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New aspects of Opial-type integral inequalities

Yasemin Başcı[1*] and Dumitru Baleanu[2,3]

*Correspondence:
basci_y@ibu.edu.tr
[1]Department of Mathematics, Faculty of Arts and Sciences, Bolu Abant Izzet Baysal University, Bolu, Turkey
Full list of author information is available at the end of the article

Abstract

In this manuscript, we prove new aspects for several Opial-type integral inequalities for the left and right Caputo–Fabrizio operators with nonsingular kernel. For this purpose we use the inequalities obtained by Andrić et al. (Integral Transforms Spec. Funct. 25(4):324–335, 2014), which is the generalization of an inequality of Agarwal and Pang (Opial Inequalities with Applications in Differential and Difference Equations, 1995). Besides, examples are presented to validate the reported results.

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1 Introduction and preliminaries

Since the discovery of Opial's inequality, it has found interesting applications. Really, Opial's inequality and its generalizations, extensions, and discretizations have been playing an important role in the study of the existence and uniqueness of initial and boundary value problems for ordinary and partial differential equations besides difference equations [3, 4, 21, 39, 42].

In 1960, Opial [43] obtained the following integral inequality:

Theorem 1.1 *Let* $x(t) \in C^1[0,h]$ *be such that* $x(0) = x(h) = 0$ *and* $x(t) > 0$ *in* $(0,h)$. *Then the following integral inequality holds*:

$$\int_0^h \left|x(t)x'(t)\right| dt \leq \frac{h}{4} \int_0^h \left(x(t)\right)^2 dt. \tag{1.1}$$

Here $\frac{h}{4}$ *is a constant best possibility.*

From that time, Opial's inequality [43] has been studied extensively by many mathematicians. This inequality has been extended, generalized in different ways, see [2, 5, 6, 22, 24–28, 39, 40, 44–47, 54]. Also, various mathematicians studied Opial-type integral inequalities for different types of fractional derivative and integral operators involving Caputo, Canavati, Riemann–Liouville, and so on, see [9, 14, 16–19, 29–31] and the references therein.

In 2000, Anastassiou [7] obtained Opial-type inequalities involving functions and their ordinary and fractional derivatives. In 2002, Anastassiou and Goldstein [12] presented the Opial-type inequalities involving fractional derivatives of different orders. The same year, Anastassiou et al. [13] studied a class of L_p-type Opial inequalities for generalized fractional derivatives for integrable functions based on the results obtained earlier by the first author in 1998. In 2004, Anastassiou [8] established the Opial-type inequalities including fractional derivatives of two functions in different order and power. In 2008, he presented Opial-type inequalities involving Riemann–Liouville fractional derivatives of two functions with applications, see [9]. Also, in 2009, he presented fractional Opial-type inequalities subject to high order boundary conditions in L_p for $p > 1$, and in 2012, he extended Opial's integral inequality using the right and left Caputo as well as Riemann–Liouville fractional derivatives, respectively, see [10, 11].

In 2013, Andrić et al. [17] obtained several Opial-type inequalities including Caputo, Canavati, and Riemann–Liouville fractional derivatives. The same year, they presented developments of composition identities for the Caputo fractional derivatives. They gave applications to Opial-type inequalities in [18]. Also, the same year, they studied some Opial-type inequalities for Riemann–Liouville fractional derivatives obtained by Fink in [34] and Pang and Agarwal in [48], see [19].

In 2014, Andrić et al. [15] gave expansions of the Opial-type integral inequalities. Also, they presented a generalization of an inequality obtained by Agarwal and Pang [4].

In 2015, Farid et al. [32] studied the Opial-type inequalities by using generalized fractional integral operator including the Mittag-Leffler function in the kernel. One year later, they presented Opial-type integral inequalities for Hilfer differential and fractional integral operators involving a generalized Mittag-Leffler function in the kernel, see [33].

In 2017, Tomovski et al. [53] gave the generalization of weighted Opial-type inequalities for fractional integral and differential operators involving generalized Mittag-Leffler functions by using Hölder's integral inequality motivated by the work of Koliha and Pečarić [38].

In 2017, Sarıkaya and Budak [50] obtained new inequalities of Opial-type for conformable integrals.

Recently, researchers have proposed different fractional-time operators from the well-known Riemann–Liouville operator, see [20, 35–37, 52]. They are defined by nonsingular memory kernels. Also, they used these new operators to generalize the usual diffusion equation. In fact, these new operators can describe better the evolution of some dynamics of complex systems which cannot be done within the standard fractional calculus operators (for more details, see Refs. [35–37] and the references therein).

The purpose of this paper is to establish some Opial-type integral inequalities for the left and right operators with nonsingular kernel. The organization of this paper is given below. The introduction is given in Sect. 1. In Sect. 2, basic definitions and theorems are introduced. Motivated by [4] and [15], we establish several Opial-type inequalities in Sect. 3. Several examples are given for our results in Sect. 4.

2 Basic definitions and theorems

In this section, we present the following theorems, corollaries, and definitions which are useful in the proofs of our results.

Let $U_1(u,K_1)$ denote the class of functions $v:[a_1,b_1]\to\mathbb{R}$ with the representation

$$v(t)=\int_{a_1}^{t}K_1(t,s)u(s)\,ds.$$

Here, the function u is continuous and K_1 is an arbitrary nonnegative kernel function such that $u(t)>0$ implies $v(t)>0$ for all $t\in[a_1,b_1]$. Similarly, let $U_2(u,K_1)$ denote the class of functions $v:[a_1,b_1]\to\mathbb{R}$ with the representation

$$v(t)=\int_{t}^{b_1}K_1(t,s)u(s)\,ds.$$

We suppose that all integrals exist. Also, they are finite.

Theorem 2.1 ([15]) *Let $\psi:[0,\infty)\to\mathbb{R}$ be a differentiable function such that, for $q_1>1$, $\psi(t^{1/q_1})$ is a convex function and $\psi(0)=0$. Also, let $v\in U_1(u,K_1)$ such that $(\int_{a_1}^{t}(K_1(t,s))^{p_1}\,ds)^{1/p_1}\le C$ and $\frac{1}{p_1}+\frac{1}{q_1}=1$. Then*

$$\begin{aligned}&\int_{a_1}^{b_1}\bigl|v(t)\bigr|^{1-q_1}\psi'\bigl(\bigl|v(t)\bigr|\bigr)\bigl|u(t)\bigr|^{q_1}\,dt\\&\quad\le\frac{q_1}{C^{q_1}}\psi\biggl(C\biggl(\int_{a_1}^{b_1}\bigl|u(t)\bigr|^{q_1}\,dt\biggr)^{1/q_1}\biggr)&&(2.1)\\&\quad\le\frac{q_1}{C^{q_1}(b_1-a_1)}\int_{a_1}^{b_1}\psi\bigl((b_1-a_1)^{1/q_1}C\bigl|u(t)\bigr|\bigr)\,dt.&&(2.2)\end{aligned}$$

If $\psi(t^{1/q_1})$ is a concave function, then reverse inequalities are valid.

When $\psi(x)=t^{p_1+q_1}$, the following corollary is obtained.

Corollary 2.1 ([15]) *Let $v\in U_1(u,K_1)$ where $(\int_{a_1}^{t}(K_1(t,s))^{p_1}\,ds)^{1/p_1}\le C$ and $\frac{1}{p_1}+\frac{1}{q_1}=1$. Then*

$$\begin{aligned}\int_{a_1}^{b_1}\bigl|v(t)\bigr|^{p_1}\bigl|u(t)\bigr|^{q_1}\,dt&\le\frac{q_1C^{p_1}}{p_1+q_1}\biggl(\int_{a_1}^{b_1}\bigl|u(t)\bigr|^{q_1}\,dt\biggr)^{(p_1+q_1)/q_1}\\&\le\frac{q_1C^{p_1}(b_1-a_1)^{p_1/q_1}}{p_1+q_1}\int_{a_1}^{b_1}\bigl|u(t)\bigr|^{p_1+q_1}\,dt.&&(2.3)\end{aligned}$$

Theorem 2.2 ([15]) *Let the function $\psi:[0,\infty)\to\mathbb{R}$ be differentiable such that, for $q_1>1$, $\psi(t^{1/q_1})$ is a convex function and $\psi(0)=0$. Let $v\in U_2(u,K_1)$ such that $(\int_{t}^{b_1}(K_1(t,s))^{p_1}\,ds)^{1/p_1}\le C$ and $\frac{1}{p_1}+\frac{1}{q_1}=1$. Then*

$$\begin{aligned}&\int_{a_1}^{b_1}\bigl|v(t)\bigr|^{1-q_1}\psi'\bigl(\bigl|v(t)\bigr|\bigr)\bigl|u(t)\bigr|^{q_1}\,dt\\&\quad\le\frac{q_1}{C^{q_1}}\psi\biggl(C\biggl(\int_{a_1}^{b_1}\bigl|u(t)\bigr|^{q_1}\,dt\biggr)^{1/q_1}\biggr)\\&\quad\le\frac{q_1}{C^{q_1}(b_1-a_1)}\int_{a_1}^{b_1}\psi\bigl((b_1-a_1)^{1/q_1}C\bigl|u(t)\bigr|\bigr)\,dt.&&(2.4)\end{aligned}$$

If $\psi(t^{1/q_1})$ is a concave function, then reverse inequalities are valid.

When $\psi(x) = t^{p_1+q_1}$, the following corollary is obtained.

Corollary 2.2 ([15]) *Let $v \in U_2(u,K_1)$ where $(\int_t^{b_1}(K_1(t,s))^{p_1}\,ds)^{1/p_1} \le C$ and $\frac{1}{p_1} + \frac{1}{q_1} = 1$. Then*

$$\int_{a_1}^{b_1} |v(t)|^{p_1} |u(t)|^{q_1}\,dt \le \frac{q_1 C^{p_1}}{p_1+q_1}\left(\int_{a_1}^{b_1} |u(t)|^{q_1}\,dt\right)^{(p_1+q_1)/q_1}$$
$$\le \frac{q_1 C^{p_1}(b_1-a_1)^{p_1/q_1}}{p_1+q_1}\int_{a_1}^{b_1} |u(t)|^{p_1+q_1}\,dt. \tag{2.5}$$

Below, we show the definitions of the left and right operators with nonsingular kernel introduced in [23]. According to [1, 23], if $g \in H^1(a_1,b_1)$, $0 < a_1 < b_1 \le \infty$, $\alpha \in (0,1)$, the left operator ${}^{CFR}_{a_1}D^{\alpha}$ is defined by

$$\left({}^{CFR}_{a_1}D^{\alpha} g\right)(t) = \frac{M(\alpha)}{1-\alpha}\frac{d}{dt}\int_{a_1}^{t} g(s)\exp\bigl(\lambda(t-s)\bigr)\,ds \tag{2.6}$$

and the right operator ${}^{CFR}D^{\alpha}_{b_1}$ is defined by

$$\left({}^{CFR}D^{\alpha}_{b_1} g\right)(t) = -\frac{M(\alpha)}{1-\alpha}\frac{d}{dt}\int_{t}^{b_1} g(s)\exp\bigl(\lambda(s-t)\bigr)\,ds, \tag{2.7}$$

with $\lambda = -\frac{\alpha}{1-\alpha}$ and $t \ge a_1$. Here $M(\alpha)$ is a normalization function depending on α. Also, the left integral operator is defined as

$$\left({}^{CF}_{a_1}I^{\alpha} g\right)(t) = \frac{1-\alpha}{B(\alpha)} g(t) + \frac{\alpha}{B(\alpha)}\int_{a_1}^{t} g(s)\,ds \tag{2.8}$$

and the right integral operator is defined as

$$\left({}^{CF}I^{\alpha}_{b_1} g\right)(t) = \frac{1-\alpha}{B(\alpha)} g(t) + \frac{\alpha}{B(\alpha)}\int_{t}^{b_1} g(s)\,ds. \tag{2.9}$$

Definition 2.1 ([49]) Let f and g be two functions that are piecewise continuous on every finite closed interval $0 \le t \le b$ and of exponential order. The function denoted by $f * g$ and defined by

$$f(t) * g(t) = \int_0^t f(s) g(t-s)\,ds$$

is called the convolution of the functions f and g.

Definition 2.2 ([51]) Let $f(x)$ and $g(x)$ be positive and be in L^1. Moreover, they are differentiable and their derivative is integrable. Then the derivative of a convolution is

$$(f * g)' = f' * g = f * g'.$$

3 Main results

In this section, we give the Opial-type integral inequalities for the left and right of the operator using the inequalities obtained by Andrić et al. [15], which is the generalization of an inequality of Agarwal and Pang [4].

The following result is obtained by using Theorem 2.1 and the left operator.

Theorem 3.1 *Let $\psi : [0,\infty) \to \mathbb{R}$ be a differentiable function such that, for $q_1 > 1$, $\psi(t^{1/q_1})$ is a convex function and $\psi(0) = 0$. Also, let $0 < \alpha < 1$, $g \in H^1(a_1, b_1)$, and let ${}^{CFR}_{a_1}D^{\alpha}$ be defined by (2.6). If $\frac{1}{p_1} + \frac{1}{q_1} = 1$, then the following inequalities hold*:

$$\begin{aligned}
&\int_{a_1}^{b_1} \bigl|\bigl({}^{CFR}_{a_1}D^{\alpha}g\bigr)(t)\bigr|^{1-q_1} \psi'\bigl(\bigl|\bigl({}^{CFR}_{a_1}D^{\alpha}g\bigr)(t)\bigr|\bigr)\bigl|g'(t)\bigr|^{q_1}\,dt \\
&\quad \le \frac{q_1}{C^{q_1}} \psi\biggl(C\biggl(\int_{a_1}^{b_1} \bigl|g'(t)\bigr|^{q_1}\,dt\biggr)^{1/q_1}\biggr) \\
&\quad \le \frac{q_1}{C^{q_1}(b_1-a_1)} \int_{a_1}^{b_1} \psi\bigl((b_1-a_1)^{1/q_1} C\bigl|g'(t)\bigr|\bigr)\,dt,
\end{aligned} \tag{3.1}$$

where

$$C = \frac{M(\alpha)}{1-\alpha}\biggl(\frac{1-\exp(p_1\lambda(b_1-a_1))}{-p_1\lambda}\biggr)^{1/p_1}. \tag{3.2}$$

If $\psi(t^{1/q_1})$ is a concave function, then reverse inequalities hold.

Proof For $t \in [a_1, b_1]$, let

$$\begin{aligned}
v(t) = \bigl({}^{CFR}_{a_1}D^{\alpha}g\bigr)(t) &= \frac{M(\alpha)}{1-\alpha}\frac{d}{dt}\int_{a_1}^{t} g(s)\exp\bigl(\lambda(t-s)\bigr)\,ds \\
&= \frac{M(\alpha)}{1-\alpha}\frac{d}{dt}\bigl(g(t) * \exp(\lambda t)\bigr) \\
&= \frac{M(\alpha)}{1-\alpha}\biggl(\frac{dg}{dt}(t) * \exp(\lambda t)\biggr) \\
&= \frac{M(\alpha)}{1-\alpha}\int_{a_1}^{t} g'(s)\exp\bigl(\lambda(t-s)\bigr)\,ds,
\end{aligned} \tag{3.3}$$

$$K_1(t,s) = \begin{cases} \frac{M(\alpha)}{1-\alpha}\exp(\lambda(t-s)), & a_1 \le s \le t; \\ 0, & t \le s \le b_1, \end{cases}$$

and

$$\phi(t) = \biggl(\int_{a_1}^{t} \bigl(K_1(t,s)\bigr)^{p_1}\,ds\biggr)^{1/p_1} = \frac{M(\alpha)}{1-\alpha}\biggl(\frac{1-\exp(p_1\lambda(t-a_1))}{-p_1\lambda}\biggr)^{1/p_1}.$$

From $\lambda < 0$, the function ϕ is increasing on $[a_1, b_1]$. Thus, we can write

$$\max_{t\in[a_1,b_1]} \phi(t) = \frac{M(\alpha)}{1-\alpha}\biggl(\frac{1-\exp(p_1\lambda(b_1-a_1))}{-p_1\lambda}\biggr)^{1/p_1} = C.$$

Then $(\int_{a_1}^{t}(K_1(t,s))^{p_1}\,ds)^{1/p_1} \le C$. Also, if it is taken as $u = g'$ and v as in (3.3), then from Theorem 2.1 it gives us (3.1) in Theorem 3.1. This completes the proof. □

When $\psi(t) = t^{p_1+q_1}$ in Theorem 3.1, the following corollary is obtained.

Corollary 3.1 *Let $0 < \alpha < 1$, $g \in H^1(a_1,b_1)$, and let ${}^{CFR}_{a_1}D^{\alpha}$ be defined by (2.6). Also let $\frac{1}{p_1} + \frac{1}{q_1} = 1$. Then the following inequalities hold:*

$$\begin{aligned}\int_{a_1}^{b_1}\bigl|\bigl({}^{CFR}_{a_1}D^{\alpha}g\bigr)(t)\bigr|^{p_1}\bigl|g'(t)\bigr|^{q_1}\,dt &\le \frac{q_1C^{p_1}}{p_1+q_1}\biggl(\int_{a_1}^{b_1}\bigl|g'(t)\bigr|^{q_1}\,dt\biggr)^{(p_1+q_1)/q_1}\\ &\le \frac{q_1C^{p_1}(b_1-a_1)^{p_1/q_1}}{p_1+q_1}\int_{a_1}^{b_1}\bigl|g'(t)\bigr|^{p_1+q_1}\,dt,\end{aligned} \tag{3.4}$$

where C is defined as in (3.2).

Theorem 3.2 *Let the function $\psi : [0,\infty) \to \mathbb{R}$ be differentiable such that, for $q_1 > 1$, $\psi(t^{1/q_1})$ is a convex function and $\psi(0) = 0$. Also, let $0 < \alpha < 1$, $g \in H^1(a_1,b_1)$, and let ${}^{CFR}D^{\alpha}_{b_1}$ be defined by (2.7). If $\frac{1}{p_1} + \frac{1}{q_1} = 1$, then the following inequalities hold:*

$$\begin{aligned}&\int_{a_1}^{b_1}\bigl|\bigl({}^{CFR}D^{\alpha}_{b_1}g\bigr)(t)\bigr|^{1-q_1}\psi'\bigl(\bigl|\bigl({}^{CFR}D^{\alpha}_{b_1}g\bigr)(t)\bigr|\bigr)\bigl|g'(t)\bigr|^{q_1}\,dt\\ &\quad\le \frac{q_1}{C^{q_1}}\psi\biggl(C\biggl(\int_{a_1}^{b_1}\bigl|g'(t)\bigr|^{q_1}\,dt\biggr)^{1/q_1}\biggr)\\ &\quad\le \frac{q_1}{C^{q_1}(b_1-a_1)}\int_{a_1}^{b_1}\psi\bigl((b_1-a_1)^{1/q_1}C\bigl|g'(t)\bigr|\bigr)\,dt,\end{aligned} \tag{3.5}$$

where C is defined as in (3.2). If $\psi(t^{1/q_1})$ is a concave function, then reverse inequalities hold.

Proof Using the same method as the proof of Theorem 3.1, inequalities follow from Theorem 2.2. □

When $\psi(t) = t^{p_1+q_1}$ in Theorem 3.2, the following corollary is obtained.

Corollary 3.2 *Let $0 < \alpha < 1$, $g \in H^1(a_1,b_1)$, and let ${}^{CFR}D^{\alpha}_{b_1}$ be defined by (2.7). Also let $\frac{1}{p_1} + \frac{1}{q_1} = 1$. Then the following inequalities hold:*

$$\begin{aligned}\int_{a_1}^{b_1}\bigl|\bigl({}^{CFR}D^{\alpha}_{b_1}g\bigr)(t)\bigr|^{p_1}\bigl|g'(t)\bigr|^{q_1}\,dt &\le \frac{q_1C^{p_1}}{p_1+q_1}\biggl(\int_{a_1}^{b_1}\bigl|g'(t)\bigr|^{q_1}\,dt\biggr)^{(p_1+q_1)/q_1}\\ &\le \frac{q_1C^{p_1}(b_1-a_1)^{p_1/q_1}}{p_1+q_1}\int_{a_1}^{b_1}\bigl|g'(t)\bigr|^{p_1+q_1}\,dt,\end{aligned} \tag{3.6}$$

where C is defined as in (3.2).

The next result is obtained by using Theorem 2.1 and the left integral operator, see for more details [41].

Theorem 3.3 *Let the function $\psi : [0,\infty) \to \mathbb{R}$ be differentiable such that, for $q_1 > 1$, $\psi(t^{1/q_1})$ is a convex function and $\psi(0) = 0$. Also, let $0 < \alpha < 1$, $g \in H^1(a_1,b_1)$, and let ${}^{CF}_{a_1}I^\alpha$ be defined by (2.8). If $\frac{1}{p_1} + \frac{1}{q_1} = 1$, then the following inequalities hold*:

$$\begin{aligned}
&\int_{a_1}^{b_1} \left| \left({}^{CF}_{a_1}I^\alpha g\right)(t) - \frac{1-\alpha}{B(\alpha)} g(t) \right|^{1-q_1} \psi'\left(\left| \left({}^{CF}_{a_1}I^\alpha g\right)(t) - \frac{1-\alpha}{B(\alpha)} g(t) \right| \right) \left|g(t)\right|^{q_1} \, dt \\
&\quad \le \frac{q_1}{C_1^{q_1}} \psi\left(C_1 \left(\int_{a_1}^{b_1} \left|g(t)\right|^{q_1} \, dt \right)^{1/q_1} \right) \\
&\quad \le \frac{q_1}{C_1^{q_1}(b_1 - a_1)} \int_{a_1}^{b_1} \psi\left((b_1 - a_1)^{1/q_1} C_1 \left|g(t)\right| \right) dt,
\end{aligned} \tag{3.7}$$

where

$$C_1 = \frac{\alpha}{B(\alpha)} (b_1 - a_1)^{1/p_1}. \tag{3.8}$$

If $\psi(t^{1/q_1})$ is a concave function, then reverse inequalities hold.

Proof For $t \in [a_1, b_1]$, let

$$\begin{aligned}
&v(t) = \left({}^{CF}_{a_1}I^\alpha g\right)(t) - \frac{1-\alpha}{B(\alpha)} g(t), \\
&K_1(t,s) = \begin{cases} \frac{\alpha}{B(\alpha)}, & a_1 \le s \le t; \\ 0, & t \le s \le b_1, \end{cases}
\end{aligned} \tag{3.9}$$

and

$$\phi(t) = \left(\int_{a_1}^{t} \left(K_1(t,s)\right)^{p_1} \, ds \right)^{1/p_1} = \frac{\alpha}{B(\alpha)} (t - a_1)^{1/p_1}.$$

From $\lambda < 0$, the function ϕ is increasing on $[a_1, b_1]$. Thus, we can write

$$\max_{t \in [a_1,b_1]} \phi(t) = \frac{\alpha}{B(\alpha)} (b_1 - a_1)^{1/p_1} = C_1.$$

Then $(\int_{a_1}^{t} (K_1(t,s))^{p_1} \, ds)^{1/p_1} \le C_1$. Also, if it is taken as $u = g$ and v as in (3.9), then from Theorem 2.1 it gives us (3.7) in Theorem 3.3. This completes the proof. □

When $\psi(t) = t^{p_1+q_1}$ in Theorem 3.3, we obtain the following corollary.

Corollary 3.3 *Let $0 < \alpha < 1$, $g \in H^1(a_1,b_1)$, and let ${}^{CF}_{a_1}I^\alpha$ be defined by (2.8). Also let $\frac{1}{p_1} + \frac{1}{q_1} = 1$. Then the following inequalities hold*:

$$\begin{aligned}
&\int_{a_1}^{b_1} \left| \left({}^{CF}_{a_1}I^\alpha g\right)(t) - \frac{1-\alpha}{B(\alpha)} g(t) \right|^{p_1} \left|g(t)\right|^{q_1} \, dt \\
&\qquad \le \frac{q_1 C_1^{p_1}}{p_1 + q_1} \left(\int_{a_1}^{b_1} \left|g(t)\right|^{q_1} \, dt \right)^{(p_1+q_1)/q_1}
\end{aligned}$$

$$\leq \frac{q_1 C_1^{p_1}(b_1-a_1)^{p_1/q_1}}{p_1+q_1}\int_{a_1}^{b_1}\left|g(t)\right|^{p_1+q_1}\,dt, \tag{3.10}$$

where C_1 is defined as in (3.8).

Theorem 3.4 *Let the function $\psi:[0,\infty)\to\mathbb{R}$ be differentiable such that, for $q_1>1$, $\psi(t^{1/q_1})$ is a convex function and $\psi(0)=0$. Also, let $0<\alpha<1$, $g\in H^1(a_1,b_1)$, and let ${}^{CF}I_{b_1}^{\alpha}$ be defined by* (2.9). *If $\frac{1}{p_1}+\frac{1}{q_1}=1$, then the following inequalities hold*:

$$\begin{aligned}
&\int_{a_1}^{b_1}\left|\left({}^{CF}I_{b_1}^{\alpha}g\right)(t)-\frac{1-\alpha}{B(\alpha)}g(t)\right|^{1-q_1}\psi'\left(\left|\left({}^{CF}I_{b_1}^{\alpha}g\right)(t)-\frac{1-\alpha}{B(\alpha)}g(t)\right|\right)\left|g(t)\right|^{q_1}\,dt\\
&\quad\leq \frac{q_1}{C_1^{q_1}}\psi\left(C_1\left(\int_{a_1}^{b_1}\left|g(t)\right|^{q_1}\,dt\right)^{1/q_1}\right)\\
&\quad\leq \frac{q_1}{C_1^{q_1}(b_1-a_1)}\int_{a_1}^{b_1}\psi\left((b_1-a_1)^{1/q_1}C_1\left|g(t)\right|\right)dt,
\end{aligned}\tag{3.11}$$

where C_1 is defined as in (3.8). *If $\psi(t^{1/q_1})$ is a concave function, then reverse inequalities hold.*

Proof Using the same method as the proof of Theorem 3.1, inequalities follow from Theorem 2.2. □

When $\psi(t)=t^{p_1+q_1}$ in Theorem 3.4, we obtain the following corollary.

Corollary 3.4 *Let $0<\alpha<1$, $g\in H^1(a_1,b_1)$, and let ${}^{CF}I_{b_1}^{\alpha}$ be defined by* (2.9). *Also let $\frac{1}{p_1}+\frac{1}{q_1}=1$. Then the following inequalities hold*:

$$\begin{aligned}
&\int_{a_1}^{b_1}\left|\left({}^{CF}I_{b_1}^{\alpha}g\right)(t)-\frac{1-\alpha}{B(\alpha)}g(t)\right|^{p_1}\left|g(t)\right|^{q_1}\,dt\\
&\quad\leq \frac{q_1 C_1^{p_1}}{p_1+q_1}\left(\int_{a_1}^{b_1}\left|g(t)\right|^{q_1}\,dt\right)^{(p_1+q_1)/q_1}\\
&\quad\leq \frac{q_1 C_1^{p_1}(b_1-a_1)^{p_1/q_1}}{p_1+q_1}\int_{a_1}^{b_1}\left|g(t)\right|^{p_1+q_1}\,dt,
\end{aligned}\tag{3.12}$$

where C_1 is defined as in (3.8).

4 Examples

Below, we will show the application of our main results with two examples.

Example 4.1 In Corollary 3.1, let $g(t)=e^t$, $\alpha=\frac{1}{2}$, $p_1=q_1=2$, $M(\alpha)=\alpha$, and $t\in[a_1,b_1]=[1,3]$. Then $\lambda=-1$, $M(\frac{1}{2})=\frac{1}{2}$, and $C=\sqrt{\frac{1-e^{-4}}{2}}$. So, we obtain

$$\begin{aligned}
\left({}_{1}^{CFR}D^{\frac{1}{2}}g\right)(t)=\left({}_{1}^{CFR}D^{\frac{1}{2}}g\right)(t)&=\frac{d}{dt}\int_1^t e^s e^{-(t-s)}\,ds=\int_1^t\frac{\partial}{\partial t}\left(e^{-t+2s}\right)ds+e^t\\
&=-\int_1^t e^{-t+2s}\,ds+e^t=\frac{e^t+e^{-t+2}}{2}.
\end{aligned}$$

Then we apply Corollary 3.1 to obtain the following inequalities:

$$\begin{aligned}\int_1^3 \left|\left({}_{1}^{CFR}D^{\frac{1}{2}}g\right)(t)\right|^2 \left|g'(t)\right|^2 dt &= \int_1^3 \left|\frac{e^t + e^{-t+2}}{2}\right|^2 e^{2t}\,dt \\ &\le \frac{1-e^{-4}}{4}\left(\int_1^3 e^{2t}\,dt\right)^2 \le \frac{1-e^{-4}}{2}\int_1^3 e^{4t}\,dt.\end{aligned}$$

Example 4.2 In Corollary 3.3, let $g(t) = \sin t$, $\alpha = \frac{1}{2}$, $p_1 = q_1 = 2$, $B(\alpha) = 1 - \alpha$, and $t \in [a_1, b_1] = [\frac{\pi}{2}, \pi]$. Then $\lambda = -1$, $B(\frac{1}{2}) = \frac{1}{2}$, and $C_1 = \sqrt{\pi - \frac{\pi}{2}}$. So, we obtain

$$\left({}_{\frac{\pi}{2}}^{CF}I^{\frac{1}{2}}g\right)(t) = \left({}_{\frac{\pi}{2}}^{CF}I^{\frac{1}{2}}\sin\right)(t) = \sin t + \int_{\frac{\pi}{2}}^{t} \sin(s)\,ds = \sin t - \cos t.$$

Then we apply Corollary 3.3 to obtain the following inequalities:

$$\begin{aligned}&\int_{\frac{\pi}{2}}^{\pi} \left|\left({}_{\frac{\pi}{2}}^{CF}I^{\frac{1}{2}}g\right)(t) - \frac{1-\alpha}{B(\alpha)}g(t)\right|^2 \left|g(t)\right|^2 dt \\ &\quad= \int_{\frac{\pi}{2}}^{\pi} |\sin t - \cos t - \sin t|^2 |\sin t|^2\,dt \\ &\quad= \int_{\frac{\pi}{2}}^{\pi} \cos^2 t \sin^2 t\,dt \le \frac{\pi - \frac{\pi}{2}}{2}\left(\int_{\frac{\pi}{2}}^{\pi} \sin^2 t\,dt\right)^2 \\ &\quad\le \frac{\pi}{4}\left(\pi - \frac{\pi}{2}\right)\int_{\frac{\pi}{2}}^{\pi} \sin^4 t\,dt.\end{aligned}$$

5 Conclusion

Caputo–Fabrizio operator has recently started to play an important role in modeling of a class of real world dissipative phenomena [35]. In fact some real data have confirmed that this operator is important for describing the dynamics of specific classes of real world problems. At the same time new mathematical generalizations of this operator were developed. In our manuscript, with the help of inequalities obtained by Andrić et al. [15], we proposed, within four theorems and their related corollaries, several Opial-type integral inequalities for Caputo–Fabrizio operators. Finally, we analyzed two illustrative examples carefully. The results reported in this manuscript can find applications within the evaluation of the existence and uniqueness of initial and boundary value problems related to diffusion process within the Caputo–Fabrizio operators.

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Author details
[1]Department of Mathematics, Faculty of Arts and Sciences, Bolu Abant Izzet Baysal University, Bolu, Turkey. [2]Department of Mathematics, Faculty of Arts and Sciences, Çankaya University, Ankara, Turkey. [3]Institutes of Space Sciences, Magurele-Bucharest, Romania.

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