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Stability analysis of nonlinear implicit fractional Langevin equation with noninstantaneous impulses

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Abstract

In this paper, we consider a nonlocal boundary value problem of nonlinear implicit fractional Langevin equation with noninstantaneous impulses. We study the existence, uniqueness and generalized Ulam–Hyers–Rassias stability of the proposed model with the help of fixed point approach, over generalized complete metric space. We give an example which supports our main result.

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1 Introduction

At Wisconsin university, Ulam raised a question about the stability of functional equations in 1940. The question of Ulam was: Under what conditions does there exist an additive mapping near an approximately additive mapping?; see [30]. In 1941, Hyers was the first mathematician who gave a partial answer to Ulam's question [12] in a Banach space. Since then, stability of such form is known as Ulam–Hyers stability. In 1978, Rassias [23] provided a remarkable generalization of the Ulam–Hyers stability of mappings by considering variables. For more information about the topic, we refer the reader to [3, 14–16, 24, 28, 31, 40, 42].

An equation of the form $m\frac{d^2X}{dt^2} = \lambda\frac{dX}{dt} + \eta(t)$ is called Langevin equation, introduced by Paul Langevin in 1908. Langevin equations have been widely used to describe stochastic problems in physics, chemistry and electrical engineering. For example, Brownian motion is well described by the Langevin equation when the random fluctuation force is assumed to be white noise. For the removal of noise, mathematicians used fractional order differential equations, which also perform well in reducing the staircase effects compared to integer order differential equations. Thus it is very important to study Langevin equations with fractional derivatives; see, for instance, [2, 10, 20, 21].

Fractional order differential equations are generalizations of the classical integer order differential equations. Fractional calculus has become a fast developing area, and its applications can be found in diverse fields ranging from physical sciences, porous media, electrochemistry, economics, electromagnetics, medicine and engineering to biological sciences. Progressively, fractional differential equations play a very important role in thermodynamics, statistical physics, viscoelasticity, nonlinear oscillation of earthquakes, defence, optics, control, electrical circuits, signal processing, astronomy, etc. There are some outstanding articles which provide the main theoretical tools for the qualitative analysis of this research field, and at the same time, show the interconnection as well as the distinction between integral models, classical and fractional differential equations; see [1, 5, 13, 17, 19, 22, 25–27, 29].

Impulsive fractional differential equations are used to describe both physical and social sciences. Also they describe many practical dynamical systems such as evolutionary processes, characterized by abrupt changes of the state at certain instants. In the last few decades, the theory of impulsive fractional differential equations were well utilized in medicine, mechanical engineering, ecology, biology and astronomy, etc. There are some remarkable monographs [8, 11, 18, 32, 33, 35–37, 39, 41], which consider fractional differential equations with impulses.

Recently, many mathematicians devoted considerable attention to the existence, uniqueness and different types of Hyers–Ulam stability of the solutions of nonlinear implicit fractional differential equations with Caputo fractional derivative, see [4, 6, 7].

Wang et al. [34] studied generalized Ulam–Hyers–Rassias stability of the following fractional differential equation

$$\begin{cases} {}^cD^{\alpha}_{0,t}x(t) = f(t,x(t)), & t \in (t_k, s_k], k = 0,1,\ldots,m, 0 < \alpha < 1, \\ x(t) = g_k(t,x(t)), & t \in (s_{k-1}, t_k], k = 1,2,\ldots,m. \end{cases}$$

Zada et al. [38] studied existence and uniqueness of solutions by using Diaz–Margolis's fixed point theorem and presented Ulam–Hyers stability, generalized Ulam–Hyers stability, Ulam–Hyers–Rassias stability, and generalized Ulam–Hyers–Rassias stability for a class of nonlinear implicit fractional differential equation with noninstantaneous integral impulses and nonlinear integral boundary condition:

$$\begin{cases} {}^cD^{\alpha}_{0,t}x(t) = f(t,x(t),{}^cD^{\alpha}_{0,t}x(t)), & t \in (t_k, s_k], k = 0,1,\ldots,m, 0 < \alpha < 1, t \in (0,1], \\ x(t) = I^{\alpha}_{s_{k-1},t_k}(\xi_k(t,x(t))), & t \in (s_{k-1}, t_k], k = 0,1,\ldots,m, \\ x(0) = \frac{1}{\Gamma\alpha}\int_0^T (T-s)^{\alpha-1}\eta(s,x(s))\,ds. \end{cases}$$

Motivated by [34, 38], we consider the following nonlocal boundary value problem of nonlinear implicit fractional Langevin equation with noninstantaneous impulses:

$$\begin{cases} {}^cD^{\alpha}_{0,t}({}^cD^{\beta}_{0,t} + \lambda)x(t) \\ \quad = f(t,x(t),{}^cD^{\alpha}_{0,t}({}^cD^{\beta}_{0,t} + \lambda)x(t)) \\ \qquad + \int_0^t \frac{(t-s)^{\sigma-1}}{\Gamma(\delta)} f(s,x(s))\,ds, & t \in (t_k, s_k], k = 0,1,\ldots,m, \\ x(t) = g_k(t,x(t)), & t \in (s_{k-1}, t_k], k = 1,2,\ldots,m, \\ x(0) = x_0, \quad x(T) = \theta \int_0^{\eta} \frac{1}{\Gamma p}(\eta - s)^{p-1}x(s)\,ds, & 0 < \eta < T, \end{cases} \tag{1.1}$$

where ${}^cD^{\alpha}_{0,t}$ and ${}^cD^{\beta}_{0,t}$ represent classical Caputo derivatives [5] of order α and β with the lower bound zero, $0 = t_0 < s_0 < t_1 < s_1 < \cdots < t_m < s_m = \tau$, τ is the free fixed number and

$\lambda \in \mathbb{R} \setminus \{0\}$, $0 < \alpha, \beta < 1$, $0 < \alpha + \beta < 2$, $\sigma, p > 0$, x_0, θ are constants, $f : [0,\tau] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and $g_k : [s_{k-1}, t_k] \times \mathbb{R} \to \mathbb{R}$ is continuous for all $k = 1, 2, \ldots, m$.

In Sect. 2, we create a uniform framework to originate appropriate formula of solutions for our proposed model. In Sect. 3, we study the concept of generalized Ulam–Hyers–Rassias stability of Eq. (1.1). Finally, we give an example to illustrate our main result.

2 Solution framework of linear impulsive fractional Langevin equation

Let $J = [0,\tau]$ and $C(J,\mathbb{R})$ be the space of all continuous functions from J to $\mathbb{R}$, and the piecewise continuous function space $PC(J,\mathbb{R}) = \{x : f \to \mathbb{R} : x \in ((t_k, t_{k-1})], \mathbb{R}), k = 0, \ldots, m$ and there exist $x(t_k^-)$ and $x(t_k^+)$, $k = 1, 2, \ldots, m$ with $x(t_k^-) = x(t_k^+)\}$.

In the current section, we create a uniform framework to originate an appropriate formula for the solution of impulsive fractional differential equation of the form:

$$\begin{cases} {}^cD_{0,t}^{\alpha}({}^cD_{0,t}^{\beta} + \lambda)x(t) = f(t), & t \in (t_k, s_k], k = 0, 1, \ldots, m, 0 < \alpha, \beta < 1, \\ x(t) = g_k(t), & t \in (s_{k-1}, t_k], k = 1, 2, \ldots, m, \\ x(0) = x_0, & x(T) = \theta I^p x(\eta) \\ \quad \text{where } I^p x(\eta) = \int_0^{\eta} \frac{1}{\Gamma p}(\eta - s)^{p-1} x(s)\,ds, & 0 < \eta < T. \end{cases} \tag{2.1}$$

We recall some definitions of fractional calculus from [17] as follows.

Definition 2.1 The fractional integral of order α from 0 to t for the function f is defined by

$$I_{0,t}^{\gamma} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t f(s)(t-s)^{\alpha-1}\,ds, \quad t > 0, \alpha > 0,$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.2 The Riemann–Liouville fractional derivative of fractional order α from 0 to t for a function f can be written as

$${}^LD_{0,t}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\alpha+1-n}}\,ds, \quad t > 0, n-1 < \alpha < n,$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.3 The Caputo derivative of fractional order α from 0 to t for a function f can be defined as

$${}^cD_{0,t}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^n(s)\,ds, \quad \text{where } n = [\alpha] + 1.$$

Definition 2.4 The general form of classical Caputo derivative of order α of a function f can be given as

$${}^cD_{0,t}^{\alpha} = {}^LD_{0,t}^{\alpha}\left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0)\right), \quad t > 0, n-1 < \alpha < n.$$

Remark 2.1

(i) If $f(\cdot) \in C^m([0,\infty),\mathbb{R})$, then

$$\begin{aligned} {}^L D_{0,t}^{\alpha} f(t) &= \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^m(s)}{(t-s)^{\alpha+1-m}}\,ds \\ &= I_{0,t}^{m-\alpha} f^{(m)}(t), \quad t>0, m-1<\alpha<m. \end{aligned}$$

(ii) In Definition 2.4, the integrable function f can be discontinuous. This fact can lead us to consider impulsive fractional problems in the sequel.

Lemma 2.1 ([22]) *Let* $\alpha>0$, $\beta>0$, *and* $f \in L^1([a,b])$. *Then*

$$I^{\alpha} I^{\beta} f(t) = I^{\alpha+\beta} f(t), {}^c D_{0,t}^{\alpha}\big({}^c D_{0,t}^{\beta} f(t)\big) = {}^c D_{0,t}^{\alpha+\beta} f(t) \quad \text{and} \quad I^{\alpha} D_{0,t}^{\alpha} f(t) = f(t), \quad t\in[a,b].$$

Lemma 2.2 *Function* $x \in PC(J,\mathbb{R})$ *is a solution of* (2.1) *if and only if*

$$x(t) = \begin{cases} \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} f(s)\,ds - \frac{\lambda}{\Gamma\beta} \int_o^t (t-s)^{\beta-1} x(s)\,ds \\ \quad - \frac{\Delta t^{\beta}}{\Gamma(\beta+1)\Gamma(\alpha+\beta)} \int_0^T (T-s)^{\alpha+\beta-1} f(s)\,ds + \frac{\lambda\Delta t^{\beta}}{\Gamma(\beta)\Gamma(\beta+1)} \int_0^T (T-s)^{\beta-1} x(s)\,ds \\ \quad + \frac{\theta\Delta t^{\beta}}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p)} \int_0^{\eta} (\eta-s)^{\alpha+\beta+p-1} f(s)\,ds \\ \quad - \frac{\theta\Delta\lambda t^{\beta}}{\Gamma(\beta+1)\Gamma(\beta+p)} \int_0^{\eta} (\eta-s)^{\beta+p-1} x(s)\,ds \\ \quad - (\frac{\Delta(\theta\eta^p-\Gamma(p+1))t^{\beta}}{\Gamma(p+1)\Gamma(\beta+1)} - 1)x_0, \quad t\in(0,s_0]; \\ \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} f(s)\,ds - \frac{\lambda}{\Gamma\beta} \int_o^t (t-s)^{\beta-1} x(s)\,ds \\ \quad + \frac{\Delta(t_k^{\beta}-t^{\beta})}{\Gamma(\beta+1)\Gamma(\alpha+\beta)} \int_0^T (T-s)^{\alpha+\beta-1} f(s)\,ds - \frac{\lambda\Delta(t_k^{\beta}-t^{\beta})}{\Gamma(\beta+1)\Gamma(\beta)} \int_0^T (T-s)^{\beta-1} x(s)\,ds \\ \quad - \frac{\theta\Delta(t_k^{\beta}-t^{\beta})}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p)} \int_0^{\eta} (\eta-s)^{\alpha+\beta+p-1} f(s)\,ds \\ \quad + \frac{\theta\Delta\lambda(t_k^{\beta}-t^{\beta})}{\Gamma(\beta+1)\Gamma(\beta+p)} \int_0^{\eta} (\eta-s)^{\beta+p-1} x(s)\,ds \\ \quad + (\Delta\frac{(t_k^{\beta}-t^{\beta})}{\Gamma(\beta+1)}(\frac{\theta\eta^p-\Gamma(p+1)}{\Gamma(p+1)}) - \lambda)\frac{1}{\Gamma(\alpha+\beta)} \int_0^{t_k} (t_k-s)^{\alpha+\beta-1} f(s)\,ds \\ \quad - (\Delta\frac{(t_k^{\beta}-t^{\beta})}{\Gamma(\beta+1)}(\frac{\theta\eta^p-\Gamma(p+1)}{\Gamma(p+1)}) - 1)\frac{\lambda}{\Gamma\beta} \int_o^{t_k} (t_k-s)^{\beta-1} x(s)\,ds \\ \quad - (\Delta\frac{(t_k^{\beta}-t^{\beta})}{\Gamma(\beta+1)}(\frac{\theta\eta^p-\Gamma(p+1)}{\Gamma(p+1)}) - 1)g_k(t_k), \quad t\in(t_k,s_k]; \\ g_k(t), \quad t\in(s_{k-1},t_k], k=1,2,\ldots,m. \end{cases}$$

Proof Let x be a solution of problem (2.1).

Case 1. For $t\in[0,s_0]$, we consider

$${}^c D_{0,t}^{\alpha}\big({}^c D_{0,t}^{\beta} + \lambda\big)x(t) = f(t) \quad \text{with } x(0) = x_0 \quad \text{and} \quad x(T) = \theta I^p x(\eta).$$

After using fractional integrals I^{α} and I^{β} for the solution of the above fractional Langevin equation, we get

$$x(t) = I^{\alpha+\beta} f(t) - \lambda I^{\beta} x(t) - \frac{c_0 t^{\beta}}{\Gamma(\beta+1)} - c. \tag{2.2}$$

Using boundary conditions, we obtain

$$\begin{aligned} x(t) ={}& \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}f(s)\,ds - \frac{\lambda}{\Gamma\beta}\int_o^t (t-s)^{\beta-1}x(s)\,ds \\ &- \frac{\Delta t^\beta}{\Gamma(\beta+1)\Gamma(\alpha+\beta)}\int_0^T (T-s)^{\alpha+\beta-1}f(s)\,ds \\ &+ \frac{\lambda\Delta t^\beta}{\Gamma(\beta)\Gamma(\beta+1)}\int_0^T (T-s)^{\beta-1}x(s)\,ds \\ &+ \frac{\theta\Delta t^\beta}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p)}\int_0^\eta (\eta-s)^{\alpha+\beta+p-1}f(s)\,ds \\ &- \frac{\theta\Delta\lambda t^\beta}{\Gamma(\beta+1)\Gamma(\beta+p)}\int_0^\eta (\eta-s)^{\beta+p-1}x(s)\,ds \\ &- \left(\frac{\Delta(\theta\eta^p-\Gamma(p+1))t^\beta}{\Gamma(p+1)\Gamma(\beta+1)} - 1\right)x_0, \quad t\in[0,s_0]. \end{aligned}$$

For $t\in(s_0,t_1]$, $x(t)=g_1(t)$.

Case 2. For $t\in(t_1,s_1]$, we consider

$$ {}^cD^\alpha_{0,t}\left({}^cD^\beta_{0,t}+\lambda\right)x(t)=f(t) \quad \text{with } x(t_1)=g_1(t_1) \quad \text{and} \quad x(T)=\theta I^p x(\eta).$$

Since $x(t_1)=g_1(t_1)$, Eq. (2.2) is of the following type:

$$g_1(t_1) = I^{\alpha+\beta}f(t_1) - \lambda I^\beta x(t_1) - \frac{c_0 t_1^\beta}{\Gamma(\beta+1)} - c. \tag{2.3}$$

Using boundary conditions, we get

$$\begin{aligned} x(t) ={}& \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}f(s)\,ds - \frac{\lambda}{\Gamma\beta}\int_o^t (t-s)^{\beta-1}x(s)\,ds \\ &+ \frac{\Delta(t_1^\beta-t^\beta)}{\Gamma(\beta+1)\Gamma(\alpha+\beta)}\int_0^T (T-s)^{\alpha+\beta-1}f(s)\,ds \\ &- \frac{\lambda\Delta(t_1^\beta-t^\beta)}{\Gamma(\beta+1)\Gamma(\beta)}\int_0^T (T-s)^{\beta-1}x(s)\,ds \\ &- \frac{\theta\Delta(t_1^\beta-t^\beta)}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p)}\int_0^\eta (\eta-s)^{\alpha+\beta+p-1}f(s)\,ds \\ &+ \frac{\theta\Delta\lambda(t_1^\beta-t^\beta)}{\Gamma(\beta+1)\Gamma(\beta+p)}\int_0^\eta (\eta-s)^{\beta+p-1}x(s)\,ds \\ &+ \left(\Delta\frac{(t_1^\beta-t^\beta)}{\Gamma(\beta+1)}\left(\frac{\theta\eta^p-\Gamma(p+1)}{\Gamma(p+1)}\right) - \lambda\right)\frac{1}{\Gamma(\alpha+\beta)}\int_0^{t_1}(t_1-s)^{\alpha+\beta-1}f(s)\,ds \\ &- \left(\Delta\frac{(t_1^\beta-t^\beta)}{\Gamma(\beta+1)}\left(\frac{\theta\eta^p-\Gamma(p+1)}{\Gamma(p+1)}\right) - 1\right)\frac{\lambda}{\Gamma\beta}\int_o^{t_1}(t_1-s)^{\beta-1}x(s)\,ds \\ &- \left(\Delta\frac{(t_1^\beta-t^\beta)}{\Gamma(\beta+1)}\left(\frac{\theta\eta^p-\Gamma(p+1)}{\Gamma(p+1)}\right) - 1\right)g_1(t_1). \end{aligned}$$

Generally speaking, for $t\in(s_{k-1},t_k]$, $x(t_k)=g_k(t)$.

Case 3. For $t \in (t_k, s_k]$, we consider

$$^cD^{\alpha}_{0,t}\left(^cD^{\beta}_{0,t} + \lambda\right)x(t) = f(t), \quad \text{with } x(t_k) = g_k(t_k) \quad \text{and} \quad x(T) = \theta I^p x(\eta).$$

By repeating again the same process, we have

$$\begin{aligned} x(t) = {} & \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}f(s)\,ds - \frac{\lambda}{\Gamma\beta}\int_o^t (t-s)^{\beta-1}x(s)\,ds \\ & + \frac{\Delta(t_k^{\beta}-t^{\beta})}{\Gamma(\beta+1)\Gamma(\alpha+\beta)}\int_0^T (T-s)^{\alpha+\beta-1}f(s)\,ds \\ & - \frac{\lambda\Delta(t_k^{\beta}-t^{\beta})}{\Gamma(\beta+1)\Gamma(\beta)}\int_0^T (T-s)^{\beta-1}x(s)\,ds \\ & - \frac{\theta\Delta(t_k^{\beta}-t^{\beta})}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p)}\int_0^{\eta} (\eta-s)^{\alpha+\beta+p-1}f(s)\,ds \\ & + \frac{\theta\Delta\lambda(t_k^{\beta}-t^{\beta})}{\Gamma(\beta+1)\Gamma(\beta+p)}\int_0^{\eta} (\eta-s)^{\beta+p-1}x(s)\,ds \\ & + \left(\Delta\frac{(t_k^{\beta}-t^{\beta})}{\Gamma(\beta+1)}\left(\frac{\theta\eta^p-\Gamma(p+1)}{\Gamma(p+1)}\right)-\lambda\right)\frac{1}{\Gamma(\alpha+\beta)}\int_0^{t_k}(t_k-s)^{\alpha+\beta-1}f(s)\,ds \\ & - \left(\Delta\frac{(t_k^{\beta}-t^{\beta})}{\Gamma(\beta+1)}\left(\frac{\theta\eta^p-\Gamma(p+1)}{\Gamma(p+1)}\right)-1\right)\frac{\lambda}{\Gamma\beta}\int_o^{t_k}(t_k-s)^{\beta-1}x(s)\,ds \\ & - \left(\Delta\frac{(t_k^{\beta}-t^{\beta})}{\Gamma(\beta+1)}\left(\frac{\theta\eta^p-\Gamma(p+1)}{\Gamma(p+1)}\right)-1\right)g_k(t_k), \end{aligned}$$

where

$$\Delta = \frac{\Gamma(\beta+1)\Gamma(\beta+p+1)}{\Gamma(\beta+p+1)\eta^{\beta} - \Gamma(\beta+1)\theta\eta^{\beta+p} + \Gamma(\beta+1)t_k^{\beta}(\frac{\theta\eta^p-\Gamma(p+1)}{\Gamma(p+1)})}$$

with $t_1^{\beta} = 0$ for $t \in [0, s_0]$ and $t_k^{\beta} \neq 0$, for $t \in (t_k, s_k], k = 2, 3, \ldots$.

Conversely, one can verify the fact by proceeding the standard steps to complete the proof. □

3 Generalized Ulam–Hyers–Rassias stability

Using the ideas of stability in [24, 31], we can generate a generalized Ulam–Hyers–Rassias stability concept for Eq. (1.1).

Let $\epsilon, \psi \geq 0$ and for a nondecreasing $\varphi \in PC(J, \mathbb{R}_+)$ consider

$$\begin{cases} |^cD^{\alpha}_{0,t}(^cD^{\beta}_{0,t} + \lambda)x(t) - f(t, x(t), {^cD^{\alpha}_{0,t}}(^cD^{\beta}_{0,t} + \lambda)x(t))| \leq \varphi(t), \\ \quad t \in (t_k, s_k], k = 0, 1, \ldots, m, 0 < \alpha, \beta < 1, \\ |x(t) - (\Delta\frac{(t_k^{\beta}-t^{\beta})}{\Gamma(\beta+1)}(\frac{\theta\eta^p-\Gamma(p+1)}{\Gamma(p+1)}) - 1)g_k(t, x(t))| \leq \psi, \quad t \in (s_{k-1}, t_k], k = 0, 1, \ldots, m. \end{cases} \tag{3.1}$$

Remark 3.1 A function $x \in PC(J, \mathbb{R})$ is a solution of the inequality (3.1) if and only if there is $G \in PC(J, \mathbb{R})$ and a sequence G_k, $k = 1, 2, \ldots, m$ (which depends on x) such that

(i) $|G(t)| \leq \varphi(t)$, $t \in J$ and $|G_k| \leq \psi$, $k = 1, 2, \ldots, m$,

(ii) ${}^cD^{\alpha}_{0,t}({}^cD^{\beta}_{0,t}+\lambda)x(t)=f(t,x(t),{}^cD^{\alpha}_{0,t}({}^cD^{\beta}_{0,t}+\lambda)x(t))+G(t)$, $t\in(t_k,s_k]$, $k=1,2,\dots,m$,
(iii) $x(t)=g_k(t,x(t))+G_k$, $t\in(s_{k-1},t_k]$, $k=1,\dots,m$.

Definition 3.1 Equation (1.1) is called generalized Ulam–Hyers–Rassias stable with respect to (φ,ψ) if there exists $c_{f,\alpha,\beta,g_i,\varphi}>0$ such that for each solution $y\in PC(J,\mathbb{R})$ of inequality (3.1) there is a solution $x\in PC(J,\mathbb{R})$ of Eq. (1.1) with

$$\bigl|y(t)-x(t)\bigr|\le c_{f,\alpha,\beta,G_i,\varphi}\bigl(\varphi(t)+\psi\bigr),\quad t\in J.$$

Remark 3.2 If $x\in PC(J,\mathbb{R})$ is a solution of inequality (3.1), then x is a solution of the following integral inequality:

$$\begin{cases}
|x(t)-(\Delta\frac{(t_k^{\beta}-t^{\beta})}{\Gamma(\beta+1)}(\frac{\theta\eta^p-\Gamma(p+1)}{\Gamma(p+1)})-1)g_k(t,x(t))|\le\psi,\\
\quad t\in(s_{k-1},t_k],k=1,2,\dots,m;\\
|x(t)-x(0)-\frac{1}{\Gamma(\alpha+\beta)}\int_0^t(t-s)^{\alpha+\beta-1}f(s,x(s),{}^cD^{\alpha}_{0,t}({}^cD^{\beta}_{0,t}+\lambda)x(s))\,ds\\
\qquad+\frac{\lambda}{\Gamma\beta}\int_o^t(t-s)^{\beta-1}x(s)\,ds\\
\qquad+\frac{\Delta t^{\beta}}{\Gamma(\beta+1)\Gamma(\alpha+\beta)}\int_0^T(T-s)^{\alpha+\beta-1}f(s,x(s),{}^cD^{\alpha}_{0,t}({}^cD^{\beta}_{0,t}+\lambda)x(s))\,ds\\
\qquad-\frac{\lambda\Delta t^{\beta}}{\Gamma(\beta)\Gamma(\beta+1)}\int_0^T(T-s)^{\beta-1}x(s)\,ds+\frac{\theta\Delta\lambda t^{\beta}}{\Gamma(\beta+1)\Gamma(\beta+p)}\int_0^{\eta}(\eta-s)^{\beta+p-1}x(s)\,ds\\
\qquad-\frac{\theta\Delta t^{\beta}}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p)}\int_0^{\eta}(\eta-s)^{\alpha+\beta+p-1}f(s,x(s),{}^cD^{\alpha}_{0,t}({}^cD^{\beta}_{0,t}+\lambda)x(s))\,ds|\\
\quad\le\frac{1}{\Gamma(\alpha+\beta)}\int_0^t(t-s)^{\alpha+\beta-1}\varphi(s)\,ds+\frac{\lambda}{\Gamma\beta}\int_o^t(t-s)^{\beta-1}\varphi(s)\,ds\\
\qquad+\frac{\Delta t^{\beta}}{\Gamma(\beta+1)\Gamma(\alpha+\beta)}\int_0^T(T-s)^{\alpha+\beta-1}\varphi(s)\,ds+\frac{\lambda\Delta t^{\beta}}{\Gamma(\beta)\Gamma(\beta+1)}\int_0^T(T-s)^{\beta-1}x(s)\,ds\\
\qquad+\frac{\theta\Delta\lambda t^{\beta}}{\Gamma(\beta+1)\Gamma(\beta+p)}\int_0^{\eta}(\eta-s)^{\beta+p-1}\varphi(s)\,ds\\
\qquad+\frac{\theta\Delta t^{\beta}}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p)}\int_0^{\eta}(\eta-s)^{\alpha+\beta+p-1}\varphi(s)\,ds,\quad t\in(0,s_0];\\
|x(t)-(1-\Delta\frac{(t_k^{\beta}-t^{\beta})}{\Gamma(\beta+1)}(\frac{\theta\eta^p-\Gamma(p+1)}{\Gamma(p+1)}))g_k(t_k,x(t_k))\\
\qquad-\frac{1}{\Gamma(\alpha+\beta)}\int_0^t(t-s)^{\alpha+\beta-1}f(s,x(s),{}^cD^{\alpha}_{0,t}({}^cD^{\beta}_{0,t}+\lambda)x(s))\,ds\\
\qquad+\frac{\lambda}{\Gamma\beta}\int_o^t(t-s)^{\beta-1}x(s)\,ds\\
\qquad-\frac{\Delta(t_k^{\beta}-t^{\beta})}{\Gamma(\beta+1)\Gamma(\alpha+\beta)}\int_0^T(T-s)^{\alpha+\beta-1}f(s,x(s),{}^cD^{\alpha}_{0,t}({}^cD^{\beta}_{0,t}+\lambda)x(s))\,ds\\
\qquad+\frac{\lambda\Delta(t_k^{\beta}-t^{\beta})}{\Gamma(\beta+1)\Gamma(\beta)}\int_0^T(T-s)^{\beta-1}x(s)\,ds-\frac{\theta\Delta\lambda(t_k^{\beta}-t^{\beta})}{\Gamma(\beta+1)\Gamma(\beta+p)}\int_0^{\eta}(\eta-s)^{\beta+p-1}x(s))\,ds\\
\qquad+\frac{\theta\Delta(t_k^{\beta}-t^{\beta})}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p)}\int_0^{\eta}(\eta-s)^{\alpha+\beta+p-1}f(s,x(s),{}^cD^{\alpha}_{0,t}({}^cD^{\beta}_{0,t}+\lambda)x(s))\,ds\\
\qquad-(\Delta\frac{(t_k^{\beta}-t^{\beta})}{\Gamma(\beta+1)}(\frac{\theta\eta^p-\Gamma(p+1)}{\Gamma(p+1)})-\lambda)\\
\qquad\times\frac{1}{\Gamma(\alpha+\beta)}\int_0^{t_k}(t_k-s)^{\alpha+\beta-1}f(s,x(s),{}^cD^{\alpha}_{0,t}({}^cD^{\beta}_{0,t}+\lambda)x(s))\,ds\\
\qquad+(\Delta\frac{(t_k^{\beta}-t^{\beta})}{\Gamma(\beta+1)}(\frac{\theta\eta^p-\Gamma(p+1)}{\Gamma(p+1)})-1)\frac{\lambda}{\Gamma\beta}\int_o^{t_k}(t_k-s)^{\beta-1}x(s)\,ds|\\
\quad\le\frac{1}{\Gamma(\alpha+\beta)}\int_0^t(t-s)^{\alpha+\beta-1}\varphi(s)\,ds+\frac{\lambda}{\Gamma\beta}\int_o^t(t-s)^{\beta-1}\varphi(s)\,ds\\
\qquad+\frac{\Delta(t_k^{\beta}-t^{\beta})}{\Gamma(\beta+1)\Gamma(\alpha+\beta)}\int_0^T(T-s)^{\alpha+\beta-1}\varphi(s)\,ds+\frac{\lambda\Delta(t_k^{\beta}-t^{\beta})}{\Gamma(\beta+1)\Gamma(\beta)}\int_0^T(T-s)^{\beta-1}\varphi(s)\,ds\\
\qquad+\frac{\theta\Delta\lambda(t_k^{\beta}-t^{\beta})}{\Gamma(\beta+1)\Gamma(\beta+p)}\int_0^{\eta}(\eta-s)^{\beta+p-1}\varphi(s)\,ds\\
\qquad+\frac{\theta\Delta(t_k^{\beta}-t^{\beta})}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p)}\int_0^{\eta}(\eta-s)^{\alpha+\beta+p-1}\varphi(s)\,ds\\
\qquad+(\Delta\frac{(t_k^{\beta}-t^{\beta})}{\Gamma(\beta+1)}(\frac{\theta\eta^p-\Gamma(p+1)}{\Gamma(p+1)})-\lambda)\frac{1}{\Gamma(\alpha+\beta)}\int_0^{t_k}(t_k-s)^{\alpha+\beta-1}\varphi(s)\,ds\\
\qquad+(\Delta\frac{(t_k^{\beta}-t^{\beta})}{\Gamma(\beta+1)}(\frac{\theta\eta^p-\Gamma(p+1)}{\Gamma(p+1)})-1)\frac{\lambda}{\Gamma\beta}\int_o^{t_k}(t_k-s)^{\beta-1}\varphi(s)\,ds+\psi,\\
\quad t\in(t_k,s_k],k=1,2,\dots,m.
\end{cases}\tag{3.2}$$

In fact, by Remark 3.1, we get

$$\begin{cases} {}^cD^{\alpha}_{0,t}({}^cD^{\beta}_{0,t} + \lambda)x(t) = f(t, x(t), {}^cD^{\alpha}_{0,t}({}^cD^{\beta}_{0,t} + \lambda)x(t)) + G(t), \\ \quad t \in (t_k, s_k], k = 0, 1, \ldots, m, 0 < \alpha, \beta < 1, \\ x(t) = (\Delta \frac{(t_k^{\beta} - t^{\beta})}{\Gamma(\beta+1)}(\frac{\theta\eta^p - \Gamma(p+1)}{\Gamma(p+1)}) - 1)g_k(t, x(t)) + G_k, \\ \quad t \in (s_{k-1}, t_k], k = 1, 2, \ldots, m. \end{cases} \tag{3.3}$$

Clearly, the solution of Eq. (3.3) is given by

$$x(t) = \begin{cases} \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1}(f(s, x(s), {}^cD^{\alpha}_{0,t}({}^cD^{\beta}_{0,t} + \lambda)x(s)) + G(s))\,ds \\ \quad - \frac{\Delta t^{\beta}}{\Gamma(\beta+1)\Gamma(\alpha+\beta)} \int_0^T (T-s)^{\alpha+\beta-1}(f(s, x(s), {}^cD^{\alpha}_{0,t}({}^cD^{\beta}_{0,t} + \lambda)x(s)) + G(s))\,ds \\ \quad + \frac{\lambda\Delta t^{\beta}}{\Gamma(\beta)\Gamma(\beta+1)} \int_0^T (T-s)^{\beta-1}x(s)\,ds - \frac{\lambda}{\Gamma\beta} \int_o^t (t-s)^{\beta-1}x(s)\,ds \\ \quad + \frac{\theta\Delta t^{\beta}}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p)} \int_0^{\eta} (\eta-s)^{\alpha+\beta+p-1}(f(s, x(s), {}^cD^{\alpha}_{0,t}({}^cD^{\beta}_{0,t} + \lambda)x(s)) + G(s))\,ds \\ \quad - \frac{\theta\Delta\lambda t^{\beta}}{\Gamma(\beta+1)\Gamma(\beta+p)} \int_0^{\eta} (\eta-s)^{\beta+p-1}x(s)\,ds - (\frac{\Delta(\theta\eta^p - \Gamma(p+1))t^{\beta}}{\Gamma(p+1)\Gamma(\beta+1)} - 1)x_0, \quad t \in (0, s_0], \\ \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1}(f(s, x(s), {}^cD^{\alpha}_{0,t}({}^cD^{\beta}_{0,t} + \lambda)x(s)) + G(s))\,ds \\ \quad - \frac{\lambda}{\Gamma\beta} \int_o^t (t-s)^{\beta-1}x(s)\,ds - \frac{\lambda\Delta(t_k^{\beta} - t^{\beta})}{\Gamma(\beta+1)\Gamma(\beta)} \int_0^T (T-s)^{\beta-1}x(s)\,ds \\ \quad + \frac{\Delta(t_k^{\beta} - t^{\beta})}{\Gamma(\beta+1)\Gamma(\alpha+\beta)} \int_0^T (T-s)^{\alpha+\beta-1}(f(s, x(s), {}^cD^{\alpha}_{0,t}({}^cD^{\beta}_{0,t} + \lambda)x(s)) + G(s))\,ds \\ \quad - \frac{\theta\Delta(t_k^{\beta} - t^{\beta})}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p)} \int_0^{\eta} (\eta-s)^{\alpha+\beta+p-1}(f(s, x(s), {}^cD^{\alpha}_{0,t}({}^cD^{\beta}_{0,t} + \lambda)x(s)) + G(s))\,ds \\ \quad + \frac{\theta\Delta\lambda(t_k^{\beta} - t^{\beta})}{\Gamma(\beta+1)\Gamma(\beta+p)} \int_0^{\eta} (\eta-s)^{\beta+p-1}x(s)\,ds + (\Delta \frac{(t_k^{\beta} - t^{\beta})}{\Gamma(\beta+1)}(\frac{\theta\eta^p - \Gamma(p+1)}{\Gamma(p+1)}) - \lambda) \\ \quad \times \frac{1}{\Gamma(\alpha+\beta)} \int_0^{t_k} (t_k-s)^{\alpha+\beta-1}(f(s, x(s), {}^cD^{\alpha}_{0,t}({}^cD^{\beta}_{0,t} + \lambda)x(s)) + G(s))\,ds \\ \quad - (\Delta \frac{(t_k^{\beta} - t^{\beta})}{\Gamma(\beta+1)}(\frac{\theta\eta^p - \Gamma(p+1)}{\Gamma(p+1)}) - 1)\frac{\lambda}{\Gamma\beta} \int_o^{t_k} (t_k-s)^{\beta-1}x(s)\,ds \\ \quad - (\Delta \frac{(t_k^{\beta} - t^{\beta})}{\Gamma(\beta+1)}(\frac{\theta\eta^p - \Gamma(p+1)}{\Gamma(p+1)}) - 1)g_k(t_k, x(t_k)) + G_k, \quad t \in (t_k, s_k], k = 0, 1, \ldots, m, \\ (\Delta \frac{(t_k^{\beta} - t^{\beta})}{\Gamma(\beta+1)}(\frac{\theta\eta^p - \Gamma(p+1)}{\Gamma(p+1)}) - 1)g_k(t, x(t)) + G_k, \quad t \in (s_{k-1}, t_k], k = 1, 2, \ldots, m. \end{cases}$$

For $t \in (t_k, s_k]$, $k = 0, 1, \ldots, m$, we get

$$\begin{aligned} &\bigg| x(t) - \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} f\big(s, x(s), {}^cD^{\alpha}_{0,t}\big({}^cD^{\beta}_{0,t} + \lambda\big)x(s)\big)\,ds \\ &\quad + \frac{\lambda}{\Gamma\beta} \int_o^t (t-s)^{\beta-1}x(s)\,ds \\ &\quad - \frac{\Delta(t_k^{\beta} - t^{\beta})}{\Gamma(\beta+1)\Gamma(\alpha+\beta)} \int_0^T (T-s)^{\alpha+\beta-1} f\big(s, x(s), {}^cD^{\alpha}_{0,t}\big({}^cD^{\beta}_{0,t} + \lambda\big)x(s)\big)\,ds \\ &\quad + \frac{\lambda\Delta(t_k^{\beta} - t^{\beta})}{\Gamma(\beta+1)\Gamma(\beta)} \int_0^T (T-s)^{\beta-1}x(s)\,ds \\ &\quad - \frac{\theta\Delta\lambda(t_k^{\beta} - t^{\beta})}{\Gamma(\beta+1)\Gamma(\beta+p)} \int_0^{\eta} (\eta-s)^{\beta+p-1}x(s))\,ds \\ &\quad + \frac{\theta\Delta(t_k^{\beta} - t^{\beta})}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p)} \int_0^{\eta} (\eta-s)^{\alpha+\beta+p-1} f\big(s, x(s), {}^cD^{\alpha}_{0,t}\big({}^cD^{\beta}_{0,t} + \lambda\big)x(s)\big)\,ds \\ &\quad - \left(\Delta \frac{(t_k^{\beta} - t^{\beta})}{\Gamma(\beta+1)}\left(\frac{\theta\eta^p - \Gamma(p+1)}{\Gamma(p+1)}\right) - \lambda\right) \end{aligned}$$

$$
\begin{aligned}
&\times \frac{1}{\Gamma(\alpha+\beta)}\int_0^{t_k}(t_k-s)^{\alpha+\beta-1}f\big(s,x(s),{}^cD_{0,t}^{\alpha}\big({}^cD_{0,t}^{\beta}+\lambda\big)x(s)\big)\,ds \\
&+\left(\Delta\frac{(t_k^{\beta}-t^{\beta})}{\Gamma(\beta+1)}\left(\frac{\theta\eta^p-\Gamma(p+1)}{\Gamma(p+1)}\right)-1\right)\frac{\lambda}{\Gamma\beta}\int_0^{t_k}(t_k-s)^{\beta-1}x(s))\,ds \\
&+\left(\Delta\frac{(t_k^{\beta}-t^{\beta})}{\Gamma(\beta+1)}\left(\frac{\theta\eta^p-\Gamma(p+1)}{\Gamma(p+1)}\right)-1\right)g_k\big(t_k,x(t_k)\big)\Bigg| \\
\le{}& \left|\frac{1}{\Gamma(\alpha+\beta)}\int_0^{t}(t-s)^{\alpha+\beta-1}G(s)\,ds\right| + \left|\frac{\lambda}{\Gamma\beta}\int_o^{t}(t-s)^{\beta-1}x(s)\,ds\right| \\
&+\left|\frac{\Delta(t_k^{\beta}-t^{\beta})}{\Gamma(\beta+1)\Gamma(\alpha+\beta)}\int_0^{T}(T-s)^{\alpha+\beta-1}G(s)\,ds\right| \\
&+\left|\frac{\lambda\Delta(t_k^{\beta}-t^{\beta})}{\Gamma(\beta+1)\Gamma(\beta)}\int_0^{T}(T-s)^{\beta-1}x(s)\,ds\right| \\
&+\left|\frac{\theta\Delta(t_k^{\beta}-t^{\beta})}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p)}\int_0^{\eta}(\eta-s)^{\alpha+\beta+p-1}G(s)\,ds\right| \\
&+\left|\frac{\theta\Delta\lambda(t_k^{\beta}-t^{\beta})}{\Gamma(\beta+1)\Gamma(\beta+p)}\int_0^{\eta}(\eta-s)^{\beta+p-1}x(s)\,ds\right| \\
&+\left|\left(\Delta\frac{(t_k^{\beta}-t^{\beta})}{\Gamma(\beta+1)}\left(\frac{\theta\eta^p-\Gamma(p+1)}{\Gamma(p+1)}\right)-\lambda\right)\frac{1}{\Gamma(\alpha+\beta)}\int_0^{t_k}(t_k-s)^{\alpha+\beta-1}G(s)\,ds\right| \\
&+\left|\left(\Delta\frac{(t_k^{\beta}-t^{\beta})}{\Gamma(\beta+1)}\left(\frac{\theta\eta^p-\Gamma(p+1)}{\Gamma(p+1)}\right)-1\right)\frac{\lambda}{\Gamma\beta}\int_0^{t_k}(t_k-s)^{\beta-1}x(s))\,ds\right| + |G_k| \\
\le{}& \frac{1}{\Gamma(\alpha+\beta)}\int_0^{t}(t-s)^{\alpha+\beta-1}\varphi(s)\,ds + \frac{\lambda}{\Gamma\beta}\int_o^{t}(t-s)^{\beta-1}x(s)\,ds \\
&+\frac{\Delta(t_k^{\beta}-t^{\beta})}{\Gamma(\beta+1)\Gamma(\alpha+\beta)}\int_0^{T}(T-s)^{\alpha+\beta-1}\varphi(s)\,ds \\
&+\frac{\lambda\Delta(t_k^{\beta}-t^{\beta})}{\Gamma(\beta+1)\Gamma(\beta)}\int_0^{T}(T-s)^{\beta-1}x(s)\,ds \\
&+\frac{\theta\Delta(t_k^{\beta}-t^{\beta})}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p)}\int_0^{\eta}(\eta-s)^{\alpha+\beta+p-1}\varphi(s)\,ds \\
&+\frac{\theta\Delta\lambda(t_k^{\beta}-t^{\beta})}{\Gamma(\beta+1)\Gamma(\beta+p)}\int_0^{\eta}(\eta-s)^{\beta+p-1}x(s))\,ds \\
&+\left(\Delta\frac{(t_k^{\beta}-t^{\beta})}{\Gamma(\beta+1)}\left(\frac{\theta\eta^p-\Gamma(p+1)}{\Gamma(p+1)}\right)-\lambda\right)\frac{1}{\Gamma(\alpha+\beta)}\int_0^{t_k}(t_k-s)^{\alpha+\beta-1}\varphi(s)\,ds \\
&+\left(\Delta\frac{(t_k^{\beta}-t^{\beta})}{\Gamma(\beta+1)}\left(\frac{\theta\eta^p-\Gamma(p+1)}{\Gamma(p+1)}\right)-1\right)\frac{\lambda}{\Gamma\beta}\int_0^{t_k}(t_k-s)^{\beta-1}x(s))\,ds+\psi.
\end{aligned}
$$

Proceeding as above, we derive

$$
\begin{aligned}
&\left|x(t)-\left(\Delta\frac{(t_k^{\beta}-t^{\beta})}{\Gamma(\beta+1)}\left(\frac{\theta\eta^p-\Gamma(p+1)}{\Gamma(p+1)}\right)-1\right)g_k\big(t,x(t)\big)\right| \le |G_k| \le \psi, \\
&t\in(s_{k-1},t_k],\ k=0,1,\dots,m,
\end{aligned}
$$

and

$$\begin{aligned}
&\left|x(t)-\left(1-\frac{\Delta(\theta\eta^p-\Gamma(p+1))t^\beta}{\Gamma(p+1)\Gamma(\beta+1)}\right)x_0-\frac{\lambda\Delta t^\beta}{\Gamma(\beta)\Gamma(\beta+1)}\int_0^T(T-s)^{\beta-1}x(s)\,ds\right.\\
&\quad+\frac{\lambda}{\Gamma\beta}\int_o^t(t-s)^{\beta-1}x(s)\,ds\\
&\quad-\frac{1}{\Gamma(\alpha+\beta)}\int_0^t(t-s)^{\alpha+\beta-1}f\bigl(s,x(s),{}^cD^\alpha_{0,t}\bigl({}^cD^\beta_{0,t}+\lambda\bigr)x(s)\bigr)\,ds\\
&\quad+\frac{\Delta t^\beta}{\Gamma(\beta+1)\Gamma(\alpha+\beta)}\int_0^T(T-s)^{\alpha+\beta-1}f\bigl(s,x(s),{}^cD^\alpha_{0,t}\bigl({}^cD^\beta_{0,t}+\lambda\bigr)x(s)\bigr)\,ds\\
&\quad-\frac{\theta\Delta t^\beta}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p)}\int_0^\eta(\eta-s)^{\alpha+\beta+p-1}f\bigl(s,x(s),{}^cD^\alpha_{0,t}\bigl({}^cD^\beta_{0,t}+\lambda\bigr)x(s)\bigr)\,ds\\
&\quad\left.+\frac{\theta\Delta\lambda t^\beta}{\Gamma(\beta+1)\Gamma(\beta+p)}\int_0^\eta(\eta-s)^{\beta+p-1}x(s)\,ds\right|\\
&\leq\left|\frac{1}{\Gamma(\alpha+\beta)}\int_0^t(t-s)^{\alpha+\beta-1}G(s)\,ds\right|\\
&\quad+\left|\frac{\Delta t^\beta}{\Gamma(\beta+1)\Gamma(\alpha+\beta)}\int_0^T(T-s)^{\alpha+\beta-1}G(s)\,ds\right|\\
&\quad+\left|\frac{\lambda\Delta t^\beta}{\Gamma(\beta)\Gamma(\beta+1)}\int_0^T(T-s)^{\beta-1}x(s)\,ds\right|\\
&\quad+\left|\frac{\lambda}{\Gamma\beta}\int_o^t(t-s)^{\beta-1}x(s)\,ds\right|+\left|\frac{\theta\Delta t^\beta}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p)}\int_0^\eta(\eta-s)^{\alpha+\beta+p-1}G(s)\,ds\right|\\
&\quad+\left|\frac{\theta\Delta\lambda t^\beta}{\Gamma(\beta+1)\Gamma(\beta+p)}\int_0^\eta(\eta-s)^{\beta+p-1}x(s)\,ds\right|\\
&\leq\frac{1}{\Gamma(\alpha+\beta)}\int_0^t(t-s)^{\alpha+\beta-1}\varphi(s)\,ds\\
&\quad+\frac{\Delta t^\beta}{\Gamma(\beta+1)\Gamma(\alpha+\beta)}\int_0^T(T-s)^{\alpha+\beta-1}\varphi(s)\,ds\\
&\quad+\frac{\lambda\Delta t^\beta}{\Gamma(\beta)\Gamma(\beta+1)}\int_0^T(T-s)^{\beta-1}x(s)\,ds\\
&\quad+\frac{\lambda}{\Gamma\beta}\int_o^t(t-s)^{\beta-1}x(s)\,ds+\frac{\theta\Delta t^\beta}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p)}\int_0^\eta(\eta-s)^{\alpha+\beta+p-1}\varphi(s)\,ds\\
&\quad+\frac{\theta\Delta\lambda t^\beta}{\Gamma(\beta+1)\Gamma(\beta+p)}\int_0^\eta(\eta-s)^{\beta+p-1}x(s)\,ds,\quad t\in(0,s_0].
\end{aligned}$$

4 Main results via fixed point methods

In order to apply a fixed point theorem of the alternative for contractions on a generalized complete metric space to achieve our main result, we want to collect the following facts.

Definition 4.1 For a nonempty set V, a function $d:V\times V\to[0,\infty]$ is called a generalized metric on V if and only if d satisfies

- ◇ $d(v_1,v_2)=0$ if and only if $v_1=v_2$;
- ◇ $d(v_1,v_2)=d(v_2,v_1)$ for all $v_1,v_2\in V$;
- ◇ $d(v_1,v_3)\leq d(v_1,v_2)+d(v_2,v_3)$ for all $v_1,v_2,v_3\in V$.

Lemma 4.1 ([9]) *Let (V,d) be a generalized complete metric space. Assume that $T: V \to V$ is a strictly contractive operator with the Lipschitz constant $L < 1$. If there exists a $k \geq 0$ such that $d(T^{k+1}v, T^k v) < \infty$ for some v in V, then the followings statements are true*:

(B_1) *The sequence $\{T^n v\}$ converges to a fixed point v^* of T;*

(B_2) *The unique fixed point of T is $v^* \in V^* = \{u \in V$ such that $d(T^k v, u) < \infty\}$;*

(B_3) *If $u \in V^*$, then $d(u, v^*) \leq \frac{1}{1-L} d(Tu, u)$.*

We can introduce some assumptions as follows:

(H_1) $f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$.

(H_2) *There exists a positive constant L_f such that*

$$\left|f(t,u_1,\bar{u_1})-f(t,u_2,\bar{u_2})\right| \leq L_{f_1}|u_1-u_2|+\bar{L}_{f_2}|\bar{u_1}-\bar{u_2}|, \quad \text{for each } t \in J \text{ and all } u_1,u_2 \in \mathbb{R}.$$

(H_3) $g_k \in C((s_{k-1}, t_k] \times \mathbb{R}, \mathbb{R})$ *and there are positive constant L_{gk}, $k = 1, 2, \dots, m$ such that*

$$\left|g_k(t,u_1) - g_k(t,u_2)\right| \leq L_{gk}|u_1 - u_2|, \quad \text{for each } t \in (s_{k-1}, t_k], \text{ and all } u_1, u_2 \in \mathbb{R}.$$

(H_4) *Let $\varphi \in C(J, \mathbb{R}_+)$ be a nondecreasing function. There exists $c_\varphi > 0$ such that*

$$\left(\int_0^t \left(\varphi(s)\right)^{\frac{1}{p}} ds\right)^p \leq C_\varphi \varphi(t) \quad \text{for each } t \in J. \tag{4.1}$$

Theorem 4.2 *Suppose that (H_1) and (H_2) are satisfied and also a function $y \in PC(J, \mathbb{R})$ satisfies (3.1). Then there exists a unique solution x of Eq. (1.1) such that*

$$x(t) = \begin{cases} \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} f(s, x(s), {}^cD_{0,t}^\alpha({}^cD_{0,t}^\beta + \lambda))x(s)\,ds \\ \quad - \frac{\lambda}{\Gamma\beta} \int_o^t (t-s)^{\beta-1} x(s)\,ds + \frac{\lambda \Delta t^\beta}{\Gamma(\beta)\Gamma(\beta+1)} \int_0^T (T-s)^{\beta-1} x(s)\,ds \\ \quad - \frac{\Delta t^\beta}{\Gamma(\beta+1)\Gamma(\alpha+\beta)} \int_0^T (T-s)^{\alpha+\beta-1} f(s, x(s), {}^cD_{0,t}^\alpha({}^cD_{0,t}^\beta + \lambda))x(s)\,ds \\ \quad + \frac{\theta \Delta t^\beta}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p)} \int_0^\eta (\eta-s)^{\alpha+\beta+p-1} f(s, x(s), {}^cD_{0,t}^\alpha({}^cD_{0,t}^\beta + \lambda))x(s)\,ds \\ \quad - \frac{\theta \Delta \lambda t^\beta}{\Gamma(\beta+1)\Gamma(\beta+p)} \int_0^\eta (\eta-s)^{\beta+p-1} x(s)\,ds - (\frac{\Delta(\theta\eta^p - \Gamma(p+1))t^\beta}{\Gamma(p+1)\Gamma(\beta+1)} - 1)x, \\ \quad t \in (0, s_0], \\ \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} f(s, x(s), {}^cD_{0,t}^\alpha({}^cD_{0,t}^\beta + \lambda))x(s)\,ds \\ \quad - \frac{\lambda}{\Gamma\beta} \int_o^t (t-s)^{\beta-1} x(s)\,ds \\ \quad + \frac{\Delta(t_k^\beta - t^\beta)}{\Gamma(\beta+1)\Gamma(\alpha+\beta)} \int_0^T (T-s)^{\alpha+\beta-1} f(s, x(s), {}^cD_{0,t}^\alpha({}^cD_{0,t}^\beta + \lambda))x(s)\,ds \\ \quad - \frac{\lambda\Delta(t_k^\beta - t^\beta)}{\Gamma(\beta+1)\Gamma(\beta)} \int_0^T (T-s)^{\beta-1} x(s)\,ds \\ \quad - \frac{\theta\Delta(t_k^\beta - t^\beta)}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p)} \int_0^\eta (\eta-s)^{\alpha+\beta+p-1} f(s, x(s), {}^cD_{0,t}^\alpha({}^cD_{0,t}^\beta + \lambda))x(s)\,ds \\ \quad + \frac{\theta\Delta\lambda(t_k^\beta - t^\beta)}{\Gamma(\beta+1)\Gamma(\beta+p)} \int_0^\eta (\eta-s)^{\beta+p-1} x(s)\,ds + (\Delta \frac{(t_k^\beta - t^\beta)}{\Gamma(\beta+1)} (\frac{\theta\eta^p - \Gamma(p+1)}{\Gamma(p+1)}) - \lambda) \\ \quad \times \frac{1}{\Gamma(\alpha+\beta)} \int_0^{t_k} (t_k-s)^{\alpha+\beta-1} f(s, x(s), {}^cD_{0,t}^\alpha({}^cD_{0,t}^\beta + \lambda))x(s)\,ds \\ \quad - (\Delta \frac{(t_k^\beta - t^\beta)}{\Gamma(\beta+1)} (\frac{\theta\eta^p - \Gamma(p+1)}{\Gamma(p+1)}) - 1) \frac{\lambda}{\Gamma\beta} \int_o^{t_k} (t_k-s)^{\beta-1} x(s)\,ds \\ \quad - (\Delta \frac{(t_k^\beta - t^\beta)}{\Gamma(\beta+1)} (\frac{\theta\eta^p - \Gamma(p+1)}{\Gamma(p+1)}) - 1) g_k(t_k, x(t_k)), \quad t \in (t_k, s_k], k = 1, 2, \dots, m, \\ g_k(t_k, x(t_k)), \quad t \in (s_{k-1}, t_k], k = 1, 2, \dots, m \end{cases} \tag{4.2}$$

and

$$\begin{aligned} \left|y(t)-x(t)\right| \leq \Bigg\{ & \frac{C_\varphi}{\Gamma(\alpha+\beta)}\left(\frac{1-r}{\alpha+\beta-r}\right)^{1-r} t^{\alpha+\beta-r} + \frac{\lambda C_\varphi}{\Gamma\beta}\left(\frac{1-r}{\beta-r}\right)^{1-r} t^{\beta-r} \\ & + \frac{\Delta C_\varphi(t_k^\beta - t^\beta)}{\Gamma(\beta+1)\Gamma(\alpha+\beta)}\left(\frac{1-r}{\alpha+\beta-r}\right)^{1-r} T^{\alpha+\beta-r} \\ & + \frac{\lambda\Delta C_\varphi(t_k^\beta - t^\beta)}{\Gamma(\beta+1)\Gamma(\beta)}\left(\frac{1-r}{\beta-r}\right)^{1-r} T^{\beta-r} \\ & + \frac{\theta\Delta C_\varphi(t_k^\beta - t^\beta)}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p)}\left(\frac{1-r}{\alpha+\beta+p-r}\right)^{1-r} \eta^{\alpha+\beta+p-r} \\ & + \frac{C_\varphi\theta\Delta\lambda(t_k^\beta - t^\beta)}{\Gamma(\beta+1)\Gamma(\beta+p)}\left(\frac{1-r}{\beta+p-r}\right)^{1-r} \eta^{\beta+p-r} \\ & + \left(\Delta\frac{(t_k^\beta - t^\beta)}{\Gamma(\beta+1)}\left(\frac{\theta\eta^p - \Gamma(p+1)}{\Gamma(p+1)}\right) - \lambda\right) \\ & \times \frac{C_\varphi}{\Gamma(\alpha+\beta)}\left(\frac{1-r}{\alpha+\beta-r}\right)^{1-r} t_k^{\alpha+\beta-r} \\ & + \left(\Delta\frac{(t_k^\beta - t^\beta)}{\Gamma(\beta+1)}\left(\frac{\theta\eta^p + \Gamma(p+1)}{\Gamma(p+1)}\right) - 1\right) \\ & \times \frac{\lambda C_\varphi}{\Gamma\beta}\left(\frac{1-r}{\beta-r}\right)^{1-r} t_k^{\beta-r} + 1\Bigg\} \\ & \times \left(\frac{\varphi(t)+\psi}{1-M}\right) \end{aligned} \tag{4.3}$$

for all $t \in J$ *if* $0 < \alpha < \beta < 1$, *with*

$$M = \max\{M_1, M_2\} < 1, \tag{4.4}$$

where

$$\begin{aligned} M_1 = \max\Bigg\{ & \frac{1}{\Gamma(\alpha+\beta)}\left(\frac{1-r}{\alpha+\beta-r}\right)^{1-r} (L_{f_1}C_\varphi + \bar{L}_{f_2}C_\varphi) s_k^{\alpha+\beta-r} \\ & + \frac{\lambda C_\varphi \varphi(t)}{\Gamma\beta}\left(\frac{1-r}{\beta-r}\right)^{1-r} s_k^{\beta-r} \\ & + \frac{\Delta(t_k^\beta - t^\beta)}{\Gamma(\beta+1)\Gamma(\alpha+\beta)}\left(\frac{1-r}{\alpha+\beta-r}\right)^{1-r} (L_{f_1}C_\varphi + \bar{L}_{f_2}C_\varphi) T^{\alpha+\beta-r} \\ & + \frac{\theta\Delta(t_k^\beta - t^\beta)}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p)}\left(\frac{1-r}{\alpha+\beta+p-r}\right)^{1-r} (L_{f_1}C_\varphi + \bar{L}_{f_2}C_\varphi) \eta^{\alpha+\beta+p-r} \\ & + \frac{C_\varphi\varphi(t)\theta\Delta\lambda(t_k^\beta - t^\beta)}{\Gamma(\beta+1)\Gamma(\beta+p)}\left(\frac{1-r}{\beta+p-r}\right)^{1-r} \eta^{\beta+p-r} \\ & + \left(\Delta\frac{(t_k^\beta - t^\beta)}{\Gamma(\beta+1)}\left(\frac{\theta\eta^p - \Gamma(p+1)}{\Gamma(p+1)}\right) - \lambda\right) \\ & \times \frac{1}{\Gamma(\alpha+\beta)}\left(\frac{1-r}{\alpha+\beta-r}\right)^{1-r} (L_{f_1}C_\varphi + \bar{L}_{f_2}C_\varphi) t_k^{\alpha+\beta-r} \end{aligned}$$

$$
\begin{aligned}
&+\frac{\lambda\Delta(t_k^{\beta}-t^{\beta})C_{\varphi}\varphi(t)}{\Gamma(\beta+1)\Gamma(\beta)}\left(\frac{1-r}{\beta-r}\right)^{1-r}T^{\beta-r}\\
&+\left(\Delta\frac{(t_k^{\beta}-t^{\beta})}{\Gamma(\beta+1)}\left(\frac{\theta\eta^{p}+\Gamma(p+1)}{\Gamma(p+1)}\right)-1\right)\\
&\times\left(\frac{\lambda C_{\varphi}\varphi(t)}{\Gamma\beta}\left(\frac{1-r}{\beta-r}\right)^{1-r}t_k^{\beta-r}+L_{gk}\right)\text{ such that } k=1,2,\dots,m\Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
M_2=\max\Bigg\{&\frac{L_{f_1}}{\Gamma(\alpha+\beta+1)}s_k^{\alpha+\beta}+\frac{\bar{L}_{f_2}}{\Gamma(\alpha+\beta+1)}s_k^{\alpha+\beta}+\frac{\lambda}{\Gamma(\beta+1)}s_k^{\beta}\\
&+\frac{\Delta(t_k^{\beta}-t^{\beta})L_{f_1}}{\Gamma(\beta+1)\Gamma(\alpha+\beta+1)}T^{\alpha+\beta}\\
&+\frac{\Delta(t_k^{\beta}-t^{\beta})\bar{L}_{f_2}}{\Gamma(\beta+1)\Gamma(\alpha+\beta+1)}T^{\alpha+\beta}+\frac{\lambda\Delta(t_k^{\beta}-t^{\beta})}{\Gamma(\beta+1)\Gamma(\beta+1)}T^{\beta}\\
&+\frac{\theta\Delta(t_k^{\beta}-t^{\beta})L_{f_1}}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p+1)}\eta^{\alpha+\beta+p}\\
&+\frac{\theta\Delta(t_k^{\beta}-t^{\beta})\bar{L}_{f_2}}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p+1)}\eta^{\alpha+\beta+p}+\frac{\theta\Delta\lambda(t_k^{\beta}-t^{\beta})}{\Gamma(\beta+1)\Gamma(\beta+p+1)}\eta^{\beta+p}\\
&+\left(\Delta\frac{(t_k^{\beta}-t^{\beta})}{\Gamma(\beta+1)}\left(\frac{\theta\eta^{p}-\Gamma(p+1)}{\Gamma(p+1)}\right)-\lambda\right)\\
&\times\left(\frac{L_{f_1}}{\Gamma(\alpha+\beta+1)}t_k^{\alpha+\beta}+\frac{\bar{L}_{f_2}}{\Gamma(\alpha+\beta+1)}t_k^{\alpha+\beta}\right)\\
&+\left(\Delta\frac{(t_k^{\beta}-t^{\beta})}{\Gamma(\beta+1)}\left(\frac{\theta\eta^{p}-\Gamma(p+1)}{\Gamma(p+1)}\right)-1\right)\\
&\times\left(\frac{\lambda}{\beta\Gamma\beta}t_k^{\beta}+L_{gk}\right)\text{ such that } k=0,1,\dots,m\Bigg\}.
\end{aligned}
$$

Proof Consider the space of piecewise continuous functions

$$V=\{g:J\to\mathbb{R}\text{ such that } g\in PC(J,\mathbb{R})\},$$

endowed with the generalized metric on V, defined by

$$
\begin{aligned}
d(g,h)=\inf\{&C_1+C_2\in[0,+\infty]\\
&\text{such that }\left|g(t)-h(t)\right|\le(C_1+C_2)\big(\varphi(t)+\psi\big)\text{ for all } t\in J\},
\end{aligned}
\tag{4.5}
$$

where

$$C_1\in\{C\in[0,\infty]\text{ such that }\left|g(t)-h(t)\right|\le C\varphi(t)\text{ for all } t\in(t_k,s_k],k=0,1,\dots,m\}$$

and

$$C_2\in\{C\in[0,\infty]\text{ such that }\left|g(t)-h(t)\right|\le C\psi\text{ for all } t\in(s_{k-1},t_k],k=1,2,\dots,m\}.$$

It is easy to verify that (V,d) is a complete generalized metric space [19].

Define an operator $\Lambda : V \to V$ by

$$(\Lambda x)(t) = \begin{cases} \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} f(s,x(s),{}^cD^\alpha_{0,t}({}^cD^\beta_{0,t}+\lambda))x(s)\,ds \\ \quad - \frac{\lambda}{\Gamma\beta}\int_o^t (t-s)^{\beta-1}x(s)\,ds + \frac{\lambda\Delta t^\beta}{\Gamma(\beta)\Gamma(\beta+1)}\int_0^T (T-s)^{\beta-1}x(s)\,ds \\ \quad - \frac{\Delta t^\beta}{\Gamma(\beta+1)\Gamma(\alpha+\beta)}\int_0^T (T-s)^{\alpha+\beta-1} f(s,x(s),{}^cD^\alpha_{0,t}({}^cD^\beta_{0,t}+\lambda))x(s)\,ds \\ \quad + \frac{\theta\Delta t^\beta}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p)}\int_0^\eta (\eta-s)^{\alpha+\beta+p-1} f(s,x(s),{}^cD^\alpha_{0,t}({}^cD^\beta_{0,t}+\lambda))x(s)\,ds \\ \quad - \frac{\theta\Delta\lambda t^\beta}{\Gamma(\beta+1)\Gamma(\beta+p)}\int_0^\eta (\eta-s)^{\beta+p-1}x(s)\,ds - (\frac{\Delta(\theta\eta^p-\Gamma(p+1))t^\beta}{\Gamma(p+1)\Gamma(\beta+1)}-1)x_0, \\ \quad t\in(0,s_0], \\ \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} f(s,x(s),{}^cD^\alpha_{0,t}({}^cD^\beta_{0,t}+\lambda))x(s)\,ds \\ \quad - \frac{\lambda}{\Gamma\beta}\int_o^t (t-s)^{\beta-1}x(s)\,ds - \frac{\lambda\Delta(t_k^\beta-t^\beta)}{\Gamma(\beta+1)\Gamma(\beta)}\int_0^T (T-s)^{\beta-1}x(s)\,ds \\ \quad + \frac{\Delta(t_k^\beta-t^\beta)}{\Gamma(\beta+1)\Gamma(\alpha+\beta)}\int_0^T (T-s)^{\alpha+\beta-1} f(s,x(s),{}^cD^\alpha_{0,t}({}^cD^\beta_{0,t}+\lambda))x(s)\,ds \\ \quad - \frac{\theta\Delta(t_k^\beta-t^\beta)}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p)}\int_0^\eta (\eta-s)^{\alpha+\beta+p-1} f(s,x(s),{}^cD^\alpha_{0,t}({}^cD^\beta_{0,t}+\lambda))x(s)\,ds \\ \quad + \frac{\theta\Delta\lambda(t_k^\beta-t^\beta)}{\Gamma(\beta+1)\Gamma(\beta+p)}\int_0^\eta (\eta-s)^{\beta+p-1}x(s)\,ds + (\Delta\frac{(t_k^\beta-t^\beta)}{\Gamma(\beta+1)}(\frac{\theta\eta^p-\Gamma(p+1)}{\Gamma(p+1)})-\lambda) \\ \quad \times \frac{1}{\Gamma(\alpha+\beta)}\int_0^{t_k}(t_k-s)^{\alpha+\beta-1} f(s,x(s),{}^cD^\alpha_{0,t}({}^cD^\beta_{0,t}+\lambda))x(s)\,ds \\ \quad - (\Delta\frac{(t_k^\beta-t^\beta)}{\Gamma(\beta+1)}(\frac{\theta\eta^p-\Gamma(p+1)}{\Gamma(p+1)})-1)\frac{\lambda}{\Gamma\beta}\int_0^{t_k}(t_k-s)^{\beta-1}x(s)\,ds \\ \quad - (\Delta\frac{(t_k^\beta-t^\beta)}{\Gamma(\beta+1)}(\frac{\theta\eta^p-\Gamma(p+1)}{\Gamma(p+1)})-1)g_k(t_k,x(t_k)), \\ \quad t\in(t_k,s_k], k=1,2,\ldots,m, \\ g_k(t_k,x(t_k)), \quad t\in(s_{k-1},t_k], k=1,2,\ldots,m \end{cases} \tag{4.6}$$

for all x belongs to V and $t \in J$. Obviously, according to (H_1), Λ is a well-defined operator.

Next we shall verify that Λ is strictly contractive on V. Note that according to definition of (V,d), for any $g,h \in V$, it is possible to find $C_1,C_2,C_3,C_4 \in [0,\infty]$ such that

$$\left|g(t)-h(t)\right| \le \begin{cases} C_1\varphi(t), & t\in(t_k,s_k], k=0,\ldots,m, \\ C_2\psi, & t\in(s_{k-1},t_k], k=1,\ldots,m, \end{cases} \tag{4.7}$$

and

$$\begin{aligned} &\left|{}^cD^\alpha_{0,t}\left({}^cD^\beta_{0,t}+\lambda\right)g(s) - {}^cD^\alpha_{0,t}\left({}^cD^\beta_{0,t}+\lambda\right)h(s)\right| \\ &\quad\le \begin{cases} C_3\zeta(t)\le C_1\varphi(t), & t\in(t_k,s_k], k=0,\ldots,m, \\ C_4\zeta(t)\le C_2\psi, & t\in(s_{k-1},t_k], k=1,\ldots,m. \end{cases} \end{aligned}$$

From the definition of Λ in Eq. (4.6), (H_2), (H_3) and (4.7), we obtain that

Case 1. For $t \in [0,s_0]$,

$$\begin{aligned} &\left|(\Lambda g)(t)-(\Lambda h)(t)\right| \\ &\quad\le \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1} \\ &\qquad\times \left|f\left(s,g(s),{}^cD^\alpha_{0,t}\left({}^cD^\beta_{0,t}+\lambda\right)g(s)\right) - f\left(s,h(s),{}^cD^\alpha_{0,t}\left({}^cD^\beta_{0,t}+\lambda\right)h(s)\right)\right|ds \\ &\qquad+ \frac{\lambda}{\Gamma\beta}\int_o^t (t-s)^{\beta-1}\left|g(s)-h(s)\right|ds + \frac{\Delta t^\beta}{\Gamma(\beta+1)\Gamma(\alpha+\beta)}\int_0^T (T-s)^{\alpha+\beta-1} \end{aligned}$$

$$
\begin{aligned}
&\times \left|f\left(s,g(s),{}^cD_{0,t}^{\alpha}\left({}^cD_{0,t}^{\beta}+\lambda\right)g(s)\right)-f\left(s,h(s),{}^cD_{0,t}^{\alpha}\left({}^cD_{0,t}^{\beta}+\lambda\right)h(s)\right)\right| ds \\
&+\frac{\lambda\Delta t^{\beta}}{\Gamma(\beta)\Gamma(\beta+1)}\int_0^T (T-s)^{\beta-1}\left|g(s)-h(s)\right| ds \\
&+\frac{\theta\Delta t^{\beta}}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p)}\int_0^{\eta} (\eta-s)^{\alpha+\beta+p-1} \\
&\times \left|f\left(s,g(s),{}^cD_{0,t}^{\alpha}\left({}^cD_{0,t}^{\beta}+\lambda\right)g(s)\right)-f\left(s,h(s),{}^cD_{0,t}^{\alpha}\left({}^cD_{0,t}^{\beta}+\lambda\right)h(s)\right)\right| ds \\
&+\frac{\theta\Delta\lambda t^{\beta}}{\Gamma(\beta+1)\Gamma(\beta+p)}\int_0^{\eta} (\eta-s)^{\beta+p-1}\left|g(s)-h(s)\right| ds \\
\le{}& \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1} \\
&\times \left[L_{f_1}\left|g(s)-h(s)\right|+\bar{L}_{f_2}\left|{}^cD_{0,t}^{\alpha}\left({}^cD_{0,t}^{\beta}+\lambda\right)g(s)-{}^cD_{0,t}^{\alpha}\left({}^cD_{0,t}^{\beta}+\lambda\right)h(s)\right|\right] ds \\
&+\frac{\lambda}{\Gamma\beta}\int_o^t (t-s)^{\beta-1}\left|g(s)-h(s)\right| ds+\frac{\Delta t^{\beta}}{\Gamma(\beta+1)\Gamma(\alpha+\beta)}\int_0^T (T-s)^{\alpha+\beta-1} \\
&\times \left[L_{f_1}\left|g(s)-h(s)\right|+\bar{L}_{f_2}\left|{}^cD_{0,t}^{\alpha}\left({}^cD_{0,t}^{\beta}+\lambda\right)g(s)-{}^cD_{0,t}^{\alpha}\left({}^cD_{0,t}^{\beta}+\lambda\right)h(s)\right|\right] ds \\
&+\frac{\lambda\Delta t^{\beta}}{\Gamma(\beta)\Gamma(\beta+1)}\int_0^T (T-s)^{\beta-1}\left|g(s)-h(s)\right| ds \\
&+\frac{\theta\Delta t^{\beta}}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p)}\int_0^{\eta} (\eta-s)^{\alpha+\beta+p-1} \\
&\times \left[L_{f_1}\left|g(s)-h(s)\right|+\bar{L}_{f_2}\left|{}^cD_{0,t}^{\alpha}\left({}^cD_{0,t}^{\beta}+\lambda\right)g(s)-{}^cD_{0,t}^{\alpha}\left({}^cD_{0,t}^{\beta}+\lambda\right)h(s)\right|\right] ds \\
&+\frac{\theta\Delta\lambda t^{\beta}}{\Gamma(\beta+1)\Gamma(\beta+p)}\int_0^{\eta} (\eta-s)^{\beta+p-1}\left|g(s)-h(s)\right| ds \\
={}& \frac{L_{f_1}}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}\left|g(s)-h(s)\right| ds+\frac{\lambda}{\Gamma\beta}\int_o^t (t-s)^{\beta-1}\left|g(s)-h(s)\right| ds \\
&+\frac{\bar{L}_{f_2}}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}\left|{}^cD_{0,t}^{\alpha}\left({}^cD_{0,t}^{\beta}+\lambda\right)g(s)-{}^cD_{0,t}^{\alpha}\left({}^cD_{0,t}^{\beta}+\lambda\right)h(s)\right| ds \\
&+\frac{L_{f_1}\Delta t^{\beta}}{\Gamma(\beta+1)\Gamma(\alpha+\beta)}\int_0^T (T-s)^{\alpha+\beta-1}\left|g(s)-h(s)\right| ds+\frac{\bar{L}_{f_2}\Delta t^{\beta}}{\Gamma(\beta+1)\Gamma(\alpha+\beta)} \\
&\times \int_0^T (T-s)^{\alpha+\beta-1}\left|{}^cD_{0,t}^{\alpha}\left({}^cD_{0,t}^{\beta}+\lambda\right)g(s)-{}^cD_{0,t}^{\alpha}\left({}^cD_{0,t}^{\beta}+\lambda\right)h(s)\right| ds \\
&+\frac{\lambda\Delta t^{\beta}}{\Gamma(\beta)\Gamma(\beta+1)}\int_0^T (T-s)^{\beta-1}\left|g(s)-h(s)\right| ds \\
&+\frac{L_{f_1}\theta\Delta t^{\beta}}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p)}\int_0^{\eta} (\eta-s)^{\alpha+\beta+p-1}\left|g(s)-h(s)\right| ds \\
&+\frac{\bar{L}_{f_2}\theta\Delta t^{\beta}}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p)} \\
&\times \int_0^{\eta} (\eta-s)^{\alpha+\beta+p-1}\left|{}^cD_{0,t}^{\alpha}\left({}^cD_{0,t}^{\beta}+\lambda\right)g(s)-{}^cD_{0,t}^{\alpha}\left({}^cD_{0,t}^{\beta}+\lambda\right)h(s)\right| ds \\
&+\frac{\theta\Delta\lambda t^{\beta}}{\Gamma(\beta+1)\Gamma(\beta+p)}\int_0^{\eta} (\eta-s)^{\beta+p-1}\left|g(s)-h(s)\right| ds
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{L_{f_1}C_1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}\bigl|\varphi(s)\bigr|\,ds + \frac{\lambda C_1}{\Gamma\beta}\int_o^t (t-s)^{\beta-1}\bigl|\varphi(s)\bigr|\,ds \\
&\quad + \frac{\bar{L}_{f_2}C_1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}\bigl|\varphi(s)\bigr|\,ds \\
&\quad + \frac{L_{f_1}C_1\Delta t^{\beta}}{\Gamma(\beta+1)\Gamma(\alpha+\beta)}\int_0^T (T-s)^{\alpha+\beta-1}\bigl|\varphi(s)\bigr|\,ds \\
&\quad + \frac{\lambda C_1\Delta t^{\beta}}{\Gamma(\beta)\Gamma(\beta+1)}\int_0^T (T-s)^{\beta-1}\bigl|\varphi(s)\bigr|\,ds \\
&\quad + \frac{\bar{L}_{f_2}C_1\Delta t^{\beta}}{\Gamma(\beta+1)\Gamma(\alpha+\beta)}\int_0^T (T-s)^{\alpha+\beta-1}\bigl|\varphi(s)\bigr|\,ds \\
&\quad + \frac{L_{f_1}C_1\theta\Delta t^{\beta}}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p)}\int_0^\eta (\eta-s)^{\alpha+\beta+p-1}\bigl|\varphi(s)\bigr|\,ds \\
&\quad + \frac{\bar{L}_{f_2}C_1\theta\Delta t^{\beta}}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p)}\int_0^\eta (\eta-s)^{\alpha+\beta+p-1}\bigl|\varphi(s)\bigr|\,ds \\
&\quad + \frac{C_1\theta\Delta\lambda t^{\beta}}{\Gamma(\beta+1)\Gamma(\beta+p)}\int_0^\eta (\eta-s)^{\beta+p-1}\bigl|\varphi(s)\bigr|\,ds \\
&\leq \frac{L_{f_1}C_1}{\Gamma(\alpha+\beta)}\biggl(\int_0^t (t-s)^{\frac{\alpha+\beta-1}{1-r}}\,ds\biggr)^{1-r}\biggl(\int_0^t \bigl(\varphi(s)\bigr)^{\frac{1}{r}}\,ds\biggr)^{r} \\
&\quad + \frac{\bar{L}_{f_2}C_1}{\Gamma(\alpha+\beta)}\biggl(\int_0^t (t-s)^{\frac{\alpha+\beta-1}{1-r}}\,ds\biggr)^{1-r}\biggl(\int_0^t \bigl(\varphi(s)\bigr)^{\frac{1}{r}}\,ds\biggr)^{r} \\
&\quad + \frac{\lambda C_1}{\Gamma\beta}\biggl(\int_o^t (t-s)^{\frac{\beta-1}{1-r}}\,ds\biggr)^{1-r}\biggl(\int_0^t \bigl(\varphi(s)\bigr)^{\frac{1}{r}}\,ds\biggr)^{r} \\
&\quad + \frac{L_{f_1}C_1\Delta t^{\beta}}{\Gamma(\beta+1)\Gamma(\alpha+\beta)}\biggl(\int_0^T (T-s)^{\frac{\alpha+\beta-1}{1-r}}\,ds\biggr)^{1-r}\biggl(\int_0^T \bigl(\varphi(s)\bigr)^{\frac{1}{r}}\,ds\biggr)^{r} \\
&\quad + \frac{\bar{L}_{f_2}C_1\Delta t^{\beta}}{\Gamma(\beta+1)\Gamma(\alpha+\beta)}\biggl(\int_0^T (T-s)^{\frac{\alpha+\beta-1}{1-r}}\,ds\biggr)^{1-r}\biggl(\int_0^T \bigl(\varphi(s)\bigr)^{\frac{1}{r}}\,ds\biggr)^{r} \\
&\quad + \frac{C_1\lambda\Delta t^{\beta}}{\Gamma(\beta)\Gamma(\beta+1)}\biggl(\int_0^T (T-s)^{\frac{\beta-1}{1-r}}\,ds\biggr)^{1-r}\biggl(\int_0^T \bigl(\varphi(s)\bigr)^{\frac{1}{r}}\,ds\biggr)^{r} \\
&\quad + \frac{L_{f_1}C_1\theta\Delta t^{\beta}}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p)}\biggl(\int_0^\eta (\eta-s)^{\frac{\alpha+\beta+p-1}{1-r}}\,ds\biggr)^{1-r}\biggl(\int_0^\eta \bigl(\varphi(s)\bigr)^{\frac{1}{r}}\,ds\biggr)^{r} \\
&\quad + \frac{\bar{L}_{f_2}C_1\theta\Delta t^{\beta}}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p)}\biggl(\int_0^\eta (\eta-s)^{\frac{\alpha+\beta+p-1}{1-r}}\,ds\biggr)^{1-r}\bigl(\varphi(s)\bigr)^{\frac{1}{r}}\,ds)^{r} \\
&\quad + \frac{\theta\Delta\lambda t^{\beta}C_1}{\Gamma(\beta+1)\Gamma(\beta+p)}\biggl(\int_0^\eta (\eta-s)^{\frac{\beta+p-1}{1-r}}\,ds\biggr)^{1-r}\biggl(\int_0^\eta \bigl(\varphi(s)\bigr)^{\frac{1}{r}}\,ds\biggr)^{r} \\
&\leq \frac{L_{f_1}C_1C_{\varphi}\varphi(t)}{\Gamma(\alpha+\beta)}\biggl(\frac{1-r}{\alpha+\beta-r}\biggr)^{1-r}t^{\alpha+\beta-r} + \frac{\bar{L}_{f_2}C_1C_{\varphi}\varphi(t)}{\Gamma(\alpha+\beta)}\biggl(\frac{1-r}{\alpha+\beta-r}\biggr)^{1-r}t^{\alpha+\beta-r} \\
&\quad + \frac{\lambda C_1C_{\varphi}\varphi(t)}{\Gamma\beta}\biggl(\frac{r-1}{\beta-r}\biggr)^{1-r}t^{\beta-r} + \frac{L_{f_1}C_1C_{\varphi}\varphi(t)\Delta t^{\beta}}{\Gamma(\beta+1)\Gamma(\alpha+\beta)}\biggl(\frac{1-r}{\alpha+\beta-r}\biggr)^{1-r}T^{\alpha+\beta-r} \\
&\quad + \frac{\bar{L}_{f_2}C_1C_{\varphi}\varphi(t)\Delta t^{\beta}}{\Gamma(\beta+1)\Gamma(\alpha+\beta)}\biggl(\frac{1-r}{\alpha+\beta-r}\biggr)^{1-r}T^{\alpha+\beta-r} + \frac{C_1C_{\varphi}\varphi(t)\lambda\Delta t^{\beta}}{\Gamma(\beta)\Gamma(\beta+1)}\biggl(\frac{1-r}{\beta-r}\biggr)^{1-r}T^{\beta-r}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{L_{f_1}C_1C_\varphi\varphi(t)\theta\Delta t^\beta}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p)}\left(\frac{1-r}{\alpha+\beta+p-r}\right)^{1-r}\eta^{\alpha+\beta+p-r}\\
&+\frac{\bar{L}_{f_2}C_1C_\varphi\varphi(t)\theta\Delta t^\beta}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p)}\left(\frac{1-r}{\alpha+\beta+p-r}\right)^{1-r}\eta^{\alpha+\beta+p-r}\\
&+\frac{\theta\Delta\lambda t^\beta C_1C_\varphi\varphi(t)}{\Gamma(\beta+1)\Gamma(\beta+p)}\left(\frac{1-r}{\beta+p-r}\right)^{1-r}\eta^{\beta+p-r}\\
\le{}&\left\{\frac{1}{\Gamma(\alpha+\beta)}\left(\frac{1-r}{\alpha+\beta-r}\right)^{1-r}(L_{f_1}+\bar{L}_{f_2})s_0^{\alpha+\beta-r}+\frac{\lambda\Delta t^\beta}{\Gamma(\beta)\Gamma(\beta+1)}\left(\frac{1-r}{\beta-r}\right)^{1-r}T^{\beta-r}\right.\\
&+\frac{\lambda}{\Gamma\beta}\left(\frac{r-1}{\beta-r}\right)^{1-r}s_0^{\beta-r}+\frac{\Delta t^\beta}{\Gamma(\beta+1)\Gamma(\alpha+\beta)}\left(\frac{1-r}{\alpha+\beta-r}\right)^{1-r}(L_{f_1}+\bar{L}_{f_2})T^{\alpha+\beta-r}\\
&+\frac{\theta\Delta t^\beta}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p)}\left(\frac{1-r}{\alpha+\beta+p-r}\right)^{1-r}(L_{f_1}+\bar{L}_{f_2})\eta^{\alpha+\beta+p-r}\\
&\left.+\frac{\theta\Delta\lambda t^\beta}{\Gamma(\beta+1)\Gamma(\beta+p)}\left(\frac{1-r}{\beta+p-r}\right)^{1-r}\eta^{\beta+p-r}\right\}C_1C_\varphi\varphi(t).
\end{aligned}
$$

Case 2. For $t\in(s_{k-1},t_k]$, we have

$$
\left|(\Lambda g)t-(\Lambda h)t\right|=\left|g_k\bigl(t,g(t)\bigr)-g_k\bigl(t,h(t)\bigr)\right|\le L_{gk}\left|g(t)-h(t)\right|\le L_{gk}C_2\psi.
$$

Case 3. For $t\in(t_k,s_k]$ and $s\in(t_k,s_k]$,

$$
\begin{aligned}
&\left|(\Lambda g)(t)-(\Lambda h)(t)\right|\\
&\quad\le\frac{1}{\Gamma(\alpha+\beta)}\int_0^t(t-s)^{\alpha+\beta-1}\\
&\qquad\times\left|f\bigl(s,g(s),{}^cD_{0,t}^\alpha\bigl({}^cD_{0,t}^\beta+\lambda\bigr)g(s)\bigr)-f\bigl(s,h(s),{}^cD_{0,t}^\alpha\bigl({}^cD_{0,t}^\beta+\lambda\bigr)h(s)\bigr)\right|ds\\
&\qquad+\frac{\lambda}{\Gamma\beta}\int_0^t(t-s)^{\beta-1}\left|g(s)-h(s)\right|ds+\frac{\Delta(t_k^\beta-t^\beta)}{\Gamma(\beta+1)\Gamma(\alpha+\beta)}\int_0^T(T-s)^{\alpha+\beta-1}\\
&\qquad\times\left|f\bigl(s,g(s),{}^cD_{0,t}^\alpha\bigl({}^cD_{0,t}^\beta+\lambda\bigr)g(s)\bigr)-f\bigl(s,h(s),{}^cD_{0,t}^\alpha\bigl({}^cD_{0,t}^\beta+\lambda\bigr)h(s)\bigr)\right|ds\\
&\qquad+\frac{\lambda\Delta(t_k^\beta-t^\beta)}{\Gamma(\beta+1)\Gamma(\beta)}\int_0^T(T-s)^{\beta-1}\left|g(s)-h(s)\right|ds\\
&\qquad+\frac{\theta\Delta(t_k^\beta-t^\beta)}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p)}\int_0^\eta(\eta-s)^{\alpha+\beta+p-1}\\
&\qquad\times\left|f\bigl(s,g(s),{}^cD_{0,t}^\alpha\bigl({}^cD_{0,t}^\beta+\lambda\bigr)g(s))\bigr)-f\bigl(s,h(s),{}^cD_{0,t}^\alpha\bigl({}^cD_{0,t}^\beta+\lambda\bigr)h(s))\bigr)\right|ds\\
&\qquad+\frac{\theta\Delta\lambda(t_k^\beta-t^\beta)}{\Gamma(\beta+1)\Gamma(\beta+p)}\int_0^\eta(\eta-s)^{\beta+p-1}\left|g(s)-h(s)\right|ds\\
&\qquad+\left(\Delta\frac{(t_k^\beta-t^\beta)}{\Gamma(\beta+1)}\left(\frac{\theta\eta^p-\Gamma(p+1)}{\Gamma(p+1)}\right)-\lambda\right)\frac{1}{\Gamma(\alpha+\beta)}\int_0^{t_k}(t_k-s)^{\alpha+\beta-1}\\
&\qquad\times\left|f\bigl(s,g(s),{}^cD_{0,t}^\alpha\bigl({}^cD_{0,t}^\beta+\lambda\bigr)h(s)\bigr)-f\bigl(s,h(s),{}^cD_{0,t}^\alpha\bigl({}^cD_{0,t}^\beta+\lambda\bigr)h(s)\bigr)\right|ds\\
&\qquad+\left(\Delta\frac{(t_k^\beta-t^\beta)}{\Gamma(\beta+1)}\left(\frac{\theta\eta^p-\Gamma(p+1)}{\Gamma(p+1)}\right)-1\right)\frac{\lambda}{\Gamma\beta}\int_0^{t_k}(t_k-s)^{\beta-1}\left|g(s)-h(s)\right|ds
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\Delta\frac{(t_k^\beta-t^\beta)}{\Gamma(\beta+1)}\left(\frac{\theta\eta^p-\Gamma(p+1)}{\Gamma(p+1)}\right)-1\right)\bigl|g\bigl(t_k,g(t_k)\bigr)-g\bigl(t_k,h(t_k)\bigr)\bigr|\\
&\leq\frac{1}{\Gamma(\alpha+\beta)}\int_0^t(t-s)^{\alpha+\beta-1}\\
&\quad\times\bigl(L_{f_1}\bigl|g(s)-h(s)\bigr|+\bar{L}_{f_2}\bigl|{}^cD_{0,t}^\alpha\bigl({}^cD_{0,t}^\beta+\lambda\bigr)g(s)-{}^cD_{0,t}^\alpha\bigl({}^cD_{0,t}^\beta+\lambda\bigr)h(s)\bigr|\bigr)\,ds\\
&\quad+\frac{\lambda}{\Gamma\beta}\int_0^t(t-s)^{\beta-1}\bigl|g(s)-h(s)\bigr|\,ds+\frac{\Delta(t_k^\beta-t^\beta)}{\Gamma(\beta+1)\Gamma(\alpha+\beta)}\int_0^T(T-s)^{\alpha+\beta-1}\\
&\quad\times\bigl(L_{f_1}\bigl|g(s)-h(s)\bigr|+\bar{L}_{f_2}\bigl|{}^cD_{0,t}^\alpha\bigl({}^cD_{0,t}^\beta+\lambda\bigr)g(s)-{}^cD_{0,t}^\alpha\bigl({}^cD_{0,t}^\beta+\lambda\bigr)h(s)\bigr|\bigr)\,ds\\
&\quad+\frac{\lambda\Delta(t_k^\beta-t^\beta)}{\Gamma(\beta+1)\Gamma(\beta)}\int_0^T(T-s)^{\beta-1}\bigl|g(s)-h(s)\bigr|\,ds\\
&\quad+\frac{\theta\Delta(t_k^\beta-t^\beta)}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p)}\int_0^\eta(\eta-s)^{\alpha+\beta+p-1}\\
&\quad\times\bigl(L_{f_1}\bigl|g(s)-h(s)\bigr|+\bar{L}_{f_2}\bigl|{}^cD_{0,t}^\alpha\bigl({}^cD_{0,t}^\beta+\lambda\bigr)g(s)-{}^cD_{0,t}^\alpha\bigl({}^cD_{0,t}^\beta+\lambda\bigr)h(s)\bigr|\bigr)\,ds\\
&\quad+\frac{\theta\Delta\lambda(t_k^\beta-t^\beta)}{\Gamma(\beta+1)\Gamma(\beta+p)}\int_0^\eta(\eta-s)^{\beta+p-1}\bigl|g(s)-h(s)\bigr|\,ds\\
&\quad+\left(\Delta\frac{(t_k^\beta-t^\beta)}{\Gamma(\beta+1)}\left(\frac{\theta\eta^p-\Gamma(p+1)}{\Gamma(p+1)}\right)-\lambda\right)\frac{1}{\Gamma(\alpha+\beta)}\int_0^{t_k}(t_k-s)^{\alpha+\beta-1}\\
&\quad\times\bigl[L_{f_1}\bigl|g(s)-h(s)\bigr|+\bar{L}_{f_2}\bigl|{}^cD_{0,t}^\alpha\bigl({}^cD_{0,t}^\beta+\lambda\bigr)g(s)-{}^cD_{0,t}^\alpha\bigl({}^cD_{0,t}^\beta+\lambda\bigr)h(s)\bigr|\,ds\\
&\quad+\left(\Delta\frac{(t_k^\beta-t^\beta)}{\Gamma(\beta+1)}\left(\frac{\theta\eta^p-\Gamma(p+1)}{\Gamma(p+1)}\right)-1\right)\frac{\lambda}{\Gamma\beta}\int_0^{t_k}(t_k-s)^{\beta-1}\bigl|g(s)-h(s)\bigr|\,ds\\
&\quad+\left(\Delta\frac{(t_k^\beta-t^\beta)}{\Gamma(\beta+1)}\left(\frac{\theta\eta^p-\Gamma(p+1)}{\Gamma(p+1)}\right)-1\right)\bigl|g\bigl(t_k,g(t_k)\bigr)-g\bigl(t_k,h(t_k)\bigr)\bigr|\\
&=\frac{L_{f_1}}{\Gamma(\alpha+\beta)}\int_0^t(t-s)^{\alpha+\beta-1}\bigl|g(s)-h(s)\bigr|\,ds+\frac{\lambda}{\Gamma\beta}\int_0^t(t-s)^{\beta-1}\bigl|g(s)-h(s)\bigr|\,ds\\
&\quad+\frac{\bar{L}_{f_2}}{\Gamma(\alpha+\beta)}\int_0^t(t-s)^{\alpha+\beta-1}\bigl|{}^cD_{0,t}^\alpha\bigl({}^cD_{0,t}^\beta+\lambda\bigr)g(s)-{}^cD_{0,t}^\alpha\bigl({}^cD_{0,t}^\beta+\lambda\bigr)h(s)\bigr|\,ds\\
&\quad+\frac{\Delta L_{f_1}(t_k^\beta-t^\beta)}{\Gamma(\beta+1)\Gamma(\alpha+\beta)}\int_0^T(T-s)^{\alpha+\beta-1}\bigl|g(s)-h(s)\bigr|\,ds+\frac{\Delta\bar{L}_{f_2}(t_k^\beta-t^\beta)}{\Gamma(\beta+1)\Gamma(\alpha+\beta)}\\
&\quad\times\int_0^T(T-s)^{\alpha+\beta-1}\bigl|{}^cD_{0,t}^\alpha\bigl({}^cD_{0,t}^\beta+\lambda\bigr)g(s)-{}^cD_{0,t}^\alpha\bigl({}^cD_{0,t}^\beta+\lambda\bigr)h(s)\bigr|\,ds\\
&\quad+\frac{\lambda\Delta(t_k^\beta-t^\beta)}{\Gamma(\beta+1)\Gamma(\beta)}\int_0^T(T-s)^{\beta-1}\bigl|g(s)-h(s)\bigr|\,ds\\
&\quad+\frac{L_{f_1}\theta\Delta(t_k^\beta-t^\beta)}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p)}\int_0^\eta(\eta-s)^{\alpha+\beta+p-1}\bigl|g(s)-h(s)\bigr|\,ds\\
&\quad+\frac{\bar{L}_{f_2}\theta\Delta(t_k^\beta-t^\beta)}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p)}\\
&\quad\times\int_0^\eta(\eta-s)^{\alpha+\beta+p-1}\bigl|{}^cD_{0,t}^\alpha\bigl({}^cD_{0,t}^\beta+\lambda\bigr)g(s)-{}^cD_{0,t}^\alpha\bigl({}^cD_{0,t}^\beta+\lambda\bigr)h(s)\bigr|\,ds\\
&\quad+\frac{\theta\Delta\lambda(t_k^\beta-t^\beta)}{\Gamma(\beta+1)\Gamma(\beta+p)}\int_0^\eta(\eta-s)^{\beta+p-1}\bigl|g(s)-h(s)\bigr|\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\Delta\frac{(t_k^\beta-t^\beta)}{\Gamma(\beta+1)}\left(\frac{\theta\eta^p-\Gamma(p+1)}{\Gamma(p+1)}\right)-\lambda\right)\\
&\times\frac{\bar{L}_{f_2}}{\Gamma(\alpha+\beta)}\int_0^{t_k}(t_k-s)^{\alpha+\beta-1}\left|{}^cD_{0,t}^\alpha\left({}^cD_{0,t}^\beta+\lambda\right)g(s)-{}^cD_{0,t}^\alpha\left({}^cD_{0,t}^\beta+\lambda\right)h(s)\right|ds\\
&+\left(\Delta\frac{(t_k^\beta-t^\beta)}{\Gamma(\beta+1)}\left(\frac{\theta\eta^p-\Gamma(p+1)}{\Gamma(p+1)}\right)-\lambda\right)\frac{L_{f_1}}{\Gamma(\alpha+\beta)}\int_0^{t_k}(t_k-s)^{\alpha+\beta-1}\left|g(s)-h(s)\right|ds\\
&+\left(\Delta\frac{(t_k^\beta-t^\beta)}{\Gamma(\beta+1)}\left(\frac{\theta\eta^p-\Gamma(p+1)}{\Gamma(p+1)}\right)-1\right)\frac{\lambda}{\Gamma\beta}\int_0^{t_k}(t_k-s)^{\beta-1}\left|g(s)-h(s)\right|ds\\
&+\left(\Delta\frac{(t_k^\beta-t^\beta)}{\Gamma(\beta+1)}\left(\frac{\theta\eta^p-\Gamma(p+1)}{\Gamma(p+1)}\right)-1\right)\left|g\left(t_k,g(t_k)\right)-g\left(t_k,h(t_k)\right)\right|\\
\leq{}&\frac{L_{f_1}C_1}{\Gamma(\alpha+\beta)}\int_0^t(t-s)^{\alpha+\beta-1}\left|\varphi(s)\right|ds+\frac{\bar{L}_{f_2}C_1}{\Gamma(\alpha+\beta)}\int_0^t(t-s)^{\alpha+\beta-1}\left|\varphi(s)\right|ds\\
&+\frac{\lambda C_1}{\Gamma\beta}\int_0^t(t-s)^{\beta-1}\left|\varphi(s)\right|ds+\frac{\Delta L_{f_1}C_1(t_k^\beta-t^\beta)}{\Gamma(\beta+1)\Gamma(\alpha+\beta)}\int_0^T(T-s)^{\alpha+\beta-1}\left|\varphi(s)\right|ds\\
&+\frac{\Delta\bar{L}_{f_2}C_1(t_k^\beta-t^\beta)}{\Gamma(\beta+1)\Gamma(\alpha+\beta)}\int_0^T(T-s)^{\alpha+\beta-1}\left|\varphi(s)\right|ds\\
&+\frac{\lambda C_1\Delta(t_k^\beta-t^\beta)}{\Gamma(\beta+1)\Gamma(\beta)}\int_0^T(T-s)^{\beta-1}\left|\varphi(s)\right|ds\\
&+\frac{L_{f_1}C_1\theta\Delta(t_k^\beta-t^\beta)}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p)}\int_0^\eta(\eta-s)^{\alpha+\beta+p-1}\left|\varphi(s)\right|ds\\
&+\frac{\bar{L}_{f_2}C_1\theta\Delta(t_k^\beta-t^\beta)}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p)}\int_0^\eta(\eta-s)^{\alpha+\beta+p-1}\left|\varphi(s)\right|ds\\
&+\frac{\theta C_1\Delta\lambda(t_k^\beta-t^\beta)}{\Gamma(\beta+1)\Gamma(\beta+p)}\int_0^\eta(\eta-s)^{\beta+p-1}\left|\varphi(s)\right|ds\\
&+\left(\Delta\frac{(t_k^\beta-t^\beta)}{\Gamma(\beta+1)}\left(\frac{\theta\eta^p-\Gamma(p+1)}{\Gamma(p+1)}\right)-\lambda\right)\frac{L_{f_1}C_1}{\Gamma(\alpha+\beta)}\int_0^{t_k}(t_k-s)^{\alpha+\beta-1}\left|\varphi(s)\right|ds\\
&+\left(\Delta\frac{(t_k^\beta-t^\beta)}{\Gamma(\beta+1)}\left(\frac{\theta\eta^p-\Gamma(p+1)}{\Gamma(p+1)}\right)-\lambda\right)\frac{\bar{L}_{f_2}C_1}{\Gamma(\alpha+\beta)}\int_0^{t_k}(t_k-s)^{\alpha+\beta-1}\left|\varphi(s)\right|ds\\
&+\left(\Delta\frac{(t_k^\beta-t^\beta)}{\Gamma(\beta+1)}\left(\frac{\theta\eta^p-\Gamma(p+1)}{\Gamma(p+1)}\right)-1\right)\frac{\lambda C_1}{\Gamma\beta}\int_0^{t_k}(t_k-s)^{\beta-1}\left|\varphi(s)\right|ds\\
&+\left(\Delta\frac{(t_k^\beta-t^\beta)}{\Gamma(\beta+1)}\left(\frac{\theta\eta^p-\Gamma(p+1)}{\Gamma(p+1)}\right)-1\right)L_{gk}\left|g(t_k)-h(t_k))\right|\\
\leq{}&\frac{L_{f_1}C_1}{\Gamma(\alpha+\beta)}\left(\int_0^t(t-s)^{\frac{\alpha+\beta-1}{1-r}}ds\right)^{1-r}\left(\int_0^t\left(\varphi(s)\right)^{\frac{1}{r}}ds\right)^r\\
&+\frac{\bar{L}_{f_2}C_1}{\Gamma(\alpha+\beta)}\left(\int_0^t(t-s)^{\frac{\alpha+\beta-1}{1-r}}ds\right)^{1-r}\left(\int_0^t\left(\varphi(s)\right)^{\frac{1}{r}}ds\right)^r\\
&+\frac{\lambda C_1}{\Gamma\beta}\left(\int_o^t(t-s)^{\frac{\beta-1}{1-r}}ds\right)^{1-r}\left(\int_0^t\left(\varphi(s)\right)^{\frac{1}{r}}ds\right)^r\\
&+\frac{\Delta(t_k^\beta-t^\beta)L_{f_1}C_1}{\Gamma(\beta+1)\Gamma(\alpha+\beta)}\left(\int_0^T(T-s)^{\frac{\alpha+\beta-1}{1-r}}ds\right)^{1-r}\left(\int_0^T\left(\varphi(s)\right)^{\frac{1}{r}}ds\right)^r
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{\Delta(t_k^\beta - t^\beta)\bar{L}_{f_2} C_1}{\Gamma(\beta+1)\Gamma(\alpha+\beta)} \left(\int_0^T (T-s)^{\frac{\alpha+\beta-1}{1-r}}\, ds \right)^{1-r} \left(\int_0^T \big(\varphi(s)\big)^{\frac{1}{r}}\, ds \right)^r \\
&+ \frac{\lambda\Delta(t_k^\beta - t^\beta) C_1}{\Gamma(\beta+1)\Gamma(\beta)} \left(\int_0^T (T-s)^{\frac{\beta-1}{1-r}}\, ds \right)^{1-r} \left(\int_0^T \big(\varphi(s)\big)^{\frac{1}{r}}\, ds \right)^r \\
&+ \frac{\theta\Delta(t_k^\beta - t^\beta) L_{f_1} C_1}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p)} \left(\int_0^\eta (\eta-s)^{\frac{\alpha+\beta+p-1}{1-r}}\, ds \right)^{1-r} \left(\int_0^\eta \big(\varphi(s)\big)^{\frac{1}{r}}\, ds \right)^r \\
&+ \frac{\theta\Delta(t_k^\beta - t^\beta)\bar{L}_{f_2} C_1}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p)} \left(\int_0^\eta (\eta-s)^{\frac{\alpha+\beta+p-1}{1-r}}\, ds \right)^{1-r} \left(\int_0^\eta \big(\varphi(s)\big)^{\frac{1}{r}}\, ds \right)^r \\
&+ \frac{C_1\theta\Delta\lambda(t_k^\beta - t^\beta)}{\Gamma(\beta+1)\Gamma(\beta+p)} \left(\int_0^\eta (\eta-s)^{\frac{\beta+p-1}{1-r}}\, ds \right)^{1-r} \left(\int_0^\eta \big(\varphi(s)\big)^{\frac{1}{r}}\, ds \right)^r \\
&+ \left(\Delta \frac{(t_k^\beta - t^\beta)}{\Gamma(\beta+1)} \left(\frac{\theta\eta^p - \Gamma(p+1)}{\Gamma(p+1)} \right) - \lambda \right) \\
&\times \frac{L_{f_1} C_1}{\Gamma(\alpha+\beta)} \left(\int_0^{t_k} (t_k-s)^{\frac{\alpha+\beta-1}{1-r}}\, ds \right)^{1-r} \left(\int_0^{t_k} \big(\varphi(s)\big)^{\frac{1}{r}}\, ds \right)^r \\
&+ \left(\Delta \frac{(t_k^\beta - t^\beta)}{\Gamma(\beta+1)} \left(\frac{\theta\eta^p - \Gamma(p+1)}{\Gamma(p+1)} \right) - \lambda \right) \\
&\times \frac{\bar{L}_{f_2} C_1}{\Gamma(\alpha+\beta)} \left(\int_0^{t_k} (t_k-s)^{\frac{\alpha+\beta-1}{1-r}}\, ds \right)^{1-r} \left(\int_0^{t_k} \big(\varphi(s)\big)^{\frac{1}{r}}\, ds \right)^r \\
&+ \left(\Delta \frac{(t_k^\beta - t^\beta)}{\Gamma(\beta+1)} \left(\frac{\theta\eta^p + \Gamma(p+1)}{\Gamma(p+1)} \right) - 1 \right) \\
&\times \frac{\lambda C_1}{\Gamma\beta} \left(\int_o^{t_k} (t_k-s)^{\frac{\beta-1}{1-r}}\, ds \right)^{1-r} \left(\int_0^{t_k} \big(\varphi(s)\big)^{\frac{1}{r}}\, ds \right)^r \\
&+ \left(\Delta \frac{(t_k^\beta - t^\beta)}{\Gamma(\beta+1)} \left(\frac{\theta\eta^p - \Gamma(p+1)}{\Gamma(p+1)} \right) - 1 \right) L_{gk} C_2 \psi \\
\le{}& \frac{L_{f_1} C_1 C_\varphi \varphi(t)}{\Gamma(\alpha+\beta)} \left(\frac{1-r}{\alpha+\beta-r} \right)^{1-r} t^{\alpha+\beta-r} + \frac{\bar{L}_{f_2} C_1 C_\varphi \varphi(t)}{\Gamma(\alpha+\beta)} \left(\frac{1-r}{\alpha+\beta-r} \right)^{1-r} t^{\alpha+\beta-r} \\
&+ \frac{\lambda C_1 C_\varphi \varphi(t)}{\Gamma\beta} \left(\frac{1-r}{\beta-r} \right)^{1-r} t^{\beta-r} + \frac{\Delta(t_k^\beta - t^\beta) L_{f_1} C_1 C_\varphi \varphi(t)}{\Gamma(\beta+1)\Gamma(\alpha+\beta)} \left(\frac{1-r}{\alpha+\beta-r} \right)^{1-r} T^{\alpha+\beta-r} \\
&+ \frac{\Delta(t_k^\beta - t^\beta)\bar{L}_{f_2} C_1 C_\varphi \varphi(t)}{\Gamma(\beta+1)\Gamma(\alpha+\beta)} \left(\frac{1-r}{\alpha+\beta-r} \right)^{1-r} T^{\alpha+\beta-r} \\
&+ \frac{\lambda\Delta(t_k^\beta - t^\beta) C_1 C_\varphi \varphi(t)}{\Gamma(\beta+1)\Gamma(\beta)} \left(\frac{1-r}{\beta-r} \right)^{1-r} T^{\beta-r} \\
&+ \frac{\theta\Delta(t_k^\beta - t^\beta) L_{f_1} C_1 C_\varphi \varphi(t)}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p)} \left(\frac{1-r}{\alpha+\beta+p-r} \right)^{1-r} \eta^{\alpha+\beta+p-r} \\
&+ \frac{\theta\Delta(t_k^\beta - t^\beta)\bar{L}_{f_2} C_1 C_\varphi \varphi(t)}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p)} \left(\frac{1-r}{\alpha+\beta+p-r} \right)^{1-r} \eta^{\alpha+\beta+p-r} \\
&+ \frac{C_1 C_\varphi \varphi(t)\theta\Delta\lambda(t_k^\beta - t^\beta)}{\Gamma(\beta+1)\Gamma(\beta+p)} \left(\frac{1-r}{\beta+p-r} \right)^{1-r} \eta^{\beta+p-r} \\
&+ \left(\Delta \frac{(t_k^\beta - t^\beta)}{\Gamma(\beta+1)} \left(\frac{\theta\eta^p - \Gamma(p+1)}{\Gamma(p+1)} \right) - \lambda \right) \frac{L_{f_1} C_1 C_\varphi \varphi(t)}{\Gamma(\alpha+\beta)} \left(\frac{1-r}{\alpha+\beta-r} \right)^{1-r} t_k^{\alpha+\beta-r}
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\Delta\frac{(t_k^\beta-t^\beta)}{\Gamma(\beta+1)}\left(\frac{\theta\eta^p-\Gamma(p+1)}{\Gamma(p+1)}\right)-\lambda\right)\frac{\bar{L}_{f_2}C_1C_\varphi\varphi(t)}{\Gamma(\alpha+\beta)}\left(\frac{1-r}{\alpha+\beta-r}\right)^{1-r}t_k^{\alpha+\beta-r}\\
&+\left(\Delta\frac{(t_k^\beta-t^\beta)}{\Gamma(\beta+1)}\left(\frac{\theta\eta^p+\Gamma(p+1)}{\Gamma(p+1)}\right)-1\right)\frac{\lambda C_1C_\varphi\varphi(t)}{\Gamma\beta}\left(\frac{1-r}{\beta-r}\right)^{1-r}t_k^{\beta-r}\\
&+\left(\Delta\frac{(t_k^\beta-t^\beta)}{\Gamma(\beta+1)}\left(\frac{\theta\eta^p-\Gamma(p+1)}{\Gamma(p+1)}\right)-1\right)L_{gk}C_2\psi\\
\le&\left\{\frac{1}{\Gamma(\alpha+\beta)}\left(\frac{1-r}{\alpha+\beta-r}\right)^{1-r}(L_{f_1}+\bar{L}_{f_2})s_0^{\alpha+\beta-r}+\frac{\lambda}{\Gamma\beta}\left(\frac{1-r}{\beta-r}\right)^{1-r}s_0^{\beta-r}\right.\\
&+\frac{\Delta(t_k^\beta-t^\beta)}{\Gamma(\beta+1)\Gamma(\alpha+\beta)}\left(\frac{1-r}{\alpha+\beta-r}\right)^{1-r}(L_{f_1}+\bar{L}_{f_2})T^{\alpha+\beta-r}\\
&+\frac{\theta\Delta(t_k^\beta-t^\beta)}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p)}\left(\frac{1-r}{\alpha+\beta+p-r}\right)^{1-r}(L_{f_1}+\bar{L}_{f_2})\eta^{\alpha+\beta+p-r}\\
&+\frac{\theta\Delta\lambda(t_k^\beta-t^\beta)}{\Gamma(\beta+1)\Gamma(\beta+p)}\left(\frac{1-r}{\beta+p-r}\right)^{1-r}\eta^{\beta+p-r}\\
&+\left(\Delta\frac{(t_k^\beta-t^\beta)}{\Gamma(\beta+1)}\left(\frac{\theta\eta^p-\Gamma(p+1)}{\Gamma(p+1)}\right)-\lambda\right)\\
&\times\frac{1}{\Gamma(\alpha+\beta)}\left(\frac{1-r}{\alpha+\beta-r}\right)^{1-r}(L_{f_1}+\bar{L}_{f_2})t_k^{\alpha+\beta-r}+\frac{\lambda\Delta(t_k^\beta-t^\beta)}{\Gamma(\beta+1)\Gamma(\beta)}\left(\frac{1-r}{\beta-r}\right)^{1-r}T^{\beta-r}\\
&\left.+\left(\Delta\frac{(t_k^\beta-t^\beta)}{\Gamma(\beta+1)}\left(\frac{\theta\eta^p+\Gamma(p+1)}{\Gamma(p+1)}\right)-1\right)\left(\frac{\lambda}{\Gamma\beta}\left(\frac{1-r}{\beta-r}\right)^{1-r}t_k^{\beta-r}+L_{gk}\right)\right\}C_\varphi\\
&\times(C_1+C_2)\big(\varphi(t)+\psi\big).
\end{aligned}
$$

Also, for $t\in(t_k,s_k]$ and $s\in(s_{k-1},t_k]$, we have

$$
\begin{aligned}
&\big|(\Lambda g)(t)-(\Lambda h)(t)\big|\\
&\le\frac{1}{\Gamma(\alpha+\beta)}\int_0^t(t-s)^{\alpha+\beta-1}\\
&\quad\times\big|f\big(s,g(s),{}^cD_{0,t}^\alpha\big({}^cD_{0,t}^\beta+\lambda\big)g(s)\big)-f\big(s,h(s),{}^cD_{0,t}^\alpha\big({}^cD_{0,t}^\beta+\lambda\big)h(s)\big)\big|\,ds\\
&\quad+\frac{\lambda}{\Gamma\beta}\int_0^t(t-s)^{\beta-1}\big|g(s)-h(s)\big|\,ds\\
&\quad+\frac{\Delta(t_k^\beta-t^\beta)}{\Gamma(\beta+1)\Gamma(\alpha+\beta)}\int_0^T(T-s)^{\alpha+\beta-1}\\
&\quad\times\big|f\big(s,g(s),{}^cD_{0,t}^\alpha\big({}^cD_{0,t}^\beta+\lambda\big)g(s)\big)-f\big(s,h(s),{}^cD_{0,t}^\alpha\big({}^cD_{0,t}^\beta+\lambda\big)h(s)\big)\big|\,ds\\
&\quad+\frac{\lambda\Delta(t_k^\beta-t^\beta)}{\Gamma(\beta+1)\Gamma(\beta)}\int_0^T(T-s)^{\beta-1}\big|g(s)-h(s)\big|\,ds\\
&\quad+\frac{\theta\Delta(t_k^\beta-t^\beta)}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p)}\int_0^\eta(\eta-s)^{\alpha+\beta+p-1}\\
&\quad\times\big|f\big(s,g(s),{}^cD_{0,t}^\alpha\big({}^cD_{0,t}^\beta+\lambda\big)g(s)\big)-f\big(s,h(s),{}^cD_{0,t}^\alpha\big({}^cD_{0,t}^\beta+\lambda\big)h(s)\big)\big|\,ds\\
&\quad+\frac{\theta\Delta\lambda(t_k^\beta-t^\beta)}{\Gamma(\beta+1)\Gamma(\beta+p)}\int_0^\eta(\eta-s)^{\beta+p-1}\big|g(s)-h(s)\big|\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\Delta\frac{(t_k^\beta-t^\beta)}{\Gamma(\beta+1)}\left(\frac{\theta\eta^p-\Gamma(p+1)}{\Gamma(p+1)}\right)-\lambda\right)\frac{1}{\Gamma(\alpha+\beta)}\int_0^{t_k}(t_k-s)^{\alpha+\beta-1}\\
&\times\left|f\left(s,g(s),{}^cD_{0,t}^\alpha\left({}^cD_{0,t}^\beta+\lambda\right)h(s)\right)-f\left(s,h(s),{}^cD_{0,t}^\alpha\left({}^cD_{0,t}^\beta+\lambda\right)h(s)\right)\right|\,ds\\
&+\left(\Delta\frac{(t_k^\beta-t^\beta)}{\Gamma(\beta+1)}\left(\frac{\theta\eta^p-\Gamma(p+1)}{\Gamma(p+1)}\right)-1\right)\frac{\lambda}{\Gamma\beta}\int_0^{t_k}(t_k-s)^{\beta-1}\left|g(s)-h(s)\right|\,ds\\
&+\left(\Delta\frac{(t_k^\beta-t^\beta)}{\Gamma(\beta+1)}\left(\frac{\theta\eta^p-\Gamma(p+1)}{\Gamma(p+1)}\right)-1\right)\left|g\left(t_k,g(t_k)\right)-g\left(t_k,h(t_k)\right)\right|\\
\le{}&\frac{1}{\Gamma(\alpha+\beta)}\int_0^t(t-s)^{\alpha+\beta-1}\\
&\times\left[L_{f_1}\left|g(s)-h(s)\right|+\bar{L}_{f_2}\left|{}^cD_{0,t}^\alpha\left({}^cD_{0,t}^\beta+\lambda\right)g(s)-{}^cD_{0,t}^\alpha\left({}^cD_{0,t}^\beta+\lambda\right)h(s)\right|\right]\,ds\\
&+\frac{\lambda}{\Gamma\beta}\int_0^t(t-s)^{\beta-1}\left|g(s)-h(s)\right|\,ds+\frac{\Delta(t_k^\beta-t^\beta)}{\Gamma(\beta+1)\Gamma(\alpha+\beta)}\int_0^T(T-s)^{\alpha+\beta-1}\\
&\times\left[L_{f_1}\left|g(s)-h(s)\right|+\bar{L}_{f_2}\left|{}^cD_{0,t}^\alpha\left({}^cD_{0,t}^\beta+\lambda\right)g(s)-{}^cD_{0,t}^\alpha\left({}^cD_{0,t}^\beta+\lambda\right)h(s)\right|\,ds\right.\\
&+\frac{\lambda\Delta(t_k^\beta-t^\beta)}{\Gamma(\beta+1)\Gamma(\beta)}\int_0^T(T-s)^{\beta-1}\left|g(s)-h(s)\right|\,ds\\
&+\frac{\theta\Delta(t_k^\beta-t^\beta)}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p)}\int_0^\eta(\eta-s)^{\alpha+\beta+p-1}\\
&\times\left[L_{f_1}\left|g(s)-h(s)\right|+\bar{L}_{f_2}\left|{}^cD_{0,t}^\alpha\left({}^cD_{0,t}^\beta+\lambda\right)g(s)-{}^cD_{0,t}^\alpha\left({}^cD_{0,t}^\beta+\lambda\right)h(s)\right|\,ds\right.\\
&+\frac{\theta\Delta\lambda(t_k^\beta-t^\beta)}{\Gamma(\beta+1)\Gamma(\beta+p)}\int_0^\eta(\eta-s)^{\beta+p-1}\left|g(s)-h(s)\right|\,ds\\
&+\left(\Delta\frac{(t_k^\beta-t^\beta)}{\Gamma(\beta+1)}\left(\frac{\theta\eta^p-\Gamma(p+1)}{\Gamma(p+1)}\right)-\lambda\right)\frac{1}{\Gamma(\alpha+\beta)}\int_0^{t_k}(t_k-s)^{\alpha+\beta-1}\\
&\times\left[L_{f_1}\left|g(s)-h(s)\right|+\bar{L}_{f_2}\left|{}^cD_{0,t}^\alpha\left({}^cD_{0,t}^\beta+\lambda\right)g(s)-{}^cD_{0,t}^\alpha\left({}^cD_{0,t}^\beta+\lambda\right)h(s)\right|\,ds\right.\\
&+\left(\Delta\frac{(t_k^\beta-t^\beta)}{\Gamma(\beta+1)}\left(\frac{\theta\eta^p-\Gamma(p+1)}{\Gamma(p+1)}\right)-1\right)\frac{\lambda}{\Gamma\beta}\int_0^{t_k}(t_k-s)^{\beta-1}\left|g(s)-h(s)\right|\,ds\\
&+\left(\Delta\frac{(t_k^\beta-t^\beta)}{\Gamma(\beta+1)}\left(\frac{\theta\eta^p-\Gamma(p+1)}{\Gamma(p+1)}\right)-1\right)\left|g\left(t_k,g(t_k)\right)-g\left(t_k,h(t_k)\right)\right|\\
={}&\frac{L_{f_1}}{\Gamma(\alpha+\beta)}\int_0^t(t-s)^{\alpha+\beta-1}\left|g(s)-h(s)\right|\,ds\\
&+\frac{\bar{L}_{f_2}}{\Gamma(\alpha+\beta)}\int_0^t(t-s)^{\alpha+\beta-1}\left|{}^cD_{0,t}^\alpha\left({}^cD_{0,t}^\beta+\lambda\right)g(s)-{}^cD_{0,t}^\alpha\left({}^cD_{0,t}^\beta+\lambda\right)h(s)\right|\,ds\\
&+\frac{\lambda}{\Gamma\beta}\int_0^t(t-s)^{\beta-1}\left|g(s)-h(s)\right|\,ds\\
&+\frac{\Delta L_{f_1}(t_k^\beta-t^\beta)}{\Gamma(\beta+1)\Gamma(\alpha+\beta)}\int_0^T(T-s)^{\alpha+\beta-1}\left|g(s)-h(s)\right|\,ds\\
&+\frac{\Delta\bar{L}_{f_2}(t_k^\beta-t^\beta)}{\Gamma(\beta+1)\Gamma(\alpha+\beta)}\\
&\times\int_0^T(T-s)^{\alpha+\beta-1}\left|{}^cD_{0,t}^\alpha\left({}^cD_{0,t}^\beta+\lambda\right)g(s)-{}^cD_{0,t}^\alpha\left({}^cD_{0,t}^\beta+\lambda\right)h(s)\right|\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{\lambda\Delta(t_k^\beta - t^\beta)}{\Gamma(\beta+1)\Gamma(\beta)} \int_0^T (T-s)^{\beta-1}\big|g(s)-h(s)\big|\,ds \\
&+ \frac{L_{f_1}\theta\Delta(t_k^\beta - t^\beta)}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p)} \int_0^\eta (\eta-s)^{\alpha+\beta+p-1}\big|g(s)-h(s)\big|\,ds \\
&+ \frac{\bar{L}_{f_2}\theta\Delta(t_k^\beta - t^\beta)}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p)} \\
&\times \int_0^\eta (\eta-s)^{\alpha+\beta+p-1}\big|{}^cD_{0,t}^\alpha\big({}^cD_{0,t}^\beta+\lambda\big)g(s) - {}^cD_{0,t}^\alpha\big({}^cD_{0,t}^\beta+\lambda\big)h(s)\big|\,ds \\
&+ \frac{\theta\Delta\lambda(t_k^\beta - t^\beta)}{\Gamma(\beta+1)\Gamma(\beta+p)} \int_0^\eta (\eta-s)^{\beta+p-1}\big|g(s)-h(s)\big|\,ds \\
&+ \left(\Delta\frac{(t_k^\beta - t^\beta)}{\Gamma(\beta+1)}\left(\frac{\theta\eta^p - \Gamma(p+1)}{\Gamma(p+1)}\right) - \lambda\right) \\
&\times \frac{\bar{L}_{f_2}}{\Gamma(\alpha+\beta)} \int_0^{t_k} (t_k-s)^{\alpha+\beta-1}\big|{}^cD_{0,t}^\alpha\big({}^cD_{0,t}^\beta+\lambda\big)g(s) - {}^cD_{0,t}^\alpha\big({}^cD_{0,t}^\beta+\lambda\big)h(s)\big|\,ds \\
&+ \left(\Delta\frac{(t_k^\beta - t^\beta)}{\Gamma(\beta+1)}\left(\frac{\theta\eta^p - \Gamma(p+1)}{\Gamma(p+1)}\right) - \lambda\right) \\
&\times \frac{L_{f_1}}{\Gamma(\alpha+\beta)} \int_0^{t_k} (t_k-s)^{\alpha+\beta-1}\big|g(s)-h(s)\big|\,ds \\
&+ \left(\Delta\frac{(t_k^\beta - t^\beta)}{\Gamma(\beta+1)}\left(\frac{\theta\eta^p - \Gamma(p+1)}{\Gamma(p+1)}\right) - 1\right)\frac{\lambda}{\Gamma\beta} \int_0^{t_k} (t_k-s)^{\beta-1}\big|g(s)-h(s)\big|\,ds \\
&+ \left(\Delta\frac{(t_k^\beta - t^\beta)}{\Gamma(\beta+1)}\left(\frac{\theta\eta^p - \Gamma(p+1)}{\Gamma(p+1)}\right) - 1\right)\big|g\big(t_k,g(t_k)\big) - g\big(t_k,h(t_k)\big)\big| \\
\le{}& \frac{L_{f_1}C_2\psi}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1}\,ds + \frac{\bar{L}_{f_2}C_2\psi}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1}\,ds \\
&+ \frac{\lambda C_2\psi}{\Gamma\beta} \int_0^t (t-s)^{\beta-1}\,ds + \frac{\Delta L_{f_1}C_2\psi(t_k^\beta - t^\beta)}{\Gamma(\beta+1)\Gamma(\alpha+\beta)} \int_0^T (T-s)^{\alpha+\beta-1}\,ds \\
&+ \frac{\Delta\bar{L}_{f_2}C_2\psi(t_k^\beta - t^\beta)}{\Gamma(\beta+1)\Gamma(\alpha+\beta)} \int_0^T (T-s)^{\alpha+\beta-1}\,ds \\
&+ \frac{\lambda C_2\psi\Delta(t_k^\beta - t^\beta)}{\Gamma(\beta+1)\Gamma(\beta)} \int_0^T (T-s)^{\beta-1}\,ds \\
&+ \frac{L_{f_1}C_2\psi\theta\Delta(t_k^\beta - t^\beta)}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p)} \int_0^\eta (\eta-s)^{\alpha+\beta+p-1}\,ds \\
&+ \frac{\bar{L}_{f_2}C_2\psi\theta\Delta(t_k^\beta - t^\beta)}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p)} \int_0^\eta (\eta-s)^{\alpha+\beta+p-1}\,ds \\
&+ \frac{\theta C_2\psi\Delta\lambda(t_k^\beta - t^\beta)}{\Gamma(\beta+1)\Gamma(\beta+p)} \int_0^\eta (\eta-s)^{\beta+p-1}\,ds \\
&+ \left(\Delta\frac{(t_k^\beta - t^\beta)}{\Gamma(\beta+1)}\left(\frac{\theta\eta^p - \Gamma(p+1)}{\Gamma(p+1)}\right) - \lambda\right)\frac{L_{f_1}C_2\psi}{\Gamma(\alpha+\beta)} \int_0^{t_k} (t_k-s)^{\alpha+\beta-1}\,ds \\
&+ \left(\Delta\frac{(t_k^\beta - t^\beta)}{\Gamma(\beta+1)}\left(\frac{\theta\eta^p - \Gamma(p+1)}{\Gamma(p+1)}\right) - \lambda\right)\frac{\bar{L}_{f_2}C_2\psi}{\Gamma(\alpha+\beta)} \int_0^{t_k} (t_k-s)^{\alpha+\beta-1}\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\Delta\frac{(t_k^{\beta}-t^{\beta})}{\Gamma(\beta+1)}\left(\frac{\theta\eta^{p}-\Gamma(p+1)}{\Gamma(p+1)}\right)-1\right)\frac{\lambda C_2\psi}{\Gamma\beta}\int_0^{t_k}(t_k-s)^{\beta-1}\,ds\\
&+\left(\Delta\frac{(t_k^{\beta}-t^{\beta})}{\Gamma(\beta+1)}\left(\frac{\theta\eta^{p}-\Gamma(p+1)}{\Gamma(p+1)}\right)-1\right)L_{gk}\big|g(t_k)-h(t_k)\big)\big|\\
\le{}&\frac{L_{f_1}C_2\psi}{\Gamma(\alpha+\beta)(\alpha+\beta)}t^{\alpha+\beta}+\frac{\bar{L}_{f_2}C_2\psi}{\Gamma(\alpha+\beta)(\alpha+\beta)}t^{\alpha+\beta}+\frac{\lambda C_2\psi}{\beta\Gamma\beta}t^{\beta}\\
&+\frac{\Delta(t_k^{\beta}-t^{\beta})L_{f_1}C_2\psi}{\Gamma(\beta+1)(\alpha+\beta)\Gamma(\alpha+\beta)}T^{\alpha+\beta}+\frac{\Delta(t_k^{\beta}-t^{\beta})\bar{L}_{f_2}C_2\psi}{\Gamma(\beta+1)(\alpha+\beta)\Gamma(\alpha+\beta)}T^{\alpha+\beta}\\
&+\frac{\lambda\Delta(t_k^{\beta}-t^{\beta})C_2\psi}{\Gamma(\beta+1)\beta\Gamma(\beta)}T^{\beta}+\frac{\theta\Delta(t_k^{\beta}-t^{\beta})L_{f_1}C_2\psi}{\Gamma(\beta+1)(\alpha+\beta+p)\Gamma(\alpha+\beta+p)}\eta^{\alpha+\beta+p}\\
&+\frac{\theta\Delta(t_k^{\beta}-t^{\beta})\bar{L}_{f_2}C_2\psi}{\Gamma(\beta+1)(\alpha+\beta+p)\Gamma(\alpha+\beta+p)}\eta^{\alpha+\beta+p}\\
&+\frac{\theta\Delta\lambda(t_k^{\beta}-t^{\beta})C_2\psi}{\Gamma(\beta+1)(\beta+p)\Gamma(\beta+p)}\eta^{\beta+p}\\
&+\left(\Delta\frac{(t_k^{\beta}-t^{\beta})}{\Gamma(\beta+1)}\left(\frac{\theta\eta^{p}-\Gamma(p+1)}{\Gamma(p+1)}\right)-1\right)\frac{\lambda C_2\psi}{\beta\Gamma\beta}t_k^{\beta}\\
&+\left(\Delta\frac{(t_k^{\beta}-t^{\beta})}{\Gamma(\beta+1)}\left(\frac{\theta\eta^{p}-\Gamma(p+1)}{\Gamma(p+1)}\right)-\lambda\right)\frac{L_{f_1}C_2\psi}{(\alpha+\beta)\Gamma(\alpha+\beta)}t_k^{\alpha+\beta}\\
&+\left(\Delta\frac{(t_k^{\beta}-t^{\beta})}{\Gamma(\beta+1)}\left(\frac{\theta\eta^{p}-\Gamma(p+1)}{\Gamma(p+1)}\right)-\lambda\right)\frac{\bar{L}_{f_2}C_2\psi}{(\alpha+\beta)\Gamma(\alpha+\beta)}t_k^{\alpha+\beta}\\
&+\left(\Delta\frac{(t_k^{\beta}-t^{\beta})}{\Gamma(\beta+1)}\left(\frac{\theta\eta^{p}-\Gamma(p+1)}{\Gamma(p+1)}\right)-1\right)L_{gk}C_2\psi\\
\le{}&\left\{\frac{L_{f_1}}{\Gamma(\alpha+\beta+1)}s_k^{\alpha+\beta}+\frac{\bar{L}_{f_2}}{\Gamma(\alpha+\beta+1)}s_k^{\alpha+\beta}+\frac{\lambda}{\Gamma(\beta+1)}s_k^{\beta}\right.\\
&+\frac{\Delta(t_k^{\beta}-t^{\beta})L_{f_1}}{\Gamma(\beta+1)\Gamma(\alpha+\beta+1)}T^{\alpha+\beta}\\
&+\frac{\Delta(t_k^{\beta}-t^{\beta})\bar{L}_{f_2}}{\Gamma(\beta+1)\Gamma(\alpha+\beta+1)}T^{\alpha+\beta}+\frac{\lambda\Delta(t_k^{\beta}-t^{\beta})}{\Gamma(\beta+1)\Gamma(\beta+1)}T^{\beta}\\
&+\frac{\theta\Delta(t_k^{\beta}-t^{\beta})L_{f_1}}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p+1)}\eta^{\alpha+\beta+p}\\
&+\frac{\theta\Delta(t_k^{\beta}-t^{\beta})\bar{L}_{f_2}}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p+1)}\eta^{\alpha+\beta+p}+\frac{\theta\Delta\lambda(t_k^{\beta}-t^{\beta})}{\Gamma(\beta+1)\Gamma(\beta+p+1)}\eta^{\beta+p}\\
&+\left(\Delta\frac{(t_k^{\beta}-t^{\beta})}{\Gamma(\beta+1)}\left(\frac{\theta\eta^{p}-\Gamma(p+1)}{\Gamma(p+1)}\right)-\lambda\right)\\
&\times\left(\frac{L_{f_1}}{\Gamma(\alpha+\beta+1)}t_k^{\alpha+\beta}+\frac{\bar{L}_{f_2}}{\Gamma(\alpha+\beta+1)}t_k^{\alpha+\beta}\right)\\
&\left.+\left(\Delta\frac{(t_k^{\beta}-t^{\beta})}{\Gamma(\beta+1)}\left(\frac{\theta\eta^{p}-\Gamma(p+1)}{\Gamma(p+1)}\right)-1\right)\left(\frac{\lambda}{\beta\Gamma\beta}t_k^{\beta}+L_{gk}\right)\right\}\\
&\times(C_1+C_2)\big(\varphi(t)+\psi\big).
\end{aligned}
$$

From above, we have

$$\left|(\Lambda g)(t)-(\Lambda h)(t)\right| \le M(C_1+C_2)\bigl(\varphi(t)+\psi\bigr), \quad t\in[0,\tau],$$

that is,

$$d(\Lambda g,\Lambda h)\le M(C_1+C_2)\bigl(\varphi(t)+\psi\bigr).$$

Hence, we conclude that

$$d(\Lambda g,\Lambda h)\le Md(g,h), \quad \text{for any } g,h\in V.$$

Since condition (4.4) is strictly contractive, continuity property is thus shown. Now we take $g_0 \in V$. From the piecewise continuity property of g_0 and Λg_0, it follows that there exists a constant $0 < G_1 < \infty$ such that

$$\begin{aligned}
&\left|(\Lambda g_0)(t)-g_0(t)\right| \\
&\quad\le \Bigg|\frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1} f(s,g_0(s),{}^cD^{\alpha}_{0,t}\bigl({}^cD^{\beta}_{0,t}+\lambda\bigr)g_0(s)\,ds \\
&\qquad -\frac{\lambda}{\Gamma\beta}\int_o^t (t-s)^{\beta-1}x(s)\,ds \\
&\qquad -\frac{\Delta t^{\beta}}{\Gamma(\beta+1)\Gamma(\alpha+\beta)}\int_0^T (T-s)^{\alpha+\beta-1} f(s,g_0(s),{}^cD^{\alpha}_{0,t}\bigl({}^cD^{\beta}_{0,t}+\lambda\bigr)g_0(s)\,ds \\
&\qquad +\frac{\lambda\Delta t^{\beta}}{\Gamma(\beta)\Gamma(\beta+1)}\int_0^T (T-s)^{\beta-1}x(s)\,ds+\frac{\theta\Delta t^{\beta}}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p)} \\
&\qquad \times\int_0^{\eta} (\eta-s)^{\alpha+\beta+p-1} f(s,g_0(s),{}^cD^{\alpha}_{0,t}\bigl({}^cD^{\beta}_{0,t}+\lambda\bigr)g_0(s)\,ds \\
&\qquad -\frac{\theta\Delta\lambda t^{\beta}}{\Gamma(\beta+1)\Gamma(\beta+p)}\int_0^{\eta} (\eta-s)^{\beta+p-1}x(s)\,ds \\
&\qquad +\left(\frac{\Delta(\theta\eta^p-\Gamma(p+1))t^{\beta}}{\Gamma(p+1)\Gamma(\beta+1)}+1\right)x_0-g_0(t)\Bigg| \\
&\quad\le G_1\varphi(t)\le G_1\bigl(\varphi(t)+\psi\bigr), \quad t\in(0,s_0].
\end{aligned}$$

There exists a constant $0 < G_2 < \infty$ such that

$$\begin{aligned}
&\left|(\Lambda g_0)(t)-g_0(t)\right| = \left|g_k\bigl(t,g_0(t)\bigr)-g_0(t)\right| \le G_2\psi \le G_2\bigl(\varphi(t)+\psi\bigr), \\
&\quad t\in(s_{k-1},t_k], k=1,2,\dots,m.
\end{aligned}$$

Also we can find a constant $0 < G_3 < \infty$ such that

$$\begin{aligned}
&\left|(\Lambda g_0)(t)-g_0(t)\right| \\
&\quad\le \Bigg|\frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1} f(s,g_0(s),{}^cD^{\alpha}_{0,t}\bigl({}^cD^{\beta}_{0,t}+\lambda\bigr)g_0(s)\,ds
\end{aligned}$$

$$
\begin{aligned}
&- \frac{\lambda}{\Gamma\beta} \int_o^t (t-s)^{\beta-1} x(s)\,ds \\
&+ \frac{\Delta(t_k^\beta - t^\beta)}{\Gamma(\beta+1)\Gamma(\alpha+\beta)} \int_0^T (T-s)^{\alpha+\beta-1} f(s, g_0(s), {}^cD_{0,t}^\alpha({}^cD_{0,t}^\beta + \lambda) g_0(s)\,ds \\
&- \frac{\lambda\Delta(t_k^\beta - t^\beta)}{\Gamma(\beta+1)\Gamma(\beta)} \int_0^T (T-s)^{\beta-1} x(s)\,ds - \frac{\theta\Delta(t_k^\beta - t^\beta)}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p)} \\
&\times \int_0^\eta (\eta-s)^{\alpha+\beta+p-1} f(s, g_0(s), {}^cD_{0,t}^\alpha({}^cD_{0,t}^\beta + \lambda) g_0(s)\,ds \\
&+ \frac{\theta\Delta\lambda(t_k^\beta - t^\beta)}{\Gamma(\beta+1)\Gamma(\beta+p)} \int_0^\eta (\eta-s)^{\beta+p-1} x(s)\,ds \\
&+ \left(\Delta \frac{(t_k^\beta - t^\beta)}{\Gamma(\beta+1)} \left(\frac{\theta\eta^p - \Gamma(p+1)}{\Gamma(p+1)} \right) - \lambda \right) \\
&\times \frac{1}{\Gamma(\alpha+\beta)} \int_0^{t_k} (t_k - s)^{\alpha+\beta-1} f(s, g_0(s), {}^cD_{0,t}^\alpha({}^cD_{0,t}^\beta + \lambda) g_0(s)\,ds \\
&- \left(\Delta \frac{(t_k^\beta - t^\beta)}{\Gamma(\beta+1)} \left(\frac{\theta\eta^p - \Gamma(p+1)}{\Gamma(p+1)} \right) - 1 \right) \frac{\lambda}{\Gamma\beta} \int_o^{t_k} (t_k - s)^{\beta-1} x(s)\,ds \\
&- \left(\Delta \frac{(t_k^\beta - t^\beta)}{\Gamma(\beta+1)} \left(\frac{\theta\eta^p - \Gamma(p+1)}{\Gamma(p+1)} \right) - 1 \right) g_k(t_k) - g_0(t) \Bigg|, \quad t \in (t_k, s_k], \\
&\big|(\Lambda g_0)(t) - g_0(t)\big| \le G_3\varphi(t) \le G_3\big(\varphi(t) + \psi\big), \quad t \in (t_k, s_k], k = 1, 2, \ldots, m.
\end{aligned}
$$

Since f, g_k and g_0 are bounded on J and $\varphi(\cdot) > 0$, Eq. (4.5) implies that $d(\Lambda g_0, g_0) < \infty$.

By using Banach fixed point theorem, there exists a continuous function $x : J \to \mathbb{R}$ such that $\Lambda^n g_0 \to x$ in (V, d) as $n \to \infty$ and $\Lambda x = y_0$, that is, x satisfies Eq. (4.2) for every $t \in J$.

Now we show that $\{g \in V$ such that $d(g_0, g) < \infty\} = V$. For any $g \in V$, since g and g_0 are bounded on J and $\min_{t\in J} \varphi(t) > 0$, there exists a constant $0 < C_g < \infty$ such that $|g_0(t) - g(t)| \le C_g(\varphi(t) + \psi)$, for any $t \in J$. Hence, we have $d(g_0, g) < \infty$ for all $g \in V$, that is, $\{g \in V$ such that $d(g_0, g) < \infty\} = V$. Thus, we determine that x is the unique continuous function satisfying Eq. (4.2). Using (3.2) and (H_4), we can write

$$
\begin{aligned}
d(y, \Lambda y) \le{}& \frac{C_\varphi}{\Gamma(\alpha+\beta)} \left(\frac{1-r}{\alpha+\beta-r} \right)^{1-r} t^{\alpha+\beta-r} + \frac{\lambda C_\varphi}{\Gamma\beta} \left(\frac{1-r}{\beta-r} \right)^{1-r} t^{\beta-r} \\
&+ \frac{\Delta C_\varphi(t_k^\beta - t^\beta)}{\Gamma(\beta+1)\Gamma(\alpha+\beta)} \left(\frac{1-r}{\alpha+\beta-r} \right)^{1-r} T^{\alpha+\beta-r} \\
&+ \frac{\lambda\Delta C_\varphi(t_k^\beta - t^\beta)}{\Gamma(\beta+1)\Gamma(\beta)} \left(\frac{1-r}{\beta-r} \right)^{1-r} T^{\beta-r} \\
&+ \frac{\theta\Delta C_\varphi(t_k^\beta - t^\beta)}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p)} \left(\frac{1-r}{\alpha+\beta+p-r} \right)^{1-r} \eta^{\alpha+\beta+p-r} \\
&+ \frac{C_\varphi\theta\Delta\lambda(t_k^\beta - t^\beta)}{\Gamma(\beta+1)\Gamma(\beta+p)} \left(\frac{1-r}{\beta+p-r} \right)^{1-r} \eta^{\beta+p-r} \\
&+ \left(\Delta \frac{(t_k^\beta - t^\beta)}{\Gamma(\beta+1)} \left(\frac{\theta\eta^p - \Gamma(p+1)}{\Gamma(p+1)} \right) - \lambda \right) \frac{C_\varphi}{\Gamma(\alpha+\beta)} \left(\frac{1-r}{\alpha+\beta-r} \right)^{1-r} t_k^{\alpha+\beta-r} \\
&+ \left(\Delta \frac{(t_k^\beta - t^\beta)}{\Gamma(\beta+1)} \left(\frac{\theta\eta^p + \Gamma(p+1)}{\Gamma(p+1)} \right) - 1 \right) \frac{\lambda C_\varphi}{\Gamma\beta} \left(\frac{1-r}{\beta-r} \right)^{1-r} t_k^{\beta-r} + 1.
\end{aligned}
$$

Summarizing, we have

$$\begin{aligned} d(y,x) &\leq \frac{d(\Lambda y,y)}{1-M} \\ &\leq \left(\frac{1}{1-M}\right)\left\{\frac{C_\varphi}{\Gamma(\alpha+\beta)}\left(\frac{1-r}{\alpha+\beta-r}\right)^{1-r} t^{\alpha+\beta-r} + \frac{\lambda C_\varphi}{\Gamma\beta}\left(\frac{1-r}{\beta-r}\right)^{1-r} t^{\beta-r}\right. \\ &\quad + \frac{\Delta C_\varphi(t_k^\beta - t^\beta)}{\Gamma(\beta+1)\Gamma(\alpha+\beta)}\left(\frac{1-r}{\alpha+\beta-r}\right)^{1-r} T^{\alpha+\beta-r} \\ &\quad + \frac{\lambda\Delta C_\varphi(t_k^\beta - t^\beta)}{\Gamma(\beta+1)\Gamma(\beta)}\left(\frac{1-r}{\beta-r}\right)^{1-r} T^{\beta-r} \\ &\quad + \frac{\theta\Delta C_\varphi(t_k^\beta - t^\beta)}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p)}\left(\frac{1-r}{\alpha+\beta+p-r}\right)^{1-r} \eta^{\alpha+\beta+p-r} \\ &\quad + \frac{C_\varphi\theta\Delta\lambda(t_k^\beta - t^\beta)}{\Gamma(\beta+1)\Gamma(\beta+p)}\left(\frac{1-r}{\beta+p-r}\right)^{1-r} \eta^{\beta+p-r} \\ &\quad + \left(\Delta\frac{(t_k^\beta - t^\beta)}{\Gamma(\beta+1)}\left(\frac{\theta\eta^p - \Gamma(p+1)}{\Gamma(p+1)}\right) - \lambda\right)\frac{C_\varphi}{\Gamma(\alpha+\beta)}\left(\frac{1-r}{\alpha+\beta-r}\right)^{1-r} t_k^{\alpha+\beta-r} \\ &\quad \left. + \left(\Delta\frac{(t_k^\beta - t^\beta)}{\Gamma(\beta+1)}\left(\frac{\theta\eta^p + \Gamma(p+1)}{\Gamma(p+1)}\right) - 1\right)\frac{\lambda C_\varphi}{\Gamma\beta}\left(\frac{1-r}{\beta-r}\right)^{1-r} t_k^{\beta-r} + 1\right\}. \end{aligned}$$

This shows that (4.3) is true for $t \in J$. □

Here, we give an example to illustrate our main result.

Example 4.3

$$\begin{cases} {}^cD_{0,t}^{\frac{1}{2}}({}^cD_{0,t}^{\frac{1}{2}} + \frac{3}{20})x(t) \\ \quad = \frac{|x(t)| + {}^cD_{0,t}^{\frac{1}{2}}({}^cD_{0,t}^{\frac{1}{2}} + \frac{3}{20})x(t)}{8+e^t+t^2+{}^cD_{0,t}^{\frac{1}{2}}({}^cD_{0,t}^{\frac{1}{2}} + \frac{3}{20})x(t)} \\ \qquad + \int_0^t \frac{(t-s)^{\frac{3}{2}}}{\Gamma\frac{5}{2}}(\frac{|x(s)|}{8+e^s+s^2})\,ds, \quad t \in (0,1] \cup (2,3], \\ x(t) = \frac{x(t)}{(3+t^2)(1+|x(t)|)}, \quad t \in (1,2], \\ x(0) = \frac{\sqrt{2}}{3}, \qquad x(1) = \frac{5}{6}\int_0^{\frac{1}{4}} \frac{(\frac{1}{4}-s)}{\Gamma\frac{4}{3}}\,ds \quad 0 < \eta < 1 \end{cases} \tag{4.8}$$

and

$$\begin{cases} |{}^cD_{0,t}^{\frac{1}{2}}({}^cD_{0,t}^{\frac{1}{2}} + \frac{3}{20})y(t) - \frac{|x(t)| + {}^cD_{0,t}^{\frac{1}{2}}({}^cD_{0,t}^{\frac{1}{2}} + \frac{3}{20})y(t)}{8+e^t+t^2+{}^cD_{0,t}^{\frac{1}{2}}({}^cD_{0,t}^{\frac{1}{2}} + \frac{3}{20})y(t)} \\ \quad - \int_0^t \frac{(t-s)^{\frac{3}{2}}}{\Gamma\frac{5}{2}}(\frac{|y(s)|}{8+e^s+s^2})\,ds| \leq e^t, \quad t \in (0,1] \cup (2,3], \\ |y(t) - \frac{y(t)}{(3+t^2)(1+|x(t)|)}| \leq 1, \quad t \in (1,2]. \end{cases}$$

Let $J = [0,3]$, $\alpha = \beta = \frac{1}{2}$, $r = \frac{1}{3}$, $\Delta = -2.70$, $\theta = \frac{5}{6}$, $p = \frac{4}{3}$, $\eta = \frac{1}{4}$ and $0 = t_0 < s_0 = 1 < t_1 = 2 < s_1 = \tau = 3$. Denote $f(t,x(t)) = \frac{|x(t)|}{8+e^t+t^2}$ with $L_f = \frac{1}{9}$ for $t \in (0,1] \cup (2,3]$ and $g_1(t,x(t)) = \frac{x(t)}{(3+t^2)(1+|x(t)|)}$ with $L_{g_k} = \frac{1}{4}$ for $t \in (1,2]$. Putting $\psi = 1$, $L_{f_1} = \bar{L}_{f_2} = \frac{1}{4}$ $\varphi(t) = e^t$ and $c_\varphi = 1$, we have $(\int_0^t (e^s)^3\,ds)^{\frac{1}{3}} \leq e^t$ and let $M_1 \approx -0.5900$, $M_2 \approx 0.9713$, so $M = 0.9713 < 1$.

By Theorem 4.2, there exists a unique solution $x:[0,3]\to\mathbb{R}$ such that

$$x(t)=\begin{cases}\int_0^t(\frac{|x(t)|+{}^cD_{0,t}^{\frac{1}{2}}({}^cD_{0,t}^{\frac{1}{2}}+\frac{3}{20})x(t)}{8+e^t+t^2+{}^cD_{0,t}^{\frac{1}{2}}({}^cD_{0,t}^{\frac{1}{2}}+\frac{3}{20})x(t)})\,ds-0.0846\int_o^t(t-s)^{\frac{-1}{2}}x(s)\,ds\\ \quad+0.0650t^{\frac{1}{2}}\int_0^1(\frac{|x(t)|+{}^cD_{0,t}^{\frac{1}{2}}({}^cD_{0,t}^{\frac{1}{2}}+\frac{3}{20})x(t)}{8+e^t+t^2+{}^cD_{0,t}^{\frac{1}{2}}({}^cD_{0,t}^{\frac{1}{2}}+\frac{3}{20})x(t)})\,ds\\ \quad-0.0901t^{\frac{1}{2}}\int_0^1(1-s)^{\frac{-1}{2}}x(s)\,ds\\ \quad-0.7454\sqrt{t}\int_0^{\frac{1}{4}}(\frac{1}{4}-s)^{\frac{4}{3}}(\frac{|x(t)|+{}^cD_{0,t}^{\frac{1}{2}}({}^cD_{0,t}^{\frac{1}{2}}+\frac{3}{20})x(t)}{8+e^t+t^2+{}^cD_{0,t}^{\frac{1}{2}}({}^cD_{0,t}^{\frac{1}{2}}+\frac{3}{20})x(t)})\,ds\\ \quad+0.1415\sqrt{t}\int_0^{\frac{1}{4}}(\frac{1}{4}-s)^{\frac{5}{6}}x(s)\,ds+(0.9476\sqrt{t}+1)x_0,\quad t\in[0,1],\\ \int_0^t(\frac{|x(t)|+{}^cD_{0,t}^{\frac{1}{2}}({}^cD_{0,t}^{\frac{1}{2}}+\frac{3}{20})x(t)}{8+e^t+t^2+{}^cD_{0,t}^{\frac{1}{2}}({}^cD_{0,t}^{\frac{1}{2}}+\frac{3}{20})x(t)})\,ds\\ \quad-0.0846\int_o^t(t-s)^{\frac{-1}{2}}x(s)\,ds\\ \quad-1.0650(\sqrt{2}-\sqrt{t})\int_0^1(\frac{|x(t)|+{}^cD_{0,t}^{\frac{1}{2}}({}^cD_{0,t}^{\frac{1}{2}}+\frac{3}{20})x(t)}{8+e^t+t^2+{}^cD_{0,t}^{\frac{1}{2}}({}^cD_{0,t}^{\frac{1}{2}}+\frac{3}{20})x(t)})\,ds\\ \quad+0.0901(\sqrt{2}-\sqrt{t})\int_0^1(1-s)^{\frac{-1}{2}}x(s)\,ds\\ \quad+0.7454(\sqrt{2}-\sqrt{t})\int_0^{\frac{1}{4}}(\frac{1}{4}-s)^{\frac{4}{3}}(\frac{|x(t)|+{}^cD_{0,t}^{\frac{1}{2}}({}^cD_{0,t}^{\frac{1}{2}}+\frac{3}{20})x(t)}{8+e^t+t^2+{}^cD_{0,t}^{\frac{1}{2}}({}^cD_{0,t}^{\frac{1}{2}}+\frac{3}{20})x(t)})\,ds\\ \quad-0.1415(\sqrt{2}-\sqrt{t})\int_0^{\frac{1}{4}}(\frac{1}{4}-s)^{\frac{5}{6}}x(s))\,ds\\ \quad+(0.9476(\sqrt{2}-\sqrt{t})-\frac{3}{20})\int_0^2(\frac{|x(t)|+{}^cD_{0,t}^{\frac{1}{2}}({}^cD_{0,t}^{\frac{1}{2}}+\frac{3}{20})x(t)}{8+e^t+t^2+{}^cD_{0,t}^{\frac{1}{2}}({}^cD_{0,t}^{\frac{1}{2}}+\frac{3}{20})x(t)})\,ds\\ \quad-(0.9476(\sqrt{2}-\sqrt{t})-1)0.846\int_o^2(2-s)^{\frac{-1}{2}}x(s)\,ds\\ \quad-(0.9476(\sqrt{2}-\sqrt{t})-1)\frac{x(t)}{(3+t^2)(1+|x(t)|)},\quad t\in(2,3]\\ \frac{x(t)}{(3+t^2)(1+|x(t)|)},\quad t\in(1,2].\end{cases}$$

Then

$$\begin{aligned}&\bigl|y(t)-x(t)\bigr|\\ &\quad\le\Bigg\{\frac{C_\varphi}{\Gamma(\alpha+\beta)}\left(\frac{1-r}{\alpha+\beta-r}\right)^{1-r}t^{\alpha+\beta-r}+\frac{\lambda C_\varphi}{\Gamma\beta}\left(\frac{1-r}{\beta-r}\right)^{1-r}t^{\beta-r}\\ &\qquad+\frac{\Delta C_\varphi(t_k^\beta-t^\beta)}{\Gamma(\beta+1)\Gamma(\alpha+\beta)}\left(\frac{1-r}{\alpha+\beta-r}\right)^{1-r}T^{\alpha+\beta-r}+\frac{\lambda\Delta C_\varphi(t_k^\beta-t^\beta)}{\Gamma(\beta+1)\Gamma(\beta)}\left(\frac{1-r}{\beta-r}\right)^{1-r}T^{\beta-r}\\ &\qquad+\frac{\theta\Delta C_\varphi(t_k^\beta-t^\beta)}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p)}\left(\frac{1-r}{\alpha+\beta+p-r}\right)^{1-r}\eta^{\alpha+\beta+p-r}\\ &\qquad+\frac{C_\varphi\theta\Delta\lambda(t_k^\beta-t^\beta)}{\Gamma(\beta+1)\Gamma(\beta+p)}\left(\frac{1-r}{\beta+p-r}\right)^{1-r}\eta^{\beta+p-r}\\ &\qquad+\left(\Delta\frac{(t_k^\beta-t^\beta)}{\Gamma(\beta+1)}\left(\frac{\theta\eta^p-\Gamma(p+1)}{\Gamma(p+1)}\right)-\lambda\right)\frac{C_\varphi}{\Gamma(\alpha+\beta)}\left(\frac{1-r}{\alpha+\beta-r}\right)^{1-r}t_k^{\alpha+\beta-r}\\ &\qquad+\left(\Delta\frac{(t_k^\beta-t^\beta)}{\Gamma(\beta+1)}\left(\frac{\theta\eta^p-\Gamma(p+1)}{\Gamma(p+1)}\right)-1\right)\frac{\lambda C_\varphi}{\Gamma\beta}\left(\frac{1-r}{\beta-r}\right)^{1-r}t_k^{\beta-r}+1\Bigg\}\left(\frac{\varphi(t)+\psi}{1-M}\right),\end{aligned}$$

which can further be reduced to

$$\begin{aligned}
&\left|y(t)-x(t)\right| \\
&\leq \Bigg\{ \frac{C_\varphi}{\Gamma(\alpha+\beta)}\left(\frac{1-r}{\alpha+\beta-r}\right)^{1-r} t^{\alpha+\beta-r} + \frac{\lambda C_\varphi}{\Gamma\beta}\left(\frac{1-r}{\beta-r}\right)^{1-r} t^{\beta-r} \\
&\quad - \frac{\Delta C_\varphi t^\beta}{\Gamma(\beta+1)\Gamma(\alpha+\beta)}\left(\frac{1-r}{\alpha+\beta-r}\right)^{1-r} T^{\alpha+\beta-r} - \frac{\lambda\Delta C_\varphi t^\beta}{\Gamma(\beta+1)\Gamma(\beta)}\left(\frac{1-r}{\beta-r}\right)^{1-r} T^{\beta-r} \\
&\quad - \frac{\theta\Delta C_\varphi t^\beta}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p)}\left(\frac{1-r}{\alpha+\beta+p-r}\right)^{1-r} \eta^{\alpha+\beta+p-r} \\
&\quad - \frac{C_\varphi\theta\Delta\lambda t^\beta}{\Gamma(\beta+1)\Gamma(\beta+p)}\left(\frac{1-r}{\beta+p-r}\right)^{1-r} \eta^{\beta+p-r} \\
&\quad - \left(\frac{\Delta t^\beta}{\Gamma(\beta+1)}\left(\frac{\theta\eta^p-\Gamma(p+1)}{\Gamma(p+1)}\right)-\lambda\right)\frac{C_\varphi}{\Gamma(\alpha+\beta)}\left(\frac{1-r}{\alpha+\beta-r}\right)^{1-r} t_k^{\alpha+\beta-r} \\
&\quad - \left(\frac{\Delta t^\beta}{\Gamma(\beta+1)}\left(\frac{\theta\eta^p-\Gamma(p+1)}{\Gamma(p+1)}\right)-1\right)\frac{\lambda C_\varphi}{\Gamma\beta}\left(\frac{1-r}{\beta-r}\right)^{1-r} t_k^{\beta-r} + 1\Bigg\}\left(\frac{\varphi(t)+\psi}{1-M}\right).
\end{aligned}$$

This implies

$$\begin{aligned}
&\left|y(t)-x(t)\right| \\
&\leq \Bigg\{ \frac{C_\varphi}{\Gamma(\alpha+\beta)}\left(\frac{1-r}{\alpha+\beta-r}\right)^{1-r} \tau^{\alpha+\beta-r} + \frac{\lambda C_\varphi}{\Gamma\beta}\left(\frac{1-r}{\beta-r}\right)^{1-r} \tau^{\beta-r} \\
&\quad - \frac{\Delta C_\varphi \tau^\beta}{\Gamma(\beta+1)\Gamma(\alpha+\beta)}\left(\frac{1-r}{\alpha+\beta-r}\right)^{1-r} T^{\alpha+\beta-r} - \frac{\lambda\Delta C_\varphi \tau^\beta}{\Gamma(\beta+1)\Gamma(\beta)}\left(\frac{1-r}{\beta-r}\right)^{1-r} T^{\beta-r} \\
&\quad - \frac{\theta\Delta C_\varphi \tau^\beta}{\Gamma(\beta+1)\Gamma(\alpha+\beta+p)}\left(\frac{1-r}{\alpha+\beta+p-r}\right)^{1-r} \eta^{\alpha+\beta+p-r} \\
&\quad - \frac{C_\varphi\theta\Delta\lambda \tau^\beta}{\Gamma(\beta+1)\Gamma(\beta+p)}\left(\frac{1-r}{\beta+p-r}\right)^{1-r} \eta^{\beta+p-r} \\
&\quad - \left(\frac{\Delta \tau^\beta}{\Gamma(\beta+1)}\left(\frac{\theta\eta^p-\Gamma(p+1)}{\Gamma(p+1)}\right)-\lambda\right)\frac{C_\varphi}{\Gamma(\alpha+\beta)}\left(\frac{1-r}{\alpha+\beta-r}\right)^{1-r} \tau^{\alpha+\beta-r} \\
&\quad - \left(\frac{\Delta \tau^\beta}{\Gamma(\beta+1)}\left(\frac{\theta\eta^p-\Gamma(p+1)}{\Gamma(p+1)}\right)-1\right)\frac{\lambda C_\varphi}{\Gamma\beta}\left(\frac{1-r}{\beta-r}\right)^{1-r} \tau^{\beta-r} + 1\Bigg\}\left(\frac{\varphi(t)+\psi}{1-M}\right).
\end{aligned}$$

Plugging-in the values, we have

$$\begin{aligned}
&\left|y(t)-x(t)\right| \\
&\leq \Bigg\{ \frac{1}{\Gamma(\frac{1}{2}+\frac{1}{2})}\left(\frac{1-\frac{1}{3}}{\frac{1}{2}+\frac{1}{2}-\frac{1}{3}}\right)^{(1-\frac{1}{3})} 3^{(\frac{1}{2}+\frac{1}{2}-\frac{1}{3})} + \frac{(0.15)}{\Gamma\frac{1}{2}}\left(\frac{1-\frac{1}{3}}{\frac{1}{2}-\frac{1}{3}}\right)^{(1-\frac{1}{3})} 3^{(\frac{1}{2}-\frac{1}{3})} \\
&\quad - \frac{(-2.7)3^{\frac{1}{2}}}{\Gamma(\frac{1}{2}+1)\Gamma(\frac{1}{2}+\frac{1}{2})}\left(\frac{1-\frac{1}{3}}{\frac{1}{2}+\frac{1}{2}-\frac{1}{3}}\right)^{(1-\frac{1}{3})} 3^{(\frac{1}{2}+\frac{1}{2}-\frac{1}{3})} \\
&\quad - \frac{(0.15)(-2.7)3^{\frac{1}{2}}}{\Gamma(\frac{1}{2}+1)\Gamma(\frac{1}{2})}\left(\frac{1-\frac{1}{3}}{\frac{1}{2}-\frac{1}{3}}\right)^{(1-\frac{1}{3})} 3^{(\frac{1}{2}-\frac{1}{3})}
\end{aligned}$$

$$
\begin{aligned}
&- \frac{(0.833)(-2.7)3^{\frac{1}{2}}}{\Gamma(\frac{1}{2}+1)\Gamma(\frac{1}{2}+\frac{1}{2}+\frac{4}{3})}\left(\frac{1-\frac{1}{3}}{\frac{1}{2}+\frac{1}{2}+\frac{4}{3}-\frac{1}{3}}\right)^{1-\frac{1}{3}}(0.25)^{\frac{1}{2}+\frac{1}{2}+\frac{4}{3}-\frac{1}{3}} \\
&- \frac{(0.833)(-2.7)(0.15)3^{\frac{1}{2}}}{\Gamma(\frac{1}{2}+1)\Gamma(\frac{1}{2}+\frac{4}{3})}\left(\frac{1-\frac{1}{3}}{\frac{1}{2}+\frac{4}{3}-\frac{1}{3}}\right)^{1-\frac{1}{3}}(0.25)^{\frac{1}{2}+\frac{4}{3}-\frac{1}{3}} \\
&- \left(\frac{(-2.7)3^{\frac{1}{2}}}{\Gamma(\frac{1}{2}+1)}\left(\frac{(0.833)(0.25)^{\frac{4}{3}}-\Gamma(\frac{4}{3}+1)}{\Gamma(\frac{4}{3}+1)}\right)-(0.15)\right) \\
&\times \frac{1}{\Gamma(\frac{1}{2}+\frac{1}{2})}\left(\frac{1-\frac{1}{3}}{\frac{1}{2}+\frac{1}{2}-\frac{1}{3}}\right)^{(1-\frac{1}{3})}3^{(\frac{1}{2}+\frac{1}{2}-\frac{1}{3})} \\
&- \left(\frac{(-2.7)3^{\frac{1}{2}}}{\Gamma(\frac{1}{2}+1)}\left(\frac{(0.833)(0.25)^{\frac{4}{3}}-\Gamma(\frac{4}{3}+1)}{\Gamma(\frac{4}{3}+1)}\right)-1\right)\frac{(0.15)}{\Gamma\frac{1}{2}}\left(\frac{1-\frac{1}{3}}{\frac{1}{2}-\frac{1}{3}}\right)^{1-\frac{1}{3}}3^{(\frac{1}{2}-\frac{1}{3})}+1\Bigg\} \\
&\times \left(\frac{e^t+1}{1-0.9714}\right) \\
&\leq 5.4846\left(\frac{e^t+1}{0.0286}\right) \\
&\leq 191.769\left(e^t+1\right),
\end{aligned}
$$

thus problem (4.8) is Ulam–Hyers–Rassias stable.

5 Conclusions

In this article, we considered a nonlocal boundary value problem of nonlinear implicit fractional Langevin equation with noninstantaneous impulses. After introduction, we built a uniform structure for the solutions of our proposed model. We studied the concept of generalized Ulam–Hyers–Rassias stability to our proposed model. And, finally, we presented a particular example for the applicability of our main result.

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Competing interests

The authors declare that they have no competing interest regarding this work.

Authors' contributions

All the authors contributed equally and significantly in writing this paper. All the authors read and approved the final manuscript.

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