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Analytical research of (3 + 1)-dimensional Rossby waves with dissipation effect in cylindrical coordinate based on Lie symmetry approach

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Abstract

Rossby waves, one of significant waves in the solitary wave, have important theoretical meaning in the atmosphere and ocean. However, the previous studies on Rossby waves commonly were carried out in the zonal area and could not be applied directly to the spherical earth. In order to overcome the problem, the research on (3 + 1)-dimensional Rossby waves in the paper is placed into the spherical area, and some new analytical solutions of (3 + 1)-dimensional Rossby waves are given through the classic Lie group method. Finally, the dissipation effect is analyzed in the sense of the above mentioned new analytical solutions. The new solutions on (3 + 1)-dimensional Rossby waves have important value for understanding the propagation of Rossby waves in the rotating earth with the influence of dissipation.

Keywords: Rossby waves; Lie group; Cylindrical coordinate; Dissipation effect

1 Introduction

As is well known, Rossby waves play a central role in the atmosphere and ocean, which depicts an essential phenomenon. The oceans's response to the atmosphere change and climate change can be determined by Rossby waves. In addition, Rossby waves have significant theoretical meaning and real value in the atmosphere and ocean. In recent years, more and more researchers have focused on the study of Rossby waves [1–4]. Many studies on Rossby waves have been conducted in the zonal area, and many meaningful results have been achieved [5, 6]. However, as everyone knows, the propagation of Rossby waves happens in the earth which is a spherical area [7], so the above mentioned achievements could not be directly applied. It is necessary to discuss the propagation characteristic of (3 + 1)-dimensional Rossby waves in the spherical area under the influence of dissipation. Here, in order to overcome the problem, our research is carried out in cylindrical coordinate, which better matches with the real condition.

With the development of soliton theory, Rossby waves have been becoming an important research direction in the field of the nonlinear partial differential [8–10]. In recent years, some weakly nonlinear models for the evolution of Rossby waves have been extensively studied [11–13]. More importantly, some methods are found to study the nonlinear models [14–18] and some significant properties are discussed [19–23]. In the past, Rossby waves were often studied in the zonal area. However, Rossby waves are prominently affected via the rotation effect of the earth. Therefore, in order to study some propagation characteristics of Rossby waves, we use the (3 + 1)-dimensional quasi-geostrophic vorticity equation with dissipation effect in the cylindrical coordinate to describe the dynamic behavior of Rossby waves.

In this paper, the (3 + 1)-dimensional Rossby waves with dissipation effect in cylindrical coordinate will be discussed through the classic Lie group method. In Sect. 2, we analyze the (3 + 1)-dimensional Rossby waves with dissipation effect in cylindrical coordinate by using the classic Lie group method. In Sect. 3, the solution of the (3 + 1)-dimensional Rossby waves with dissipation effect in cylindrical coordinate can be obtained. In addition, some conclusions are placed in Sect. 4.

2 Symmetry analysis for the (3 + 1)-dimensional Rossby waves with dissipation effect in cylindrical coordinate

In Ref. [24], the (2 + 1)-dimensional model for Rossby waves is researched based on the classical Lie group approach. Then, the (3 + 1)-dimensional model for Rossby waves is analyzed by Myagkov [25]. In Ref. [26], the authors adopt the plane polar coordinate to study the (2 + 1)-dimensional model

$$\left[\frac{\partial}{\partial t}+\left(\frac{1}{r}\frac{\partial \Psi}{\partial r}\frac{\partial}{\partial \theta}-\frac{1}{r}\frac{\partial \Psi}{\partial \theta}\frac{\partial}{\partial r}\right)\right]\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Psi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 \Psi}{\partial \theta^2}\right]+\frac{\beta}{r}\frac{\partial \Psi}{\partial \theta}=0,$$

and discuss the dynamic characteristics of the model in rotating barotropic atmosphere. But the dissipation effect is ignored.

In the paper, we analyze the (3 + 1)-dimensional quasi-geostrophic vorticity equation with dissipation effect in cylindrical coordinate

$$\begin{aligned}&\left[\frac{\partial}{\partial t}+\frac{1}{\rho}\frac{\partial \phi}{\partial \rho}\frac{\partial}{\partial \theta}-\frac{1}{\rho}\frac{\partial \phi}{\partial \theta}\frac{\partial}{\partial \rho}\right]\left[\frac{\partial^2 \phi}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial \phi}{\partial \rho}+\frac{1}{\rho^2}\frac{\partial^2 \phi}{\partial \theta^2}+\frac{\partial^2 \phi}{\partial z^2}\right]\\&\quad+\beta\left[\cos\theta\frac{\partial \phi}{\partial \rho}-\frac{\sin\theta}{\rho}\frac{\partial \phi}{\partial \theta}\right]=-\alpha\left[\frac{\partial^2 \phi}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial \phi}{\partial \rho}+\frac{1}{\rho^2}\frac{\partial^2 \phi}{\partial \theta^2}+\frac{\partial^2 \phi}{\partial z^2}\right],\end{aligned} \tag{1}$$

where ϕ describes the dimensionless stream function, $\beta = \beta_0(L^2/U)$, and $\beta_0 = (\omega_0/R_0)\cos\varphi_0$, in which ω_0 and R_0 are the angular frequency of the Earth's rotation and the Earth's radius, respectively, L is the characteristic horizontal length, φ_0 depicts the latitude, U is velocity scales and α depicts the dissipation coefficient.

In addition, we introduce the vector field

$$\begin{aligned}V={}&\xi(\rho,\theta,z,t,\phi)\frac{\partial}{\partial \rho}+\eta(\rho,\theta,z,t,\phi)\frac{\partial}{\partial \theta}+\lambda(\rho,\theta,z,t,\phi)\frac{\partial}{\partial z}+\tau(\rho,\theta,z,t,\phi)\frac{\partial}{\partial t}\\&+\psi(\rho,\theta,z,t,\phi)\frac{\partial}{\partial \phi}.\end{aligned}$$

The first order prolongation operator and the second order prolongation operator can be given as follows:

$$\begin{aligned}\mathrm{Pr}^{(1)}V &= V + \psi^{\rho}\frac{\partial}{\partial\phi_{\rho}} + \psi^{\theta}\frac{\partial}{\partial\phi_{\theta}} + \psi^{z}\frac{\partial}{\partial\phi_{z}} + \psi^{t}\frac{\partial}{\partial\phi_{t}},\\ \mathrm{Pr}^{(2)}V &= \mathrm{Pr}^{(1)}V + \psi^{\rho\rho}\frac{\partial}{\partial\phi_{\rho\rho}} + \psi^{\rho\theta}\frac{\partial}{\partial\phi_{\rho\theta}} + \psi^{\theta\theta}\frac{\partial}{\partial\phi_{\rho\rho}} + \psi^{\rho z}\frac{\partial}{\partial\phi_{\rho z}} + \psi^{zz}\frac{\partial}{\partial\phi_{zz}}\\ &\quad + \psi^{\theta z}\frac{\partial}{\partial\phi_{\theta z}} + \psi^{\rho t}\frac{\partial}{\partial\phi_{\rho t}} + \psi^{\theta t}\frac{\partial}{\partial\phi_{\theta t}} + \psi^{zt}\frac{\partial}{\partial\phi_{zt}} + \psi^{tt}\frac{\partial}{\partial\phi_{tt}}.\end{aligned}$$

Similarly, the third order prolongation operator can be defined as

$$\begin{aligned}\mathrm{Pr}^{(3)}V &= \mathrm{Pr}^{(2)}V + \psi^{\rho\rho\rho}\frac{\partial}{\partial\phi_{\rho\rho\rho}} + \psi^{\rho\rho\theta}\frac{\partial}{\partial\phi_{\rho\rho\theta}} + \psi^{\rho\theta\theta}\frac{\partial}{\partial\phi_{\rho\theta\theta}} + \psi^{\theta\theta\theta}\frac{\partial}{\partial\phi_{\theta\theta\theta}}\\ &\quad + \psi^{\rho\rho z}\frac{\partial}{\partial\phi_{\rho\rho z}} + \psi^{\rho zz}\frac{\partial}{\partial\phi_{\rho zz}} + \psi^{zzz}\frac{\partial}{\partial\phi_{zzz}} + \cdots + \psi^{\theta\theta z}\frac{\partial}{\partial\phi_{\theta\theta z}} + \cdots.\end{aligned} \tag{2}$$

Substituting Eq. (2) into Eq. (1), it is easy to conclude that

$$\begin{cases}\psi^{\rho\rho\rho} + \frac{1}{\rho}\psi^{\rho t} + \frac{1}{\rho^2}\psi^{\theta\theta t} + \psi^{zzt} - \frac{1}{\rho^2}\xi\phi_{pt} - \frac{2}{\rho^3}\xi\phi_{\theta\theta t} - \frac{1}{\rho}\psi^{\rho\rho\rho}\phi_{\theta} + \frac{1}{\rho}\psi^{\rho\rho\theta}\phi_{\rho}\\ \quad - \frac{1}{\rho}\psi^{\theta}\phi_{\rho} + \frac{1}{\rho}\psi^{\rho}\phi_{\rho\rho\theta} + \frac{1}{\rho^2}\xi\phi_{\theta}\phi_{\rho\rho\rho} - \frac{1}{\rho^2}\xi\phi_{\rho}\phi_{\rho\rho\theta} + \frac{1}{\rho^3}\psi^{\rho}\phi_{\theta} + \frac{1}{\rho^3}\psi^{\theta}\phi_{\rho}\\ \quad - \frac{3}{\rho^4}\xi\phi_{\theta}\phi_{\rho} - \frac{1}{\rho^2}\psi^{\rho\rho}\phi_{\theta} - \frac{1}{\rho^2}\psi^{\theta}\phi_{\rho\rho} - \frac{2}{\rho^3}\xi\phi_{\theta}\phi_{\rho\rho} + \frac{1}{\rho^2}\psi^{\rho\theta}\phi_{\rho} + \frac{1}{\rho^2}\psi^{\rho}\phi_{\rho\theta}\\ \quad - \frac{2}{\rho^3}\xi\phi_{\rho}\phi_{\rho\theta} + \frac{2}{\rho^4}\psi^{\rho}\phi_{\theta} + \frac{2}{\rho^4}\psi^{\theta}\phi_{\rho} - \frac{8}{\rho^5}\xi\phi_{\theta}\phi_{\rho} - \frac{1}{\rho^3}\psi^{\rho\theta\theta}\phi_{\theta} - \frac{1}{\rho^3}\psi^{\theta}\phi_{\rho\theta\theta}\\ \quad + \frac{3}{\rho^4}\xi\phi_{\theta}\phi_{\rho\theta\theta} + \frac{1}{\rho^3}\psi_{\theta\theta\theta}\phi_{\rho} + \frac{1}{\rho^3}\psi^{\rho}\phi_{\theta\theta\theta} - \frac{3}{\rho^4}\xi\phi_{\rho}\phi_{\theta\theta\theta} - \frac{1}{\rho}\psi_{zz\rho}\phi^{\theta}\\ \quad - \frac{1}{\rho}\psi^{\theta}\phi_{\rho zz} + \frac{1}{\rho^2}\xi\phi_{\theta}\phi_{\rho zz} + \frac{1}{\rho}\psi^{\theta zz}\phi_{\rho} + \frac{1}{\rho}\psi^{\rho}\phi_{\theta zz} - \frac{1}{\rho^2}\xi\phi_{\rho}\phi_{\theta zz}\\ \quad + \alpha\psi^{\rho\rho} - \frac{\alpha}{\rho^2}\xi\phi_{\rho} + \frac{\alpha}{\rho}\psi^{\rho} - \frac{2\alpha}{\rho^3}\xi\phi_{\theta\theta} + \frac{\alpha}{\rho^2}\psi^{\theta\theta} + \alpha\psi^{zz} + (p\cos\theta)\xi\psi^{\rho}\\ \quad - (\rho\sin\theta)\eta\phi_{\rho} - \frac{\beta\sin\theta}{\rho}\psi^{\theta} + \frac{\beta\sin\theta}{\rho^2}\xi\phi_{\theta} - \frac{\beta\cos\theta}{\rho^2}\eta\phi_{\theta} + \beta\psi^{z} = 0,\end{cases}$$

based on the Lie group method.

By comparing the coefficient of ϕ_{ρ}, ϕ_{θ}, ϕ_{z}, ϕ_{t}, $\phi_{\rho\theta}$, $\phi_{\rho z}$, $\phi_{\rho t}$, $\phi_{\rho\theta z}$, we get

$$\begin{cases}\xi = C_1 + C_5\rho + C_7(\int g_2(t)\,dt)\cos\theta + C_{10}(\int g_5(t)\,dt)\sin\theta,\\ \eta = C_2 - C_7\frac{(\int g_2(t)\,dt)\sin\theta}{\rho} - C_{10}\frac{(\int g_5(t)\,dt)\cos\theta}{\rho},\\ \lambda = C_3 + C_5 z,\\ \tau = C_4 - C_5 t,\\ \varphi = 3C_5\phi + C_6 g_1(t) - C_7 g_2(t)\rho\sin\theta + C_8 g_3(z) + C_9 g_4(t)z\\ \qquad + C_{10}(g_5(t)\rho\cos\theta - \frac{\beta}{2}g_5(t)z^2 + \frac{\alpha}{2}e^{-t}z^2),\end{cases} \tag{3}$$

where $g_1(t), g_2(t), \ldots, g_5(t)$ are arbitrary functions of t and $C_1, C_2, \ldots, C_{10}$ are arbitrary constants.

Therefore, the Lie algebra of Eq. (1) can be given by

$$\begin{cases} V_1 = \frac{\partial}{\partial\rho}, \quad V_2 = \frac{\partial}{\partial\theta}, \quad V_3 = \frac{\partial}{\partial z}, \quad V_4 = \frac{\partial}{\partial t}, \\ V_5 = \rho\frac{\partial}{\partial\rho} + z\frac{\partial}{\partial z} - t\frac{\partial}{\partial t} + 3\phi\frac{\partial}{\partial\phi}, \quad V_6 = g_1(t)\frac{\partial}{\partial\phi}, \\ V_7 = (\int g_2(t)\,dt)\cos\theta\frac{\partial}{\partial\rho} - \frac{(\int g_2(t)\,dt)\sin\theta}{\rho}\frac{\partial}{\partial\theta} - g_2(t)\rho\sin\theta\frac{\partial}{\partial\phi}, \\ V_8 = g_3(z)\frac{\partial}{\partial\phi}, \quad V_9 = g_4(t)z\frac{\partial}{\partial\phi}, \\ V_{10} = (\int g_5(t)\,dt)\sin\theta\frac{\partial}{\partial\rho} - \frac{(\int g_5(t)\,dt)\cos\theta}{\rho}\frac{\partial}{\partial\theta} \\ \qquad + (g_5(t)\rho\cos\theta - \frac{\beta}{2}g_5(t)z^2 + \frac{\alpha}{2}e^{-t}z^2)\frac{\partial}{\partial\phi}, \end{cases} \tag{4}$$

where g_1, g_2, g_3, g_4, g_5 are continuous functions and the vector field V is the Lie symmetry group generator. The calculation and proof of Eq. (4) can be taken into account and all applications of the symmetry group below are proved by substituting into Eq. (1) directly.

Assume $\phi_s(t,\rho,\theta,z)$ is a solution of Eq. (1). According to operators V_6, V_7, V_8, V_9, V_{10}, the new solution can be written as follows:

$$\begin{cases} \phi_{\text{new}}(t,\rho,\theta,z) = g_1(t) + \phi_s(t,\rho,\theta,z), \\ \phi_{\text{new}}(t,\rho,\theta,z) = \phi_s(t,\rho + \int g_2(t)\,dt\cos\theta, \theta + \frac{\int g_2(t)\,dt\sin\theta}{\rho}, z) \\ \qquad - g_2(t)\rho\sin\theta, \\ \phi_{\text{new}}(t,\rho,\theta,z) = g_3(z) + \phi_s(t,\rho,\theta,z), \\ \phi_{\text{new}}(t,\rho,\theta,z) = g_4(t)z + \phi_s(t,\rho,\theta,z), \\ \phi_{\text{new}}(t,\rho,\theta,z) = \phi_s(t,\rho + \int g_5(t)\,dt\sin\theta, \theta + \frac{\int g_5(t)\,dt\cos\theta}{\rho}, z) \\ \qquad + g_5(t)\rho\cos\theta - \frac{\beta}{2}g_5(t)z^2 + \frac{\alpha}{2}e^{-t}z^2. \end{cases} \tag{5}$$

Thus, we have

$$\begin{aligned} \phi_{\text{new}}(t,\rho,\theta,z) = {}& -g_2(t)\rho\sin\theta + g_4(t)z + g_5(t)\rho\cos\theta - \frac{\beta}{2}g_5(t)z^2 + \frac{\alpha}{2}e^{-t}z^2 \\ & + \phi_s\biggl(t, \rho + \int g_5(t)\,dt\sin\theta + \int g_2(t)\,dt\cos\theta, \theta + \frac{\int g_5(t)\,dt\cos\theta}{\rho} \\ & + \frac{\int g_2(t)\,dt\sin\theta}{\rho}, z\biggr), \end{aligned} \tag{6}$$

according to the nontrivial transformation of Eq. (5). It is well known that new exact solutions for the differential equation can be found through the classic Lie symmetry group when a particular solution is known [27].

3 The new solution of the (3 + 1)-dimensional Rossby waves with dissipation effect in cylindrical coordinate

In order to obtain the solution of the (3 + 1)-dimensional dissipation Rossby waves in cylindrical coordinate, it is important to seek the solution of the (2 + 1)-dimensional dissipation Rossby waves in cylindrical coordinate. Next, we discuss the (2 + 1)-dimensional quasi-geostrophic vorticity equation with dissipation effect in cylindrical coordinate

$$\left[\frac{\partial}{\partial t}+\frac{1}{\rho}\frac{\partial\phi}{\partial\rho}\frac{\partial}{\partial\theta}-\frac{1}{\rho}\frac{\partial\phi}{\partial\theta}\frac{\partial}{\partial\rho}\right]\left[\frac{\partial^2\phi}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial\phi}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2\phi}{\partial\theta^2}\right] + \beta\left[\cos\theta\frac{\partial\phi}{\partial\rho}-\frac{\sin\theta}{\rho}\frac{\partial\phi}{\partial\theta}\right]=-\alpha\left[\frac{\partial^2\phi}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial\phi}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2\phi}{\partial\theta^2}\right]. \tag{7}$$

When dissipation does not exist, we first suppose

$$r=\rho\cos\theta-ct, \tag{8}$$

where c is a constant and denotes the phase speed of the wave. Equation (8) implies that

$$\begin{cases}\frac{\partial}{\partial t}=\frac{\partial}{\partial r}\frac{\partial r}{\partial t}=-c\frac{\partial}{\partial r},\\ \frac{\partial}{\partial\rho}=\frac{\partial}{\partial r}\frac{\partial r}{\partial\rho}=\cos\theta\frac{\partial}{\partial r},\\ \frac{\partial}{\partial\theta}=\frac{\partial}{\partial r}\frac{\partial r}{\partial\theta}=-\rho\sin\theta\frac{\partial}{\partial r}.\end{cases}$$

Based on the above transform, Eq. (7) becomes

$$-c\frac{\partial}{\partial r}\left[\frac{\partial^2\phi}{\partial r^2}+\frac{\cos\theta}{\rho}\frac{\partial\phi}{\partial r}-\frac{\beta}{c}\phi\right]=0. \tag{9}$$

According to a complex calculation, we acquire

$$\phi=C_1e^{\frac{r}{2}\frac{-c\cos\theta+\sqrt{c^2(\cos\theta)^2+4\rho^2\beta c}}{\rho c}}+C_2e^{-\frac{r}{2}\frac{c\cos\theta+\sqrt{c^2(\cos\theta)^2+4\rho^2\beta c}}{\rho c}}, \tag{10}$$

where C_1, C_2, c express arbitrary constants. Substituting (8) into (10), we have

$$\phi=C_1e^{\frac{\rho\cos\theta-ct}{2}\frac{-c\cos\theta+\sqrt{c^2(\cos\theta)^2+4\rho^2\beta c}}{\rho c}}+C_2e^{-\frac{\rho\cos\theta-ct}{2}\frac{c\cos\theta+\sqrt{c^2(\cos\theta)^2+4\rho^2\beta c}}{\rho c}}.$$

In the following, we respect the influence of dissipation on Eq. (7). Defining $\alpha\ll 1$ and $\alpha\ll\beta$, we suppose

$$s=\rho\cos\theta-\int_0^t\frac{c_1\phi_0}{2}\,dt, \tag{11}$$

where $\phi_0=\phi_0(\alpha t)$ varies slowly with time. Substituting Eq. (11) into Eq. (7), we get

$$\frac{\partial}{\partial t}\left[\frac{\partial^2\phi}{\partial s^2}+\frac{\cos\theta}{\rho}\frac{\partial\phi}{\partial s}\right]-\frac{c_1\phi_0}{2}\frac{\partial}{\partial s}\left[\frac{\partial^2\phi}{\partial s^2}+\frac{\cos\theta}{\rho}\frac{\partial\phi}{\partial s}\right]+\beta\frac{\partial\phi}{\partial s} =-\alpha\left[\frac{\partial^2\phi}{\partial s^2}+\frac{\cos\theta}{\rho}\frac{\partial\phi}{\partial s}\right]. \tag{12}$$

Suppose

$$T=t,\qquad \gamma=\alpha t, \tag{13}$$

and

$$\phi(s,T) = \phi_1(s,T,\gamma) + \alpha\phi_2(s,T) + \cdots, \tag{14}$$

we have

$$\alpha^0 : \frac{\partial}{\partial t}\left[\frac{\partial^2\phi_1}{\partial s^2} + \frac{\cos\theta}{\rho}\frac{\partial\phi_1}{\partial s}\right] - \frac{c_1\phi_0}{2}\frac{\partial}{\partial s}\left[\frac{\partial^2\phi_1}{\partial s^2} + \frac{\cos\theta}{\rho}\frac{\partial\phi_1}{\partial s}\right] + \beta\frac{\partial\phi_1}{\partial s} = 0, \tag{15}$$

$$\begin{aligned} \alpha^1 : {} & \frac{\partial}{\partial t}\left[\frac{\partial^2\phi_2}{\partial s^2} + \frac{\cos\theta}{\rho}\frac{\partial\phi_2}{\partial s}\right] - \frac{c_1 u_0}{2}\frac{\partial}{\partial s}\left[\frac{\partial^2\phi_2}{\partial s^2} + \frac{\cos\theta}{\rho}\frac{\partial\phi_2}{\partial s}\right] + \beta\frac{\partial\phi_2}{\partial s} \\ & = -\left[\frac{\partial^2\phi_1}{\partial s^2} + \frac{\cos\theta}{\rho}\frac{\partial\phi_1}{\partial s}\right], \end{aligned} \tag{16}$$

by substituting Eq. (14) into Eq. (12).

Then, setting

$$\delta = s + \frac{c_1\phi_0}{2}T, \tag{17}$$

we have

$$\alpha^0 : \frac{\partial}{\partial T}\left[\frac{\partial^2\phi_1}{\partial s^2} + \frac{\cos\theta}{\rho}\frac{\partial\phi_1}{\partial s}\right] + \beta\frac{\partial\phi_1}{\partial s} = 0, \tag{18}$$

$$\alpha^1 : \frac{\partial}{\partial T}\left[\frac{\partial^2\phi_2}{\partial s^2} + \frac{\cos\theta}{\rho}\frac{\partial\phi_2}{\partial s}\right] + \beta\frac{\partial\phi_2}{\partial s} = -\left[\frac{\partial^2\phi_1}{\partial s^2} + \frac{\cos\theta}{\rho}\frac{\partial\phi_1}{\partial s}\right]. \tag{19}$$

It is easy to derive that the solution of Eq. (19) has the following form

$$\phi_1 = C_1 e^{\frac{\delta - \frac{c_1\phi_0}{2}T}{2}\frac{-c\cos\theta + \sqrt{c^2(\cos\theta)^2 + 4\rho^2\beta c}}{\rho c}} + C_2 e^{-\frac{\delta - \frac{c_1\phi_0}{2}T}{2}\frac{c\cos\theta + \sqrt{c^2(\cos\theta)^2 + 4\rho^2\beta c}}{\rho c}}, \tag{20}$$

where $\phi_0 = C_1 C_2$. Based on Eq. (11) and Eq. (17), we acquire

$$\begin{aligned} \phi_1 = {} & \frac{\phi_0}{C_2} e^{\frac{\rho\cos\theta - \int_0^t \frac{c_1\phi_0}{2}\,dt}{2}\frac{-c\cos\theta + \sqrt{c^2(\cos\theta)^2 + 4\rho^2\beta c}}{\rho c}} \\ & + \frac{\phi_0}{C_1} e^{-\frac{\rho\cos\theta - \int_0^t \frac{c_1\phi_0}{2}\,dt}{2}\frac{c\cos\theta + \sqrt{c^2(\cos\theta)^2 + 4\rho^2\beta c}}{\rho c}}. \end{aligned} \tag{21}$$

Next, let

$$\phi_2 = D(\xi), \qquad \xi = \delta - \varepsilon T, \tag{22}$$

and substituting Eq. (22) into Eq. (19), we acquire

$$-\varepsilon\frac{\partial}{\partial\xi}\left[\frac{\partial^2 D}{\partial\xi^2} + \frac{\cos\theta}{\rho}\frac{\partial D}{\partial\xi}\right] + \beta\frac{\partial D}{\partial\xi} = N(\phi_1), \tag{23}$$

where

$$N(\phi_1) = -\left[\frac{\partial^2\phi_1}{\partial\xi^2} + \frac{\cos\theta}{\rho}\frac{\partial\phi_1}{\partial\xi}\right].$$

The solvability condition of Eq. (23) is

$$\int_{-\infty}^{+\infty} M(\xi) N(\phi_1)\, d\xi = 0, \tag{24}$$

where

$$-\varepsilon \frac{\partial}{\partial \xi}\left[\frac{\partial^2 M}{\partial \xi^2} + \frac{\cos\theta}{\rho}\frac{\partial M}{\partial \xi}\right] + \beta \frac{\partial M}{\partial \xi} = 0. \tag{25}$$

The solution of Eq. (25) is easy to acquire in the following form

$$M(\xi) = \frac{\phi_0}{C_2} e^{\frac{\xi}{2}\frac{-c\cos\theta+\sqrt{c^2(\cos\theta)^2+4\rho^2\beta c}}{\rho c}} + \frac{\phi_0}{C_1} e^{-\frac{\xi}{2}\frac{c\cos\theta+\sqrt{c^2(\cos\theta)^2+4\rho^2\beta c}}{\rho c}}, \tag{26}$$

in the case of $G(\pm\infty) = 0$. Equation (26) can be rewritten as

$$M(\xi) = \frac{\phi_0}{C_2} e^{\frac{\delta-\varepsilon T}{2}\frac{-c\cos\theta+\sqrt{c^2(\cos\theta)^2+4\rho^2\beta c}}{\rho c}} + \frac{\phi_0}{C_1} e^{-\frac{\delta-\varepsilon T}{2}\frac{c\cos\theta+\sqrt{c^2(\cos\theta)^2+4\rho^2\beta c}}{\rho c}}. \tag{27}$$

Substituting Eq. (27) into Eq. (24), we obtain

$$\phi_0 = \bar{\phi}_0 e^{-\alpha t}, \tag{28}$$

where $\bar{\phi}_0 = \phi_0$. Hence, we have

$$\begin{aligned}\phi &= \frac{\bar{\phi}_0 e^{-\alpha t}}{C_2} e^{\frac{\rho\cos\theta-\int_0^t \frac{c_1\bar{\phi}_0 e^{-\alpha t}}{2}\,dt}{2}\frac{-c\cos\theta+\sqrt{c^2(\cos\theta)^2+4\rho^2\beta c}}{\rho c}} \\ &\quad + \frac{\bar{\phi}_0 e^{-\alpha t}}{C_1} e^{-\frac{\rho\cos\theta-\int_0^t \frac{c_1\bar{\phi}_0 e^{-\alpha t}}{2}\,dt}{2}\frac{c\cos\theta+\sqrt{c^2(\cos\theta)^2+4\rho^2\beta c}}{\rho c}}.\end{aligned} \tag{29}$$

It is simple to prove that Eq. (29) is the approximate analytical solution of Eq. (7). Thus, the new (3 + 1)-dimensional dissipation Rossby waves solution in cylindrical coordinate can be given by

$$\begin{aligned}\phi &= -g_2(t)\rho\sin\theta + g_4(t)z + g_5(t)\rho\cos\theta - \frac{\beta}{2}g_5(t)z^2 + \frac{\alpha}{2}e^{-t}z^2 \\ &\quad + \frac{\bar{\phi}_0 e^{-\alpha t}}{C_2} e^{\frac{(\rho+\tilde{\rho})\cos(\theta+\tilde{\theta})-\int_0^t \frac{c_1\bar{\phi}_0 e^{-\alpha t}}{2}\,dt}{2}\frac{-c\cos(\theta+\tilde{\theta})+\sqrt{c^2(\cos(\theta+\tilde{\theta}))^2+4\rho^2\beta c}}{\rho c}} \\ &\quad + \frac{\bar{\phi}_0 e^{-\alpha t}}{C_1} e^{-\frac{(\rho+\tilde{\rho})\cos(\theta+\tilde{\theta})-\int_0^t \frac{c_1\bar{\phi}_0 e^{-\alpha t}}{2}\,dt}{2}\frac{c\cos(\theta+\tilde{\theta})+\sqrt{c^2(\cos(\theta+\tilde{\theta}))^2+4\rho^2\beta c}}{\rho c}},\end{aligned} \tag{30}$$

where

$$\begin{cases}\tilde{\theta} = \frac{\int g_5(t)\,dt\cos\theta}{\rho} + \frac{\int g_2(t)\,dt\sin\theta}{\rho}, \\ \tilde{\rho} = \int g_5(t)\,dt\sin\theta + \int g_2(t)\,dt\cos\theta.\end{cases}$$

4 Discussion and conclusion

In this section, the solution of the (3 + 1)-dimensional dissipation Rossby waves can be discussed relying on Eq. (30).

When $g_2(t) = \cos t$, $g_4(t) = \cos t$, $g_5(t) = \sin t$, Eq. (30) can be rewritten as

$$\begin{aligned}\phi = {}& -\rho \cos t \sin\theta + z\cos t + \rho \sin t \cos\theta - \frac{\beta}{2} z^2 \sin t + \frac{\alpha}{2} z^2 e^{-t} \\ & + \frac{\bar{\phi}_0 e^{-\alpha t}}{C_2} e^{\frac{(\rho+\tilde{\rho})\cos(\theta+\tilde{\theta}) - \int_0^t \frac{c_1 \bar{\phi}_0 e^{-\alpha t}}{2}\,dt}{2} \frac{-c\cos(\theta+\tilde{\theta}) + \sqrt{c^2(\cos(\theta+\tilde{\theta}))^2 + 4\rho^2\beta c}}{\rho c}} \\ & + \frac{\bar{\phi}_0 e^{-\alpha t}}{C_1} e^{-\frac{(\rho+\tilde{\rho})\cos(\theta+\tilde{\theta}) - \int_0^t \frac{c_1 \bar{\phi}_0 e^{-\alpha t}}{2}\,dt}{2} \frac{c\cos(\theta+\tilde{\theta}) + \sqrt{c^2(\cos(\theta+\tilde{\theta}))^2 + 4\rho^2\beta c}}{\rho c}},\end{aligned} \tag{31}$$

where

$$\begin{cases} \tilde{\theta} = \frac{\cos(\theta+t)}{\rho}, \\ \tilde{\rho} = \sin(\theta - t). \end{cases}$$

Obviously, Rossby waves were established in the zonal area and could not be used directly to the spherical earth in the previous research. However, in this paper, we get the solution of the (3 + 1)-dimensional dissipation Rossby waves by using the classic Lie group method in cylindrical coordinate, and the new solution overcomes the problem. According to theoretical analysis, we can make the following conclusion: In the spherical earth, the dissipation effect could give rise to a decrease in amplitude $e^{-\alpha t}$, where α denotes the dissipation coefficient from Eq. (31).

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Competing interests
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Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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