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Hyers–Ulam stability and existence criteria for coupled fractional differential equations involving p-Laplacian operator

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Abstract

In this article, by using nonlinear Leray–Schauder-type alternative and Banach's fixed point theorem, we investigate existence and uniqueness of solutions. We also prove Hyers–Ulam stability for the proposed coupled system of fractional differential equations (FDEs) with the nonlinear p-Laplacian operator and Riemann–Liouville integral boundary conditions (IBCs). An illustrative example is presented to demonstrate our main results.

Keywords: Fractional differential equations; Riemann–Liouville integral boundary conditions; p-Laplacian operator; Hyers–Ulam stability; Banach's fixed point theorem

1 Introduction

Fractional calculus is a generalization of classical calculus. FDEs are proved to be valuable tools compared to integer order differential equations. Different physical and natural phenomena were modeled using FDEs in various disciplines, such as combustion theory, papulation biology, non-Newtonian mechanics, control theory, aerodynamics, hydro- and electro-dynamics, economics, bioengineering, networking, system of Monge–Kantorovich partial differential equations, image and signal processing, viscoelasticity, blood flow, game theory, chemistry; for details we refer to [1–10].

Recently, differential equations involving the nonlinear p-Laplacian operator gained attention from researchers. Articles dealing with ordinary differential equations and partial differential equations involving the nonlinear p-Laplacian operator have been studied. In the last few years, the mentioned area was extended to FDEs using different fractional order integrals and differential operators. By different mathematical approaches FDEs were investigated for existence and uniqueness of solutions, as well as for multiple positive solutions [11–20].

Further, FDEs involving the nonlinear p-Laplacian operator with integral boundary conditions have attracted researchers of various disciplines, because such systems are increasingly used in modeling [21–24]. Zhi et al. [14] have studied FDEs with the p-Laplacian nonlinear operator aiming to show existence of positive solutions for a nonlocal boundary value problem and have given a valuable example to demonstrate the results. The problem is given by

$$\begin{aligned}
&\big(\varphi_p\big(\mathcal{D}^{\theta_1}\omega(x)\big)\big)'' = \mathcal{F}\big(x,\omega(x),\mathcal{D}^{\theta_2}\omega(x)\big), \quad x \in (0,1),\\
&\omega(x)|_{x=0} = \omega''(x)\big|_{x=0} = 0, \qquad \omega(1) = \int_0^1 g(\theta)\omega(\theta)\,d\theta,\\
&\big(\varphi_p\big(\mathcal{D}^{\theta_1}\big)\omega(0)\big)' = \xi_1\big(\varphi_p\big(\mathcal{D}^{\theta_1}\big)\omega(a_1)\big)',\\
&\varphi_p\big(\mathcal{D}^{\theta_1}\big)\omega(1) = \xi_2\big(\varphi_p\big(\mathcal{D}^{\theta_1}\big)\omega(b_2)\big),
\end{aligned}$$

where $\mathcal{D}^{\theta_1}$, $\mathcal{D}^{\theta_2}$ represent derivatives of order θ_1, θ_2 in Caputo sense, where $2 < \theta_1 \leq 3$, $1 < \theta_2 < \theta_1 - 1 < 2$, $0 \leq \xi_1, \xi_2 < 1$, $0 < a_1 \leq b_2 < 1$, and φ_p is the nonlinear p-Laplacian operator. Hu and Zhang [25] investigated a coupled system of FDEs with the nonlinear p-Laplacian operator and infinite boundary conditions, namely the following problem:

$$\begin{aligned}
&\mathcal{D}^{\theta_1^*}\varphi_p\big(\mathcal{D}^{\theta_1}\omega_1(r)\big) = h\big(r,\omega_2(r),\mathcal{D}^{\theta_2-1},\mathcal{D}^{\theta_2-1}\omega_2(r),\dots,\mathcal{D}^{\theta_2-(n-1)}\omega_2(r)\big), \quad r \in (0,1),\\
&\mathcal{D}^{\theta_2^*}\varphi_p\big(\mathcal{D}^{\theta_2}y(r)\big) = g\big(r,\omega_1(r),\mathcal{D}^{\theta_1-1},\mathcal{D}^{\theta_1-1}\omega_1(r),\dots,\mathcal{D}^{\theta_1-(n-1)}\omega_1(r)\big), \quad r \in (0,1),\\
&\omega_1'(0) = \cdots = \omega_1^{(n-1)}(0) = \mathcal{D}^{\theta_1}\omega_1(0) = 0, \qquad \omega_1(0) = \sum_{i=1}^{\infty} a_i\omega_1(\mu_i),\\
&\omega_2'(0) = \cdots = \omega_2^{(n-1)}(0) = \mathcal{D}^{\theta_2}\omega_2(0) = 0, \qquad \omega_2(0) = \sum_{i=1}^{\infty} b_i\omega_2(\nu_i),
\end{aligned}$$

where $\mathcal{D}^{\theta_i^*}$, $\mathcal{D}^{\theta_i}$ for $i = 1, 2$, are Caputo fractional derivatives, $0 < \theta_1^*, \theta_2^* < 1$, $n-1 < \theta_1, \theta_2 < n$, $0 < \mu_1 < \mu_2 < \cdots < \mu_i < \cdots < 1$, $0 < \nu_1 < \nu_2 < \cdots < \nu_i < \cdots < 1$, $\sum_{i=1}^{\infty}|a_i| < \infty$, $\sum_{i=1}^{\infty}|b_i| < \infty$, $\sum_{i=1}^{\infty} a_i = \sum_{i=1}^{\infty} b_i = 1$, and h, g are real-valued continuous functions.

Recently, Ali et al. [26] investigated a coupled system of fractional differential equations with noninteger order integral boundary conditions for the existence and uniqueness of solutions, and furthermore, checked Hyers–Ulam stability. By using the topological degree theory, some special conditions were developed to show stability. As an application, an expressive example was provided to demonstrate the considered problem, which is given below:

$$\begin{aligned}
&\mathcal{D}^{\theta_1}\omega_1(r) - f\big(r,\omega_2(r)\big) = 0, \quad r \in [0,1],\\
&\mathcal{D}^{\theta_2}\omega_2(r) - g\big(r,\omega_1(r)\big) = 0, \quad r \in [0,1],\\
&\omega_1(0) = 0, \qquad \omega_1(r)|_{t=1} = \frac{1}{\Gamma(\gamma)}\int_0^T (r-s)^{\gamma-1}p\big(\omega_1,(s)\big)\,ds,\\
&\omega_2(0) = 0, \qquad \omega_2(r)|_{t=1} = \frac{1}{\Gamma(\delta)}\int_0^T (r-s)^{\delta-1}q\big(\omega_2(s)\big)\,ds,
\end{aligned}$$

where $\mathcal{D}^{\theta_1}$, $\mathcal{D}^{\theta_2}$ are Caputo fractional derivatives, $\theta_1, \theta_2, \gamma, \delta \in (1,2]$, $p, q \in L[0,1]$ and $f, g \in J \times \mathcal{R} \to \mathcal{R}$, the functions involved in fractional IBCs are continuous and also satisfy certain growth conditions.

Inspired by the aforementioned work, we investigate existence and uniqueness of solutions and prove Hyres–Ulam stability for the coupled suggested problem FDEs involving the nonlinear p-Laplacian operator with integral boundary conditions, namely

$$\begin{aligned}
&\mathcal{D}^{\theta_1^*}\varphi_p\big(\mathcal{D}^{\theta_1}\omega_1(r)\big) = f\big(r,\omega_1(r),\omega_2(r)\big)\big), \quad r\in[0,1],\\
&\mathcal{D}^{\theta_2^*}\varphi_p\big(\mathcal{D}^{\theta_2}\omega_2(r)\big) = g\big(r,\omega_1(r),\omega_2(r)\big)\big), \quad r\in[0,1],\\
&\varphi_p\big(\mathcal{D}^{\theta_1}\omega_1(r)\big)\big|_{r=0} = \omega_1'(r)\big|_{r=0} = 0, \qquad \omega_1(r)|_{r=0} = \gamma I^{\alpha-1}\omega_1(\eta),\\
&\varphi_p\big(\mathcal{D}^{\theta_2}\omega_2(r)\big)\big|_{r=0} = \omega_2'(r)|_{r=0} = 0, \qquad \omega_2(r)\big|_{r=0} = \delta I^{\beta-1}\omega_2(\xi),
\end{aligned} \tag{1.1}$$

where $\mathcal{D}^{\theta_i}$, $\mathcal{D}^{\theta_i^*}$ for $i = 1,2$ are Caputo fractional derivatives, $\theta_i \in (1,2]$, $\theta_i^* \in (0,1]$, $f,g \in C([0,1]\times\mathcal{R}^2,\mathcal{R})$, $i = 1,2$ are continuous functions and $\alpha,\beta \geq 1$, $\gamma,\delta \in [-1,0]$ $\varphi_p(\vartheta) = |\vartheta|^{p-2}\vartheta$ is the p-Laplacian operator where $1/p + 1/q = 1$, φ_q denotes the inverse of p-Laplacian. For existence and uniqueness of solutions, employing nonlinear Leray–Schauder-type alternative and Banach's fixed point theorem, new results are obtained for the coupled considered FDEs involving the nonlinear p-Laplacian operator with IBCs. The important aspect of this article is to check stability for the coupled considered FDEs involving the nonlinear p-Laplacian operator. In the literature, different types of stability were presented for functional, differential and integral equations, for example, Lyapunov and exponential stability [27, 28]. But checking such stability is difficult and time consuming due to calculation of Lyapunov functions. An interesting and motivating stability method was introduced by Ulam and then by Hyers in 1941, which is known as Hyers–Ulam stability [29, 30]. Such stability has outstanding applications in integer order and fractional order differential equations appearing in physics, optimization, numerical analysis, biological phenomena, economic, biochemistry, etc. For the details of using Hyres–Ulam stability, we suggest [9, 31–33]. In the following sections, we provide necessary definitions, lemmas, assumptions, as well as decribe the stability method and an example for the coupled considered FDEs with the nonlinear p-Laplacian operator with integral boundary conditions.

2 Background material and auxiliary results

Let us introduce $\mathcal{X} = \{\omega_1(r)|\omega_1(r)\in\mathcal{C}^1([0,1])\}$ as the space all continuous functions, endowed with a norm $\|\omega_1\| = \max\{\omega_1(r), r\in[0,1]\}$, here $(\mathcal{X},\|\cdot\|)$ is obviously a Banach space, let $\mathcal{Y} = \{\omega_2(r)|\omega_2(r)\in\mathcal{C}^1\}$. Then the product space denoted by $(\mathcal{X}\times\mathcal{Y},\|(\omega_1,\omega_2)\|)$, equipped with the norm $\|(\omega_1,\omega_2)\| = \|\omega_1\| + \|\omega_2\|$, is also a Banach space. This will be used throughout in the considered coupled FDEs with the nonlinear p-Laplacian with IBCs. Now recall the following definition which can be traced to [34–36].

Definition 2.1 Let $\digamma$ be a given function on a closed interval $[0,b]$. Then the non-integer order derivative in the Caputo sense of $\digamma$ is defined by

$$\mathcal{D}^{\theta_1}\digamma(r) = \int_0^r \frac{(r-\tau)^{n-\theta_1-1}}{\Gamma(n-\theta_1)}\left(\frac{d^n}{d\tau^n}\digamma(\tau)\right)d\tau, \quad \theta_1\in(n-1,n],$$

where $n-1 = [\theta_1]$. In particular, if $\digamma$ is defined on the interval $[0,b]$ and $\theta_1\in(0,1)$, then

$$\mathcal{D}^{\theta_1}\digamma(r) = \frac{1}{\Gamma(1-\theta_1)}\int_0^r \frac{\digamma'(\tau)}{(r-\tau)^{\theta_1}}\,d\tau, \quad \text{where } \varphi'(s) = \frac{d\varphi(s)}{ds}.$$

It is to be noted that the integral on the right-hand side is pointwise defined on $\mathcal{R}^+$.

Definition 2.2 The integral of arbitrary order $\theta_1 \in \mathcal{R}^+$ in the Riemann–Liouville sense for a function $\varphi : \mathcal{R}^+ \to \mathcal{R}$ is given as

$$I^{\theta_1} \digamma(r) = \frac{1}{\Gamma(\theta_1)} \int_0^r (r-\tau)^{\theta_1 - 1} \digamma(\tau)\, d\tau,$$

so that the integral on the right-hand side is pointwise defined on $\mathcal{R}^+$.

Lemma 2.3 *Let $\theta_1 > 0$ and $\omega_1 \in \mathcal{C}(0,1) \cap \mathcal{L}^1(0,1)$. Then the general solution of FDE*

$$D^{\theta_1} \omega_1(r) = y(r),$$

is given by

$$\omega_1(r) = I^{\theta_1} y(r) + \rho_0 + \rho_1 t + \rho_2 t^2 + \cdots + \rho_{m-1} t^{m-1},$$

for some $\rho_i \in \mathcal{R}$, $i = 0, 1, 2, \ldots, m-1$, where m is the smallest integer such that $m \geq \theta_1$.

Lemma 2.4 ([37]) *Let φ_p be the p-Laplacian operator.*

(i) *If $1 < p \leq 2$, $\zeta_1 \xi_2 > 0$ and $|\zeta_1|, |\xi_2| \geq m > 0$, then*

$$\left|\varphi_p(\zeta_1) - \varphi_p(\zeta_2)\right| \leq (p-1) m^{p-2} |\zeta_1 - \zeta_2|.$$

(ii) *If $p > 2$, and $|\zeta_1|, |\zeta_2| \leq M$, then*

$$\left|\varphi_p(\zeta_1) - \varphi_p(\zeta_2)\right| \leq (p-1) M^{p-2} |\zeta_1 - \zeta_2|.$$

Definition 2.5 Let $\mathcal{T} : \mathcal{L} \to \mathcal{L}$. Then the operator equation given by

$$\mathcal{T}\omega_1(r) = \omega_1(r), r \in [0,1],$$

is called Hyers–Ulam stable if, for any $\xi > 0$, the inequality

$$\|\omega_1 - \mathcal{T}\omega_1\| \leq \xi, \omega_1 \in [0,1],$$

has a unique fixed point, say $\omega_1{}^*$, with constant $D > 0$ such that $\|\omega_1 - \omega_1{}^*\| \leq D\xi$ holds for all $\omega_1 \in [0,1]$.

To proceed further, let the following hypothesis hold:

$(\mathcal{H}_1)$ The nonlocal functions f, g, where $\omega_1, \omega_2, x, y \in \mathcal{R}$, satisfy the inequalities:

$$\left|f(\kappa, \omega_1, \omega_2) - f(\kappa, x, y)\right| \leq \mathcal{K}_f\big(|\omega_1 - x| + |\omega_2 - y|\big),$$
$$\left|g(\kappa, \omega_1, \omega_2) - g(\kappa, x, y)\right| \leq \mathcal{K}_g\big(|\omega_1 - x| + |\omega_2 - y|\big),$$

where $\mathcal{K}_f, \mathcal{K}_g \in [0,1)$.

Theorem 2.6 *Let $\gamma \neq \frac{\eta^{(\theta_1+\alpha-1)}}{\Gamma(\theta_1+\alpha)}$. Then for a given $g \in \mathcal{C}([0,1],\mathcal{R})$, the solution of the fractional differential equation*

$$\mathcal{D}^{\theta_1^*}\varphi_p\big(\mathcal{D}^{\theta_1}x(r)\big) - g(r) = 0, \quad \theta_1 \in (1,2], \theta_1^* \in (0,1], \tag{2.1}$$

with the boundary condition

$$\varphi_p\big(\mathcal{D}^{\theta_1}x(r)\big)\big|_{r=0} = x'(r)\big|_{r=0} = 0, \qquad x(r)|_{r=0} = \gamma I^{\alpha-1}h_1(\eta), \tag{2.2}$$

has a unique solution given by

$$\begin{aligned} x(r) &= \frac{1}{\Gamma(\theta_1)}\int_0^r (r-s)^{\theta_1-1}\varphi_q\big(I^{\theta_1^*}g(s)\big)\,ds \\ &\quad + \frac{\gamma}{\Delta_1\Gamma(\theta_1+\alpha-1)}\int_0^\eta (\eta-s)^{\theta_1+\alpha-2}\varphi_q\big(I^{\theta_1^*}g(s)\big)\,ds. \end{aligned} \tag{2.3}$$

Proof Applying the operator $I^{\theta_1^*}$ on (2.1) and using Lemma 2.3, we get from (2.1) the following equivalent integral form:

$$\varphi_p\big(\mathcal{D}^{\theta_1}x(r)\big) = A_0 + I^{\theta_1^*}g(r), \tag{2.4}$$

and then, by using condition $\varphi_p(\mathcal{D}^{\theta_1}x(r))|_{r=0} = 0$, we get $A_0 = 0$. From (2.4), we have

$$\mathcal{D}^{\theta_1}x(r) = \varphi_q\big(I^{\theta_1^*}g(r)\big). \tag{2.5}$$

Applying the operator $I_0^{\theta_1}$ on (2.5) and using Lemma 2.3 again, we get from (2.5) the following equivalent integral form:

$$x(r) = A_1 + A_2 r + I^{\theta_1}\big(\varphi_q I^{\theta_1^*}g(r)\big). \tag{2.6}$$

By using the condition $x'(r)|_{r=0} = 0$ in (2.6), we obtain $A_2 = 0$. Also in view of condition $x(r)|_{r=0} = \gamma I^{\alpha-1}x(\eta)$ in (2.6), we get

$$A_1 = \frac{\gamma}{\Delta_1}I^{\theta_1+\alpha-1}\big(\varphi_q I^{\theta_1^*}g(\eta)\big), \tag{2.7}$$

where $\Delta_1 = (1 - \frac{\gamma}{\Gamma\alpha})$. By substituting the values of A_1 and A_2 in (2.6), we get (2.3). □

With the help of Theorem 2.6, our coupled FDEs involving the p-Laplacian with integral boundary conditions are equivalent to the following Hammerstein-type integral system:

$$\begin{cases} \omega_1(r) = \frac{1}{\Gamma(\theta_1)}\int_0^r (r-s)^{\theta_1-1}\varphi_q(I^{\theta_1^*}f(s,\omega_1(s),\omega_2(s))))\,ds \\ \qquad + \frac{\gamma}{\Delta_1\Gamma(\theta_1+\alpha-1)}\int_0^\eta (\eta-s)^{\theta_1+\alpha-2}\varphi_q(I^{\theta_1^*}f(s,\omega_1(s),\omega_2(s)))\,ds, \\ \omega_2(r) = \frac{1}{\Gamma(\theta_2)}\int_0^r (r-s)^{\theta_2-1}\varphi_q(I^{\theta_2^*}g(s,\omega_1(s),\omega_2(s))))\,ds \\ \qquad + \frac{\delta}{\Delta_2\Gamma(\theta_2+\beta-1)}\int_0^\xi (\xi-s)^{\theta_2+\beta-2}\varphi_q(I^{\theta_2^*}g(s,\omega_1(s),\omega_2(s)))\,ds. \end{cases} \tag{2.8}$$

3 Existence and uniqueness

For the sake of convenience, we set

$$M_1 = \left(\frac{1}{\Gamma(\theta_1^*+1)}\right)^{q-1}\left[\frac{1}{\Gamma(\theta_1+1)} + \frac{\gamma\eta^{\theta_1+\alpha-1}}{\Delta_1\Gamma(\theta_1+\alpha)}\right], \tag{3.1}$$

$$M_2 = \left(\frac{1}{\Gamma(\theta_1^*+1)}\right)^{q-1}\left[\frac{1}{\Gamma(\theta_2+1)} + \frac{\delta\xi^{\theta_2+\beta-1}}{\Delta_2\Gamma(\theta_2+\beta)}\right], \tag{3.2}$$

$$M_0 = \min\bigl\{1-(M_1k_1+M_2\lambda_1), 1-(M_1k_2+M_2\lambda_2)\bigr\}, \tag{3.3}$$

$$\blacktriangle_f = \frac{(q-1)J_1^{q-2}}{\Gamma(\theta_1^*+1)}\left[\frac{1}{\Gamma(\theta_1+1)} + \frac{\gamma\eta^{\theta_1+\alpha-1}}{\Delta_1\Gamma(\theta_1+\alpha)}\right], \tag{3.4}$$

$$\blacktriangle_g = \frac{(q-1)J_2^{q-2}}{\Gamma(\theta_1^*+1)}\left[\frac{1}{\Gamma(\theta_2+1)} + \frac{\delta\xi^{\theta_2+\beta-1}}{\Delta_2\Gamma(\theta_2+\beta)}\right]. \tag{3.5}$$

We define operators $T_1, T_2 : \mathcal{X}\times\mathcal{Y}\to\mathcal{X}\times\mathcal{Y}$ as

$$\begin{aligned} T_1(\omega_1,\omega_2)(r) &= \frac{1}{\Gamma(\theta_1)}\int_0^r (r-s)^{\theta_1-1}\varphi_q\bigl(I^{\theta_1^*}f\bigl(s,\omega_1(s),\omega_2(s)\bigr)\bigr)\,ds \\ &\quad + \frac{\gamma}{\Delta_1\Gamma(\theta_1+\alpha-1)}\int_0^\eta (\eta-s)^{\theta_1+\alpha-2}\varphi_q\bigl(I^{\theta_1^*}f\bigl(s,\omega_1(s),\omega_2(s)\bigr)\bigr)\,ds, \\ T_2(\omega_1,\omega_2)(r) &= \frac{1}{\Gamma(\theta_2)}\int_0^r (r-s)^{\theta_2-1}\varphi_q\bigl(I^{\theta_2^*}g\bigl(s,\omega_1(s),\omega_2(s)\bigr)\bigr)\,ds \\ &\quad + \frac{\delta}{\Delta_2\Gamma(\theta_2+\beta-1)}\int_0^\xi (\xi-s)^{\theta_2+\beta-2}\varphi_q\bigl(I^{\theta_2^*}g\bigl(s,\omega_1(s),\omega_2(s)\bigr)\bigr)\,ds. \end{aligned} \tag{3.6}$$

Lemma 3.1 ([9, 26, 30]) *Let $\mathcal{F}:\mathcal{A}\to\mathcal{A}$ be a completely continuous operator (i.e., a map which, restricted to any bounded set in $\mathcal{A}$, is compact). Let*

$$\varepsilon(\mathcal{F}) = \bigl\{x\in\mathcal{A} : x = \lambda\mathcal{F}(x), \textit{ for some } 0<\lambda<1\bigr\}. \tag{3.7}$$

Then either the set $\varepsilon(F)$ is unbounded, or $\mathcal{F}$ has at least one fixed point.

Theorem 3.2 *Suppose that $\gamma \neq \frac{\eta^{(\theta_1+\alpha-1)}}{\Gamma(\theta_1+\alpha)}$ and $\delta \neq \frac{\xi^{(\theta_2+\beta-1)}}{\Gamma(\theta_2+\beta)}$. Assume that there exist real constants $k_i, \lambda_i \geq 0$ $(i=1,2)$ and $k_0>0$, $\lambda_0>0$ such that for all $x_i\in\mathcal{R}$ $(i=1,2)$, we have*

$$\begin{aligned} \bigl|f(r,x_1,x_2)\bigr| &\leq \varphi_p\bigl(k_0+k_1|x_1|+k_2|x_2|\bigr), \\ \bigl|g(r,x_1,x_2)\bigr| &\leq \varphi_p\bigl(\lambda_0+\lambda_1|x_1|+\lambda_2|x_2|\bigr). \end{aligned} \tag{3.8}$$

In addition, it is assumed that

$$M_1k_1+M_2\lambda_1<1 \quad \textit{and} \quad M_1k_2+M_2\lambda_2<1,$$

where M_1 and M_2 are given by (3.1) and (3.2) respectively. Then the boundary value problem (1.1) has at least one solution.

Proof First, we show that the operator $T:\mathcal{X}\times\mathcal{Y}\to\mathcal{X}\times\mathcal{Y}$ is completely continuous. By the continuity of functions f and g, the operator T is continuous. Let $\Omega\subset\mathcal{X}\times\mathcal{Y}$ be bounded. Then there exist positive constants L_1 and L_2 such that

$$\left|f\left(r,\omega_1(r),\omega_2(r)\right)\right| \le \varphi_p(L_1), \qquad \left|g\left(r,\omega_1(r),\omega_2(r)\right)\right| \le \varphi_p(L_2), \quad \forall(\omega_1,\omega_2)\in\Omega. \tag{3.9}$$

Then for any $(\omega_1,,\omega_2)\in\Omega$, we have

$$\begin{aligned}
&\left|T_1(\omega_1,\omega_2)(r)\right| \\
&\quad= \left|\frac{1}{\Gamma(\theta_1)}\int_0^r (r-s)^{\theta_1-1}\varphi_q\left(I^{\theta_1^*}f\left(s,\omega_1(s),\omega_2(s)\right)\right)ds \right.\\
&\qquad \left. + \frac{\gamma}{\Delta_1\Gamma(\theta_1+\alpha-1)}\int_0^\eta (\eta-s)^{\theta_1+\alpha-2}\varphi_q\left(I^{\theta_1^*}f\left(s,\omega_1(s),\omega_2(s)\right)\right)ds\right| \\
&\quad\le \frac{1}{\Gamma(\theta_1)}\int_0^r (r-s)^{\theta_1-1}\varphi_q\left(\frac{1}{\Gamma(\theta_1^*)}\int_0^r (r-s)^{\theta_1^*-1}\left|f\left(s,\omega_1(s),\omega_2(s)\right)\right|ds\right) \\
&\qquad + \frac{\gamma}{\Delta_1\Gamma(\theta_1+\alpha-1)} \\
&\qquad \times \int_0^\eta (\eta-s)^{\theta_1+\alpha-2}\varphi_q\left(\frac{1}{\Gamma(\theta_1^*)}\int_0^\eta (\eta-s)^{\theta_1^*-1}\left|f\left(s,\omega_1(s),\omega_2(s)\right)\right|\right)ds \\
&\quad\le \frac{1}{\Gamma(\theta_1)}\int_0^r (r-s)^{\theta_1-1}\varphi_q\left(\frac{1}{\Gamma(\theta_1^*)}\int_0^r (r-s)^{\theta_1^*-1}\varphi_p(L_1)\,ds\right) \\
&\qquad + \frac{\gamma}{\Delta_1\Gamma(\theta_1+\alpha-1)}\int_0^\eta (\eta-s)^{\theta_1+\alpha-2}\varphi_q\left(\frac{1}{\Gamma(\theta_1^*)}\int_0^\eta (\eta-s)^{\theta_1^*-1}\varphi_p(L_1)\right)ds \\
&\quad\le L_1\left(\frac{1}{\Gamma(\theta_1^*+1)}\right)^{q-1}\left[\frac{1}{\Gamma(\theta_1+1)} + \frac{\gamma\eta^{\theta_1+\alpha-1}}{\Delta_1\Gamma(\theta_1+\alpha)}\right] = L_1M_1.
\end{aligned} \tag{3.10}$$

And also,

$$\begin{aligned}
&\left|T_2(\omega_1,\omega_2)(r)\right| \\
&\quad= \left|\frac{1}{\Gamma(\theta_2)}\int_0^r (r-s)^{\theta_2-1}\varphi_q\left(I^{\theta_2^*}g\left(s,\omega_1(s),\omega_2(s)\right)\right)ds \right.\\
&\qquad \left. + \frac{\delta}{\Delta_2\Gamma(\theta_2+\beta-1)}\int_0^\xi (\xi-s)^{\theta_2+\beta-2}\varphi_q\left(I^{\theta_2^*}g\left(s,\omega_1(s),\omega_2(s)\right)\right)ds\right| \\
&\quad\le \frac{1}{\Gamma(\theta_2)}\int_0^r (r-s)^{\theta_2-1}\varphi_q\left(\frac{1}{\Gamma(\theta_2 2)}\int_0^r (r-s)^{\theta_2^*-1}\left|g\left(s,\omega_1(s),\omega_2(s)\right)\right|ds\right) \\
&\qquad + \frac{\delta}{\Delta_2\Gamma(\theta_2+\beta-1)} \\
&\qquad \times \int_0^\xi (\xi-s)^{\theta_2+\beta-2}\varphi_q\left(\frac{1}{\Gamma(\theta_2^*)}\int_0^\xi (\xi-s)^{\theta_2^*-1}\left|g\left(s,\omega_1(s),\omega_2(s)\right)\right|\right)ds \\
&\quad\le \frac{1}{\Gamma(\theta_2)}\int_0^r (r-s)^{\theta_2-1}\varphi_q\left(\frac{1}{\Gamma(\theta_2^*)}\int_0^r (r-s)^{\theta_2^*-1}\varphi_p(L_2)\,ds\right) \\
&\qquad + \frac{\delta}{\Delta_2\Gamma(\theta_2+\beta-1)}\int_0^\xi (\xi-s)^{\theta_2+\beta-2}\varphi_q\left(\frac{1}{\Gamma(\theta_2^*)}\int_0^\xi (\xi-s)^{\theta_2^*-1}\varphi_p(L_2)\right)ds \\
&\quad\le L_2\left(\frac{1}{\Gamma(\theta_2^*+1)}\right)^{q-1}\left[\frac{1}{\Gamma(\theta_2+1)} + \frac{\delta\xi^{\theta_2+\beta-1}}{\Delta_2\Gamma(\theta_2+\beta)}\right] = L_2M_2.
\end{aligned} \tag{3.11}$$

Thus, it follows from the above inequalities that the operator T is uniformly bounded. Next we show that T is equicontinuous. Let $0 \leq r_1 \leq r_2 \leq 1$. Then we have

$$\begin{aligned}
&\bigl|T_1\bigl(\omega_1(r_2),\omega_2(r_2)\bigr) - T_1\bigl(\omega_1(r_1),\omega_2(r_1)\bigr)\bigr| \\
&\quad= \biggl|\frac{1}{\Gamma(\theta_1)}\int_0^{r_2}(r_2-s)^{\theta_1-1}\varphi_q\biggl(\frac{1}{\Gamma(\theta_1^*)}\int_0^r (r-s)^{\theta_1^*-1}f\bigl(s,\omega_1(s),\omega_2(s)\bigr)\biggr)\,ds \\
&\qquad - \frac{1}{\Gamma(\theta_1)}\int_0^{r_1}(r_1-s)^{\theta_1-1}\varphi_q\biggl(\frac{1}{\Gamma(\theta_1^*)}\int_0^r (r-s)^{\theta_1^*-1}f\bigl(s,\omega_1(s),\omega_2(s)\bigr)\biggr)\,ds\biggr| \\
&\quad\leq \biggl|\frac{1}{\Gamma(\theta_1)}\int_0^{r_2}\bigl[(r_2-s)^{\theta_1^*-1} - (r_1-s)^{\theta_1^*-1}\bigr] \\
&\qquad\times \varphi_q\biggl(\frac{1}{\Gamma(\theta_1^*)}\int_0^r (r-s)^{\theta_1^*-1}f\bigl(s,\omega_1(s),\omega_2(s)\bigr)\biggr)\,ds \\
&\qquad - \frac{1}{\Gamma(\theta_1)}\int_{r_1}^{r_2}(r_2-s)^{\theta_1-1}\varphi_q\biggl(\frac{1}{\Gamma(\theta_1^*)}\int_0^r (r-s)^{\theta_1^*-1}f\bigl(s,\omega_1(s),\omega_2(s)\bigr)\biggr)\,ds\biggr| \\
&\quad\leq \frac{L_1}{\Gamma(\theta_1+1)(\Gamma\theta_1^*+1)^{q-1}}\bigl(r_1^{\theta_1} - r_2^{\theta_1}\bigr), \qquad (3.12)
\end{aligned}$$

$$\begin{aligned}
&\bigl|T_2\bigl(\omega_1(r_2),\omega_2(r_2)\bigr) - T_2\bigl(\omega_1(r_1),\omega_2(r_1)\bigr)\bigr| \\
&\quad= \biggl|\frac{1}{\Gamma(\theta_2)}\int_0^{r_2}(r_2-s)^{\theta_2-1}\varphi_q\biggl(\frac{1}{\Gamma(\theta_2^*)}\int_0^r (r-s)^{\theta_2^*-1}f\bigl(s,\omega_1(s),\omega_2(s)\bigr)\biggr)\,ds \\
&\qquad - \frac{1}{\Gamma(\theta_2)}\int_0^{r_1}(r_1-s)^{\theta_2-1}\varphi_q\biggl(\frac{1}{\Gamma(\theta_2^*)}\int_0^r (r-s)^{\theta_2^*-1}f\bigl(s,\omega_1(s),\omega_2(s)\bigr)\biggr)\,ds\biggr| \\
&\quad\leq \biggl|\frac{1}{\Gamma(\theta_2)}\int_0^{r_2}\bigl[(r_2-s)^{\theta_2^*-1} - (r_1-s)^{\theta_2^*-1}\bigr] \\
&\qquad\times \varphi_q\biggl(\frac{1}{\Gamma(\theta_2^*)}\int_0^r (r-s)^{\theta_2^*-1}f\bigl(s,\omega_1(s),\omega_2(s)\bigr)\biggr)\,ds \\
&\qquad - \frac{1}{\Gamma(\theta_2)}\int_{r_1}^{r_2}(r_2-s)^{\theta_2-1}\varphi_q\biggl(\frac{1}{\Gamma(\theta_2^*)}\int_0^r (r-s)^{\theta_2^*-1}f\bigl(s,\omega_1(s),\omega_2(s)\bigr)\biggr)\,ds\biggr| \\
&\quad\leq \frac{L_1}{\Gamma(\theta_2+1)(\Gamma\theta_2^*+1)^{q-1}}\bigl(r_1^{\theta_2} - r_2^{\theta_2}\bigr). \qquad (3.13)
\end{aligned}$$

Therefore, the operator $T(\omega_1,\omega_2)$ is equicontinuous, and thus the operator $T(\omega_1,\omega_2)$ is completely continuous. Finally, it will be verified that the set $\varepsilon = \{(\omega_1,\omega_2) \in \mathcal{X} \times \mathcal{Y} | (\omega_1,\omega_2) = \lambda T(\omega_1,\omega_2), 0 \leq \lambda \leq 1\}$ is bounded. Let $(\omega_1,\omega_2) \in \varepsilon$, then $(\omega_1,\omega_2) = \lambda T(\omega_1,\omega_2)$. For any $r \in [0,1]$, we have

$$\omega_1(r) = \lambda r_1(\omega_1,\omega_2), \qquad \omega_2(r) = \lambda T_2(\omega_1,\omega_2).$$

Then

$$\begin{aligned}
\bigl|\omega_1(r)\bigr| &= \biggl(\frac{1}{\Gamma(\theta_1^*+1)}\biggr)^{q-1}\biggl[\frac{1}{\Gamma(\theta_1+1)} + \frac{\gamma\eta^{\theta_1+\alpha-1}}{\Delta_1\Gamma(\theta_1+\alpha)}\biggr] \\
&\quad\times \bigl(k_0 + k_1\bigl|\omega_1(r)\bigr| + k_2\bigl|\omega_2(r)\bigr|\bigr) \qquad (3.14)
\end{aligned}$$

and

$$\left|\omega_2(r)\right| = \left(\frac{1}{\Gamma(\theta_2^* + 1)}\right)^{q-1}\left[\frac{1}{\Gamma(\theta_2 + 1)} + \frac{\delta\xi^{\theta_2+\beta-1}}{\Delta_2\Gamma(\theta_2 + \beta)}\right] \times \left(\lambda_0 + \lambda_1\left|\omega_1(r)\right| + \lambda_2\left|\omega_2(r)\right|\right). \tag{3.15}$$

Hence we have

$$\begin{aligned} \|\omega_1\| &= M_1\left(k_0 + k_1\left\|\omega_1(r)\right\| + k_2\left\|\omega_2(r)\right\|\right) \quad \text{and} \\ \|\omega_2\| &= M_2\left(\lambda_0 + \lambda_1\left\|\omega_1(r)\right\| + \lambda_2\left\|\omega_2(r)\right\|\right). \end{aligned} \tag{3.16}$$

From (3.16) we have

$$\|\omega_1\| + \|\omega_2\| = (M_1k_0 + M_2\lambda_0) + (M_1k_1 + M_2\lambda_1)\|\omega_1\| + (M_1k_2 + M_2\lambda_2)\|\omega_2\|. \tag{3.17}$$

Consequently,

$$\left\|(\omega_1, \omega_2)\right\| \le \frac{M_1k_0 + M_2\lambda_0}{M_0}, \tag{3.18}$$

for any $r \in [0, 1]$, where M_0 is defined in (3.3), which proves that ε is bounded. Thus, by Lemma 3.2, operator T has at leat one fixed point. Hence, the boundary value problem (1.1) has at least one solution. □

Theorem 3.3 *Assume that $f, g : [0, 1] \times \mathcal{R}^2 \to \mathcal{R}$ are continuous functions and there exist constants m_i, n_i, $i = 1, 2$ such that for all $r \in [0, 1]$ and $\omega_1, \omega_2, \hbar_1, \hbar_2 \in \mathcal{R}$,*

$$\begin{aligned} \left|f(r, \omega_1, \hbar_1) - f(r, \omega_2, \hbar_2)\right| &\le m_1|\omega_1 - \omega_2| + n_1|\hbar_1 - \hbar_2|, \\ \left|g(r, \omega_1, \hbar_1) - g(r, \omega_2, \hbar_2)\right| &\le m_2|\omega_1 - \omega_2| + n_2|\hbar_1 - \hbar_2|. \end{aligned} \tag{3.19}$$

In addition, assume that

$$\blacktriangle_f(m_1 + m_2) + \blacktriangle_g(n_1 + n_2) < 1,$$

where $\blacktriangle_f$ and $\blacktriangle_g$ are given by (3.4) and (3.5), respectively. Then the boundary value problem (1.1) has a unique solution.

Proof Consider a bounded set $\|T(\omega_1, \omega_2)(r)\| \le r$. For $(\hbar_1, \hbar_2), (\omega_1, \omega_2) \in \mathcal{X} \times \mathcal{Y}$, and for any $r \in [0, 1]$, we get

$$\begin{aligned} &\left|r_1(\hbar_1, \hbar_2)(r) - r_1(\omega_1, \omega_2)(r)\right| \\ &\quad = \left|\frac{1}{\Gamma(\theta_1)}\int_0^r (r - s)^{\theta_1-1}\varphi_q\left(\frac{1}{\Gamma(\theta_1^*)}\int_0^r (r - s)^{\theta_1^*-1}f\left(s, \hbar_1(s), \hbar_2(s)\right)\right) ds\right. \\ &\qquad + \frac{\gamma}{\Delta_1\Gamma(\theta_1 + \alpha - 1)}\int_0^\eta (\eta - s)^{\theta_1+\alpha-2}\varphi_q\left(\frac{1}{\Gamma(\theta_1^*)}\int_0^r (r - s)^{\theta_1^*-1}f\left(s, \hbar_1(s), \hbar_2(s)\right)\right) ds \\ &\qquad - \frac{1}{\Gamma(\theta_1)}\int_0^r (r - s)^{\theta_1-1}\varphi_q\left(\frac{1}{\Gamma(\theta_1^*)}\int_0^r (r - s)^{\theta_1^*-1}f\left(s, \omega_1(s), \omega_2(s)\right)\right) ds \end{aligned}$$

$$
\begin{aligned}
&- \frac{\gamma}{\Delta_1 \Gamma(\theta_1+\alpha-1)} \int_0^\eta (\eta-s)^{\theta_1+\alpha-2} \varphi_q \left(\frac{1}{\Gamma(\theta_1^*)} \int_0^r (r-s)^{\theta_1^*-1} f\bigl(s,\omega_1(s),\omega_2(s)\bigr)\right) ds \Bigg| \\
&\leq \frac{(q-1)J_1^{q-2}}{\Gamma(\theta_1)} \int_0^r \bigl|(r-s)^{\theta_1-1}\bigr| \frac{1}{\Gamma(\theta_1^*)} \int_0^r \bigl|(r-s)^{\theta_1^*-1}\bigr| \bigl|f\bigl(s,\hbar_1(s),\hbar_2(s)\bigr) \\
&-f\bigl(s,\omega_1(s),\omega_2(s)\bigr)\bigr| \, ds \\
&+ \frac{(q-1)J_1^{q-2}\gamma}{\Delta_1 \Gamma(\theta_1+\alpha-1)} \int_0^\eta \bigl|(\eta-s)^{\theta_1+\alpha-2}\bigr| \frac{1}{\Gamma(\theta_1^*)} \int_0^r \bigl|(r-s)^{\theta_1^*-1}\bigr| \bigl|f\bigl(s,\hbar_1(s),\hbar_2(s)\bigr) \\
&-f\bigl(s,\omega_1(s),\omega_2(s)\bigr)\bigr| \, ds \\
&\leq \frac{(q-1)J_1^{q-2}}{\Gamma(\theta_1^*+1)} \left[\frac{1}{\Gamma(\theta_1+1)} + \frac{\gamma \eta^{\theta_1+\alpha-1}}{\Delta_1 \Gamma(\theta_1+\alpha)} \right] \bigl(m_1|\hbar_1-\omega_1| + m_2|\hbar_2-\omega_2|\bigr) \\
&\leq \blacktriangle_f (m_1+m_2)\bigl(|\hbar_1-\omega_1| + |\hbar_2-\omega_2|\bigr).
\end{aligned} \tag{3.20}
$$

Similarly, we have

$$
\begin{aligned}
&\bigl|T_2(\hbar_1,\hbar_2)(r) - T_2(\omega_1,\omega_2)(r)\bigr| \\
&= \Bigg| \frac{1}{\Gamma(\theta_2)} \int_0^r (r-s)^{\theta_2-1} \varphi_q \left(\frac{1}{\Gamma(\theta_2^*)} \int_0^r (r-s)^{\theta_2^*-1} g\bigl(s,\hbar_1(s),\hbar_2(s)\bigr)\right) ds \\
&+ \frac{\delta}{\Delta_2 \Gamma(\theta_2+\beta-1)} \int_0^\xi (\xi-s)^{\theta_2+\beta-2} \varphi_q \left(\frac{1}{\Gamma(\theta_2^*)} \int_0^r (r-s)^{\theta_2^*-1} g\bigl(s,\hbar_1(s),\hbar_2(s)\bigr)\right) ds \\
&- \frac{1}{\Gamma(\theta_2)} \int_0^r (r-s)^{\theta_2-1} \varphi_q \left(\frac{1}{\Gamma(\theta_2^*)} \int_0^r (r-s)^{\theta_2^*-1} g\bigl(s,\omega_1(s),\omega_2(s)\bigr)\right) ds \\
&- \frac{\delta}{\Delta_2 \Gamma(\theta_2+\beta-1)} \int_0^\xi (\xi-s)^{\theta_2+\beta-2} \varphi_q \left(\frac{1}{\Gamma(\theta_2^*)} \int_0^r (r-s)^{\theta_2^*-1} g\bigl(s,\omega_1(s),\omega_2(s)\bigr)\right) ds \\
&- \frac{1}{\Gamma(\theta_2)} \int_0^r (r-s)^{\theta_2-1} \varphi_q \left(\frac{1}{\Gamma(\theta_2^*)} \int_0^r (r-s)^{\theta_2^*-1} g\bigl(s,\omega_1(s),\omega_2(s)\bigr)\right) ds \\
&- \frac{\delta}{\Delta_2 \Gamma(\theta_2+\beta-1)} \int_0^\xi (\xi-s)^{\theta_2+\beta-2} \varphi_q \left(\frac{1}{\Gamma(\theta_2^*)} \int_0^r (r-s)^{\theta_2^*-1} g\bigl(s,\omega_1(s),\omega_2(s)\bigr)\right) ds \Bigg| \\
&\leq \frac{(q-1)J_2^{q-2}}{\Gamma(\theta_2)} \int_0^r \bigl|(r-s)^{\theta_2-1}\bigr| \frac{1}{\Gamma(\theta_2^*)} \int_0^r \bigl|(r-s)^{\theta_2^*-1}\bigr| \bigl|g\bigl(s,\hbar_1(s),\hbar_2(s)\bigr) \\
&-g\bigl(s,\omega_1(s),\omega_2(s)\bigr)\bigr| \, ds \\
&+ \frac{(q-1)J_2^{q-2}\delta}{\Delta_2 \Gamma(\theta_2+\beta-1)} \int_0^\xi \bigl|(\xi-s)^{\theta_2+\beta-2}\bigr| \frac{1}{\Gamma(\theta_2^*)} \int_0^r \bigl|(r-s)^{\theta_2^*-1}\bigr| \bigl|g\bigl(s,\hbar_1(s),\hbar_2(s)\bigr) \\
&-g\bigl(s,\omega_1(s),\omega_2(s)\bigr)\bigr| \, ds \\
&\leq \frac{(q-1)J_2^{(q-2)}}{\Gamma(\theta_2^*+1)} \left[\frac{1}{\Gamma(\theta_2+1)} + \frac{\delta \xi^{\theta_2+\beta-1}}{\Delta_2 \Gamma(\theta_2+\beta)} \right] \bigl(n_1|\hbar_1-\omega_1| + n_2|\hbar_2-\omega_2|\bigr) \\
&\leq \blacktriangle_g (n_1+n_2)\bigl(\|\hbar_1-\omega_1\| + \|\hbar_2-\omega_2\|\bigr).
\end{aligned} \tag{3.21}
$$

Therefore, by (3.20) and (3.21), we have

$$
\begin{aligned}
&\bigl\|T(\hbar_1,\hbar_2)(r) - T(\omega_1,\omega_2)(r)\bigr\| \\
&\leq \bigl[\blacktriangle_f (m_1+m_2) + \blacktriangle_g (n_1+n_2)\bigr]\bigl(\|\hbar_1-\omega_1\| + \|\hbar_2-\omega_2\|\bigr).
\end{aligned} \tag{3.22}
$$

Hence $\blacktriangle_f(m_1 + m_2) + \blacktriangle_g(n_1 + n_2) < 1$, and therefore T is a contraction operator. So by Banach's fixed point theorem, the operator T has a unique fixed point, which is the unique solution of problem (1.1). □

4 Hyers–Ulam stability of the coupled system

Definition 4.1 The coupled system of Hammerstein-type integral equations (2.8) is Hyres–Ulam stable if there exist positive constants $D_i > 0$ ($i = 1, 2$) satisfying:

For $\varrho_i > 0$, $i = 1, 2$, if

$$\begin{aligned} &\biggl|\omega_1(r) - \frac{1}{\Gamma(\theta_1)}\int_0^r (r-s)^{\theta_1-1}\varphi_q\bigl(I^{\theta_1^*}f\bigl(s,\omega_1(s),\omega_2(s)\bigr)\bigr)\,ds \\ &\quad - \frac{\gamma}{\Delta_1\Gamma(\theta_1+\alpha-1)}\int_0^\eta (\eta-s)^{\theta_1+\alpha-2}\varphi_q\bigl(I^{\theta_1^*}f\bigl(s,\omega_1(s),\omega_2(s)\bigr)\bigr)\,ds\biggr| \le \varrho_1, \\ &\biggl|\omega_2(r) - \frac{1}{\Gamma(\theta_2)}\int_0^r (r-s)^{\theta_2-1}\varphi_q\bigl(I^{\theta_2^*}g\bigl(s,\omega_1(s),\omega_2(s)\bigr)\bigr)\,ds \\ &\quad - \frac{\delta}{\Delta_2\Gamma(\theta_2+\beta-1)}\int_0^\xi (\xi-s)^{\theta_2+\beta-2}\varphi_q\bigl(I^{\theta_2^*}g\bigl(s,\omega_1(s),\omega_2(s)\bigr)\bigr)\,ds\biggr| \le \varrho_2, \end{aligned} \tag{4.1}$$

there exist $(\omega_1^*(r), \omega_2^*(r))$, satisfying

$$\begin{aligned} \omega_1^*(r) &= \frac{1}{\Gamma(\theta_1)}\int_0^r (r-s)^{\theta_1-1}\varphi_q\bigl(I^{\theta_1^*}f\bigl(s,\omega_1^*(s),\omega_2^*(s)\bigr)\bigr)\,ds \\ &\quad + \frac{\gamma}{\Delta_1\Gamma(\theta_1+\alpha-1)}\int_0^\eta (\eta-s)^{\theta_1+\alpha-2}\varphi_q\bigl(I^{\theta_1^*}f\bigl(s,\omega_1^*(s),\omega_2^*(s)\bigr)\bigr)\,ds, \\ \omega_2^*(r) &= \frac{1}{\Gamma(\theta_2)}\int_0^r (r-s)^{\theta_2-1}\varphi_q\bigl(I^{\theta_2^*}g\bigl(s,\omega_1^*(s),\omega_2^*(s)\bigr)\bigr)\,ds \\ &\quad + \frac{\delta}{\Delta_2\Gamma(\theta_2+\beta-1)}\int_0^\xi (\xi-s)^{\theta_2+\beta-2}\varphi_q\bigl(I^{\theta_2^*}g\bigl(s,\omega_1^*(s),\omega_2^*(s)\bigr)\bigr)\,ds, \end{aligned} \tag{4.2}$$

such that

$$\begin{aligned} &\bigl|\omega_1(r) - \omega_1^*(r)\bigr| \le D_1\varrho_1, x \in [0,1], \\ &\bigl|\omega_2(r) - \omega_2^*(r)\bigr| \le D_2\varrho_2, x \in [0,1]. \end{aligned} \tag{4.3}$$

In the present section, we derive the Hyers–Ulam type stability for the solution of the considered problem.

Theorem 4.2 *By the assumption that* $f, g : [0,1] \times \mathcal{R}^2 \to \mathcal{R}$ *are continuous functions and there exist constants* m_i, n_i, $i = 1, 2$ *such that for all* $r \in [0,1]$ *and* $\omega_1, \omega_2, \hbar_1, \hbar_2 \in \mathcal{R}$,

$$\begin{aligned} &\bigl|f(r,\omega_1,\hbar_1) - f(r,\omega_2,\hbar_2)\bigr| \le m_1|\omega_1 - \omega_2| + n_1|\hbar_1 - \hbar_2|, \\ &\bigl|g(r,\omega_1,\hbar_1) - g(r,\omega_2,\hbar_2)\bigr| \le m_2|\omega_1 - \omega_2| + n_2|\hbar_1 - \hbar_2|, \end{aligned} \tag{4.4}$$

system (1.1) *is Hyers–Ulam stable.*

Proof By Theorem 3.3 and Definition 4.1, let $(\omega_1(r), \omega_2(r))$ be the exact solution, and $(\omega_1^*(r), \omega_2^*(r))$ be any other solution of system (2.8). Then, with the help of (2.8), we have

$$\begin{aligned}
\left|\omega_1(r) - \omega_1^*(r)\right| &\leq \left| \frac{1}{\Gamma(\theta_1)} \int_0^r (r-s)^{\theta_1 - 1} \varphi_q\left(\frac{1}{\Gamma(\theta_1^*)} \int_0^r (r-s)^{\theta_1^* - 1} f\bigl(s, \omega_1(s), \omega_2(s)\bigr) \right) ds \right. \\
&\quad + \frac{\gamma}{\Delta_1 \Gamma(\theta_1 + \alpha - 1)} \int_0^\eta (\eta - s)^{\theta_1 + \alpha - 2} \\
&\quad \times \varphi_q\left(\frac{1}{\Gamma(\theta_1^*)} \int_0^r (r-s)^{\theta_1^* - 1} f\bigl(s, \omega_1(s), \omega_2(s)\bigr) \right) ds \\
&\quad - \frac{1}{\Gamma(\theta_1)} \int_0^r (r-s)^{\theta_1 - 1} \varphi_q\left(\frac{1}{\Gamma(\theta_1^*)} \int_0^r (r-s)^{\theta_1^* - 1} f\bigl(s, \omega_1^*(s), \omega_2^*(s)\bigr) \right) ds \\
&\quad - \frac{\gamma}{\Delta_1 \Gamma(\theta_1 + \alpha - 1)} \int_0^\eta (\eta - s)^{\theta_1 + \alpha - 2} \\
&\quad \left. \times \varphi_q\left(\frac{1}{\Gamma(\theta_1^*)} \int_0^r (r-s)^{\theta_1^* - 1} f\bigl(s, \omega_1^*(s), \omega_2^*(s)\bigr) \right) ds \right| \\
&\leq \frac{1}{\Gamma(\theta_1)} \int_0^r \left|(r-s)^{\theta_1 - 1}\right| \left| \varphi_q\left(\frac{1}{\Gamma(\theta_1^*)} \int_0^r (r-s)^{\theta_1^* - 1} g\bigl(s, \omega_1(s), \omega_2(s)\bigr)\, ds \right) \right. \\
&\quad \left. - \varphi_q\left(\frac{1}{\Gamma(\theta_1^*)} \int_0^r (r-s)^{\theta_1^* - 1} g\bigl(s, \omega_1^*(s), \omega_2^*(s)\bigr)\, ds \right) \right| \\
&\quad + \frac{\gamma}{\Delta_1 \Gamma(\theta_1 + \alpha - 1)} \int_0^\eta \left|(\eta - s)^{\theta_1 + \alpha - 2}\right| \\
&\quad \times \left| \varphi_q\left(\frac{1}{\Gamma(\theta_1^*)} \int_0^r (r-s)^{\theta_1^* - 1} g\bigl(s, \omega_1(s), \omega_2(s)\bigr)\, ds \right) \right. \\
&\quad \left. - \varphi_q\left(\frac{1}{\Gamma(\theta_1^*)} \int_0^r (r-s)^{\theta_1^* - 1} g\bigl(s, \omega_1^*(s), \omega_2^*(s)\bigr)\, ds \right) \right| \\
&\leq \frac{(q-1)J_1^{q-2}}{\Gamma(\theta_1)} \int_0^r \left|(r-s)^{\theta_1 - 1}\right| \frac{1}{\Gamma(\theta_1^*)} \int_0^r \left|(r-s)^{\theta_1^* - 1}\right| \bigl| f\bigl(s, \omega_1(s), \omega_2(s)\bigr) \\
&\quad - f\bigl(s, \omega_1^*(s), \omega_2^*(s)\bigr) \bigr|\, ds \\
&\quad + \frac{(q-1)J_1^{q-2}\gamma}{\Delta_1 \Gamma(\theta_1 + \alpha - 1)} \int_0^\eta \left|(\eta - s)^{\theta_1 + \alpha - 2}\right| \\
&\quad \times \frac{1}{\Gamma(\theta_1^*)} \int_0^r \left|(r-s)^{\theta_1^* - 1}\right| \bigl| f\bigl(s, \omega_1(s), \omega_2(s)\bigr) - f\bigl(s, \omega_1^*(s), \omega_2^*(s)\bigr) \bigr|\, ds \\
&\leq \frac{(q-1)J_1^{q-2}\mathcal{K}_f}{\Gamma(\theta_1 + 1)\Gamma(\theta_1^* + 1)} \bigl(\left|\omega_1(r) - \omega_1^*(r)\right| + \left|\omega_2(r) - \omega_2^*(r)\right| \bigr) \\
&\quad + \frac{(q-1)J_1^{q-2}\gamma \eta^{\theta_1 + \alpha - 1}\mathcal{K}_f}{\Delta_1 \Gamma(\theta_1 + \alpha)\Gamma(\theta_1^* + 1)} \bigl(\left|\omega_1(r) - \omega_1^*(r)\right| + \left|\omega_2(r) - \omega_2^*(r)\right| \bigr),
\end{aligned} \tag{4.5}$$

which implies that

$$\begin{aligned}
\left\|\omega_1 - \omega_1^*\right\| &\leq \frac{(q-1)J_1^{q-2}\mathcal{K}_f}{\Gamma(\theta_1^* + 1)} \left[\frac{1}{\Gamma(\theta_1 + 1)} + \frac{\gamma \eta^{\theta_1 + \alpha - 1}}{\Delta_1 \Gamma(\theta_1 + \alpha)} \right] \\
&\quad \times \bigl(\left\|\omega_1 - \omega_1^*\right\| + \left\|\omega_2 - \omega_2^*\right\| \bigr) \\
&\leq D_1 \varrho_1,
\end{aligned} \tag{4.6}$$

where $D_1 = \frac{(q-1)J_1^{q-2}\mathcal{K}_f}{\Gamma(\theta_1^*+1)}[\frac{1}{\Gamma(\theta_1+1)} + \frac{\gamma\eta^{\theta_1+\alpha-1}}{\Delta_1\Gamma(\theta_1+\alpha)}]$. Similarly, we further have

$$\begin{aligned}
\left|\omega_2(r) - \omega_2^*(r)\right| \leq{}& \left| \frac{1}{\Gamma(\theta_2)} \int_0^r (r-s)^{\theta_2-1} \varphi_q\left(\frac{1}{\Gamma(\theta_2^*)} \int_0^r (r-s)^{\theta_2^*-1} g\big(s, \omega_1(s), \omega_2(s)\big) \right) ds \right. \\
&+ \frac{\delta}{\Delta_2 \Gamma(\theta_2+\beta-1)} \\
&\times \int_0^\xi (\xi-s)^{\theta_2+\beta-2} \varphi_q\left(\frac{1}{\Gamma(\theta_2^*)} \int_0^r (r-s)^{\theta_2^*-1} g\big(s, \omega_1(s), \omega_2(s)\big) \right) ds \\
&- \frac{1}{\Gamma(\theta_2)} \int_0^r (r-s)^{\theta_2-1} \varphi_q\left(\frac{1}{\Gamma(\theta_2^*)} \int_0^r (r-s)^{\theta_2^*-1} g\big(s, \omega_1^*(s), \omega_2^*(s)\big) \right) ds \\
&- \frac{\delta}{\Delta_2 \Gamma(\theta_2+\beta-1)} \int_0^\xi (\xi-s)^{\theta_2+\beta-2} \\
&\left. \times \varphi_q\left(\frac{1}{\Gamma(\theta_2^*)} \int_0^r (r-s)^{\theta_2^*-1} g\big(s, \omega_1^*(s), \omega_2^*(s)\big) \right) ds \right| \\
\leq{}& \frac{1}{\Gamma(\theta_2)} \int_0^r \left|(r-s)^{\theta_2-1}\right| \left| \varphi_q\left(\frac{1}{\Gamma(\theta_2^*)} \int_0^r (r-s)^{\theta_2^*-1} g\big(s, \omega_1(s), \omega_2(s)\big)\, ds \right) \right. \\
&\left. - \varphi_q\left(\frac{1}{\Gamma(\theta_2^*)} \int_0^r (r-s)^{\theta_2^*-1} g\big(s, \omega_1^*(s), \omega_2^*(s)\big)\, ds \right) \right| \\
&+ \frac{\delta}{\Delta_2 \Gamma(\theta_2+\beta-1)} \\
&\times \int_0^\xi \left|(\xi-s)^{\theta_2+\beta-2}\right| \left| \varphi_q\left(\frac{1}{\Gamma(\theta_2^*)} \int_0^r (r-s)^{\theta_2^*-1} g\big(s, \omega_1(s), \omega_2(s)\big)\, ds \right) \right. \\
&\left. - \varphi_q\left(\frac{1}{\Gamma(\theta_2^*)} \int_0^r (r-s)^{\theta_2^*-1} g\big(s, \omega_1^*(s), \omega_2^*(s)\big)\, ds \right) \right| \\
\leq{}& \frac{(q-1)J_2^{q-2}}{\Gamma(\theta_2)} \int_0^r \left|(r-s)^{\theta_2-1}\right| \frac{1}{\Gamma(\theta_2^*)} \int_0^r \left|(r-s)^{\theta_2^*-1}\right| \big| g\big(s, \omega_1(s), \omega_2(s)\big) \\
&- g\big(s, \omega_1^*(s), \omega_2^*(s)\big) \big|\, ds \\
&+ \frac{(q-1)J_2^{q-2}\delta}{\Delta_2 \Gamma(\theta_2+\beta-1)} \int_0^\xi \left|(\xi-s)^{\theta_2+\beta-2}\right| \frac{1}{\Gamma(\theta_2^*)} \\
&\times \int_0^r \left|(r-s)^{\theta_2^*-1}\right| \big| g\big(s, \omega_1(s), \omega_2(s)\big) - g\big(s, \omega_1^*(s), \omega_2^*(s)\big) \big|\, ds \\
\leq{}& \frac{(q-1)J_2^{q-2}\mathcal{K}_g}{\Gamma(\theta_2+1)\Gamma(\theta_2^*+1)} \big(\left|\omega_1(r) - \omega_1^*(r)\right| + \left|\omega_2(r) - \omega_2^*(r)\right| \big) \\
&+ \frac{(q-1)J_2^{q-2}\delta\xi^{\theta_2+\alpha-1}\mathcal{K}_g}{\Delta_2 \Gamma(\theta_2+\beta)\Gamma(\theta_2^*+1)} \big(\left|\omega_1(r) - \omega_1^*(r)\right| + \left|\omega_2(r) - \omega_2^*(r)\right| \big), \qquad (4.7)
\end{aligned}$$

which implies that

$$\begin{aligned}
\left\|\omega_2 - \omega_2^*\right\| \leq{}& \frac{(q-1)J_2^{q-2}\mathcal{K}_g}{\Gamma(\theta_2^*+1)} \left[\frac{1}{\Gamma(\theta_2+1)} + \frac{\delta\xi^{\theta_2+\beta-1}}{\Delta_2 \Gamma(\theta_2+\beta)} \right] \\
&\times \big(\left\|\omega_1 - \omega_1^*\right\| + \left\|\omega_2 - \omega_2^*\right\| \big) \\
\leq{}& D_2 \varrho_2,
\end{aligned}$$

where $D_2 = \frac{(q-1)J_2^{q-2}\mathcal{K}_g}{\Gamma(\theta_2^*+1)}[\frac{1}{\Gamma(\theta_2+1)} + \frac{\delta\xi^{\theta_2+\beta-1}}{\Delta_2\Gamma(\theta_2+\beta)}]$. Hence in view of (4.5) and (4.7), the system of integral equations (2.8) is Hyers–Ulam stable, and consequently, the solution of system (1.1) is Hyers–Ulam stable. □

5 Illustrative example

Example 5.1 Consider the following coupled FDEs involving the nonlinear p-Laplacian operator with IBCs:

$$\begin{aligned} &\mathcal{D}^{\frac{1}{2}}\varphi_3\big(\mathcal{D}^{\frac{3}{2}}\omega_1(r)\big) = \frac{e^{-2r}\omega_1(r)}{100} + \frac{\sin|\omega_1(r)| + \cos|\omega_2(r)|}{2(25+r^2)}, \quad r \in [0,1], \\ &\mathcal{D}^{\frac{1}{2}}\varphi_3\big(\mathcal{D}^{\frac{3}{2}}\omega_2(r)\big) = \frac{\sin|\omega_1(r)| + |\omega_2(r)|}{10(r+1)} + \frac{e^{-2r}\omega_2(r)}{20}, \quad r \in [0,1], \\ &\varphi_3\big(\mathcal{D}^{\frac{3}{2}}\omega_1(r)\big)\big|_{r=0} = \omega_1'(r)\big|_{r=0} = 0, \qquad \omega_1(r)|_{r=0} = \sqrt{2}I^{\frac{1}{3}}\omega_1\left(\frac{1}{2}\right), \\ &\varphi_3\big(\mathcal{D}^{\frac{3}{2}}\omega_2(r)\big)\big|_{r=0} = \nu'(r)\big|_{r=0} = 0, \qquad \omega_2(r)|_{r=0} = \sqrt{3}I^{\frac{1}{2}}\omega_2\left(\frac{1}{3}\right). \end{aligned} \tag{5.1}$$

The suggested parameter values are $\theta_i = 3/2$, $\theta_i^* = 1/2$ $(i = 1,2)$, $p = 3$, $\gamma = \sqrt{2}$, $\delta = \sqrt{3}$, $\eta = 1/2$, $\xi = 1/3$, $p = 4/3$, $q = 3/2$, $f(r,\omega_1,\omega_2) = \frac{e^{-3r}\omega_1(r)}{30} + \frac{\sin|\omega_1(r)|+\cos|\omega_2(r)|}{(50+r^2)}$ and $g(r,\omega_1,\omega_2) = \frac{r^3+\sin|\omega_1(r)|+|\omega_2(r)|}{25} + \frac{e^{-2r}\omega_1(r)}{20}$. Further, we have

$$\begin{aligned} \big|f(r,\omega_1,\omega_2) - f(r,\omega_1,\omega_2)\big| &\le \frac{1}{50}|\omega_1 - \omega_2| + \frac{1}{50}|\omega_1 - \omega_2|, \\ \big|g(r,\omega_1,\omega_2) - g(r,\omega_1,\omega_2)\big| &\le \frac{1}{10}|\omega_1 - \omega_2| + \frac{1}{10}|\omega_1 - \omega_2|. \end{aligned}$$

Therefore, we have $m_1 = 0.3 = m_2$, $n_1 = 0.2 = n_2$

$$\begin{aligned} \blacktriangle_f(m_1+m_2) + \blacktriangle_g(n_1+n_2) &= \frac{(\frac{3}{2}-1)2^{\frac{-1}{2}}}{\Gamma\frac{3}{2}}\left[\frac{1}{\Gamma\frac{5}{2}} + \frac{\sqrt{2}(\frac{1}{2})^{\frac{11}{6}}}{(1-\frac{\sqrt{2}}{\Gamma\frac{4}{3}})\Gamma\frac{17}{6}}\right](0.3+0.3) \\ &\quad + \frac{(\frac{3}{2}-1)2^{\frac{-1}{2}}}{\Gamma\frac{3}{2}}\left[\frac{1}{\Gamma\frac{5}{2}} + \frac{\sqrt{3}(\frac{1}{3})^{\frac{11}{6}}}{(1-\frac{\sqrt{3}}{\Gamma\frac{4}{3}})\Gamma\frac{17}{6}}\right](0.2+0.2) \\ &< 1. \end{aligned}$$

Hence all the conditions of Theorem 3.3 are satisfied, and so the coupled system (5.1) has a unique solution and is Hyres–Ulam stable.

6 Conclusion

In this paper we investigated existence and uniqueness of solutions for coupled fractional differential equations involving the nonlinear p-Laplacian operator with integral boundary conditions, by using nonlinear Leray–Schauder-type alternative and Banach's fixed point theorem. We have also developed some conditions to prove Hyres–Ulam stability. An illustrative example was provided to demonstrate the results. For further studies, we suggest investigating our problem (1.1) for multiplicity results and exponential stability. Readers may also consider the problem for the new established derivative known as ABC fractional derivative.

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