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Solvability for some class of multi-order nonlinear fractional systems

Yige Zhao[1*], Xinyi Hou[1], Yibing Sun[1] and Zhanbing Bai[2]

*Correspondence:
zhaoeager@126.com
[1]School of Mathematical Sciences, University of Jinan, Jinan, P.R. China
Full list of author information is available at the end of the article

Abstract

The existence of some class of multi-order nonlinear fractional systems is investigated in this paper. Some sufficient conditions of solutions for multi-order nonlinear systems are obtained based on fixed point theorems. Our results in this paper improve some known results.

MSC: 34A08; 34B18

Keywords: Nonlinear fractional differential system; Fractional Green's Function; Fixed point theorem; Multi-order

1 Introduction

Fractional calculus has drawn people's attention extensively. This is because of its extensive development of the theory and its applications in various fields, such as physics, engineering, chemistry and biology; see [1–9]. To be compared with integer derivatives, fractional derivatives are used for a better description of considered material properties, and the design of mathematical models by the differential equations of fractional order can more accurately describe the characteristics of the real-world phenomena; see [4, 7, 8]. Recently, many papers about the solvability for fractional equations have appeared; see [10–18].

Furthermore, the study of fractional systems has also been a topic focused on; see [19–25]. Although the coupled systems of fractional boundary value problems have been considered by some authors, coupled systems with multi-order fractional orders are seldom discussed. The orders of the nonlinear fractional systems which are considered in the existing papers belong to the same interval $(n, n+1]$ $(n \in \mathbb{N}^+)$; see [19–24].

Zhao et al. [25] investigated the solvability for nonlinear systems with mixed fractional orders via the Guo–Krasnosel'skii fixed point theorem

$$\begin{cases} -{}^{RL}D_{0^+}^{\alpha} x(t) = f(t, y(t)), \quad 0 < t < 1, \\ {}^{RL}D_{0^+}^{\beta} y(t) = g(t, x(t)), \quad 0 < t < 1, \\ x(0) = x(1) = x'(0) = y(0) = y(1) = y'(0) = y'(1) = 0, \end{cases} \tag{1}$$

where $2 < \alpha \leq 3$, $3 < \beta \leq 4$, ${}^{RL}D_{0^+}^{\alpha}$, ${}^{RL}D_{0^+}^{\beta}$ are the standard Riemann–Liouville fractional derivatives, and $f, g : (0,1] \times [0,+\infty) \to [0,+\infty)$ are continuous, $f(t,0) \equiv 0$, $g(t,0) \equiv 0$.

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However, we can see the fact, in Remark 3.2 [25] that the conditions $f(t,0) \equiv 0$ and $g(t,0) \equiv 0$ are too strong for the nonlinear systems. Therefore, we will study some new results for the problem (1) without the conditions $f(t,0) \equiv 0$ and $g(t,0) \equiv 0$.

Motivated by all the work above, in this paper we consider the existence of boundary value problem for multi-order nonlinear differential system (1) without the conditions $f(t,0) \equiv 0$ and $g(t,0) \equiv 0$. Our analysis relies on the Schauder fixed point theorem and the Banach contraction principle. Some sufficient conditions of the existence of boundary value problem for the multi-order nonlinear fractional differential systems are given. Our results in this paper improve some well-known results in [25]. Finally, we present examples to demonstrate our results.

The plan of the paper is as follows. Section 2 gives some preliminaries to prove our main results. Section 3 considers the solvability of multi-order nonlinear system (1) by the Schauder fixed point theorem and the Banach contraction principle. Section 4 presents illustrative examples to verify our new results, which is followed by a brief conclusion in Sect. 5.

2 Preliminaries

In this section, we give some definitions and lemmas about fractional calculus; see [25–27].

Definition 2.1 ([26]) The Riemann–Liouville fractional derivative of order $\gamma > 0$ of a continuous function $f : (0,+\infty) \to \mathbb{R}$ is denoted by

$$ {}^{RL}D_{0^+}^{\gamma} x(t) = \frac{1}{\Gamma(n-\gamma)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{x(s)}{(t-s)^{\alpha-n+1}}\, ds, $$

where $n = [\gamma] + 1$, $[\gamma]$ denotes the integer part of the number γ.

Definition 2.2 ([26]) The Riemann–Liouville fractional integral of order $\gamma > 0$ of a function $f : (0,+\infty) \to \mathbb{R}$ is denoted by

$$ I_{0^+}^{\gamma} x(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} x(s)\, ds. $$

For the solutions of fractional equations which are expressed based on Green's function refer to Lemma 2.3 and Lemma 2.5 in [25].

Lemma 2.1 *The function* $G_1(t,s)$ *defined by* (2.3) *in* [25] *has the following properties*:

($C1$) $G_1(t,s) > 0$, *for* $t,s \in (0,1)$;

($C2$) $q_1(t)k_1(s) \le \Gamma(\alpha)G_1(t,s) \le (\alpha-1)k_1(s)$, *for* $t,s \in (0,1)$, *where* $q_1(t) = t^{\alpha-1}(1-t)$, $k_1(s) = s(1-s)^{\alpha-1}$.

Lemma 2.2 *The function* $G_2(t,s)$ *defined by* (2.6) *in* [25] *has the following properties*:

($D1$) $G_2(t,s) > 0$, *for* $t,s \in (0,1)$;

($D2$) $(\beta-2)q_2(t)k_2(s) \le \Gamma(\beta)G_2(t,s) \le M_0 k_2(s)$, *for* $t,s \in (0,1)$, *where* $M_0 = \max\{\beta-1, (\beta-2)^2\}$, $q_2(t) = t^{\beta-2}(1-t)^2$, $k_2(s) = s^2(1-s)^{\beta-2}$.

We recall the following fixed point theorem for our main results.

Lemma 2.3 ([27]) *Let E be a Banach space with $C \subset E$ close and convex. Assume U is a relatively open subset of C with $0 \in U$ and $A : \overline{U} \to C$ is a continuous compact map. Then either*

($E1$) *A has a fixed point in U; or*

($E2$) *there exist a $u \in \partial U$, and a $\lambda \in (0, 1)$ with $u = \lambda Au$.*

3 Main results

In this section, we establish the existence of multi-order nonlinear fractional systems (1).

$I = [0, 1]$, and $C(I)$ denotes the space of all continuous real functions defined on I. $P = \{x(t)|x \in C(I)\}$ denotes a Banach space endowed with the norm $\|x\|_P = \max_{t\in I} |x(t)|$. We define the norm by $\|(x, y)\|_{P\times P} = \max\{\|x\|_P, \|y\|_P\}$ for $(x, y) \in P \times P$, then $(P \times P, \|\cdot\|_{P\times P})$ is a Banach space.

Consider the following system:

$$\begin{cases} x(t) = \int_0^1 G_1(t,s)f(s,y(s))\,ds, \\ y(t) = \int_0^1 G_2(t,s)g(s,x(s))\,ds. \end{cases} \tag{2}$$

Then we have the following results.

Lemma 3.1 *Suppose that $f, g : I \times [0, +\infty) \to [0, +\infty)$ are continuous. Then $(x, y) \in P \times P$ is a solution of (1) if and only if $(x, y) \in P \times P$ is a solution of system (2).*

This proof can be referred to that of Lemma 3.3 in [24], so it is omitted.

Let $T : P \times P \to P \times P$ be the operator defined by

$$\begin{aligned} T(x,y)(t) &= \left(\int_0^1 G_1(t,s)f\bigl(s,y(s)\bigr)\,ds, \int_0^1 G_2(t,s)g\bigl(s,x(s)\bigr)\,ds \right) \\ &=: \bigl(T_1y(t), T_2x(t)\bigr), \quad t \in I. \end{aligned}$$

By the continuity of the functions G_1, G_2, f and g, it implies that T is continuous. Furthermore, from Lemma 3.1, the fixed point of T is equivalent to the solution of system (1).

Next define the following notation:

$$\bar{A} = \left(\int_0^1 \frac{(\alpha-1)k_1(s)}{\Gamma(\alpha)}\,ds \right)^{-1}, \qquad \bar{B} = \left(\int_0^1 \frac{M_0k_2(s)}{\Gamma(\beta)}\,ds \right)^{-1}.$$

Theorem 3.1 *Let $f, g : I \times [0, +\infty) \to [0, +\infty)$ be continuous functions. Assume that the following conditions are satisfied:*

(H_1) *There exist two nonnegative functions $a_1(t), b_1(t) \in L(0, 1)$ and two nonnegative continuous functions $p(x), q(x) : [0, +\infty) \to [0, +\infty)$ such that $f(t, x) \le a_1(t) + p(x)$, $g(t, x) \le b_1(t) + q(x)$;*

(H_2) $\lim_{x\to+\infty} \frac{p(x)}{x} < \bar{A}$, $\lim_{x\to+\infty} \frac{q(x)}{x} < \bar{B}$.

Then the system (1) has a solution.

Proof Let $e_1 = \frac{1}{2}(\bar{A} - \lim_{x\to+\infty} \frac{p(x)}{x})$. By hypothesis (H_2), we find that there exists $c_1 > 0$ such that

$$p(x) \le (\bar{A} - e_1)x, \quad \text{for } x \ge c_1.$$

Set $M = \max\{p(x) : x \in [0, c_1]\}$. Then there exists $c_2 > c_1$ such that $\frac{M}{c_2} \le \bar{A} - e_1$, so we get

$$p(x) \le (\bar{A} - e_1)c_2, \quad \text{for } x \in [0, c_2].$$

Thus, for any $c \ge c_2$ and $u \in [0, c]$, we obtain

$$p(x) \le (\bar{A} - e_1)c.$$

Let $e_2 = \frac{1}{2}(\bar{B} - \lim_{x\to+\infty} \frac{q(x)}{x})$. In the same way, there exists $c_3 > 0$ such that, for any $c \ge c_3$ and $x \in [0, c]$, we get

$$q(x) \le (\bar{B} - e_2)c.$$

Define

$$X = \bigl\{\bigl(x(t), y(t)\bigr) | x(t), y(t) \in P, \bigl\| \bigl(x(t), y(t)\bigr) \bigr\|_{P\times P} \le c, t \in I\bigr\},$$

where

$$c = \max\left\{c_2, c_3, \frac{\bar{A}}{e_1}h_1, \frac{\bar{B}}{e_2}h_2\right\}$$

and

$$h_1 = \max_{t\in I} \int_0^1 G_1(t,s)a_1(s)\,ds, \qquad h_2 = \max_{t\in I} \int_0^1 G_2(t,s)b_1(s)\,ds.$$

Observe that X is a ball in the Banach space $P \times P$. Moreover, for any $(x, y) \in X$, $f(t, y(t))$, $g(t, x(t))$ are bounded, and $p(y(t)) \le (\bar{A} - e_1)c$, $q(x(t)) \le (\bar{B} - e_2)c$.

Now we verify that $T : X \to X$. From hypothesis (H_2), for any $(x, y) \in X$, we obtain

$$\begin{aligned}
\bigl|T_1 y(t)\bigr| &= \left|\int_0^1 G_1(t,s) f\bigl(s, y(s)\bigr)\,ds\right| \\
&= \int_0^1 G_1(t,s) f\bigl(s, y(s)\bigr)\,ds \\
&\le \int_0^1 G_1(t,s)a_1(s)\,ds + \int_0^1 G_1(t,s)p\bigl(y(s)\bigr)\,ds \\
&\le h_1 + (\bar{A} - e_1)c \int_0^1 G_1(t,s)\,ds \\
&\le h_1 + (\bar{A} - e_1)c \int_0^1 \frac{(\alpha - 1)k_1(s)}{\Gamma(\alpha)}\,ds
\end{aligned}$$

$$
\begin{aligned}
&\le h_1 + (\bar{A} - e_1)cA^{-1} \\
&\le \frac{e_1}{\bar{A}}c + \left(1 - \frac{e_1}{\bar{A}}\right)c = c.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\left|T_2x(t)\right| &\le h_2 + (\bar{B} - e_2)c\int_0^1 G_2(t,s)\,ds \\
&\le h_2 + (\bar{B} - e_2)c\int_0^1 \frac{M_0k_2(s)}{\Gamma(\beta)}\,ds \\
&\le h_2 + (\bar{B} - e_2)c\bar{B}^{-1} \\
&\le \frac{e_2}{\bar{B}}c + \left(1 - \frac{e_2}{\bar{B}}\right)c = c.
\end{aligned}
$$

Thus, $\|T_1y\|_P \le c$, $\|T_2x\|_P \le c$. That is, we get $\|T(x,y)\|_{P\times P} \le c$. Notice that $T_1y(t)$, $T_2x(t)$ are continuous on I. Therefore, we obtain $T : X \to X$.

Next we verify T is a completely continuous operator. In fact, we fix

$$
M = \max_{t\in I} f\bigl(t, y(t)\bigr), \qquad N = \max_{t\in I} g\bigl(t, x(t)\bigr).
$$

For $(x,y) \in X$, $t,\tau \in I$, $t < \tau$, we get

$$
\begin{aligned}
&\left|T_1y(t) - T_1y(\tau)\right| \\
&\quad= \left|\int_0^1 \bigl(G_1(t,s) - G_1(\tau,s)\bigr)f\bigl(s,y(s)\bigr)\,ds\right| \\
&\quad\le M\biggl[\int_0^t \left|G_1(t,s) - G_1(\tau,s)\right|ds + \int_t^\tau \left|G_1(t,s) - G_1(\tau,s)\right|ds \\
&\qquad+ \int_\tau^1 \left|G_1(t,s) - G_1(\tau,s)\right|ds\biggr] \\
&\quad\le M\left(\int_0^1 \frac{(1-s)^{\alpha-1}(\tau - t)}{\Gamma(\alpha)}\,ds + \int_0^\tau \frac{(\tau - s)^{\alpha-1}}{\Gamma(\alpha)}\,ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\,ds\right) \\
&\quad= \frac{M}{\Gamma(\alpha+1)}\bigl((\tau - t) + \bigl(\tau^\alpha - t^\alpha\bigr)\bigr).
\end{aligned}
$$

Similarly,

$$
\left|T_2x(t) - T_2x(\tau)\right| \le \frac{N}{\Gamma(\beta+1)}\bigl(\bigl(\tau^{\beta-2} - t^{\beta-2}\bigr) + 2\bigl(\tau^{\beta-1} - t^{\beta-1}\bigr) + \bigl(\tau^\beta - t^\beta\bigr)\bigr).
$$

Since the functions $t^\alpha, t^\beta, t^{\beta-1}, t^{\beta-2}$ are uniformly continuous on I, from the above analysis, TX is an equicontinuous set. Furthermore, $TX \subset X$. Therefore, T is a completely continuous operator. Hence, by Schauder fixed point theorem, the system (1) has one solution. □

Theorem 3.2 *Let $f,g : I \times [0,+\infty) \to [0,+\infty)$ be continuous functions. Assume that one of the following conditions is satisfied*:

(H_3) *There exist two nonnegative functions* $a_2(t), b_2(t) \in L(0,1)$ *such that* $f(t,x) \leq a_2(t) + d_1|x|^{\rho_1}$, $g(t,x) \leq b_2(t) + d_2|x|^{\rho_2}$, *where* $d_i \geq 0$, $0 < \rho_i < 1$ *for* $i = 1,2$;

(H_4) $f(t,x) \leq d_1|x|^{\rho_1}$, $g(t,x) \leq d_2|x|^{\rho_2}$ *where* $d_i > 0$, $\rho_i > 1$ *for* $i = 1,2$.

Then the system (1) *has a solution.*

Proof Let (H_4) be valid. Then we define

$$Y = \{(x(t), y(t)) | x(t), y(t) \in P, \|(x(t), y(t))\|_{P\times P} \leq r, t \in I\},$$

where

$$r \geq \max\left\{\left(\frac{2d_1}{\bar{A}}\right)^{\frac{1}{1-\rho_1}}, \left(\frac{2d_2}{\bar{B}}\right)^{\frac{1}{1-\rho_2}}, 2l_1, 2l_2\right\}$$

and

$$l_1 = \max_{t\in I} \int_0^1 G_1(t,s)a_2(s)\,ds, \qquad l_2 = \max_{t\in I} \int_0^1 G_2(t,s)b_2(s)\,ds.$$

Next, we verify $T : Y \to Y$. By hypothesis (H_3), for any $(x,y) \in Y$, we obtain

$$\begin{aligned}
|T_1 y(t)| &= \left|\int_0^1 G_1(t,s) f(s, y(s))\,ds\right| \\
&= \int_0^1 G_1(t,s) f(s, y(s))\,ds \\
&\leq \int_0^1 G_1(t,s)a_2(s)\,ds + d_1 r^{\rho_1} \int_0^1 G_1(t,s)\,ds \\
&\leq l_1 + d_1 r^{\rho_1} \int_0^1 \frac{(\alpha-1)k_1(s)}{\Gamma(\alpha)}\,ds \\
&= l_1 + d_1 r^{\rho_1} \bar{A}^{-1} \\
&\leq \frac{r}{2} + \frac{r}{2} = r.
\end{aligned}$$

Similarly,

$$\begin{aligned}
|T_2 x(t)| &\leq l_2 + d_2 r^{\rho_2} \int_0^1 G_2(t,s)\,ds \\
&\leq l_2 + d_2 r^{\rho_2} \int_0^1 \frac{M_0 k_2(s)}{\Gamma(\beta)}\,ds \\
&= l_2 + d_2 r^{\rho_2} c\bar{B}^{-1} \\
&\leq \frac{r}{2} + \frac{r}{2} = r.
\end{aligned}$$

Therefore, $\|T_1 y\|_P \leq r$, $\|T_2 x\|_P \leq r$. That is, we have $\|T(x,y)\|_{P\times P} \leq r$. Notice that $T_1 y(t)$, $T_2 x(t)$ are continuous on I. Thus, we get $T : Y \to Y$.

Next, let (H_4) be valid. Then we choose

$$0 < r \leq \min\left\{\left(\frac{\bar{A}}{d_1}\right)^{\frac{1}{\rho_1-1}}, \left(\frac{\bar{B}}{d_2}\right)^{\frac{1}{\rho_2-1}}\right\}.$$

Similarly, we can obtain

$$\|T_1 y\|_P \le d_1 r^{\rho_1} \bar{A}^{-1} \le r, \qquad \|T_2 x\|_P \le d_2 r^{\rho_2} \bar{B}^{-1} \le r.$$

That is, we have $\|T(x,y)\|_{P\times P} \le r$. And $T_1y(t)$, $T_2x(t)$ are continuous on I. Thus, we get $T: Y \to Y$. By Theorem 3.1, we see that T is a completely continuous operator. Hence, by the Schauder fixed point theorem, the system (1) has one solution. □

Theorem 3.3 *Let* $f, g : I \times [0,+\infty) \to [0,+\infty)$ *be continuous functions. Assume that one of the following conditions is satisfied*:

(H_5) *There exist two nonnegative functions* $a_3(t), b_3(t) \in L(0,1)$ *such that* $|f(t,x_1) - f(t,x_2)| \le a_3(t)|x_1 - x_2|$, $|g(t,x_1) - g(t,x_2)| \le b_3(t)|x_1 - x_2|$, $t \in [0,1]$ *and* f, g *satisfies* $f(0,0) = 0$, $g(0,0) = 0$.

(H_6) *Suppose that* $\lambda = \max\{\lambda_1, \lambda_2\} < 1$, *where*

$$\lambda_1 = \int_0^1 \frac{(\alpha-1)k_1(s)a_3(s)}{\Gamma(\alpha)}\,ds, \qquad \lambda_2 = \int_0^1 \frac{M_0 k_2(s)b_3(s)}{\Gamma(\beta)}\,ds.$$

Then the system (1) *has a unique solution*.

Proof For any $(x_1,y_1),(x_2,y_2) \in P \times P$, we get

$$\begin{aligned}
\bigl|T_1y_1(t) - T_1y_2(t)\bigr| &= \left|\int_0^1 G_1(t,s)f\bigl(s,y_1(s)\bigr)\,ds - \int_0^1 G_1(t,s)f\bigl(s,y_2(s)\bigr)\,ds\right| \\
&= \int_0^1 G_1(t,s)\bigl|f\bigl(s,y_1(s)\bigr) - f\bigl(s,y_2(s)\bigr)\bigr|\,ds \\
&\le \int_0^1 G_1(t,s)a_3(s)\|y_1 - y_2\|_P\,ds \\
&\le \int_0^1 \frac{(\alpha-1)k_1(s)a_3(s)}{\Gamma(\alpha)}\,ds\|y_1 - y_2\|_P \\
&= \lambda_1\|y_1 - y_2\|_P.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\bigl|T_2x_1(t) - T_2x_2(t)\bigr| &= \left|\int_0^1 G_2(t,s)g\bigl(s,x_1(s)\bigr)\,ds - \int_0^1 G_2(t,s)g\bigl(s,x_2(s)\bigr)\,ds\right| \\
&= \int_0^1 G_2(t,s)\bigl|g\bigl(s,x_1(s)\bigr) - g\bigl(s,x_2(s)\bigr)\bigr|\,ds \\
&\le \int_0^1 G_2(t,s)b_3(s)\|x_1 - x_2\|_P\,ds \\
&\le \int_0^1 \frac{M_0k_2(s)b_3(s)}{\Gamma(\beta)}\,ds\|x_1 - x_2\|_P \\
&= \lambda_2\|x_1 - x_2\|_P.
\end{aligned}$$

Thus, $\|T_1y_1 - T_1y_2\|_P \le \lambda_1\|y_1 - y_2\|_P$, $\|T_2x_1 - T_2x_2\|_P \le \lambda_2\|x_1 - x_2\|_P$.

Therefore, for the Euclidean distance d on $\mathbb{R}^2$, we have

$$\begin{aligned} d\big(T(x_1,y_1),T(x_2,y_2)\big) &= \sqrt{(T_1y_1-T_1y_2)^2+(T_2x_1-T_2x_2)^2} \\ &= \sqrt{\|T_1y_1-T_1y_2\|_P^2+\|T_2x_1-T_2x_2\|_P^2} \\ &\le \sqrt{\big(\lambda_1\|y_1-y_2\|_P\big)^2+\big(\lambda_2\|x_1-x_2\|_P\big)^2} \\ &\le \lambda\sqrt{\|y_1-y_2\|_P^2+\|x_1-x_2\|_P^2} \\ &= \lambda d\big((x_1,y_1),(x_2,y_2)\big). \end{aligned}$$

Thus, T is a contraction since $\lambda < 1$.

By the Banach contraction principle, T has a unique fixed point which is a solution of the system (1). □

Remark 3.1 In this paper, we give some new results for the system (1) without conditions $f(t,0) \equiv 0$ and $g(t,0) \equiv 0$. Our results in this paper improve some well-known results in [25].

4 Example

In this section, we will present examples to illustrate the main results.

Example 4.1 Consider the following system:

$$\begin{cases} -{}^{RL}D_{0^+}^{\frac{5}{2}}x(t) = 2t + 28(t-\frac{1}{2})^2y, \quad 0<t<1, \\ {}^{RL}D_{0^+}^{\frac{7}{2}}y(t) = t^2 + 100(t-\frac{1}{2})^2x, \quad 0<t<1, \\ x(0)=x(1)=x'(0)=y(0)=y(1)=y'(0)=y'(1)=0. \end{cases} \tag{3}$$

Choose $a_1(t) = 3t$, $b_1(t) = 2t^2$ and $p(x) = 7x$, $q(x) = 25x$. So (H_1) holds. Since $\bar{A} = 7.7545$, $\bar{B} = 26.1714$, thus (H_2) holds. By Theorem 3.1, the system (3) has a solution.

Remark 4.1 In Example 4.1 and Example 4.2 of [25], the systems (4.1) and (4.2) with conditions $f(t,0) \equiv 0$ and $g(t,0) \equiv 0$ are considered. However, in Example 4.1 of this paper, $f(t,y) = 2t + 28(t-\frac{1}{2})^2$, $g(t,y) = t^2 + 100(t-\frac{1}{2})^2$, we can easily see that $f(t,0) \not\equiv 0$ and $g(t,0) \not\equiv 0$. Thus, it is clear that one cannot deal with the system (3) of this paper by the method presented [25].

Example 4.2 Consider the following system:

$$\begin{cases} -{}^{RL}D_{0^+}^{\frac{5}{2}}x(t) = (t-\frac{1}{4})^4(y(t))^{\rho_1}, \quad 0<t<1, \\ {}^{RL}D_{0^+}^{\frac{7}{2}}y(t) = (t-\frac{1}{4})^4(x(t))^{\rho_2}, \quad 0<t<1, \\ x(0)=x(1)=x'(0)=y(0)=y(1)=y'(0)=y'(1)=0, \end{cases} \tag{4}$$

where $0 < \rho_i < 1$ or $\rho_i \ge 1$ for $i = 1,2$.

Note that $a_2(t) = b_2(t) = 0$ and $d_1 = d_2 = \frac{81}{256}$. By Theorem 3.2, the system (4) has a solution.

5 Conclusion

We have considered the solvability of some class of multi-order nonlinear fractional differential systems in this paper. Some sufficient conditions for multi-order nonlinear differential systems have been established by fixed point theorems. Our results improve the work presented in [25].

In future work, one can study the stability and the stabilization problems for multi-order nonlinear fractional differential systems which concern the existence of solutions.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

Author details
[1]School of Mathematical Sciences, University of Jinan, Jinan, P.R. China. [2]College of Mathematics and System Science, Shandong University of Science and Technology, Qingdao, P.R. China.

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