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New uniqueness results for fractional differential equation with dependence on the first order derivative

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Abstract

In this paper, we study the uniqueness of solutions for a fractional differential equation with dependence on the first order derivative. By means of Banach's contraction mapping principle and a weighted norm in product space, sufficient conditions for the uniqueness of solutions are investigated. An example is given to illustrate the main results.

Keywords: Fractional differential equation; Uniqueness results

1 Introduction

In this paper, we consider the following Dirichlet boundary value problem for fractional differential equation:

$$\begin{cases} D_{0+}^{\alpha}u(t) + f(t, u(t), u'(t)) = 0, & t \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{1.1}$$

where $1 < \alpha \le 2$ and $f \in C([0,1] \times \mathbb{R}^2, \mathbb{R})$. Here, $D_{0+}^{\alpha}u(t)$ denotes the standard Riemann–Liouville fractional derivative of $u : [0,1] \to \mathbb{R}$ defined by

$$D_{0+}^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_0^t \frac{u(s)}{(t-s)^{\alpha-n+1}}\,ds,$$

where $n - 1 \le \alpha < n$, provided that the right-hand side is pointwise defined on $(0, \infty)$.

The problem of the existence of solutions for fractional differential equation with various boundary conditions has received an increased attention by using variational methods and critical point theory, the theory of coincidence degree, some well-known fixed point theorems, upper and lower solution method; see the monographs of Kilbas et al. [17], Miller and Ross [21], Podlubny [23], the papers [1–5, 12, 14–16, 18, 19, 24–27, 29–31, 33, 35, 36, 38, 40–42], and the references therein. For example, Bai and Lü [5] considered the special case of BVP (1.1) that f does not contain first order derivative term u':

$$\begin{cases} D_{0+}^{\alpha}u(t) + f(t, u(t)) = 0, & t \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{1.2}$$

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where $1 < \alpha \leq 2$ and $f \in C([0,1] \times [0,+\infty), [0,+\infty))$. The authors obtained the existence and multiplicity of positive solutions by means of the Krasnosel'skii fixed point theorem and the Leggett–Williams fixed point theorem. In [18], the existence of at least one solution for BVP (1.1) is proved by the Leray–Schauder continuation principle. In [2], the authors investigated the fractional differential equations

$$\begin{cases} D_{0+}^{\alpha} u(t) + f(t, u(t), D_{0+}^{\mu} u(t)) = 0, & t \in (0,1), \\ u(0) = u(1) = 0, \end{cases}$$

where $1 < \alpha < 2$, $\mu > 0$ are real numbers, $\alpha - \mu \geq 1$, f is a Carathéodory function, and $f(t,x,y)$ is singular at $x = 0$. The authors obtained the existence of positive solutions based on regularization and sequential techniques. Recently, the authors of [9] proved uniqueness results for BVP (1.2) by means of Banach's contraction mapping principle and the theory of linear operator.

At present, many papers are devoted to the uniqueness results for BVP; see [6–11, 13, 20, 22, 28, 34, 37, 39]. Some nonlinear analytical techniques have been used to study the uniqueness of solutions for differential equation and differential systems such as the method of Banach's contraction mapping principle, fixed point theorems for mixed monotone operators, the maximal principle, u_0-positive operator, and linear operator theory. On the other hand, there are some papers studying fractional differential equations and fractional differential systems in which the fractional orders are involved in the nonlinearity, we refer the reader to [2, 18, 32]. Motivated by the results above, utilizing Banach's contraction mapping principle, we investigate the uniqueness result for solution of BVP (1.1).

It should noted here that our main result has various new system features. First of all, BVP (1.1) is reformulated as a fixed point problem for system of integral equations. Second, a weighted norm in product space is introduced. Third, the first order derivative is involved in the nonlinear terms.

Throughout the paper, we assume that the following condition holds:

(H) $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function and there exist constants $A, B > 0$ such that

$$\left| f(t,u_1,v_1) - f(t,u_2,v_2) \right| \leq A|u_1 - u_2| + B|v_1 - v_2|, \quad t \in [0,1],$$

for all $u_1, u_2, v_1, v_2 \in \mathbb{R}$.

2 Preliminaries

Define two functions G, G_1 as follows:

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} (t(1-s))^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \leq s \leq t \leq 1, \\ (t(1-s))^{\alpha-1}, & 0 \leq t \leq s \leq 1, \end{cases} \tag{2.1}$$

$$G_1(t,s) = \frac{\alpha - 1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-2}(1-s)^{\alpha-1} - (t-s)^{\alpha-2}, & 0 \leq s \leq t \leq 1, \\ t^{\alpha-2}(1-s)^{\alpha-1}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{2.2}$$

Lemma 2.1 ([5]) *Let G be as in* (2.1). *Then*

$$0 \le G(t,s) \le \frac{1}{\Gamma(\alpha)} t^{\alpha-1}(1-s)^{\alpha-1}, \quad t,s \in [0,1].$$

After routine calculation we get the following four inequalities:

$$\begin{aligned}
&\Gamma(\alpha)\int_0^1 G(t,s)s^{\alpha-1}\,ds \\
&\quad= \int_0^1 \big(t(1-s)\big)^{\alpha-1}s^{\alpha-1}\,ds - \int_0^t (t-s)^{\alpha-1}s^{\alpha-1}\,ds \\
&\quad= t^{\alpha-1}B(\alpha,\alpha) - t^{2\alpha-1}\int_0^1 (1-s)^{\alpha-1}s^{\alpha-1}\,ds \\
&\quad= B(\alpha,\alpha)t^{\alpha-1}\big(1-t^{\alpha}\big) \le B(\alpha,\alpha)t^{\alpha-1}, \qquad (2.3)
\end{aligned}$$

$$\begin{aligned}
&\Gamma(\alpha)\int_0^1 G(t,s)s^{\alpha-2}\,ds \\
&\quad= \int_0^1 \big(t(1-s)\big)^{\alpha-1}s^{\alpha-2}\,ds - \int_0^t (t-s)^{\alpha-1}s^{\alpha-2}\,ds \\
&\quad= t^{\alpha-1}B(\alpha,\alpha-1) - t^{2\alpha-2}B(\alpha,\alpha-1) \le B(\alpha,\alpha-1)t^{\alpha-1}, \qquad (2.4)
\end{aligned}$$

$$\begin{aligned}
&\frac{\Gamma(\alpha)}{\alpha-1}\int_0^1 \big|G_1(t,s)\big|s^{\alpha-1}\,ds \\
&\quad\le t^{\alpha-2}\int_0^1 (1-s)^{\alpha-1}s^{\alpha-1}\,ds + \int_0^t (t-s)^{\alpha-2}s^{\alpha-1}\,ds \\
&\quad= t^{\alpha-2}B(\alpha,\alpha) + t^{2\alpha-2}B(\alpha-1,\alpha) \\
&\quad\le \big(B(\alpha,\alpha) + B(\alpha-1,\alpha)\big)t^{\alpha-2}, \qquad (2.5)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\Gamma(\alpha)}{\alpha-1}\int_0^1 \big|G_1(t,s)\big|s^{\alpha-2}\,ds \\
&\quad\le t^{\alpha-2}\int_0^1 (1-s)^{\alpha-1}s^{\alpha-2}\,ds + \int_0^t (t-s)^{\alpha-2}s^{\alpha-2}\,ds \\
&\quad= t^{\alpha-2}B(\alpha,\alpha-1) + t^{2\alpha-3}B(\alpha-1,\alpha-1) \\
&\quad\le \big(B(\alpha,\alpha-1) + B(\alpha-1,\alpha-1)\big)t^{\alpha-2}. \qquad (2.6)
\end{aligned}$$

Here, $B(\alpha,\beta)$ is the beta function defined by the Euler integral:

$$B(\alpha,\beta) = \int_0^1 s^{\alpha-1}(1-s)^{\beta-1}\,ds.$$

Lemma 2.2 ([5, 18]) *Let* $h \in C[0,1]$. *Then*

$$u(t) = \int_0^1 G(t,s)h(s)\,ds$$

is the unique solution of

$$\begin{cases} D_{0+}^{\alpha} u(t) + h(t) = 0, & t \in (0,1), \\ u(0) = u(1) = 0. \end{cases}$$

Moreover, $u' \in C(0,1] \cap AC_{\mathrm{loc}}(0,1]$, $\lim_{t\to 0^+} t^{2-\alpha} u'(t)$ *exists and satisfies*

$$u'(t) = \int_0^1 G_1(t,s)h(s)\,ds.$$

We set $E_1 = C[0,1]$ with the usual maximum norm denoted by $\|u\|_{E_1} = \max_{t\in(0,1]} |u(t)|$. Consider the Banach space

$$E_2 = \Bigl\{ v \in C(0,1] : \lim_{t\to 0^+} t^{2-\alpha} v(t) \text{ exists} \Bigr\}$$

with the norm $\|v\|_{E_2} = \sup_{t\in(0,1]} |t^{2-\alpha} v(t)|$. Then $E_1 \times E_2$ is a Banach space with the norm

$$\bigl\|(u,v)\bigr\|_{E_1\times E_2} = \max\bigl\{ \|u\|_{E_1}, \|v\|_{E_2} \bigr\}.$$

According to Lemma 2.2, BVP (1.1) has a solution $u = u(t)$ if and only if $(u,v) \in E_1 \times E_2$ solves the following integral equations:

$$\begin{cases} u(t) = \int_0^1 G(t,s) f(s,u(s),v(s))\,ds, \\ v(t) = \int_0^1 G_1(t,s) f(s,u(s),v(s))\,ds, \end{cases}$$

with $v = u'$. Define an operator T by

$$T(u,v) = \bigl(T_1(u,v), T_2(u,v)\bigr), \quad (u,v) \in E_1 \times E_2,$$

where operators T_1, T_2 are defined by

$$T_1(u,v)(t) = \int_0^1 G(t,s) f\bigl(s,u(s),v(s)\bigr)\,ds, \quad (u,v) \in E_1 \times E_2,$$
$$T_2(u,v)(t) = \int_0^1 G_1(t,s) f\bigl(s,u(s),v(s)\bigr)\,ds, \quad (u,v) \in E_1 \times E_2,$$

respectively. For $(u,v) \in E$, by Lemma 2.1 and (H), we have

$$\begin{aligned} &\bigl|T_1(u,v)(t)\bigr| \\ &\quad \le \int_0^1 G(t,s)\bigl|f\bigl(s,u(s),v(s)\bigr) - f(s,0,0)\bigr|\,ds + \int_0^1 G(t,s)\bigl|f(s,0,0)\bigr|\,ds \\ &\quad \le \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 \bigl(A\bigl|u(s)\bigr| + B\bigl|v(s)\bigr|\bigr)(1-s)^{\alpha-1}\,ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 \bigl|f(s,0,0)\bigr|\,ds \\ &\quad \le \frac{A\|u\|_{E_1}}{\Gamma(\alpha+1)} t^{\alpha-1} + \frac{B\|v\|_{E_2}}{\Gamma(\alpha)} \int_0^1 s^{\alpha-2}(1-s)^{\alpha-1}\,ds \cdot t^{\alpha-1} \end{aligned}$$

$$+\frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}\big|f(s,0,0)\big|\,ds$$

$$=\left(\frac{A\|u\|_{E_1}}{\Gamma(\alpha+1)}+\frac{B\Gamma(\alpha-1)\|v\|_{E_2}}{\Gamma(2\alpha-1)}+\frac{1}{\Gamma(\alpha)}\int_0^1\big|f(s,0,0)\big|\,ds\right)t^{\alpha-1}, \tag{2.7}$$

which implies that T_1 is well defined on E and $T_1(u,v)\in E_1$. In the same way, we can prove that T_2 is well defined on E and $T_2(u,v)\in E_2$ for all $(u,v)\in E$. Thus, the existence of a solution of BVP (1.1) is equivalent to the existence of a fixed point of T on $E_1\times E_2$.

It follows from (2.7) that T_1 maps all of $E_1\times E_2$ into the following vector subspace of E_1:

$$E_3=\left\{u\in E_1:\frac{|u(t)|}{t^{\alpha-1}}\text{ are bounded for }t\in[0,1]\right\}.$$

Clearly, E_3 is a Banach space with the norm

$$\|u\|_{E_3}=\sup_{t\in(0,1]}t^{1-\alpha}\big|u(t)\big|.$$

Hence, in the following we only need to consider the fixed points of T in the Banach space $E=E_3\times E_2$ with the weighted norm

$$\big\|(u,v)\big\|_E=\max\left\{\|u\|_{E_3},\frac{\|v\|_{E_2}}{\theta}\right\}$$

with a constant $\theta>0$.

Lemma 2.3 *Let $a,d\in[0,1)$, $b,c\in[0,+\infty)$ with $(1-d)(1-a)>bc$. Then the system of inequalities*

$$\begin{cases}a+b\theta\le\lambda,\\ \frac{c}{\theta}+d\le\lambda\end{cases} \tag{2.8}$$

has a solution (λ,θ) with $\lambda\in(0,1)$ and $\theta>0$.

Proof For the case $bc=0$, we may take $\lambda=\max\{\frac{d+1}{2},\frac{d+1}{2}\}$. So it remains to consider the case $bc\neq0$. Let

$$\phi(x)=(x-d)(x-a)-bc,\quad x\in\mathbb{R}.$$

It follows from the derivative of $\phi(x)$ that $\phi(x)$ is increasing on $[\frac{a+d}{2},1]$. With the help of the locally sign-preserving property of $\phi(x)$, we conclude that there exists $\lambda\in[\frac{a+d}{2},1)$ such that

$$(\lambda-d)(\lambda-a)\ge bc.$$

The above inequality is equivalent to $\frac{c}{\lambda-d}\le\frac{\lambda-a}{b}$. Therefore (2.8) holds for $\theta\in[\frac{c}{\lambda-d},\frac{\lambda-a}{b}]$. □

3 Main results

Set

$$a_{11}=\frac{A\Gamma(\alpha)}{\Gamma(2\alpha)},\qquad a_{21}=\frac{A(\alpha-1)(B(\alpha,\alpha)+B(\alpha-1,\alpha))}{\Gamma(\alpha)},$$
$$a_{12}=\frac{B\Gamma(\alpha-1)}{\Gamma(2\alpha-1)},\qquad a_{22}=\frac{B(\alpha-1)(B(\alpha,\alpha-1)+B(\alpha-1,\alpha-1))}{\Gamma(\alpha)}.$$

Now, we show that a uniqueness result follows from Banach's contraction mapping principle.

Theorem 3.1 *Suppose that condition* (H) *holds. In addition, we assume that the four constants* $a_{11}, a_{12}, a_{21}, a_{22}$ *satisfy*

$$a_{11}<1,\qquad a_{22}<1,$$

and

$$(1-a_{11})(1-a_{22})>a_{12}a_{21}.$$

Then BVP (1.1) *has a unique solution.*

Proof It follows from Lemma 2.3 that there exist $\lambda\in(0,1)$ and $\theta>0$ such that

$$\begin{cases} a_{11}+a_{12}\theta\le\lambda,\\ \frac{a_{21}}{\theta}+a_{22}\le\lambda.\end{cases}\tag{3.1}$$

We shall apply Banach's contraction mapping principle in $E=E_3\times E_2$ endowed with the weighted norm

$$\|(u,v)\|_E=\max\left\{\|u\|_{E_3},\frac{\|v\|_{E_2}}{\theta}\right\}$$

with $\theta>0$ as in (3.1). More precisely, we prove that

$$\|T(u_1,v_1)-T(u_2,v_2)\|_E\le\lambda\|(u_1,v_1)-(u_2,v_2)\|_E$$

for all $(u_1,v_1),(u_2,v_2)\in E$. In fact, by (2.3), (2.4), and (3.1), we have

$$\begin{aligned}&\left|T_1(u_1,v_1)(t)-T_1(u_2,v_2)(t)\right|\\ &\quad\le\int_0^1 G(t,s)\left|f\left(s,u_1(s),v_1(s)\right)-f\left(s,u_2(s),v_2(s)\right)\right|ds\\ &\quad\le\int_0^1 G(t,s)\left(A\left|u_1(s)-u_2(s)\right|+B\left|v_1(s)-v_2(s)\right|\right)ds\\ &\quad\le\int_0^1 G(t,s)\left(A\|u_1-u_2\|_{E_3}s^{\alpha-1}+B\|v_1-v_2\|_{E_2}s^{\alpha-2}\right)ds\end{aligned}$$

$$\leq \left(a_{11}\|u_1-u_2\|_{E_3}+a_{12}\theta\frac{\|v_1-v_2\|_{E_2}}{\theta}\right)\cdot t^{\alpha-1}$$
$$\leq \lambda\big\|(u_1,v_1)-(u_2,v_2)\big\|_E\cdot t^{\alpha-1}.$$

Hence, from the definition of $\|\cdot\|_{E_3}$, we have

$$\big\|T_1(u_1,v_1)-T_1(u_2,v_2)\big\|_{E_3}\leq \lambda\big\|(u_1,v_1)-(u_2,v_2)\big\|_E. \tag{3.2}$$

Similar, we have

$$\frac{1}{\theta}\big|T_2(u_1,v_1)(t)-T_2(u_2,v_2)(t)\big|$$
$$\leq \frac{1}{\theta}\int_0^1\big|G_1(t,s)\big|\big|f\big(s,u_1(s),v_1(s)\big)-f\big(s,u_2(s),v_2(s)\big)\big|\,ds$$
$$\leq \frac{1}{\theta}\int_0^1\big|G_1(t,s)\big|\big(A\big|u_1(s)-u_2(s)\big|+B\big|v_1(s)-v_2(s)\big|\big)\,ds$$
$$\leq \frac{1}{\theta}\int_0^1\big|G_1(t,s)\big|\big(A\|u_1-u_2\|_{E_3}s^{\alpha-1}+B\|v_1-v_2\|_{E_2}s^{\alpha-2}\big)\,ds$$
$$\leq \left(\frac{1}{\theta}\cdot a_{21}\|u_1-u_2\|_{E_3}+a_{22}\frac{\|v_1-v_2\|_{E_2}}{\theta}\right)\cdot t^{\alpha-2}$$
$$\leq \lambda\big\|(u_1,v_1)-(u_2,v_2)\big\|_E\cdot t^{\alpha-2}.$$

Thus, we have

$$\frac{1}{\theta}\big\|T_2(u_1,v_1)-T_2(u_2,v_2)\big\|_{E_2}\leq \lambda\big\|(u_1,v_1)-(u_2,v_2)\big\|_E. \tag{3.3}$$

Now, both inequalities (3.2) and (3.3) can be rewritten equivalently as

$$\big\|T(u_1,v_1)-T(u_2,v_2)\big\|_E\leq \lambda\big\|(u_1,v_1)-(u_2,v_2)\big\|_E.$$

Note that $\lambda\in(0,1)$. The uniqueness result follows from Banach's contraction mapping principle. □

In what follows, we give an example to illustrate the application of our results.

Example 3.1 Consider the BVP

$$\begin{cases} D_{0+}^{\frac{3}{2}}u(t)+\frac{1}{3\sqrt{\pi}}\sin(6u(t)+u'(t))+h(t)=0, \\ u(0)=u(1)=0, \end{cases} \tag{3.4}$$

where $t\in[0,1]$, $\alpha=\frac{3}{2}$, and $h\in C[0,1]$. Let $f(t,u,v)=\frac{1}{3\sqrt{\pi}}\sin(6u+v)+h(t)$. It is easy to see that $|f(t,u_1,v_1)-f(t,u_2,v_2)|\leq \frac{2}{\sqrt{\pi}}|u_1-u_2|+\frac{1}{3\sqrt{\pi}}|v_1-v_2|$ for all $t\in[0,1]$, $u_1,u_2,v_1,v_2\in\mathbb{R}$. Then, for $\alpha=\frac{3}{2}$, $A=\frac{2}{\sqrt{\pi}}$, and $B=\frac{1}{3\sqrt{\pi}}$, we have that $a_{11}=\frac{1}{2}$, $a_{12}=\frac{1}{6}$, $a_{21}=\frac{5}{4}$, and $a_{22}=\frac{1}{2}$. So $(1-a_{11})(1-a_{22})>a_{12}a_{21}$ holds, and from Theorem 3.1 BVP (3.4) has a unique solution.

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Authors' contributions
The authors have made the same contribution. All authors read and approved the final manuscript.

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