Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

# Multiple periodic solutions of high order differential delay equations with $2k-1$ lags

Lin Li[1], Huafei Sun[1*] and Weigao Ge[1]

*Correspondence:
3120170667@bit.edu.cn
[1]School of Mathematics and Statistics, Beijing Institute of Technology, Beijing, P.R. China

**Abstract**

In this paper, we study the periodic solutions of high order differential delay equations with $2k-1$ lags. The $4k$-periodic solutions are obtained by using the variational method and the method of Kaplan–Yorke coupling system. These are new types of differential delay equations compared with all previous research. And it provides a precise counting method for the number of periodic solutions. Two examples are given to demonstrate our main results.

**Keywords:** High order differential delay equation; Periodic solutions; Critical point theory; Variational method

## 1 Introduction

Given $f \in C^0(R,R)$ with $f(-x) = -f(x)$, $xf(x) > 0$, $x \neq 0$. Kaplan and Yorke [13] studied the existence of 4-periodic and 6-periodic solutions to the differential delay equations

$$x'(t) = -f\big(x(t-1)\big) \tag{1}$$

and

$$x'(t) = -f\big(x(t-1)\big) - f\big(x(t-2)\big), \tag{2}$$

respectively. The method they applied is transforming the two equations into adequate ordinary differential equations by regarding the retarded functions $x(t-1)$ and $x(t-2)$ as independent variables. They guessed that the existence of $2(n+1)$-periodic solution to the equation

$$x'(t) = -\sum_{i=1}^{n} f\big(x(t-i)\big) \tag{3}$$

could be studied under the restriction

$$x\big(t-(n+1)\big) = -x(t),$$

which was proved by Nussbaum [20] in 1978 by the use of a fixed point theorem on cones.

After that a lot of papers [3–12, 14–18] discussed the existence and multiplicity of $2(n+1)$-periodic solutions to Eq. (3) and its extension

$$x'(t) = -\sum_{i=1}^{n} \nabla F\bigl(x(t-i)\bigr), \tag{4}$$

where $F \in C^1(R^N, R)$, $\nabla F(-x) = -\nabla F(x)$, $F(0) = 0$. But they all studied first order equations.

In this paper, we study the periodic orbits to two types of high order differential delay equations with $2k-1$ lags in the form

$$x^{(2s+1)}(t) = -\sum_{i=1}^{2k-1} f\bigl(x(t-i)\bigr), \tag{5}$$

and

$$x^{(2s+1)}(t) = -\sum_{i=1}^{2k-1} (-1)^{i+1} f\bigl(x(t-i)\bigr), \tag{6}$$

which are different from (3) and can be regarded as a new extension of (3). The method applied in this paper is the variational approach in the critical point theory [1, 2, 19].

We suppose that

$$f \in C^0(R, R), \quad f(-x) = -f(x) \tag{7}$$

and there are $\alpha, \beta \in R$ such that

$$\lim_{x\to 0} \frac{f(x)}{x} = \alpha, \qquad \lim_{x\to\infty} \frac{f(x)}{x} = \beta. \tag{8}$$

Let $F(x) = \int_0^x f(s)\,ds$. Then $F(-x) = F(x)$ and $F(0) = 0$. For convenience, we make the following assumptions:

($S_1$) $f$ satisfies (7) and (8),

($S_2$) there exist $M > 0$ and a function $r \in C^0(R^+, R^+)$ satisfying $r(s) \to \infty$, $r(s)/s \to 0$ as $s \to \infty$ such that

$$\left| F(x) - \frac{1}{2}\beta x^2 \right| > r\bigl(|x|\bigr) - M,$$

($S_3^{\pm}$) $\pm[F(x) - \frac{1}{2}\beta x^2] > 0$, $|x| \to \infty$,

($S_4^{\pm}$) $\pm[F(x) - \frac{1}{2}\alpha x^2] > 0$, $0 < |x| \ll 1$.

In this paper, we need the following lemma as the base of our discussion.

Let $X$ be a Hilbert space, $L : X \to X$ be a linear operator, and $\Phi : X \to R$ be a differentiable functional.

**Lemma 1.1** ([3], Lemma 2.4) *Assume that there are two closed $s^1$-invariant linear subspaces, $X^+$ and $X^-$, and $r > 0$ such that*

(a) *$X^+ \cup X^-$ is closed and of finite codimensions in $X$,*

(b) *$\widehat{L}(X^-) \subset X^-$, $\widehat{L} = L + P^{-1}A_0$ or $\widehat{L} = L + P^{-1}A_\infty$,*

(c) *there exists* $c_0 \in R$ *such that*

$$\inf_{x\in X^+} \Phi(x) \geq c_0,$$

(d) *there is* $c_\infty \in R$ *such that*

$$\Phi(x) \leq c_\infty < \Phi(0) = 0, \quad \forall x \in X^- \cap S_r = \{x \in X^- : \|x\| = r\},$$

(e) $\Phi$ *satisfies the* $(P.S)_c$*-condition*, $c_0 < c < c_\infty$, *i.e., every sequence* $\{x_n\} \subseteq X$ *with* $\Phi(x_n) \to c$ *and* $\Phi'(x_n) \to 0$ *possesses a convergent subsequence.*

*Then* $\Phi$ *has at least* $\frac{1}{2}[\dim(X^+ \cap X^-) - \operatorname{codim}_X(X^+ \cup X^-)]$ *generally different critical orbits in* $\Phi^{-1}([c_0, c_\infty])$ *if* $[\dim(X^+ \cap X^-) - \operatorname{codim}_X(X^+ \cup X^-)] > 0$.

**Definition 1.2** We say that $\Phi$ satisfies the $(P.S)$-condition if every sequence $\{x_n\}$ with $\Phi(x_n)$ is bounded and $\Phi'(x_n) \to 0$ possesses a convergent subsequence.

*Remark* 1.3 We may use the $(P.S)$-condition to replace condition (e) in Lemma 1.1 since the $(P.S)$-condition implies that the $(P.S)_c$-condition holds for each $c \in R$.

## 2 Space $X$, functional $\Phi$, and its differential $\Phi'$

We are concerned with the $4k$-periodic solutions to (5) and (6) and suppose

$$x(t-2k) = -x(t), \quad k \geq 1. \tag{9}$$

Let

$$\begin{aligned}
\widehat{X} &= \{x \in L^2 : x(t-2k) = -x(t)\} \\
&= \left\{\sum_{i=0}^{\infty}\left(a_i \cos\frac{(2i+1)\pi t}{2k} + b_i \sin\frac{(2i+1)\pi t}{2k}\right) : a_i, b_i \in R\right\}, \\
X &= \operatorname{cl}\left\{\sum_{i=0}^{\infty}\left(a_i \cos\frac{(2i+1)\pi t}{2k} + b_i \sin\frac{(2i+1)\pi t}{2k}\right) : a_i, b_i \in R,\right. \\
&\quad \left.\sum_{i=0}^{\infty}(2i+1)\left(a_i^2 + b_i^2\right) < \infty\right\} \subset \widehat{X},
\end{aligned}$$

and define $P : X \to L^2$ by

$$\begin{aligned}
\mathrm{Px}(t) &= P\left(\sum_{i=0}^{\infty}\left(a_i \cos\frac{(2i+1)\pi t}{2k} + b_i \sin\frac{(2i+1)\pi t}{2k}\right)\right) \\
&= \sum_{i=0}^{\infty}(2i+1)^{2s+1}\left(a_i \cos\frac{(2i+1)\pi t}{2k} + b_i \sin\frac{(2i+1)\pi t}{2k}\right).
\end{aligned} \tag{10}$$

Let

$$P^{-1}x(t) = \sum_{i=0}^{\infty}\frac{1}{(2i+1)^{2s+1}}\left(a_i \cos\frac{(2i+1)\pi t}{2k} + b_i \sin\frac{(2i+1)\pi t}{2k}\right).$$

Then the inverse $P^{-1}$ of $P$ exists. For $x \in X$, define

$$\langle x, y\rangle = \int_0^{4k} \bigl(\mathrm{P}x(t), y(t)\bigr)\,dt, \quad \|x\| = \sqrt{\langle x, x\rangle},$$
$$\langle x, y\rangle_2 = \int_0^{4k} \bigl(x(t), y(t)\bigr)\,dt, \quad \|x\|_2 = \sqrt{\langle x, x\rangle_2}.$$

Therefore $(X, \|\cdot\|)$ is an $H^{\frac{1}{2}}$ space.

For (5), define a functional $\Phi : X \to R$ by

$$\Phi(x) = \frac{1}{2}\langle Lx, x\rangle + \int_0^{4k} F\bigl(x(t)\bigr)\,dt, \tag{11}$$

where

$$Lx = -P^{-1}\sum_{i=1}^{2k-1}(-1)^{i+1}x^{(2s+1)}(t-i). \tag{12}$$

Let $m = k - 1$ and

$$X(i) = \left\{x(t) = a_i \cos\frac{(2i+1)\pi t}{2k} + b_i \sin\frac{(2i+1)\pi t}{2k} : a_i, b_i \in R\right\}.$$

We have

$$X = \sum_{l=0}^{\infty}\left[\sum_{i=0}^{m}\bigl(X(2lk+i) + X(2lk+2k-i-1)\bigr)\right]. \tag{13}$$

If $x_i(t) = a_i \cos\frac{(2i+1)\pi t}{2k} + b_i \sin\frac{(2i+1)\pi t}{2k} \in X(i)$, $i \in N$, we have

$$Lx = (-1)^{s+1}\left(\frac{\pi}{2k}\right)^{2s+1}\left(\sum_{i=0}^{\infty} x_i \tan\frac{(2i+1)\pi}{4k}\right). \tag{14}$$

Obviously, $L|_{X(i)} : X(i) \to X(i)$ is invertible.

Based on the theorem given by Mawhin and Willem [19, Theorem 1.4] the differential of functional $\Phi$ is differentiable, and its differential is

$$P^{-1}\Phi'(x) = Lx + K(x), \tag{15}$$

where $K(x) = P^{-1}f(x)$. It is easy to prove that $K : (X, \|x\|^2) \to (X, \|x\|_2^2)$ is compact.

Therefore, from (14) we have that if

$$x(t) = \sum_{i=0}^{\infty}\left(a_i \cos\frac{(2i+1)\pi t}{2k} + b_i \sin\frac{(2i+1)\pi t}{2k}\right),$$

then

$$
\begin{aligned}
\langle Lx,x\rangle &= (-1)^{s+1}\sum_{i=0}^{\infty}\left(\frac{\pi}{2k}\right)^{2s+1}2k(2i+1)^{2s+1}\left(a_i^2+b_i^2\right)\tan\frac{(2i+1)\pi}{4k} \\
&= \sum_{l=0}^{\infty}\Bigg[\sum_{i=0}^{m}(-1)^{s+1}\left(\frac{\pi}{2k}\right)^{2s+1}2k(4lk+2i+1)^{2s+1}\left(a_{2lk+i}^2+b_{2lk+i}^2\right)\tan\frac{(2i+1)\pi}{4k} \\
&\quad -\sum_{i=0}^{m}(-1)^{s+1}\left(\frac{\pi}{2k}\right)^{2s+1}2k(4lk+4k-2i-1)^{2s+1} \\
&\quad \times\left(a_{2lk+2k-i-1}^2+b_{2lk+2k-i-1}^2\right)\tan\frac{(2i+1)\pi}{4k}\Bigg].
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\langle P^{-1}\beta x,x\rangle &= \sum_{i=0}^{\infty}2k\beta\left(a_i^2+b_i^2\right) \\
&= \sum_{l=0}^{\infty}\left[\sum_{i=0}^{m}2k\beta\left(a_{2lk+i}^2+b_{2lk+i}^2\right)+\sum_{i=0}^{m}2k\beta\left(a_{2lk+2k-i-1}^2+b_{2lk+2k-i-1}^2\right)\right], \\
\langle P^{-1}\alpha x,x\rangle &= \sum_{i=0}^{\infty}2k\alpha\left(a_i^2+b_i^2\right) \\
&= \sum_{l=0}^{\infty}\left[\sum_{i=0}^{m}2k\alpha\left(a_{2lk+i}^2+b_{2lk+i}^2\right)+\sum_{i=0}^{m}2k\alpha\left(a_{2lk+2k-i-1}^2+b_{2lk+2k-i-1}^2\right)\right].
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&\left\langle\left(L+P^{-1}\beta\right)x,x\right\rangle \\
&\quad = 2k\sum_{l=0}^{\infty}\Bigg[\sum_{i=0}^{m}\left((-1)^{s+1}\left(\frac{\pi}{2k}\right)^{2s+1}(4lk+2i+1)^{2s+1}\tan\frac{(2i+1)\pi}{4k}+\beta\right)\left(a_{2lk+i}^2+b_{2lk+i}^2\right) \\
&\qquad +\sum_{i=0}^{m}\left(-(-1)^{s+1}\left(\frac{\pi}{2k}\right)^{2s+1}(4lk+4k-2i-1)^{2s+1}\tan\frac{(2i+1)\pi}{4k}+\beta\right) \\
&\qquad \times\left(a_{2lk+2k-i-1}^2+b_{2lk+2k-i-1}^2\right)\Bigg], \\
&\left\langle\left(L+P^{-1}\alpha\right)x,x\right\rangle \\
&\quad = 2k\sum_{l=0}^{\infty}\Bigg[\sum_{i=0}^{m}\left((-1)^{s+1}\left(\frac{\pi}{2k}\right)^{2s+1}(4lk+2i+1)^{2s+1}\tan\frac{(2i+1)\pi}{4k}+\alpha\right)\left(a_{2lk+i}^2+b_{2lk+i}^2\right) \\
&\qquad +\sum_{i=0}^{m}\left(-(-1)^{s+1}\left(\frac{\pi}{2k}\right)^{2s+1}(4lk+4k-2i-1)^{2s+1}\tan\frac{(2i+1)\pi}{4k}+\alpha\right) \\
&\qquad \times\left(a_{2lk+2k-i-1}^2+b_{2lk+2k-i-1}^2\right)\Bigg].
\end{aligned}
$$

For (6), define the functional $\Psi : X \to R$ by

$$\Psi(x) = \frac{1}{2}\langle Lx, x\rangle + \int_0^{4k} F\bigl(x(t)\bigr)\,dt, \tag{16}$$

where

$$Lx = -P^{-1}\sum_{i=1}^{2k-1} x^{(2s+1)}(t-i). \tag{17}$$

Let $m = k-1$ and

$$X(i) = \left\{x(t) = a_i \cos\frac{(2i+1)\pi t}{2k} + b_i \sin\frac{(2i+1)\pi t}{2k} : a_i, b_i \in R\right\}.$$

We have

$$X = \sum_{l=0}^{\infty}\left[\sum_{i=0}^{m}\bigl(X(2lk+i) + X(2lk+2k-i-1)\bigr)\right]. \tag{18}$$

If $x_i(t) = a_i \cos\frac{(2i+1)\pi t}{2k} + b_i \sin\frac{(2i+1)\pi t}{2k} \in X(i)$, $i \in N$, we have

$$Lx = (-1)^{s+1}\left(\frac{\pi}{2k}\right)^{2s+1}\left(\sum_{i=0}^{\infty} x_i \cot\frac{(2i+1)\pi}{4k}\right). \tag{19}$$

Obviously, $L|_{X(i)} : X(i) \to X(i)$ is invertible.

Based on the theorem given by Mawhin and Willem [16, Theorem 1.4] the differential of functional $\Phi$ is differentiable, and its differential is

$$P^{-1}\Psi'(x) = Lx + K(x), \tag{20}$$

where $K(x) = P^{-1}f(x)$. It is easy to prove that $K : (X, \|x\|^2) \to (X, \|x\|_2^2)$ is compact.

Therefore, from (14) we have that if

$$x(t) = \sum_{i=0}^{\infty}\left(a_i \cos\frac{(2i+1)\pi t}{2k} + b_i \sin\frac{(2i+1)\pi t}{2k}\right),$$

then

$$\begin{aligned}
\langle Lx, x\rangle &= (-1)^{s+1}\sum_{i=0}^{\infty}\left(\frac{\pi}{2k}\right)^{2s+1} 2k(2i+1)^{2s+1}\bigl(a_i^2 + b_i^2\bigr)\cot\frac{(2i+1)\pi}{4k} \\
&= \sum_{l=0}^{\infty}\left[\sum_{i=0}^{m}(-1)^{s+1}\left(\frac{\pi}{2k}\right)^{2s+1} 2k(4lk+2i+1)^{2s+1}\bigl(a_{2lk+i}^2 + b_{2lk+i}^2\bigr)\cot\frac{(2i+1)\pi}{4k}\right. \\
&\quad - \sum_{i=0}^{m}(-1)^{s+1}\left(\frac{\pi}{2k}\right)^{2s+1} 2k(4lk+4k-2i-1)^{2s+1}\bigl(a_{2lk+2k-i-1}^2 + b_{2lk+2k-i-1}^2\bigr) \\
&\quad \left.\times \cot\frac{(2i+1)\pi}{4k}\right].
\end{aligned}$$

On the other hand,

$$\begin{aligned}\langle P^{-1}\beta x,x\rangle &= \sum_{i=0}^{\infty} 2k\beta\left(a_i^2+b_i^2\right)\\ &= \sum_{l=0}^{\infty}\left[\sum_{i=0}^{m} 2k\beta\left(a_{2lk+i}^2+b_{2lk+i}^2\right)+\sum_{i=0}^{m} 2k\beta\left(a_{2lk+2k-i-1}^2+b_{2lk+2k-i-1}^2\right)\right].\\ \langle P^{-1}\alpha x,x\rangle &= \sum_{i=0}^{\infty} 2k\alpha\left(a_i^2+b_i^2\right)\\ &= \sum_{l=0}^{\infty}\left[\sum_{i=0}^{m} 2k\alpha\left(a_{2lk+i}^2+b_{2lk+i}^2\right)+\sum_{i=0}^{m} 2k\alpha\left(a_{2lk+2k-i-1}^2+b_{2lk+2k-i-1}^2\right)\right].\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\left\langle\left(L+P^{-1}\beta\right)x,x\right\rangle\\ &\quad= 2k\sum_{l=0}^{\infty}\left[\sum_{i=0}^{m}\left((-1)^{s+1}\left(\frac{\pi}{2k}\right)^{2s+1}(4lk+2i+1)^{2s+1}\cot\frac{(2i+1)\pi}{4k}+\beta\right)\right.\\ &\qquad\times\left(a_{2lk+i}^2+b_{2lk+i}^2\right)\\ &\qquad+\sum_{i=0}^{m}\left(-(-1)^{s+1}\left(\frac{\pi}{2k}\right)^{2s+1}(4lk+4k-2i-1)^{2s+1}\cot\frac{(2i+1)\pi}{4k}+\beta\right)\\ &\qquad\left.\times\left(a_{2lk+2k-i-1}^2+b_{2lk+2k-i-1}^2\right)\right],\\ &\left\langle\left(L+P^{-1}\alpha\right)x,x\right\rangle\\ &\quad= 2k\sum_{l=0}^{\infty}\left[\sum_{i=0}^{m}\left((-1)^{s+1}\left(\frac{\pi}{2k}\right)^{2s+1}(4lk+2i+1)^{2s+1}\cot\frac{(2i+1)\pi}{4k}+\alpha\right)\right.\\ &\qquad\times\left(a_{2lk+i}^2+b_{2lk+i}^2\right)\\ &\qquad+\sum_{i=0}^{m}\left(-(-1)^{s+1}\left(\frac{\pi}{2k}\right)^{2s+1}(4lk+4k-2i-1)^{2s+1}\cot\frac{(2i+1)\pi}{4k}+\alpha\right)\\ &\qquad\left.\times\left(a_{2lk+2k-i-1}^2+b_{2lk+2k-i-1}^2\right)\right].\end{aligned}$$

**Lemma 2.1** *Each critical point of the functional $\Phi$ is a $4k$-periodic classical solution of Eq.* (5) *satisfying* (9).

*Proof* Let $x$ be a critical point of the functional $\Phi$. Then $x(t)$ satisfies

$$-\sum_{i=1}^{2k-1}(-1)^{i+1}x^{(2s+1)}(t-i)+f\left(x(t)\right)=0,\quad \text{a.e. } t\in[0,4k]. \tag{21}$$

Consequently,

$$-\sum_{i=1}^{2k-1}(-1)^{i+1}x^{(2s+1)}(t-i-1)+f\bigl(x(t-1)\bigr)=0, \tag{21.1}$$

$$-\sum_{i=1}^{2k-1}(-1)^{i+1}x^{(2s+1)}(t-i-2)+f\bigl(x(t-2)\bigr)=0, \tag{21.2}$$

$$-\sum_{i=1}^{2k-1}(-1)^{i+1}x^{(2s+1)}(t-i-3)+f\bigl(x(t-3)\bigr)=0, \tag{21.3}$$

$$\vdots$$

$$-\sum_{i=0}^{2k-1}(-1)^{i+1}x^{(2s+1)}\bigl(t-i-(2k-1)\bigr)+f\bigl(x\bigl(t-(2k-1)\bigr)\bigr)=0. \tag{21.(2k-1)}$$

Calculating (21.1) + (21.2) + (21.3) + ⋯ + (21.(2k-1)), we can get

$$x^{(2s+1)}(t)+\sum_{i=1}^{2k-1}f\bigl(x(t-i)\bigr)=0, \quad \text{a.e. } t\in[0,4k],$$

namely

$$x^{(2s+1)}(t)=-\sum_{i=1}^{2k-1}f\bigl(x(t-i)\bigr), \quad \text{a.e. } t\in[0,4k],$$

which implies that $x$ satisfies the above equation for all $t\in[0,4k]$ since the function on the right-hand side is continuous, then $x$ is a classical solution to (5). □

**Lemma 2.2** *Each critical point of the functional $\Psi$ is a $4k$-periodic classical solution of Eq.* (6) *satisfying* (9).

*Proof* Let $x$ be a critical point of the functional $\Psi$. Then $x(t)$ satisfies

$$-\sum_{i=1}^{2k-1}x^{(2s+1)}(t-i)+f\bigl(x(t)\bigr)=0, \quad \text{a.e. } t\in[0,4k]. \tag{22}$$

Consequently,

$$-\sum_{i=1}^{2k-1}x^{(2s+1)}(t-i-1)+f\bigl(x(t-1)\bigr)=0, \tag{22.1}$$

$$-\sum_{i=1}^{2k-1}x^{(2s+1)}(t-i-2)+f\bigl(x(t-2)\bigr)=0, \tag{22.2}$$

$$-\sum_{i=1}^{2k-1}x^{(2s+1)}(t-i-3)+f\bigl(x(t-3)\bigr)=0, \tag{22.3}$$

$$\vdots$$

$$-\sum_{i=1}^{2k-1} x^{(2s+1)}\bigl(t-i-(2k-1)\bigr)+f\bigl(x\bigl(t-(2k-1)\bigr)\bigr)=0. \qquad (22.(2k\text{-}1))$$

Calculating (22.1) − (22.2) + (22.3) − ··· + (22.(2k-1)), we can get

$$x^{(2s+1)}(t)+\sum_{i=1}^{2k-1}(-1)^{i+1}f\bigl(x(t-i)\bigr)=0, \quad \text{a.e. } t\in[0,4k],$$

namely

$$x^{(2s+1)}(t)=-\sum_{i=1}^{2k-1}(-1)^{i+1}f\bigl(x(t-i)\bigr), \quad \text{a.e. } t\in[0,4k],$$

which implies that $x$ satisfies the above equation for all $t\in[0,4k]$ since the function on the right-hand side is continuous, then $x$ is a classical solution to (6). □

## 3 Partition of space $X$ and symbols

For (5), let

$$\begin{aligned}
X_\infty^+ = \biggl\{ & X(2lk+i): l\geq 0, 0\leq i\leq m, \\
& (-1)^{s+1}\left(\frac{(4lk+2i+1)\pi}{2k}\right)^{2s+1}\tan\frac{(2i+1)\pi}{4k}+\beta>0\biggr\} \\
& \cup \biggl\{X(2lk+2k-i-1): l\geq 0, 0\leq i\leq m, \\
& -(-1)^{s+1}\left(\frac{(4lk+4k-2i-1)\pi}{2k}\right)^{2s+1}\tan\frac{(2i+1)\pi}{4k}+\beta>0\biggr\}, \\
X_\infty^- = \biggl\{ & X(2lk+i): l\geq 0, 0\leq i\leq m, \\
& (-1)^{s+1}\left(\frac{(4lk+2i+1)\pi}{2k}\right)^{2s+1}\tan\frac{(2i+1)\pi}{4k}+\beta<0\biggr\} \\
& \cup \biggl\{X(2lk+2k-i-1): l\geq 0, 0\leq i\leq m, \\
& -(-1)^{s+1}\left(\frac{(4lk+4k-2i-1)\pi}{2k}\right)^{2s+1}\tan\frac{(2i+1)\pi}{4k}+\beta<0\biggr\}, \\
X_0^+ = \biggl\{ & X(2lk+i): l\geq 0, 0\leq i\leq m, \\
& (-1)^{s+1}\left(\frac{(4lk+2i+1)\pi}{2k}\right)^{2s+1}\tan\frac{(2i+1)\pi}{4k}+\alpha>0\biggr\} \\
& \cup \biggl\{X(2lk+2k-i-1): l\geq 0, 0\leq i\leq m, \\
& -(-1)^{s+1}\left(\frac{(4lk+4k-2i-1)\pi}{2k}\right)^{2s+1}\tan\frac{(2i+1)\pi}{4k}+\alpha>0\biggr\},
\end{aligned}$$

$$\begin{aligned}X_0^- = &\Big\{X(2lk+i) : l \geq 0, 0 \leq i \leq m,\\
&(-1)^{s+1}\left(\frac{(4lk+2i+1)\pi}{2k}\right)^{2s+1} \tan\frac{(2i+1)\pi}{4k} + \alpha < 0\Big\}\\
&\cup \Big\{X(2lk+2k-i-1) : l \geq 0, 0 \leq i \leq m,\\
&-(-1)^{s+1}\left(\frac{(4lk+4k-2i-1)\pi}{2k}\right)^{2s+1} \tan\frac{(2i+1)\pi}{4k} + \alpha < 0\Big\}.\end{aligned}$$

On the other hand,

$$\begin{aligned}X_\infty^0 = &\Big\{X(2lk+i) : l \geq 0, 0 \leq i \leq m,\\
&(-1)^{s+1}\left(\frac{(4lk+2i+1)\pi}{2k}\right)^{2s+1} \tan\frac{(2i+1)\pi}{4k} + \beta = 0\Big\}\\
&\cup \Big\{X(2lk+2k-i-1) : l \geq 0, 0 \leq i \leq m,\\
&-(-1)^{s+1}\left(\frac{(4lk+4k-2i-1)\pi}{2k}\right)^{2s+1} \tan\frac{(2i+1)\pi}{4k} + \beta = 0\Big\},\\
X_0^0 = &\Big\{X(2lk+i) : l \geq 0, 0 \leq i \leq m,\\
&(-1)^{s+1}\left(\frac{(4lk+2i+1)\pi}{2k}\right)^{2s+1} \tan\frac{(2i+1)\pi}{4k} + \alpha = 0\Big\}\\
&\cup \Big\{X(2lk+2k-i-1) : l \geq 0, 0 \leq i \leq m,\\
&-(-1)^{s+1}\left(\frac{(4lk+4k-2i-1)\pi}{2k}\right)^{2s+1} \tan\frac{(2i+1)\pi}{4k} + \alpha = 0\Big\}.\end{aligned}$$

Obviously, $\dim X_\infty^0 < \infty$ and $\dim X_0^0 < \infty$.

**Lemma 3.1** *Under assumptions* $(S_1)$ *and* $(S_2)$, *there is* $\sigma > 0$ *such that*

$$\begin{aligned}&\langle (L + P^{-1}\beta)x, x\rangle > \sigma\|x\|^2, \quad x \in X_\infty^+ \quad \textit{and}\\
&\langle (L + P^{-1}\beta)x, x\rangle < -\sigma\|x\|^2, \quad x \in X_\infty^-.\end{aligned} \tag{23}$$

*Proof* First, we have that, for $\beta \geq 0$ and $s$ = even, $(-1)^{s+1} = -1$, $i \in \{0, 1, \dots, m\}$,

$$\begin{aligned}&-\left(\frac{(4lk+2i+1)\pi}{2k}\right)^{2s+1} \tan\frac{(2i+1)\pi}{4k} + \beta\\
&\quad > -\left(\frac{(4l^+(i)k+2i+1)\pi}{2k}\right)^{2s+1} \tan\frac{(2i+1)\pi}{4k} + \beta > 0,\end{aligned}$$

where $l^+(i) = \max\{l \in N : -(\frac{(4lk+2i+1)\pi}{2k})^{2s+1} \tan \frac{(2i+1)\pi}{4k} + \beta > 0\}$ and

$$-\left(\frac{(4lk+2i+1)\pi}{2k}\right)^{2s+1} \tan \frac{(2i+1)\pi}{4k} + \beta < -\left(\frac{(4l^-(i)k+2i+1)\pi}{2k}\right)^{2s+1} \tan \frac{(2i+1)\pi}{4k} + \beta < 0,$$

where $l^-(i) = \min\{l \in N : -(\frac{(4lk+2i+1)\pi}{2k})^{2s+1} \tan \frac{(2i+1)\pi}{4k} + \beta < 0\}$.

In this case, we may choose

$$\sigma_i = \min\left\{-\left(\frac{\pi}{2k}\right)^{2s+1} \tan \frac{(2i+1)\pi}{4k} + \frac{\beta}{(4l^+(i)k+2i+1)^{2s+1}}, \left(\frac{\pi}{2k}\right)^{2s+1} \tan \frac{(2i+1)\pi}{4k} - \frac{\beta}{(4l^-(i)k+2i+1)^{2s+1}}\right\} > 0,$$

and let $\sigma = \min\{\sigma_0, \sigma_1, \dots, \sigma_m\} > 0$.

Second, we have that, for $\beta \geq 0$ and $s$ = odd, $(-1)^{s+1} = 1$, $i \in \{0, 1, \dots, m\}$,

$$-\left(\frac{(4lk+4k-2i-1)\pi}{2k}\right)^{2s+1} \tan \frac{(2i+1)\pi}{4k} + \beta > -\left(\frac{(4l^+(i)k+4k-2i-1)\pi}{2k}\right)^{2s+1} \tan \frac{(2i+1)\pi}{4k} + \beta > 0,$$

where $l^+(i) = \max\{l \in N : -(\frac{(4lk+4k-2i-1)\pi}{2k})^{2s+1} \tan \frac{(2i+1)\pi}{4k} + \beta > 0\}$ and

$$-\left(\frac{(4lk+4k-2i-1)\pi}{2k}\right)^{2s+1} \tan \frac{(2i+1)\pi}{4k} + \beta < -\left(\frac{(4l^-(i)k+4k-2i-1)\pi}{2k}\right)^{2s+1} \tan \frac{(2i+1)\pi}{4k} + \beta < 0,$$

where $l^-(i) = \min\{l \in N : -(\frac{(4lk+4k-2i-1)\pi}{2k})^{2s+1} \tan \frac{(2i+1)\pi}{4k} + \beta < 0\}$.

In this case, we may choose

$$\sigma_i = \min\left\{-\left(\frac{\pi}{2k}\right)^{2s+1} \tan \frac{(2i+1)\pi}{4k} + \frac{\beta}{(4l^+(i)k+4k-2i-1)^{2s+1}}, \left(\frac{\pi}{2k}\right)^{2s+1} \tan \frac{(2i+1)\pi}{4k} - \frac{\beta}{(4l^-(i)k+4k-2i-1)^{2s+1}}\right\} > 0,$$

and let $\sigma = \min\{\sigma_0, \sigma_1, \dots, \sigma_m\} > 0$. The proof for the case $\beta < 0$ is similar. We omit it. The inequalities in (23) are proved. □

**Lemma 3.2** *Under conditions* $(S_1)$ *and* $(S_2)$, *the functional* $\Phi$ *defined by* (11) *satisfies the* $(P.S)$*-condition.*

*Proof* Let $\Pi$, $\mathbb{N}$, $\mathbb{Z}$ be the orthogonal projections from $X$ onto $X_\infty^+$, $X_\infty^-$, $X_\infty^0$, respectively. From the second condition in (8) it follows that

$$\left|\left\langle P^{-1}\big(f(x)-\beta x\big),x\right\rangle\right| < \frac{\sigma}{2}\|x\|^2 + M, \quad x\in X \tag{24}$$

for some $M > 0$.

Assume that $\{x_n\}\subset X$ is a subsequence such that $\Phi'(x_n)\to 0$ and $\Phi(x_n)$ is bounded. Let $w_n = \Pi x_n$, $y_n = \mathbb{N}x_n$, $z_n = \mathbb{Z}x_n$. Then we have

$$\Pi\big(L+P^{-1}\beta\big) = \big(L+P^{-1}\beta\big)\Pi, \qquad \mathbb{N}\big(L+P^{-1}\beta\big) = \big(L+P^{-1}\beta\big)\mathbb{N}. \tag{25}$$

From

$$\left\langle \Phi'(x_n),x_n\right\rangle = \left\langle Lx_n + P^{-1}f(x_n),x_n\right\rangle = \left\langle \big(L+P^{-1}\beta\big)x_n,x_n\right\rangle + \left\langle P^{-1}\big(f(x_n)-\beta x_n\big),x_n\right\rangle$$

and (25), we have

$$\begin{aligned}\left\langle \Pi\Phi'(x_n),x_n\right\rangle &= \left\langle \Pi\big(L+P^{-1}\beta\big)x_n,x_n\right\rangle + \left\langle \Pi P^{-1}\big(f(x_n)-\beta x_n\big),x_n\right\rangle\\ &= \left\langle \big(L+P^{-1}\beta\big)w_n,w_n\right\rangle + \left\langle \Pi P^{-1}\big(f(x_n)-\beta x_n\big),w_n\right\rangle,\end{aligned}$$

and then, by (23), we have

$$\left\langle \big(L+P^{-1}\beta\big)w_n,w_n\right\rangle + \left\langle \Pi P^{-1}\big(f(x_n)-\beta x_n\big),w_n\right\rangle > \frac{\sigma}{2}\|w_n\|^2 - M\|w_n\|,$$

which, together with $\Pi\Phi'(x_n)\to 0$, implies the boundedness of $w_n$. Similarly we have the boundedness of $y_n$. At the same time, $(S_2)$ yields

$$\begin{aligned}\Phi(x_n) &= \frac{1}{2}\left\langle \big(L+P^{-1}\beta\big)x_n,x_n\right\rangle + \int_0^{4k} F(x_n)\,dt - \frac{\beta}{2}\|x_n\|_2^2\\ &= \frac{1}{2}\left\langle \big(L+P^{-1}\beta\big)w_n,w_n\right\rangle + \frac{1}{2}\left\langle \big(L+P^{-1}\beta\big)y_n,y_n\right\rangle\\ &\quad + \int_0^{4k} F(x_n)\,dt - \frac{\beta}{2}\big(\|w_n\|_2^2 + \|y_n\|_2^2 + \|z_n\|_2^2\big).\end{aligned}$$

Then the boundedness of $\Phi(x)$ implies that $\|z_n\|_2$ is bounded. Consequently $\|z_n\|$ is bounded since $X_\infty^0$ is finite-dimensional. Therefore, $\|x_n\|$ is bounded.

It follows from (15) that

$$\begin{aligned}(\Pi + N)\Phi'(x_n) &= (\Pi+\mathbb{N})Lx_n + (\Pi+\mathbb{N})K(x_n)\\ &= L(w_n+y_n) + (\Pi+\mathbb{N})K(x_n).\end{aligned}$$

From the compactness of operator $K$ and the boundedness of $x_n$, we have that $K(x_n)\to u$. Then

$$L|_{x_\infty^+ + x_\infty^-}(w_n+y_n) \to -(\Pi+\mathbb{N})u. \tag{26}$$

The finite-dimensionality of $X_\infty^0$ and the boundedness of $z_n = \mathbb{Z}x_n$ imply $z_n \to \varphi \in X_\infty^0$. Therefore,

$$x_n = z_n + w_n + y_n \to \varphi - (L|_{x_\infty^+ + x_\infty^-})^{-1}(\Pi + \mathbb{N})u,$$

which implies the $(P.S)$-condition.

For (6), let

$$\begin{aligned}
X_\infty^+ = {} & \Bigg\{X(2lk+i) : l \ge 0, 0 \le i \le m, \\
& (-1)^{s+1}\left(\frac{(4lk+2i+1)\pi}{2k}\right)^{2s+1} \cot\frac{(2i+1)\pi}{4k} + \beta > 0\Bigg\} \\
& \cup \Bigg\{X(2lk+2k-i-1) : l \ge 0, 0 \le i \le m, \\
& -(-1)^{s+1}\left(\frac{(4lk+4k-2i-1)\pi}{2k}\right)^{2s+1} \cot\frac{(2i+1)\pi}{4k} + \beta > 0\Bigg\}, \\
X_\infty^- = {} & \Bigg\{X(2lk+i) : l \ge 0, 0 \le i \le m, \\
& (-1)^{s+1}\left(\frac{(4lk+2i+1)\pi}{2k}\right)^{2s+1} \cot\frac{(2i+1)\pi}{4k} + \beta < 0\Bigg\} \\
& \cup \Bigg\{X(2lk+2k-i-1) : l \ge 0, 0 \le i \le m, \\
& -(-1)^{s+1}\left(\frac{(4lk+4k-2i-1)\pi}{2k}\right)^{2s+1} \cot\frac{(2i+1)\pi}{4k} + \beta < 0\Bigg\}, \\
X_0^+ = {} & \Bigg\{X(2lk+i) : l \ge 0, 0 \le i \le m, \\
& (-1)^{s+1}\left(\frac{(4lk+2i+1)\pi}{2k}\right)^{2s+1} \cot\frac{(2i+1)\pi}{4k} + \alpha > 0\Bigg\} \\
& \cup \Bigg\{X(2lk+2k-i-1) : l \ge 0, 0 \le i \le m, \\
& -(-1)^{s+1}\left(\frac{(4lk+4k-2i-1)\pi}{2k}\right)^{2s+1} \cot\frac{(2i+1)\pi}{4k} + \alpha > 0\Bigg\}, \\
X_0^- = {} & \Bigg\{X(2lk+i) : l \ge 0, 0 \le i \le m, \\
& (-1)^{s+1}\left(\frac{(4lk+2i+1)\pi}{2k}\right)^{2s+1} \cot\frac{(2i+1)\pi}{4k} + \alpha < 0\Bigg\} \\
& \cup \Bigg\{X(2lk+2k-i-1) : l \ge 0, 0 \le i \le m, \\
& -(-1)^{s+1}\left(\frac{(4lk+4k-2i-1)\pi}{2k}\right)^{2s+1} \cot\frac{(2i+1)\pi}{4k} + \alpha < 0\Bigg\}.
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
X_\infty^0 = {} & \biggl\{ X(2lk+i) : l \geq 0, 0 \leq i \leq m, \\
& (-1)^{s+1}\left(\frac{(4lk+2i+1)\pi}{2k}\right)^{2s+1}\cot\frac{(2i+1)\pi}{4k}+\beta = 0 \biggr\} \\
& \cup \biggl\{ X(2lk+2k-i-1) : l \geq 0, 0 \leq i \leq m, \\
& -(-1)^{s+1}\left(\frac{(4lk+4k-2i-1)\pi}{2k}\right)^{2s+1}\cot\frac{(2i+1)\pi}{4k}+\beta = 0 \biggr\}, \\
X_0^0 = {} & \biggl\{ X(2lk+i) : l \geq 0, 0 \leq i \leq m, \\
& (-1)^{s+1}\left(\frac{(4lk+2i+1)\pi}{2k}\right)^{2s+1}\cot\frac{(2i+1)\pi}{4k}+\alpha = 0 \biggr\} \\
& \cup \biggl\{ X(2lk+2k-i-1) : l \geq 0, 0 \leq i \leq m, \\
& -(-1)^{s+1}\left(\frac{(4lk+4k-2i-1)\pi}{2k}\right)^{2s+1}\cot\frac{(2i+1)\pi}{4k}+\alpha = 0 \biggr\}.
\end{aligned}
$$

Obviously, $\dim X_\infty^0 < \infty$ and $\dim X_0^0 < \infty$. □

**Lemma 3.3** *Under assumptions* $(S_1)$ *and* $(S_2)$, *there is* $\sigma > 0$ *such that*

$$
\begin{aligned}
& \bigl\langle (L+P^{-1}\beta)x, x \bigr\rangle > \sigma \|x\|^2, \quad x \in X_\infty^+ \quad \text{and} \\
& \bigl\langle (L+P^{-1}\beta)x, x \bigr\rangle < -\sigma \|x\|^2, \quad x \in X_\infty^-.
\end{aligned}
\tag{27}
$$

*Proof* The proof is similar to Lemma 3.1, we omit it. □

**Lemma 3.4** *Under conditions* $(S_1)$ *and* $(S_2)$, *the functional* $\Psi$ *defined by* (16) *satisfies the* $(P.S)$*-condition.*

*Proof* The proof is similar to Lemma 3.2, we omit it. □

## 4 Notations and main results of this paper

We first give some notations.

For (5), if $(-1)^{s+1} = -1$, denote

$$
N_1(\alpha) = \begin{cases} -\sum_{i=0}^m \operatorname{card}\{l \geq 0 : 0 < (\frac{(4lk+4k-2i-1)\pi}{2k})^{2s+1}\tan\frac{(2i+1)\pi}{4k} < -\alpha\}, & \alpha < 0, \\ \sum_{i=0}^m \operatorname{card}\{l \geq 0 : 0 < (\frac{(4lk+2i+1)\pi}{2k})^{2s+1}\tan\frac{(2i+1)\pi}{4k} < \alpha\}, & \alpha \geq 0. \end{cases}
$$

$$
N_1(\beta) = \begin{cases} -\sum_{i=0}^m \operatorname{card}\{l \geq 0 : 0 < (\frac{(4lk+4k-2i-1)\pi}{2k})^{2s+1}\tan\frac{(2i+1)\pi}{4k} < -\beta\}, & \beta < 0, \\ \sum_{i=0}^m \operatorname{card}\{l \geq 0 : 0 < (\frac{(4lk+2i+1)\pi}{2k})^{2s+1}\tan\frac{(2i+1)\pi}{4k} < \beta\}, & \beta \geq 0. \end{cases}
$$

And

$$N_1^0(\alpha_-)=\sum_{i=0}^{m}\operatorname{card}\left\{l\geq 0:0<\left(\frac{(4lk+4k-2i-1)\pi}{2k}\right)^{2s+1}\tan\frac{(2i+1)\pi}{4k}=-\alpha\right\},\quad \alpha<0,$$

$$N_1^0(\alpha_+)=\sum_{i=0}^{m}\operatorname{card}\left\{l\geq 0:0<\left(\frac{(4lk+2i+1)\pi}{2k}\right)^{2s+1}\tan\frac{(2i+1)\pi}{4k}=\alpha\right\},\quad \alpha\geq 0,$$

$$N_1^0(\beta_-)=\sum_{i=0}^{m}\operatorname{card}\left\{l\geq 0:0<\left(\frac{(4lk+4k-2i-1)\pi}{2k}\right)^{2s+1}\tan\frac{(2i+1)\pi}{4k}=-\beta\right\},\quad \beta<0,$$

$$N_1^0(\beta_+)=\sum_{i=0}^{m}\operatorname{card}\left\{l\geq 0:0<\left(\frac{(4lk+2i+1)\pi}{2k}\right)^{2s+1}\tan\frac{(2i+1)\pi}{4k}=\beta\right\},\quad \beta\geq 0.$$

Alternatively, if $(-1)^{s+1}=1$, denote

$$N_1(\alpha)=\begin{cases}-\sum_{i=0}^{m}\operatorname{card}\{l\geq 0:0<(\frac{(4lk+2i+1)\pi}{2k})^{2s+1}\tan\frac{(2i+1)\pi}{4k}<-\alpha\}, & \alpha<0,\\ \sum_{i=0}^{m}\operatorname{card}\{l\geq 0:0<(\frac{(4lk+4k-2i-1)\pi}{2k})^{2s+1}\tan\frac{(2i+1)\pi}{4k}<\alpha\}, & \alpha\geq 0.\end{cases}$$

$$N_1(\beta)=\begin{cases}-\sum_{i=0}^{m}\operatorname{card}\{l\geq 0:0<(\frac{(4lk+2i+1)\pi}{2k})^{2s+1}\tan\frac{(2i+1)\pi}{4k}<-\beta\}, & \beta<0,\\ \sum_{i=0}^{m}\operatorname{card}\{l\geq 0:0<(\frac{(4lk+4k-2i-1)\pi}{2k})^{2s+1}\tan\frac{(2i+1)\pi}{4k}<\beta\}, & \beta\geq 0.\end{cases}$$

And

$$N_1^0(\alpha_-)=\sum_{i=0}^{m}\operatorname{card}\left\{l\geq 0:0<\left(\frac{(4lk+2i+1)\pi}{2k}\right)^{2s+1}\tan\frac{(2i+1)\pi}{4k}=-\alpha\right\},\quad \alpha<0,$$

$$N_1^0(\alpha_+)=\sum_{i=0}^{m}\operatorname{card}\left\{l\geq 0:0<\left(\frac{(4lk+4k-2i-1)\pi}{2k}\right)^{2s+1}\tan\frac{(2i+1)\pi}{4k}=\alpha\right\},\quad \alpha\geq 0,$$

$$N_1^0(\beta_-)=\sum_{i=0}^{m}\operatorname{card}\left\{l\geq 0:0<\left(\frac{(4lk+2i+1)\pi}{2k}\right)^{2s+1}\tan\frac{(2i+1)\pi}{4k}=-\beta\right\},\quad \beta<0,$$

$$N_1^0(\beta_+)=\sum_{i=0}^{m}\operatorname{card}\left\{l\geq 0:0<\left(\frac{(4lk+4k-2i-1)\pi}{2k}\right)^{2s+1}\tan\frac{(2i+1)\pi}{4k}=\beta\right\},\quad \beta\geq 0.$$

For (6), if $(-1)^{s+1}=-1$, denote

$$N_2(\alpha)=\begin{cases}-\sum_{i=0}^{m}\operatorname{card}\{l\geq 0:0<(\frac{(4lk+4k-2i-1)\pi}{2k})^{2s+1}\cot\frac{(2i+1)\pi}{4k}<-\alpha\}, & \alpha<0,\\ \sum_{i=0}^{m}\operatorname{card}\{l\geq 0:0<(\frac{(4lk+2i+1)\pi}{2k})^{2s+1}\cot\frac{(2i+1)\pi}{4k}<\alpha\}, & \alpha\geq 0.\end{cases}$$

$$N_2(\beta)=\begin{cases}-\sum_{i=0}^{m}\operatorname{card}\{l\geq 0:0<(\frac{(4lk+4k-2i-1)\pi}{2k})^{2s+1}\cot\frac{(2i+1)\pi}{4k}<-\beta\}, & \beta<0,\\ \sum_{i=0}^{m}\operatorname{card}\{l\geq 0:0<(\frac{(4lk+2i+1)\pi}{2k})^{2s+1}\cot\frac{(2i+1)\pi}{4k}<\beta\}, & \beta\geq 0.\end{cases}$$

And

$$N_2^0(\alpha_-)=\sum_{i=0}^{m}\operatorname{card}\left\{l\geq 0:0<\left(\frac{(4lk+4k-2i-1)\pi}{2k}\right)^{2s+1}\cot\frac{(2i+1)\pi}{4k}=-\alpha\right\},\quad \alpha<0,$$

$$N_2^0(\alpha_+) = \sum_{i=0}^{m} \operatorname{card}\left\{ l \geq 0 : 0 < \left( \frac{(4lk+2i+1)\pi}{2k} \right)^{2s+1} \cot \frac{(2i+1)\pi}{4k} = \alpha \right\}, \quad \alpha \geq 0,$$

$$N_2^0(\beta_-) = \sum_{i=0}^{m} \operatorname{card}\left\{ l \geq 0 : 0 < \left( \frac{(4lk+4k-2i-1)\pi}{2k} \right)^{2s+1} \cot \frac{(2i+1)\pi}{4k} = -\beta \right\}, \quad \beta < 0,$$

$$N_2^0(\beta_+) = \sum_{i=0}^{m} \operatorname{card}\left\{ l \geq 0 : 0 < \left( \frac{(4lk+2i+1)\pi}{2k} \right)^{2s+1} \cot \frac{(2i+1)\pi}{4k} = \beta \right\}, \quad \beta \geq 0.$$

Alternatively, if $(-1)^{s+1} = 1$, denote

$$N_2(\alpha) = \begin{cases} -\sum_{i=0}^{m} \operatorname{card}\{l \geq 0 : 0 < (\frac{(4lk+2i+1)\pi}{2k})^{2s+1} \cot \frac{(2i+1)\pi}{4k} < -\alpha\}, & \alpha < 0, \\ \sum_{i=0}^{m} \operatorname{card}\{l \geq 0 : 0 < (\frac{(4lk+4k-2i-1)\pi}{2k})^{2s+1} \cot \frac{(2i+1)\pi}{4k} < \alpha\}, & \alpha \geq 0. \end{cases}$$

$$N_2(\beta) = \begin{cases} -\sum_{i=0}^{m} \operatorname{card}\{l \geq 0 : 0 < (\frac{(4lk+2i+1)\pi}{2k})^{2s+1} \cot \frac{(2i+1)\pi}{4k} < -\beta\}, & \beta < 0, \\ \sum_{i=0}^{m} \operatorname{card}\{l \geq 0 : 0 < (\frac{(4lk+4k-2i-1)\pi}{2k})^{2s+1} \cot \frac{(2i+1)\pi}{4k} < \beta\}, & \beta \geq 0. \end{cases}$$

And

$$N_2^0(\alpha_-) = \sum_{i=0}^{m} \operatorname{card}\left\{ l \geq 0 : 0 < \left( \frac{(4lk+2i+1)\pi}{2k} \right)^{2s+1} \cot \frac{(2i+1)\pi}{4k} = -\alpha \right\}, \quad \alpha < 0,$$

$$N_2^0(\alpha_+) = \sum_{i=0}^{m} \operatorname{card}\left\{ l \geq 0 : 0 < \left( \frac{(4lk+4k-2i-1)\pi}{2k} \right)^{2s+1} \cot \frac{(2i+1)\pi}{4k} = \alpha \right\}, \quad \alpha \geq 0,$$

$$N_2^0(\beta_-) = \sum_{i=0}^{m} \operatorname{card}\left\{ l \geq 0 : 0 < \left( \frac{(4lk+2i+1)\pi}{2k} \right)^{2s+1} \cot \frac{(2i+1)\pi}{4k} = -\beta \right\}, \quad \beta < 0,$$

$$N_2^0(\beta_+) = \sum_{i=0}^{m} \operatorname{card}\left\{ l \geq 0 : 0 < \left( \frac{(4lk+4k-2i-1)\pi}{2k} \right)^{2s+1} \cot \frac{(2i+1)\pi}{4k} = \beta \right\}, \quad \beta \geq 0.$$

Now we give the main results of this paper.

**Theorem 4.1** *Suppose that* $(S_1)$ *and* $(S_2)$ *hold. Then Eq.* (5) *possesses at least*

$$n = \max\{N_1(\beta) - N_1(\alpha) - N_1^0(\beta_-) - N_1^0(\alpha_+), N_1(\alpha) - N_1(\beta) - N_1^0(\alpha_-) - N_1^0(\beta_+)\}$$

$4k$*-periodic solutions satisfying* $x(t-2k) = -x(t)$ *provided that* $n > 0$.

**Theorem 4.2** *Suppose that* $(S_1)$, $(S_2)$, $(S_3^+)$, *and* $(S_4^-)$ *hold. Then Eq.* (5) *possesses at least*

$$n = N_1(\beta) - N_1(\alpha) + N_1^0(\beta_+) + N_1^0(\alpha_-)$$

$4k$*-periodic solutions satisfying* $x(t-2k) = -x(t)$ *provided that* $n > 0$.

**Theorem 4.3** *Suppose that* $(S_1)$, $(S_2)$, $(S_3^-)$, *and* $(S_4^+)$ *hold. Then Eq.* (5) *possesses at least*

$$n = N_1(\alpha) - N_1(\beta) + N_1^0(\alpha_+) + N_1^0(\beta_-)$$

$4k$*-periodic solutions satisfying* $x(t-2k) = -x(t)$ *provided that* $n > 0$.

**Theorem 4.4** *Suppose that* $(S_1)$ *and* $(S_2)$ *hold. Then Eq.* (6) *possesses at least*

$$n = \max\bigl\{N_2(\beta) - N_2(\alpha) - N_2^0(\beta_-) - N_2^0(\alpha_+), N_2(\alpha) - N_2(\beta) - N_2^0(\alpha_-) - N_2^0(\beta_+)\bigr\}$$

$4k$*-periodic solutions satisfying* $x(t-2k) = -x(t)$ *provided that* $n > 0$.

**Theorem 4.5** *Suppose that* $(S_1)$, $(S_2)$, $(S_3^+)$, *and* $(S_4^-)$ *hold. Then Eq.* (6) *possesses at least*

$$n = N_2(\beta) - N_2(\alpha) + N_2^0(\beta_+) + N_2^0(\alpha_-)$$

$4k$*-periodic solutions satisfying* $x(t-2k) = -x(t)$ *provided that* $n > 0$.

**Theorem 4.6** *Suppose that* $(S_1)$, $(S_2)$, $(S_3^-)$, *and* $(S_4^+)$ *hold. Then Eq.* (6) *possesses at least*

$$n = N_2(\alpha) - N_2(\beta) + N_2^0(\alpha_+) + N_2^0(\beta_-)$$

$4k$*-periodic solutions satisfying* $x(t-2k) = -x(t)$ *provided that* $n > 0$.

## 5 Proof of main results of this paper

*Proof of Theorem* 4.1 Suppose without loss of generality that

$$n = N_1(\beta) - N_1(\alpha) - N_1^0(\beta_-) - N_1^0(\alpha_+).$$

Let $X^+ = X_\infty^+$ and $X^- = X_0^-$. Then

$$X \setminus \bigl(X^+ \cup X^-\bigr) = X \setminus \bigl(X_\infty^+ \cup X_0^-\bigr) \subseteq X_\infty^0 \cup X_0^0 \cup \bigl(X_\infty^+ \cap X_0^-\bigr).$$

Obviously,

$$\operatorname{codim}_X\bigl(X^+ + X^-\bigr) \le \dim X_\infty^0 + \dim X_0^0 + \dim\bigl(X_\infty^+ \cap X_0^-\bigr) < \infty,$$

which implies that condition (a) in Lemma 1.1 holds. Let $A_\infty = \beta$. Then condition (b) in Lemma 1.1 holds since, for each $j \in N$, we have that $x \in X(j)$ yields $(L + P^{-1}\beta)x \in X(j)$.

At the same time, Lemma 3.2 gives the $(P.S)$-condition.

Now it suffices to show that conditions (c) and (d) in Lemma 1.1 hold under assumptions $(S_1)$ and $(S_2)$.

In fact, we have shown in Lemma 3.1 that there is $\sigma > 0$ such that $\langle (L+P^{-1}\beta)x, x\rangle > \sigma\|x\|^2$, $x \in X_\infty^+$. And the second condition in (8) implies that $|F(x) - \frac{1}{2}\beta x^2| < \frac{1}{4}\sigma|x|^2 + M_1$, $x \in R$ for some $M_1 > 0$.

Then

$$\begin{aligned}\Phi(x) &= \frac{1}{2}\langle Lx, x\rangle + \int_0^{4k} F\bigl(x(t)\bigr)\,dt \\ &= \frac{1}{2}\bigl\langle \bigl(L + P^{-1}\beta\bigr)x, x\bigr\rangle + \int_0^{4k}\left[F\bigl(x(t)\bigr) - \frac{1}{2}\beta\bigl|x(t)\bigr|^2\right]dt\end{aligned}$$

$$\geq \frac{1}{2}\sigma\|x\|^2 - \frac{1}{4}\sigma\|x\|^2 - 4kM_1$$
$$\geq \frac{1}{4}\sigma\|x\|^2 - 4kM_1$$

if $x \in X^+$. Clearly, there is $c_0 \in R$ such that

$$\inf_{x\in X^+} \Phi(x) \geq c_0.$$

On the other hand, we have shown in Lemma 3.1 that there is $\sigma > 0$ such that $\langle (L + P^{-1}\beta)x, x\rangle < -\sigma\|x\|^2$, $x \in X_\infty^-$. And we can show that there are $r, \sigma > 0$ such that $|F(x) - \frac{1}{2}\beta x^2| < \frac{1}{4}\sigma|x|^2$, $\|x\| = r$. So

$$\begin{aligned}
\Phi(x) &= \frac{1}{2}\langle Lx, x\rangle + \int_0^{4k} F\bigl(x(t)\bigr)\,dt \\
&= \frac{1}{2}\bigl\langle (L + P^{-1}\beta)x, x\bigr\rangle + \int_0^{4k}\left[F\bigl(x(t)\bigr) - \frac{1}{2}\beta\bigl|x(t)\bigr|^2\right]dt \\
&\leq -\frac{1}{2}\sigma\|x\|^2 + \frac{1}{4}\sigma\|x\|^2 \\
&\leq -\frac{1}{4}\sigma\|x\|^2.
\end{aligned}$$

That is, there are $r > 0$ and $c_\infty < 0$ such that

$$\Phi(x) \leq c_\infty < 0 = \Phi(0), \quad \forall x \in X^- \cap S_r = \bigl\{x \in X : \|x\| = r\bigr\}.$$

Our last task is to compute the value of

$$\begin{aligned}
n &= \frac{1}{2}\Bigl[\dim\Bigl(X^+ \cap X^-\Bigr) - \operatorname{codim}_X\bigl(X^+ + X^-\bigr)\Bigr] \\
&= \frac{1}{2}\Bigl[\dim\Bigl(X_\infty^+ \cap X_0^-\Bigr) - \operatorname{codim}_X\bigl(X_\infty^+ + X_0^-\bigr)\Bigr] \\
&= \frac{1}{2}\sum_{j=0}^{\infty}\Bigl[\dim\Bigl(X_\infty^+(j) \cap X_0^-(j)\Bigr) - \operatorname{codim}_{X(j)}\bigl(X_\infty^+(j) + X_0^-(j)\bigr)\Bigr].
\end{aligned}$$

By computation we get that, for each $i \in \{0, 1, \ldots, m\}$,

$$\begin{aligned}
&\bigl\langle (L + P^{-1}\beta)x, x\bigr\rangle = \left((-1)^{s+1}\left(\frac{\pi}{2k}\right)^{2s+1}\tan\frac{(2i+1)\pi}{4k} + \frac{\beta}{(4lk + 2i + 1)^{2s+1}}\right)\|x\|^2, \\
&\quad x \in X(2lk + i), \\
&\bigl\langle (L + P^{-1}\beta)x, x\bigr\rangle = \left(-(-1)^{s+1}\left(\frac{\pi}{2k}\right)^{2s+1}\tan\frac{(2i+1)\pi}{4k} + \frac{\beta}{(4lk + 4k - 2i - 1)^{2s+1}}\right)\|x\|^2, \\
&\quad x \in X(2lk + 2k - i - 1),
\end{aligned}$$

and

$$\begin{aligned}
&\bigl\langle \bigl(L+P^{-1}\alpha\bigr)x,x\bigr\rangle = \left((-1)^{s+1}\left(\frac{\pi}{2k}\right)^{2s+1}\tan\frac{(2i+1)\pi}{4k}+\frac{\alpha}{(4lk+2i+1)^{2s+1}}\right)\|x\|^2,\\
&\quad x\in X(2lk+i),\\
&\bigl\langle \bigl(L+P^{-1}\alpha\bigr)x,x\bigr\rangle = \left(-(-1)^{s+1}\left(\frac{\pi}{2k}\right)^{2s+1}\tan\frac{(2i+1)\pi}{4k}+\frac{\alpha}{(4lk+4k-2i-1)^{2s+1}}\right)\|x\|^2,\\
&\quad x\in X(2lk+2k-i-1).
\end{aligned}$$

Therefore,

$X_\infty^+(2lk+i)=X_\infty^+\cap X(2lk+i)=\emptyset$,
$X_\infty^+(2lk+2k-i-1)=X_\infty^+\cap X(2lk+2k-i-1)=X(2lk+2k-i-1)$,
$X_0^-(2lk+i)=X_0^-\cap X(2lk+i)=X(2lk+i)$,
$X_0^-(2lk+2k-i-1)=X_0^-\cap X(2lk+2k-i-1)=\emptyset$

if $i\in\{0,1,\ldots,m\}$ and $(-1)^{s+1}=1$ and $l\geq 0$ is large enough, which means that there is $M>0$ such that $\dim(X_\infty^+(j)\cap X_0^-(j))-\operatorname{codim}_X(X_\infty^+(j)+X_0^-(j))=0$, $j>M$, from which it follows that

$$\begin{aligned}
n &= \frac{1}{2}\sum_{j=0}^{M}\Bigl[\dim\Bigl(X_\infty^+(j)\cap X_0^-(j)\Bigr)-\operatorname{codim}_{X(j)}\bigl(X_\infty^+(j)+X_0^-(j)\bigr)\Bigr]\\
&= \frac{1}{2}\sum_{j=0}^{M}\bigl[\dim X_\infty^+(j)+\dim X_0^-(j)-2\bigr]\\
&= \frac{1}{2}\sum_{j=0}^{M}\bigl[\dim X_\infty^+(j)+\dim X_0^-(j)\bigr]-(M+1).
\end{aligned}$$

Then we have

$$\begin{aligned}
&\sum_{j=0}^{M}\dim\bigl(X_\infty^+(j)\bigr)\\
&\quad = 2\begin{cases} N_1(\beta)+\operatorname{card}\{2lk+2k-i-1:0\leq 2lk+2k-i-1\leq M\}, & \beta\geq 0,\\ N_1(\beta)-N_1^0(\beta_-)+\operatorname{card}\{2lk+2k-i-1:0\leq 2lk+2k-i-1\leq M\}, & \beta<0,\end{cases}\\
&\sum_{j=0}^{M}\dim\bigl(X_0^-(j)\bigr) = 2\begin{cases} -N_1(\alpha)-N_1^0(\alpha_+)+\operatorname{card}\{2lk+i:0\leq 2lk+i\leq M\}, & \alpha\geq 0,\\ -N_1(\alpha)+\operatorname{card}\{2lk+i:0\leq 2lk+i\leq M\}, & \alpha<0,\end{cases}
\end{aligned}$$

and

$$\sum_{j=0}^{M}\bigl[\dim X_\infty^+(j)+\dim X_0^-(j)\bigr]=2\bigl[N_1(\beta)-N_1(\alpha)-N_1^0(\beta_-)-N_1^0(\alpha_+)\bigr]+2(M+1). \tag{28}$$

Therefore

$$n=N_1(\beta)-N_1(\alpha)-N_1^0(\beta_-)-N_1^0(\alpha_+).$$

The proof for the case $(-1)^{s+1}=-1$ is similar.

Theorem 4.1 is proved. □

*Proof of Theorem* 4.2 *and Theorem* 4.3 Since the proof for the two theorems is similar, we prove only Theorem 4.2.

Let $X^+ = X^+_\infty + X^0_\infty$, $X^- = X^0_- + X^0_0$. Then, as in the proof of Theorem 4.1, we check conditions (a), (b), (c), (d), and (e). In the present case, we may suppose that (28) still holds for some $M > 0$. Let $X^0_\infty(i) = X^0_\infty \cap X(i)$, $X^0_0(i) = X^0_0 \cap X(i)$. Then

$$\begin{aligned} n &= \frac{1}{2}\sum_{i=0}^{M}\Big[\dim\Big(X^+_\infty(i)\cap X^-_0(i)\Big) - \operatorname{codim}_{X(i)}\big(X^+_\infty(i)+X^-_0(i)\big)\Big] + \big(\dim X^0_\infty + \dim X^0_0\big) \\ &= \frac{1}{2}\sum_{i=0}^{M}\big[\dim X^+_\infty(i) + \dim X^-_0(i) - 2\big] + \big(\dim X^0_\infty + \dim X^0_0\big) \\ &= \frac{1}{2}\sum_{i=0}^{M}\big[\dim X^+_\infty(i) + \dim X^-_0(i)\big] - (M+1) + \big(\dim X^0_\infty + \dim X^0_0\big) \\ &= N(\beta) - N(\alpha) - N^0(\beta_-) - N^0(\alpha_+) + \big(N^0(\beta_+) + N^0(\beta_-) + N^0(\alpha_+) + N^0(\alpha_-)\big) \\ &= N(\beta) - N(\alpha) + N^0(\beta_+) + N^0(\alpha_-). \qquad \square \end{aligned}$$

*Proof of Theorem* 4.4, *Theorem* 4.5, *and Theorem* 4.6 Since the proof of Theorem 4.4 is similar to that of Theorem 4.1, and the proofs of Theorems 4.5 and 4.6 are similar to that of Theorem 4.2, we omit it. Our proof is completed. $\square$

## 6 Examples

*Example* 6.1 Suppose that $f \in C^0(R,R)$ satisfies

$$f(x) = \begin{cases} 5\pi^3 x + x^{\frac{1}{5}}, & |x| \gg 1, \\ -\pi^3 x - x^5, & |x| \ll 1. \end{cases}$$

We are to discuss the multiplicity of 8-periodic solutions of the equation

$$x^{(3)}(t) = -\sum_{i=1}^{3} f\big(x(t-i)\big). \tag{29}$$

In this case, $s = 1$, $(-1)^{s+1} = 1$, $k = 2$, $m = 1$, $\alpha = -\pi^3$, $\beta = 5\pi^3$. This yields that

$$\begin{aligned} N_1(\alpha) = {} & -\operatorname{card}\left\{l \ge 0 : 0 < \left(\frac{(8l+1)\pi}{4}\right)^3 \tan\frac{\pi}{8} < \pi^3\right\} \\ & -\operatorname{card}\left\{l \ge 0 : 0 < \left(\frac{(8l+3)\pi}{4}\right)^3 \tan\frac{3\pi}{8} < \pi^3\right\} = -1, \\ N_1(\beta) = {} & \operatorname{card}\left\{l \ge 0 : 0 < \left(\frac{(8l+8-1)\pi}{4}\right)^3 \tan\frac{\pi}{8} < 5\pi^3\right\} \\ & +\operatorname{card}\left\{l \ge 0 : 0 < \left(\frac{(8l+8-2-1)\pi}{4}\right)^3 \tan\frac{3\pi}{8} < 5\pi^3\right\} = 2, \end{aligned}$$

and $N^0_1(\alpha_+) = N^0_1(\beta_-) = N^0_1(\alpha_-) = N^0_1(\beta_+) = 0$.

Applying Theorem 4.2, we conclude that Eq. (29) possesses at least three different 8-periodic orbits satisfying $x(t-4) = -x(t)$.

*Example* 6.2 Suppose that $f \in C^0(R,R)$ satisfies

$$f(x) = \begin{cases} 3\pi x + x^{\frac{1}{3}}, & |x| \gg 1, \\ \pi x - x^3, & |x| \ll 1. \end{cases}$$

We are to discuss the multiplicity of 12-periodic solutions of the equation

$$x'(t) = -\sum_{i=1}^{5}(-1)^{i+1} f\bigl(x(t-i)\bigr). \tag{30}$$

In this case, $s = 0$, $(-1)^{s+1} = -1$, $k = 3$, $m = 2$, $\alpha = \pi$, $\beta = 3\pi$. This yields that

$$\begin{aligned} N_2(\alpha) = {} & \operatorname{card}\left\{ l \geq 0 : 0 < \frac{(12l+1)\pi}{6} \cot\frac{\pi}{12} < \pi \right\} \\ & + \operatorname{card}\left\{ l \geq 0 : 0 < \frac{(12l+3)\pi}{6} \cot\frac{3\pi}{12} < \pi \right\} \\ & + \operatorname{card}\left\{ l \geq 0 : 0 < \frac{(12l+5)\pi}{6} \cot\frac{5\pi}{12} < \pi \right\} = 4, \\ N_2(\beta) = {} & \operatorname{card}\left\{ l \geq 0 : 0 < \frac{(12l+1)\pi}{6} \cot\frac{\pi}{12} < 3\pi \right\} \\ & + \operatorname{card}\left\{ l \geq 0 : 0 < \frac{(12l+3)\pi}{6} \cot\frac{3\pi}{12} < 3\pi \right\} \\ & + \operatorname{card}\left\{ l \geq 0 : 0 < \frac{(12l+5)\pi}{6} \cot\frac{5\pi}{12} < 3\pi \right\} = 9, \end{aligned}$$

and $N_2^0(\alpha_+) = N_2^0(\beta_-) = N_2^0(\alpha_-) = N_2^0(\beta_+) = 0$.

Applying Theorem 4.5, we conclude that Eq. (30) possesses at least five different 12-periodic orbits satisfying $x(t-6) = -x(t)$.

**Acknowledgements**
The present research is supported by the National Natural Science Foundations of China (No. 61179031). The authors thank the referees for carefully reading the manuscript and for their valuable suggestions which have significantly improved the paper.

**Funding**
Not applicable.

**Availability of data and materials**
Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

## Publisher's Note



Received: 21 September 2018 Accepted: 9 December 2018 Published online: 05 January 2019

**References**


1. Benci, V.: On critical point theory for indefinite functionals in the presence of symmetries. Transl. Am. Math. Soc. **274**(2), 533–572 (1982)
2. Fannio, L.: Multiple periodic solutions of Hamiltonian systems with strong resonance at infinity. Discrete Contin. Dyn. Syst. **3**, 251–264 (1997)
3. Fei, G.: Multiple periodic solutions of differential delay equations via Hamiltonian systems (I). Nonlinear Anal. **65**, 25–39 (2006)
4. Fei, G.: Multiple periodic solutions of differential delay equations via Hamiltonian systems (II). Nonlinear Anal. **65**, 40–58 (2006)
5. Ge, W.: On the existence of periodic solutions of differential delay equations with multiple lags. Acta Math. Appl. Sin. **17**(2), 173–181 (1994) (in Chinese)
6. Ge, W.: Two existence theorems of periodic solutions for differential delay equations. Chin. Ann. Math. **15**(B2), 217–224 (1994)
7. Ge, W.: Periodic solutions of the differential delay equation $x'(t) = -f(x(t-1))$. Acta Math. Sin. New Ser. **12**, 113–121 (1996)
8. Ge, W.: Oscillatory periodic solutions of differential delay equations with multiple lags. Chin. Sci. Bull. **42**(6), 444–447 (1997)
9. Ge, W., Zhang, L.: Multiple periodic solutions of delay differential systems with $2k-1$ lags via variational approach. Discrete Contin. Dyn. Syst. **36**(9), 4925–4943 (2016)
10. Ge, W., Zhang, L.: Multiple periodic solutions of delay differential systems with $2k$ lags via variational approach. Preprint
11. Guo, Z., Yu, J.: Multiple results for periodic solutions to delay differential equations via critical point theory. J. Differ. Equ. **218**, 15–35 (2005)
12. Guo, Z., Yu, J.: Multiple results on periodic solutions to higher dimensional differential equations with multiple delays. J. Dyn. Differ. Equ. **23**, 1029–1052 (2011)
13. Kaplan, J., Yorke, J.: Ordinary differential equations which yield periodic solution of delay equations. J. Math. Anal. Appl. **48**, 317–324 (1974)
14. Li, J., He, X.: Multiple periodic solutions of differential delay equations created by asymptotically linear Hamiltonian systems. Nonlinear Anal. TMA **31**, 45–54 (1998)
15. Li, J., He, X.: Proof and generalization of Kaplan–Yorke conjecture under the condition $f'(0) > 0$ on periodic solution of differential delay equations. Sci. China Ser. A **42**(9), 957–964 (1999)
16. Li, J., He, X., Liu, Z.: Hamiltonian symmetric group and multiple periodic solutions of differential delay equations. Nonlinear Anal. TMA **35**, 957–964 (1999)
17. Li, L., Xue, C., Ge, W.: Periodic orbits to Kaplan–Yorke like differential delay equations with two lags of ratio. Adv. Differ. Equ. **2016**, 247 (2016)
18. Li, S., Liu, J.: Morse theory and asymptotically linear Hamiltonian systems. J. Differ. Equ. **78**, 53–73 (1989)
19. Mawhen, J., Willem, M.: Critical Point Theory and Hamiltonian Systems. Springer, New York (1989)
20. Nussbaum, R.: Periodic solutions of special differential delay equations: an example in nonlinear functional analysis. Proc. R. Soc. Edinb., Sect. A, Math. **81**(1–2), 131–151 (1977)