2\widehat{u}-2w\widehat{w}\big)+\delta q^2(w+\widehat{w}+q). \tag{2.2b}$$

2.1 Exponential case

Q1(1) has an exponential-type solution, cf. [15]

$$w_{n,m}^{0SS}=\frac{1}{2}\big(A\alpha^n\beta^m+A^{-1}\alpha^{-n}\beta^{-m}\big), \tag{2.3}$$

where A is an arbitrary non-zero constant, and α, β are connected to p, q through

$$p=\frac{(1-\alpha)^2}{2\alpha},\qquad q=\frac{(1-\beta)^2}{2\beta}. \tag{2.4}$$

Inserting $w_{n,m}^{0SS}$ into (2.2a)–(2.2b) with $\delta=1$ and extracting out the common factors, the BT (2.2a)–(2.2b) are reduced to

$$Z_{n,m}u-Z_{n+1,m}\widetilde{u}=\frac{1-\alpha}{2}\big(A\alpha^n\beta^m+A^{-1}\alpha^{-n}\beta^{-m}\big)+Z_{n,m}-Z_{n+1,m}-P,$$

$$Z_{n,m}u-Z_{n,m+1}\widehat{u}=\frac{1-\beta}{2}\big(A\alpha^n\beta^m+A^{-1}\alpha^{-n}\beta^{-m}\big)+Z_{n,m}-Z_{n,m+1}-Q,$$

where

$$\begin{aligned}&Z_{n,m}=\frac{A\alpha^n\beta^m-1}{A\alpha^n\beta^m+1},\qquad P=\frac{(\alpha^2-1)(\alpha^2-4\alpha+1)}{4\alpha^2},\\&Q=\frac{(\beta^2-1)(\beta^2-4\beta+1)}{4\beta^2}.\end{aligned} \tag{2.5}$$

By integrating the above two equations w.r.t. n and m, respectively, one gets

$$\begin{aligned}&Z_{0,m}u_{0,m}-Z_{n,m}u_{n,m}\\&\quad=\frac{1}{2}\big(-A\alpha^n\beta^m+A^{-1}\alpha^{-n}\beta^{-m}\big)+\frac{1}{2}\big(A\beta^m-A^{-1}\beta^{-m}\big)+Z_{0,m}-Z_{n,m}-Pn,\\&Z_{n,0}u_{n,0}-Z_{n,m}u_{n,m}\\&\quad=\frac{1}{2}\big(-A\alpha^n\beta^m+A^{-1}\alpha^{-n}\beta^{-m}\big)+\frac{1}{2}\big(A\alpha^n-A^{-1}\alpha^{-n}\big)+Z_{n,0}-Z_{n,m}-Qm.\end{aligned}$$

Then a combination of them yields a solution of Q2

$$u_{n,m}^{0SS}=\frac{1}{2}\big(A\alpha^n\beta^m+A^{-1}\alpha^{-n}\beta^{-m}+4\big)+Z_{n,m}^{-1}(Pn+Qm+r), \tag{2.6}$$

where r is another constant.

2.2 Rational case

Again based on the BT (2.2a)–(2.2b), solutions of Q2 can be derived using rational solutions of Q1(δ). For simplicity, we only consider the two lowest order rational solutions of Q1(δ) [17]. Higher order rational solutions can in theory be used, but with more complications. Introduce a linear function

$$x_i = a^i n + b^i m + \zeta_i, \tag{2.7}$$

where

$$p = a^2, \qquad q = b^2 \tag{2.8}$$

and ζ_i is a constant. One imposes the following relation between u and w:

$$u = y + \frac{w^2}{\delta^2}, \tag{2.9}$$

where δ is a parameter and y is to be determined. It follows from the BT (2.2a)–(2.2b) that

$$\widetilde{y} = \frac{\widetilde{w} - w - \delta p}{\widetilde{w} - w + \delta p} y + \frac{1}{\delta^2}(w + \delta p - \widetilde{w})(w + \delta p + \widetilde{w}), \tag{2.10a}$$

$$\widehat{y} = \frac{\widehat{w} - w - \delta q}{\widehat{w} - w + \delta q} y + \frac{1}{\delta^2}(w + \delta q - \widehat{w})(w + \delta q + \widehat{w}). \tag{2.10b}$$

Submitting a Q1(δ) solution[a] [17] $w = \delta(x_1^2 + c_0)$ (c_0 is a constant) into (2.10a)–(2.10b) yields

$$\widetilde{y} = \frac{x_1}{\widetilde{x}_1} y - 4ax_1\big(x_1\widetilde{x}_1 + a^2 + c_0\big), \qquad \widehat{y} = \frac{x_1}{\widehat{x}_1} y - 4bx_1\big(x_1\hat{x}_1 + b^2 + c_0\big).$$

Introducing a new variable $z = yx_1$, then the above system can be reduced to

$$\widetilde{z} = z - 4ax_1\widetilde{x}_1\big(x_1\widetilde{x}_1 + a^2 + c_0\big), \qquad \widehat{z} = z - 4bx_1\widehat{x}_1\big(x_1\widehat{x}_1 + b^2 + c_0\big),$$

which can be integrated

$$z = -\frac{4}{5}x_1^5 - \frac{4}{3}c_0x_1^3 + \frac{4}{3}c_0x_3 + \frac{4}{5}x_5. \tag{2.11}$$

As a result, one obtains a rational solution of Q2 via (2.9)

$$u = \frac{1}{5}x_1^4 + \frac{2}{3}c_0x_1^2 + \frac{4c_0x_3}{3x_1} + \frac{4x_5}{5x_1} + c_0^2. \tag{2.12}$$

Using another rational solution of Q1(δ) [17]

$$w = \frac{x_1^3 - x_3}{3} - \delta^2 x_1, \tag{2.13}$$

then y satisfies

$$\widetilde{y} = \frac{(x_1 - \delta)(\widetilde{x}_1 + \delta)}{(x_1 + \delta)(\widetilde{x}_1 - \delta)} y - \frac{a}{\delta^2}(x_1 - \delta)(\widetilde{x}_1 + \delta)\big(w + \widetilde{w} + \delta a^2\big),$$

$$\widetilde{y} = \frac{(x_1-\delta)(\widehat{x}_1+\delta)}{(x_1+\delta)(\widehat{x}_1-\delta)}y - \frac{b}{\delta^2}(x_1-\delta)(\widehat{x}_1+\delta)\big(w+\widehat{w}+\delta b^2\big),$$

which can be integrated by introducing $z = \frac{x_1-\delta}{x_1+\delta}y$. The equation for z reads

$$\widetilde{z} = z - \frac{a}{\delta^2}(x_1-\delta)(\widetilde{x}_1-\delta)\big(w+\widetilde{w}+\delta a^2\big),$$
$$\widehat{z} = z - \frac{b}{\delta^2}(x_1-\delta)(\widehat{x}_1-\delta)\big(w+\widehat{w}+\delta b^2\big),$$

which leads to

$$\begin{aligned} z = {} & -\frac{1}{9\delta^2}x_1^6 + \frac{4}{15\delta}x_1^5 + \frac{1}{3}x_1^4 + \frac{2}{9\delta^2}x_1^3x_3 - \frac{4\delta}{3}x_1^3 - \frac{2}{3\delta}x_1^2x_3 \\ & + \delta^2x_1^2 + \frac{2}{3}x_1x_3 - \frac{1}{9\delta^2}x_3^2 - \frac{2\delta}{3}x_3 + \frac{2}{5\delta}x_5. \end{aligned} \tag{2.14}$$

Therefore, we obtain a second rational solution of Q2 via (2.9)

$$\begin{aligned} u = {} & \frac{1}{45\delta(x_1-\delta)}\big[30\big(x_1^3-x_3\big)\delta^3 - 15\big(x_1^4+2x_1x_3\big)\delta^2 \\ & + 3\big(-x_1^5+10x_1^2x_3+6x_5\big)\delta + 2x_1^6 + 18x_1x_5 - 10x_3^2 - 10x_1^3x_3\big]. \end{aligned} \tag{2.15}$$

Both (2.12) and (2.15) go to infinity when $n, m \to \pm\infty$, which is depicted in Fig. 2. There is a third polynomial solution of Q1(δ), cf. [15]

$$w = \alpha n + \beta m + \gamma, \tag{2.16}$$

obeying the parametrizations

$$p = \frac{c_0}{a^2-\delta^2}, \qquad q = \frac{c_0}{b^2-\delta^2}, \qquad \alpha = pa, \qquad \beta = qb, \tag{2.17}$$

with γ, δ, c_0 being constants. Using the ansatz (2.9), through (2.10a)–(2.10b), one has

$$y = -\frac{wc_0}{\delta^3} + \frac{1}{2\delta^4}c_0^2 + y_0, \tag{2.18}$$

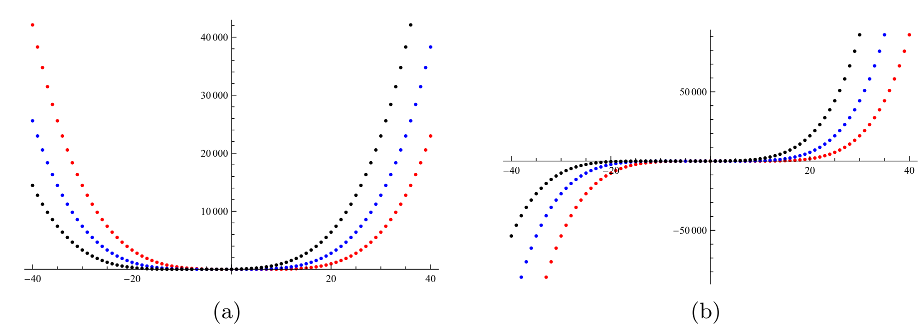

Figure 2 Background solutions of rational type. (**a**) Solution (2.12), (**b**) solution (2.15); where $a = 0.5$, $b = 0.5$, $c_0 = 2$, $\zeta_1 = \zeta_3 = \zeta_5 = 1$, $\delta = 1$, $m \in [-40, 40]$, waves in red, blue and black stand for $n = -5, 0, 5$, respectively

with y_0 satisfying

$$\widetilde{y}_0 = \frac{a-(\pm\delta)}{a+(\pm\delta)}y_0, \qquad \widehat{y}_0 = \frac{b-(\pm\delta)}{b+(\pm\delta)}y_0.$$

The function y_0 can be integrated as discrete exponentials, which leads to a Q2 solution through (2.9) and (2.18). However, it is interesting that one can verify the following:

$$u = \frac{w^2}{\delta^2} + \frac{c_0^2}{4\delta^4} + \left(\frac{a-\delta}{a+\delta}\right)^n \left(\frac{b-\delta}{b+\delta}\right)^m \gamma_0^+ + \left(\frac{a+\delta}{a-\delta}\right)^n \left(\frac{b+\delta}{b-\delta}\right)^m \gamma_0^- + \frac{4\gamma_0^+\gamma_0^-\delta^4}{c_0^2}, \quad (2.19)$$

is a solution of Q2 as well, where $\gamma_0^\pm$ are constants and a reparametrization $\gamma \to \gamma + \frac{c_0}{\delta^2}$ is used. We note that the above solution with $c_0 = \delta^2$ coincides with the seed solution (zero-soliton solution) of Q2 derived as a degeneration of Q3(δ) (see (5.8) in [9]). Inversely, we find this solution as a seed solution generate the same 1SS as obtained in [9] in the auto-BT process, and hence will not be considered in the next section.

Besides, note that there is a degeneration approach from Q2 equation (1.1) to Q1(δ) equation (2.1), via [9, 18]

$$u \to \frac{\delta^2}{4\epsilon^2} + \frac{w}{\epsilon}, \quad \epsilon \to 0. \quad (2.20)$$

Imposing the above limit on (2.12) together with taking $c_0 \to \frac{\delta}{2\epsilon} + c_0$, (2.12) leads to

$$w = \delta\left(\frac{x_1^3 + 2x_3}{3x_1} + c_0\right). \quad (2.21)$$

This is a new solution to Q1(δ), which was not found in [14, 17, 18]. For the solution (2.15), we again make use of (2.20) and together setting $\delta \to \frac{\delta}{\epsilon}$, $c_3 \to \frac{3\delta}{8\epsilon} + c_3$, (2.15) turns to be

$$w = -\frac{\delta(4x_1^3 - 4x_3 - 3x_1)}{6},$$

which is nothing new but Q1($\frac{1}{2}$) (2.13) multiplied by δ. For (2.19), through (2.20) and

$$\gamma \to \frac{\gamma'\delta^2}{\epsilon} + \gamma, \qquad \gamma^+ \to \frac{\gamma^+}{\epsilon}, \qquad \gamma^- \to \frac{\gamma^-}{\epsilon}, \quad \text{where } \gamma'^2 + \frac{4\delta^2\gamma^+\gamma^-}{c_0^2} = \frac{1}{4},$$

from (2.19) we arrive at

$$w = 2\gamma'(\alpha n + \beta m + \gamma) + \left(\frac{a-\delta}{a+\delta}\right)^n \left(\frac{b-\delta}{b+\delta}\right)^m \gamma_0^+ + \left(\frac{a+\delta}{a-\delta}\right)^n \left(\frac{b+\delta}{b-\delta}\right)^m \gamma_0^-,$$

which is the same as obtained in [9].

2.3 Fixed-point solution

Fixed-point solutions, stationary solutions of quad-equations along a third lattice direction, can also be used to solve the equations themselves [19]. This idea has been applied to the equations in the ABS list [9, 15]. In this idea $\overline{u}$ is considered to be u, i.e. k is not significant in generating solitons and then u is solved usually as a background solution (seed solution) of solitons. By this idea the BT (1.2a)–(1.2b) of Q2 yields its fixed point version

$$(u-\widetilde{u})^2 = p(p-k)\big(2u+2\widetilde{u}-p^2+pk-k^2\big), \tag{2.22a}$$

$$(u-\widehat{u})^2 = q(q-k)\big(2u+2\widehat{u}-k^2+kq-q^2\big). \tag{2.22b}$$

Solving them with parametrization (2.17) and $k=-c_0/\delta^2$ yields the Q2 solution (2.19) with $\gamma_0^{\pm}=0$.

3 One-soliton solutions of Q2

In this section, the solutions (2.6), (2.12) and (2.15) are used as seed solutions in the auto BT approach to generating 1SSs.

3.1 General procedure

Suppose u (denoted by u_θ conventionally, cf. [15, 19–22]) is a seed solution of Q2 and denote $\overline{u}_\theta$ as a shifted u in the third direction in the light of the CAC property. In fact, the CAC property of Q2 indicates its solution $u(n,m)$ can be consistently embedded into a 3-dimension cube (see Fig. 1(b)). Although there is no explicit independent variable l in $u(n,m)$, one can introduce a bar shift (shift in l-direction) for it according to $\widetilde{\ }$ or $\widehat{\ }$ shifts. For example, for w defined in (2.3), we have $\widetilde{w}=\frac{1}{2}(A\alpha^n\beta^m\alpha+A^{-1}\alpha^{-n}\beta^{-m}\alpha^{-1})$ and α is related to p (the spacing parameter of n-direction) as in (2.4). Then $\overline{w}$ should be accordingly defined as $\overline{w}=\frac{1}{2}(A\alpha^n\beta^m s+A^{-1}\alpha^{-n}\beta^{-m}s^{-1})$ and s is related to the spacing parameter k of l-direction by $k=(1-s)^2/2s$, which is coincident with (2.4). For x_i defined in (2.7), we have $\overline{x}_i=x_i+c^i$ where we suppose $c^2=k$ to coincide with (2.8).

One imposes $\overline{u}$ as a solution of Q2 in the form[b]

$$\overline{u}=\overline{u}_\theta+v. \tag{3.1}$$

Then the BT (1.2a)–(1.2b) is reduced to difference equations for v

$$\widetilde{v}=\frac{Ev}{v+F},\qquad \widehat{v}=\frac{Gv}{v+H}, \tag{3.2}$$

where

$$E=\widetilde{u}-\widetilde{\overline{u}}_\theta+\frac{k}{p}(u-\widetilde{u})+k(k-p),\qquad F=-u+\overline{u}_\theta+\frac{k}{p}(u-\widetilde{u})-k(k-p), \tag{3.3a}$$

$$G=\widehat{u}-\widehat{\overline{u}}_\theta+\frac{k}{q}(u-\widehat{u})+k(k-q),\qquad H=-u+\overline{u}_\theta+\frac{k}{q}(u-\widehat{u})-k(k-q). \tag{3.3b}$$

The difference system (3.2) can be linearized using $v=\frac{f}{g}$, which leads to

$$\widetilde{\Phi}=\Lambda\begin{pmatrix}E & 0\\ 1 & F\end{pmatrix}\Phi,\qquad \widehat{\Phi}=\Lambda\begin{pmatrix}G & 0\\ 1 & H\end{pmatrix}\Phi,\qquad \Phi=\begin{pmatrix}f\\ g\end{pmatrix}, \tag{3.4}$$

where Λ is some "balancing" factor to be determined to guarantee the compatibility $\widehat{\widetilde{\Phi}}=\widetilde{\widehat{\Phi}}$. Then it remains to integrate the linear difference system (3.4) so that $v=\frac{f}{g}$, hence $\bar{u}$, can be derived. The so-obtained $\bar{u}$ gives a 1SS of Q2.

3.2 1SS from exponential seed solution (2.6)

Using the background form (2.6) and its shift we have

$$\overline{u}_\theta = \frac{1}{2}\big(A\alpha^n\beta^m s + A^{-1}\alpha^{-n}\beta^{-m}s^{-1} + 4\big) + \frac{A\alpha^n\beta^m s + 1}{A\alpha^n\beta^m s - 1}(Pn + Qm + K + r). \tag{3.5}$$

Similar to the parametrizations of α, β and P, Q, s and K satisfy

$$k = \frac{(1-s)^2}{2s}, \qquad K = \frac{(s^2-1)(s^2-4s+1)}{4s^2}, \tag{3.6}$$

as k is understood as the lattice parameter in the $\bar{}$ direction. Following the ansatz described above (3.1)–(3.4), direct but hard computations lead to the difference system

$$\widetilde{\Phi} = \begin{pmatrix} S\cdot\frac{U_{n+1,m}V_{n,m}}{U_{n,m}V_{n+1,m}} & 0 \\ \frac{1}{U_{n,m}} & \Delta\cdot\frac{V_{n+1,m}}{V_{n,m}} \end{pmatrix}\Phi, \qquad \widehat{\Phi} = \begin{pmatrix} T\cdot\frac{U_{n,m+1}V_{n,m}}{U_{n,m}V_{n,m+1}} & 0 \\ \frac{1}{U_{n,m}} & \Omega\cdot\frac{V_{n,m+1}}{V_{n,m}} \end{pmatrix}\Phi, \tag{3.7}$$

where

$$U_{n,m} = \frac{A\alpha^n\beta^m + \frac{1}{A\alpha^n\beta^m} - 4}{2Z_{n,m}} + \frac{2(Pn+Qm+r)}{(A\alpha^n\beta^m-1)(\frac{1}{A\alpha^n\beta^m}-1)}, \qquad V_{n,m} = \frac{A\alpha^n\beta^m s - 1}{A\alpha^n\beta^m - 1},$$
$$S = \frac{(1-s)(1-\alpha s)}{(1-\alpha)s}, \qquad \Delta = \frac{(1-s)(\alpha - s)}{(1-\alpha)s},$$
$$T = \frac{(1-s)(1-\beta s)}{(1-\beta)s}, \qquad \Omega = \frac{(1-s)(\beta-s)}{(1-\beta)s},$$

and the balancing factor takes $\Lambda = 1/U_{n,m}$. Integrating the first equation in (3.7), one gets

$$f_{n,m} = \frac{U_{n,m}V_{0,m}}{U_{0,m}V_{n,m}}S^n f_{0,m},$$
$$g_{n,m} = \frac{1}{s}V_{n,m}\frac{V_{0,m}}{U_{0,m}}\left(\frac{S^n-\Delta^n}{1-s^2} + \frac{S^n}{A\alpha^n\beta^m s - 1} - \frac{\Delta^n}{A\beta^m s - 1}\right)f_{0,m} + \frac{V_{n,m}}{V_{0,m}}\Delta^n g_{0,m},$$

where the expression of $g_{n,m}$ is obtained using the identity

$$\frac{1}{V_{n-j,m}V_{n-1-j,m}} = \frac{1}{s^2}\left[1 + \frac{1-s}{1-\alpha}\left(\frac{1-\alpha s}{A\alpha^{n-j}\beta^m s - 1} - \frac{\alpha - s}{A\alpha^{n-1-j}\beta^m s - 1}\right)\right].$$

Similarly expressions can be derived solving the second equation in (3.7). Let

$$\rho_{n,m} = \left(\frac{S}{\Delta}\right)^n\left(\frac{T}{\Omega}\right)^m\rho_{0,0} = \left(\frac{1-\alpha s}{\alpha - s}\right)^n\left(\frac{1-\beta s}{\beta - s}\right)^m\rho_{0,0}, \tag{3.8}$$

we can obtain v in the form

$$v_{n,m} = \frac{\frac{U_{n,m}V_{0,0}}{U_{0,0}V_{n,m}}v_{0,0}\rho_{n,m}/\rho_{0,0}}{\frac{V_{n,m}V_{0,0}}{sU_{0,0}}\big(-\frac{1-\rho_{n,m}/\rho_{0,0}}{1-s^2} + \frac{\rho_{n,m}/\rho_{0,0}}{A\alpha^n\beta^m s - 1} - \frac{1}{As-1}\big)v_{0,0} + \frac{V_{n,m}}{V_{0,0}}}. \tag{3.9}$$

After a reparametrization of $\rho_{0,0}$, we get

$$v_{n,m} = \frac{U_{n,m}}{V_{n,m}^2} \cdot \frac{\rho_{n,m}}{1 + \frac{1}{1-s^2} \cdot \frac{A\alpha^n\beta^m - s}{A\alpha^n\beta^m s - 1} \cdot \rho_{n,m}}, \tag{3.10}$$

which, together with (3.5), forms a 1SS of Q2. Reparametrising A to $\frac{A}{s}$ one obtains

$$\begin{aligned} u_{n,m}^{1SS} = {} & \frac{1}{2}\left(\frac{1}{A\alpha^n\beta^m} + A\alpha^n\beta^m + 4 + \left(\frac{s}{A\alpha^n\beta^m} + \frac{A\alpha^n\beta^m}{s} - 4\right)\left(\frac{A^2\alpha^{2n}\beta^{2m}}{s^2} - 1\right)\Gamma_{n,m}\right) \\ & + (Pn + Qm + r)\left(1 + 2\frac{(A\alpha^n\beta^m - 1)(1 - s^2 + s\rho_{n,m})}{(1 - s^2)\rho_{n,m}}\Gamma_{n,m}\right) \\ & + \frac{A\alpha^n\beta^m + 1}{A\alpha^n\beta^m - 1}K, \end{aligned} \tag{3.11}$$

where

$$\Gamma_{n,m} = \frac{1}{A\alpha^n\beta^m - 1} \cdot \frac{(1 - s^2)\rho_{n,m}}{(A\alpha^n\beta^m - 1)(1 - s^2) + (A\alpha^n\beta^m s^{-1} - s)\rho_{n,m}}. \tag{3.12}$$

3.3 1SSs from rational seed solutions (2.12) and (2.15)

We continue computing 1SSs of Q2 using the rational seed solutions (2.12) and (2.15). We will skip computational details and just put the main results.

From the seed solution (2.12), the terms defined in (3.3a)–(3.3b) and (3.4) are in the forms

$$\begin{aligned} & E = SU_{n+1,m}\frac{\varphi_{n,m}}{\varphi_{n+1,m}}, \qquad F = \Delta U_{n,m}\frac{\varphi_{n+1,m}}{\varphi_{n,m}}, \\ & G = TU_{n,m+1}\frac{\varphi_{n,m}}{\varphi_{n,m+1}}, \qquad H = \Omega U_{n,m}\frac{\varphi_{n,m+1}}{\varphi_{n,m}}, \\ & S = \frac{s(a+s)}{a}, \qquad T = \frac{s(b+s)}{b}, \qquad \Delta = \frac{s(s-a)}{a}, \qquad \Omega = \frac{s(s-b)}{b}, \qquad \Lambda = 1/U_{n,m}, \end{aligned} \tag{3.13}$$

where

$$\varphi_{n,m} = \frac{x_1 + s}{x_1}, \qquad U_{n,m} = \frac{z_{n,m}}{x_1^2},$$

with $z_{n,m}$ (2.11) and $k = s^2$. Then, by solving the system (3.4), we have

$$v_{n,m} = \frac{f_{n,m}}{g_{n,m}} = \frac{2sU_{n,m}\rho_{n,m}}{\varphi_{n,m}^2(1 + \frac{x_1-s}{x_1+s}\rho_{n,m})}, \qquad \rho_{n,m} = \left(\frac{s+a}{s-a}\right)^n\left(\frac{s+b}{s-b}\right)^m\rho_{0,0}. \tag{3.14}$$

Thus the so-obtained 1SS of Q2 is

$$u_{n,m}^{1SS} = \overline{u}_\theta + \frac{2sz_{n,m}\rho_{n,m}}{(x_1^2 - s^2)\rho_{n,m} + (x_1 + s)^2}, \tag{3.15}$$

where $\overline{u}_\theta$ is a bar-shifted u (2.12) and $z_{n,m}$ given in (2.11). We depict its shape and motion, after removing the background $\overline{u}_\theta$, in Fig. 3(a). One can find the wave asymptotically is governed by zero on one direction and by $\frac{2sz_{n,m}}{x_1^2-s^2}$ on the other direction.

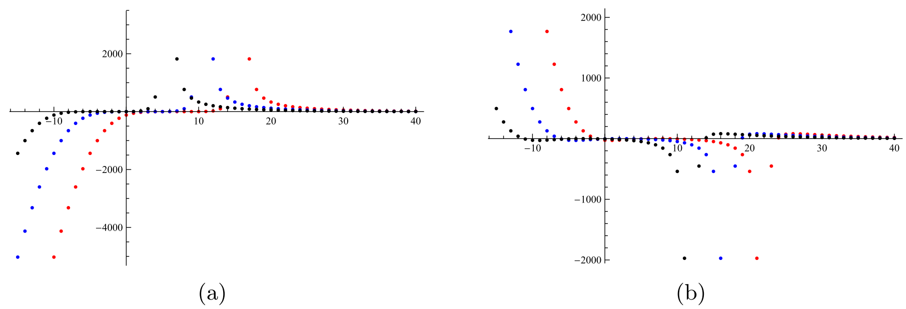

Figure 3 One-soliton solutions of Q2 without backgrounds. (**a**) Solution (3.15), (**b**) solution (3.17); where $a = -0.5$, $b = -0.5$, $s = 4.5$, $\rho_{0,0} = 0.5$, $c_0 = 1$, $\zeta_1 = \zeta_3 = \zeta_5 = 1$, $\delta = -3$, $m \in [-15, 40]$, waves in red, blue and black stand for $n = -5, 0, 5$, respectively

For the seed solution (2.15), one obtains similar expressions (3.13) and

$$S = \frac{2s\delta(a+s)}{a}, \qquad T = \frac{2s\delta(b+s)}{b}, \qquad \Delta = \frac{2s\delta(s-a)}{a},$$
$$\Omega = \frac{2s\delta(s-b)}{b}, \qquad \Lambda = 1/U_{n,m},$$

where

$$\varphi_{n,m} = \frac{x_1 + s - \delta}{x_1 - \delta}, \qquad U_{n,m} = \frac{z_{n,m}}{(x_1 - \delta)^2}, \qquad k = s^2,$$

with $z_{n,m}$ (2.14). Solving (3.4) yields

$$v_{n,m} = \frac{4\delta s U_{n,m}\rho_{n,m}}{\varphi_{n,m}^2(1 + \frac{x_1 - s - \delta}{x_1 + s - \delta}\rho_{n,m})}, \qquad \rho_{n,m} = \left(\frac{s+a}{s-a}\right)^n \left(\frac{s+b}{s-b}\right)^m \rho_{0,0}, \tag{3.16}$$

which leads to another 1SS of Q2 in the form

$$u_{n,m}^{1SS} = \overline{u}_\theta + \frac{4\delta s z_{n,m}\rho_{n,m}}{[(x_1-\delta)^2 - s^2]\rho_{n,m} + (x_1 - \delta + s)^2}, \tag{3.17}$$

where $\overline{u}_\theta$ is a bar-shifted u (2.15) and $z_{n,m}$ given in (2.14). We depict its shape and motion, after removing the background $\overline{u}_\theta$, in Fig. 3(b). This is a moving wave asymptotically governed by zero on one direction and by $\frac{4\delta s z_{n,m}}{(x_1-\delta)^2 - s^2}$ on the other direction.

4 Conclusions

In this short paper, we manage to provide explicit formulas of solutions of the Q2 equation (1.1) in the ABS list, at the cost of a considerable computational effort (in particular, to determine the balancing factor Λ). We derive seed solutions of Q2 in Sect. 2 using solutions of Q1(δ) and a non-auto BT connecting them. The results are then used in Sect. 3 to derive 1SSs of Q2 using an auto BT approach. Both the auto BT and the non-auto BT are realizations of the CAC property. The seed and the associated soliton solutions belong to either exponential type or rational type of solutions. They are essentially new solutions, i.e. (2.12), (2.15), (3.15) and (3.17), as they cannot be obtained as degenerated cases of known solutions.

Here we have relaxed the definition of "solitons" as it is hard to remove background (seed solution) from an 1SS and leave a pure soliton form. As an example we may have an extra glance at the 1SS (3.15), in which $\overline{u}_\theta$ is a background. The remaining soliton part takes a form

$$v = \frac{\frac{2sz_{n,m}}{(x_1+s)^2}\rho_{n,m}}{1+\frac{x_1-s}{x_1+s}\rho_{n,m}},$$

with the soliton parameter s, which looks much neater than (3.10) derived from the exponential case. This is again different from the pure soliton form $\frac{s\rho_{n,m}}{1+\rho_{n,m}}$ (cf. (3.15) in [15]). In addition, although rational solutions can usually be interpreted as some limits of solitons, so far in the literature we do not know any solutions that can yield the above form by imposing suitable limits.

In comparison with other equations in the ABS list, Q2 is rather special and needs further investigations. For instance, its bilinear form, continuous counterpart, geometric interpretations or physical significance have not been fully understood. In particular, a systematic approach to generating N-soliton solutions of given type is yet to be understood, into which we hope our results could provide insight.

Funding
This project is supported by the NSFC (Nos. 11371241, 11631007, 11601312, 11875040, 11801289) and Shanghai Young Eastern Scholar scheme (2016–2019).

Competing interests
The authors declare that there is no conflict of interest regarding the publication of this paper.

Authors' contributions
All authors contributed equally to the manuscript and read and approved the final manuscript.

Author details
[1] Department of Mathematics, Shanghai University, Shanghai, P.R. China. [2] Faculty of Science, Ningbo University, Ningbo, P.R. China.

Endnotes
[a] This leads to a δ-free u (2.12).
[b] With this special form v will satisfy the linear system (3.4) which then will be handled more conveniently. More examples one can refer to [15, 20–22].

Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 18 August 2018 Accepted: 16 January 2019 Published online: 25 January 2019

References
1. Hietarinta, J., Joshi, N., Nijhoff, F.W.: Discrete Systems and Integrablity. Cambridge University Press, Cambridge (2016)
2. Nijhoff, F.W., Walker, A.J.: The discrete and continuous Painlevé VI hierarchy and the Garnier systems. Glasg. Math. J. **43A**, 109–123 (2001)
3. Nijhoff, F.W.: Lax pair for the Adler (lattice Krichever–Novikov) system. Phys. Lett. A **297**, 49–58 (2002)
4. Bobenko, A.I., Suris, Y.B.: Integrable systems on quad-graphs. Int. Math. Res. Not. **2002**(11), 573–611 (2002)
5. Adler, V.E., Bobenko, A.I., Suris, Y.B.: Classification of integrable equations on quad-graphs, the consistency approach. Commun. Math. Phys. **233**, 513–543 (2002)
6. Krichever, I.M., Novikov, S.P.: Holomorphic fiberings and nonlinear equations: finite zone solutions of rank 2. Sov. Math. Dokl. **20**, 650–654 (1979)
7. Adler, V.E.: Bäcklund transformation for the Krichever–Novikov equation. Int. Math. Res. Not. **1998**, 1–4 (1998)
8. Nijhoff, F.W., Quispel, G.R.W., Capel, H.W.: Direct linearization of nonlinear difference–difference equations. Phys. Lett. A **97**, 125–128 (1983)
9. Nijhoff, F.W., Atkinson, J., Hietarinta, J.: Soliton solutions for ABS lattice equations. I. Cauchy matrix approach. J. Phys. A, Math. Theor. **42**, 404005 (2009)
10. Nijhoff, F.W., Capel, H.W.: The discrete Korteweg–de Vries equation. Acta Appl. Math. **39**, 133–158 (1995)

11. Zhang, D.J., Zhao, S.L.: Solutions to ABS lattice equations via generalized Cauchy matrix approach. Stud. Appl. Math. **131**, 72–103 (2013)
12. Nijhoff, F.W., Atkinson, J.: Elliptic *N*-soliton solutions of ABS lattice equations. Int. Math. Res. Not. **2010**, 3837–3895 (2010)
13. Butler, S.: Multidimensional inverse scattering of integrable lattice equations. Nonlinearity **25**, 1613–1634 (2012)
14. Zhang, D.D., Zhang, D.J.: Rational solutions to the ABS list: transformation approach. SIGMA **13**, 078 (2017)
15. Hietarinta, J., Zhang, D.J.: Soliton solutions for ABS lattice equations. II. Casoratians and bilinearization. J. Phys. A, Math. Theor. **42**, 404006 (2009)
16. Atkinson, J.: Bäcklund transformations for integrable lattice equations. J. Phys. A, Math. Theor. **41**, 135202 (2008)
17. Zhang, D.D., Zhang, D.J.: On decomposition of the ABS lattice equations and related Bäcklund transformations. J. Nonlinear Math. Phys. **25**, 34–53 (2018)
18. Zhao, S.L., Zhang, D.J.: Rational solutions to Q3(δ) in the Adler–Bobenko–Suris list and degenerations. J. Nonlinear Math. Phys. **26**, 107–132 (2019)
19. Atkinson, J., Hietarinta, J., Nijhoff, F.: Seed and soliton solutions of Adler's lattice equation. J. Phys. A, Math. Theor. **40**, F1–F8 (2007)
20. Atkinson, J., Hietarinta, J., Nijhoff, F.W.: Soliton solutions for Q3. J. Phys. A, Math. Theor. **41**, 142001 (2008)
21. Hietarinta, J., Zhang, D.J.: Multisoliton solutions to the lattice Boussinesq equation. J. Math. Phys. **51**, 033505 (2010)
22. Hietarinta, J., Zhang, D.J.: Soliton taxonomy for a modification of the lattice Boussinesq equation. SIGMA **7**, 061 (2011)

Submit your manuscript to a SpringerOpen journal and benefit from:

- Convenient online submission
- Rigorous peer review
- Open access: articles freely available online
- High visibility within the field
- Retaining the copyright to your article

Submit your next manuscript at ► springeropen.com