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Robust adaptive control for fractional-order chaotic systems with system uncertainties and external disturbances

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Abstract

This paper studies the robust adaptive control of fractional-order chaotic systems with system uncertainties and bounded external disturbances. Based on a proposed lemma, quadratic Lyapunov functions are used in the stability analysis and fractional-order adaptation laws are designed to update the controller parameters. By employing the fractional-order expansion of classical Lyapunov stability method, a robust controller is designed for fractional-order chaotic systems. The system states asymptotically converge to the origin and all signals in the closed-loop system remain bounded. A counterexample is constructed to show that the fractional-order derivative of a function is less than zero does not mean that the function monotonically decreases (this property appears in many references). Finally, simulation results are presented to confirm our theoretical results.

Keywords: Robust control; Fractional-order chaotic system; Fractional-order adaptation law

1 Introduction

Chaos phenomenon is very common in physical systems (also biology, economics, etc.), and chaotic systems have become one of the hot topics in the nonlinear systems [1–14]. Since Lorenz discovered the first chaotic attractor, the research on the control and synchronization of chaotic systems has been widely used in many fields. In the past 20 years, many chaos control and synchronization methods have been proposed [15–17]. Fractional-order calculus has almost the same long history as integer calculus. It was found that the complex chaotic behavior appeared if one introduced the fractional differential operator into the integer order chaotic systems. In fact, fractional-order calculus provides new mathematical tools for many practical systems, especially for chaotic systems in physics, because it is very suitable for describing the dynamic behavior of some physical systems that are very sensitive to the initial state values [18–28]. There are many methods (such as drive-response control, Lyapunov function method, sliding mode control, generalized synchronization control, active control, nonlinear feedback control) on synchronization and control of fractional-order chaotic systems. However, some new problems run into the case of fractional-order chaotic systems. Firstly, as the tiny initial value changes will cause the system's trajectory shape and affect its stability, it is difficult to control or synchronize the chaotic systems. Secondly, although the Lyapunov second method

of fractional-order systems is proposed in [29], and the control and stability analysis of fractional order nonlinear systems has gradually become a research focus, it is hard to use the squared Lyapunov function in the stability analysis of fractional-order systems because the fractional-order derivatives of square functions have a very complex form (as pointed out in [30, 31]). In order to get over it, many efforts have been made. In 2015, Liu et al. [32, 33] proposed a method to realize adaptive fuzzy synchronous control of uncertain fractional-order chaotic systems with unknown asymptotic control gain. This is really a relatively universal construction method which can be used in many kinds of control and synchronization of chaotic systems [26, 32, 34–50].

We must point out that the technical part of the successful use of this method is Remark 1 (i.e., Lemma 5 in [32] or Lemma 7 in [33]) which is indeed questionable. In this paper, we proposed some corrections and verified that the revised theory can still work in control of chaotic systems. Our contributions are as follows:

- To coin a counterexample which shows that the fractional-order derivative of a function is less than zero does not mean that the function monotonically decreases. Then we correct Lemma 5 in [32] and Lemma 7 in [33] to Lemma 4 in this paper.
- To further verify the effectiveness of the new Lemma 4 in the above relatively universal fractional-order Lyapunov second method, we study the robust adaptive control of fractional-order chaotic systems with bounded external disturbances and system uncertainties.

2 Preliminaries

In this section we present some notions and lemmas which are needed in this paper.

Definition 1 (1) Suppose $f : [0, +\infty) \longrightarrow R$ (the set of all real numbers) and f' are bounded and continuous almost everywhere. The Caputo fractional-order derivative and Caputo fractional-order integral of f with order α ($\alpha \in (0, 1)$) are defined by the following equalities (1) and (2), respectively [51]:

$$D^\alpha(f)(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\tau)\, d\tau}{(t-\tau)^\alpha}, \tag{1}$$

$$I^\alpha(f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)\, d\tau}{(t-\tau)^{1-\alpha}}, \tag{2}$$

where $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt$ ($z \in \mathbb{C}$, $\mathrm{Re}(z) > 0$) is the gamma function, and $\mathbb{C}$ is the set of all complex numbers.

(2) Assume that $f : [0, +\infty) \longrightarrow R$ is piecewise continuous and satisfies $|f(t)| \le Me^{ct}$ for some $M > 0$ and $c \ge 0$. Then the integral $\int_0^\infty f(t)e^{-zt}\, dt$ is convergent in $\mathbb{C}_c = \{z \in \mathbb{C} \mid \mathrm{Re}(z) > c\}$, and hence we get a mapping $F : \mathbb{C}_c \longrightarrow \mathbb{C}$ (called the Laplace transformation of f, written also as $\mathscr{L}(f)$). It has been shown that there exists unique $g : [0, +\infty) \longrightarrow R$ (called the inverse Laplace transformation of F, written as $\mathscr{L}^{-1}(F)$) such that $\mathscr{L}(g) = F$ (i.e., $g = f$). It can be shown that the Laplace transform of the Caputo fractional derivative is

$$\int_0^\infty e^{-st} D^\alpha(f)(t)\, dt = s^\alpha F(s) - s^{\alpha-1} f(0). \tag{3}$$

(3) The convolution G of two functions f and g is defined by $G(x) = f(x) * g(x) = \int_{-\infty}^{+\infty} f(u)g(x-u)\,du$, and the Mittag-Leffler function with two parameters is defined by

$$E_{\alpha,\beta}(s) = \sum_{k=0}^{\infty} \frac{s^k}{\Gamma(\alpha k + \beta)} \quad (s \in \mathbb{C}), \tag{4}$$

where $\alpha, \beta > 0$. Note that $E_{1,1}(s) = e^s$. The Laplace transform of the function $f(t) = t^{\beta-1}E_{\alpha,\beta}(-at^\alpha)$ is [52]

$$\mathscr{L}(f)(s) = \frac{s^{\alpha-\beta}}{s^\alpha + a}. \tag{5}$$

Lemma 1 ([29]) *Let* $\mathbf{0} : [0, +\infty) \longrightarrow R^n$ *be the mapping taking constant value* $\mathbf{0} = [0, 0, \ldots, 0]$, *where* R^n *is the ordinary n-dimensional Euclidean space. If* $\mathbf{0}$ *is an equilibrium of the following fractional-order nonlinear system (where* $\mathbf{x} : [0, +\infty) \longrightarrow R^n$):

$$D^\alpha(\mathbf{x})(t) = f(\mathbf{x}(t)), \tag{6}$$

and there exist a Lyapunov function $V(t, \mathbf{x})$ *and three class-k functions* g_1, g_2, g_3 *such that*

$$g_1(\|\mathbf{x}\|) \le V(t, \mathbf{x}) \le g_2(\|\mathbf{x}\|), \tag{7}$$

$$D^\beta[V(t, \mathbf{x})] \le -g_3(\|\mathbf{x}\|), \tag{8}$$

where $0 < \beta < 1$, *and* $\|\cdot\|$ *is the Euclidean norm, then the equilibrium point* $\mathbf{0}$ *of system* (6) *is Mittag-Leffler stable (and thus asymptotically stable).*

Lemma 2 ([52, 53]) *Suppose that* $x(t) \in C^1[0, T]$ *where T is a positive constant, then the following two equations hold*:

$$I^\alpha D^\alpha x(t) = x(t) - x(0), \tag{9}$$

$$D^\alpha I^\alpha x(t) = x(t). \tag{10}$$

Lemma 3 ([51]) *If* $\mathbf{x} : [0, +\infty) \longrightarrow R^n$ *is continuously derivable, then*

$$\frac{1}{2}D^\alpha[\mathbf{x}^T(t)\mathbf{x}(t)] \le \mathbf{x}^T(t)D^\alpha(\mathbf{x})(t) \quad (\forall t \in [0, +\infty)). \tag{11}$$

3 Controller design and stability analysis

In this paper, the fractional-order chaotic system model is considered. This mathematical model describes a fractional-order system by n-directional nonlinear fractional-order differential equations. The considered model can be expressed as follows:

$$D^\alpha x_i(t) = f_i(\mathbf{x}(t)), \tag{12}$$

where $x_i(t)$ is the state variable. Assume that system (12) has some equilibrium points, one of them is noted as $\mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_n^*)^T$.

According to (12), the controlled model can be expressed as follows:

$$D^\alpha x_i(t) = f_i\big(\mathbf{x}(t)\big) + \Delta f_i\big(\mathbf{x}(t)\big) + d_i(t) + u_i(t), \tag{13}$$

where $\Delta f_i(\mathbf{x}(t))$ is system uncertainty, $d_i(t)$ is unknown external disturbance, and $u_i(t)$ is the control input.

Let the error state be

$$e_i(t) = x_i(t) - x_i^*. \tag{14}$$

In the closed-loop system, we will design the control input $u_i(t)$ to make sure that the error states converge to the origin and all signals remain bounded. To reach this goal, the following assumptions are needed.

Assumption 1 The system uncertainty $\Delta f_i(\mathbf{x}(t))$ is Lipschitz continuous, and there exists a positive constant γ_i such that

$$\big|\Delta f_i\big(\mathbf{x}(t)\big)\big| \le \gamma_i \big\|\mathbf{x}(t)\big\|, \tag{15}$$

where $\|\cdot\|$ denotes the Euclidean norm.

Assumption 2 The external disturbance $d_i(t)$ is a bounded continuous function, i.e., $d_i(t)$ satisfies the following inequality:

$$\big|d_i(t)\big| \le \bar{d}_i, \tag{16}$$

where d_i is a positive constant.

Since the Caputo derivative of a constant is zero, from (13) and (14) we have

$$D^\alpha e_i(t) = f_i\big(\mathbf{x}(t)\big) + \Delta f_i\big(\mathbf{x}(t)\big) + d_i(t) + u_i(t). \tag{17}$$

For each $i \in \{1, 2, \ldots, n\}$, multiplying $e_i(t)$ to both sides of (17), by Assumptions 1 and 2, we have

$$\begin{aligned}
e_i(t)D^\alpha e_i(t) &= e_i(t)f_i\big(\mathbf{x}(t)\big) + e_i(t)\Delta f_i\big(\mathbf{x}(t)\big) + e_i(t)d_i(t) + e_i(t)u_i(t) \\
&\le e_i(t)f_i\big(\mathbf{x}(t)\big) + e_i(t)u_i(t) + \gamma_i\big\|\mathbf{x}(t)\big\|\big|e_i(t)\big| + \bar{d}_i\big|e_i(t)\big|.
\end{aligned} \tag{18}$$

Then we design a controller $u_i(t)$ as follows:

$$u_i(t) = -f_i\big(\mathbf{x}(t)\big) + \text{sign}\big(e_i(t)\big)\big(-k_i\big|e_i(t)\big| - \hat{\gamma}_i(t)\big\|\mathbf{x}(t)\big\| - \hat{\bar{d}}_i(t)\big), \tag{19}$$

where k_i is a positive design parameter, $\hat{\gamma}_i(t)$ is the estimation of γ_i, and $\hat{\bar{d}}_i(t)$ is the estimation of $\bar{d}_i$.

Substituting (19) into (18) yields

$$e_i(t)D^\alpha e_i(t) \le \big|e_i(t)\big|\big(\big(-k_i\big|e_i(t)\big| - \hat{\gamma}_i(t)\big\|\mathbf{x}(t)\big\| - \hat{\bar{d}}_i(t)\big) + \gamma_i\big\|\mathbf{x}(t)\big\| + \bar{d}_i\big)$$

$$= -k_i\big|e_i(t)\big|^2 - \big|e_i(t)\big|\tilde{\gamma}_i(t)\big\|\mathbf{x}(t)\big\| - \big|e_i(t)\big|\tilde{\bar{d}}_i(t), \tag{20}$$

where

$$\tilde{\gamma}_i(t) = \hat{\gamma}_i(t) - \gamma_i \tag{21}$$

and

$$\tilde{\bar{d}}_i(t) = \hat{\bar{d}}_i(t) - \bar{d}_i \tag{22}$$

are the estimation errors of unknown parameters γ_i and $\bar{d}_i$, respectively.

Remark 1 In [32], there exists such a result: Suppose that $f' : [0, +\infty) \longrightarrow R$ is continuous. Then $f : [0, +\infty) \longrightarrow R$ is monotone increasing (resp., monotone decreasing) if $D^\alpha(f)(t) \ge 0$ (resp., $D^\alpha(f)(t) \le 0$) for all $t \in [0, +\infty)$. However, this result is not proper. In this section we will construct a counter-example to show this and meanwhile give a corrected form (i.e., Lemma 4).

Example 1 Let $\alpha \in (0, 1)$. We are going to construct the needed counter-example in 11 steps.
   Step 1. Let $a = \max\{1, \frac{1}{4^{1-\alpha} - 2^{1-\alpha}}\}$, and

$$h_1(t) = t^{1-\alpha} - (t-2)^{1-\alpha}. \tag{23}$$

From

$$h_1'(t) = (1-\alpha)\left[\frac{1}{t^\alpha} - \frac{1}{(t-2)^\alpha}\right] \le 0 \quad \big(\forall t \in [3,4]\big), \tag{24}$$

we have

$$h_1(t) \ge h_1(4) = 4^{1-\alpha} - 2^{1-\alpha} \ge \frac{1}{a} \quad \big(\forall t \in [3,4]\big). \tag{25}$$

Step 2. Let

$$I(t) = \int_0^2 \frac{1}{(t-\tau)^\alpha}\, d\tau + \int_3^4 \frac{-1}{(t-\tau)^\alpha}\, d\tau$$

$$= \frac{1}{1-\alpha}\big[t^{1-\alpha} - (t-2)^{1-\alpha} - (t-3)^{1-\alpha} + (t-4)^{1-\alpha}\big] \quad \big(t \in [4, +\infty)\big). \tag{26}$$

By L'Hospital's rule, we have

$$\lim_{t \to +\infty} t^\beta - (t-n)^\beta = \lim_{t \to +\infty} \frac{1 - (1 - \frac{n}{t})^\beta}{t^{-\beta}} = \lim_{t \to +\infty} \frac{n\beta(1 - \frac{n}{t})^{\beta-1}\frac{1}{t^2}}{\beta t^{-\beta-1}}$$

$$= \lim_{t \to +\infty} \frac{n\beta(1 - \frac{n}{t})^{\beta-1}}{\beta t^{1-\beta}} = 0 \quad (n \text{ is a natural number}, 0 < \beta < 1). \tag{27}$$

From (26) and (27), we can know

$$\lim_{t \to +\infty} I(t) = 0. \tag{28}$$

Step 3. Since $h_2(t) = t^{-\alpha}$ is decreasing in $(0, +\infty)$, $\frac{1}{(t-3)^\alpha} \leq \frac{1}{(t-4)^\alpha}$ ($\forall t \in (4, +\infty)$), thus

$$\frac{1}{2}\left[\frac{2}{t^\alpha} - \frac{1}{(t-3)^\alpha}\right] + \frac{1}{2}\left[\frac{2}{t^\alpha} - \frac{1}{(t-4)^\alpha}\right] \geq \frac{2}{t^\alpha} - \frac{1}{(t-4)^\alpha} \quad \left(\forall t \in (4, +\infty)\right) \tag{29}$$

and

$$\frac{1}{(t-4)^{1-\alpha}} \geq \frac{1}{t^{1-\alpha}} \quad \left(\forall t \in (4, +\infty)\right). \tag{30}$$

Step 4.

$$\lim_{t \to +\infty}\left[2(t-4)^\alpha - t^\alpha\right] = \lim_{t \to +\infty} t^\alpha\left[2\left(1 - \frac{4}{t}\right)^\alpha - 1\right] = +\infty. \tag{31}$$

Step 5. As $I(t)$ is continuous in $[4, +\infty)$ and $\lim_{t \to +\infty} I(t) = 0$ (see (28)), there exists $b > 0$ such that $|I(t)| \leq b$ ($\forall t \in [4, +\infty)$).

Step 6. Given $t \in (4, +\infty)$, we note

$$u(\tau) = \frac{1}{(t-\tau)^\alpha} \quad \left(\forall \tau \in [0, t)\right). \tag{32}$$

As $u''(\tau) = (-\alpha)(-\alpha - 1)(t - \tau)^{-\alpha-2} \geq 0$, $u(\tau) = \frac{1}{(t-\tau)^\alpha}$ is convex downwards in $[0, t)$, thus

$$\int_3^4 u(\tau)\,d\tau = \int_0^1 u\big(3\lambda + 4(1 - \lambda)\big)\,d\lambda \leq \int_0^1 \big[\lambda u(3) + (1 - \lambda)u(4)\big]\,d\lambda$$
$$= \frac{1}{2}\big[u(3) + u(4)\big] = \frac{1}{2}\left[\frac{1}{(t-3)^\alpha} + \frac{1}{(t-4)^\alpha}\right]. \tag{33}$$

Step 7. From (29) and (33) we know ($\forall t \in (4, +\infty)$)

$$I(t) \geq \int_0^2 \frac{1}{t^\alpha}\,d\tau - \int_3^4 u(\tau)\,d\tau \geq \frac{2}{t^\alpha} - \frac{1}{2}\left[\frac{1}{(t-3)^\alpha} + \frac{1}{(t-4)^\alpha}\right]$$
$$= \frac{1}{2}\left[\frac{2}{t^\alpha} - \frac{1}{(t-3)^\alpha}\right] + \frac{1}{2}\left[\frac{2}{t^\alpha} - \frac{1}{(t-4)^\alpha}\right] \geq \frac{2}{t^\alpha} - \frac{1}{(t-4)^\alpha}. \tag{34}$$

Step 8. There exists $t_0 \in (4, +\infty)$ such that $I(t) \geq 0$ ($\forall t \in [t_0, +\infty)$). In fact, $\lim_{t \to +\infty}[2(t - 4)^\alpha - t^\alpha] = +\infty$ by (27). As $2(t - 4)^\alpha - t^\alpha$ is continuous,

$$2(t - 4)^\alpha - t^\alpha \geq 0 \quad \left(\forall t \in [t_0, +\infty)\right) \tag{35}$$

for some $t_0 \in (4, +\infty)$, which together with (34) implies $I(t) \geq 0$ ($\forall t \in [t_0, +\infty)$).

Step 9. Let $c = \max\{0, b[\int_{2.25}^{2.75} \frac{d\tau}{(t_0 - \tau)^\alpha}]^{-1}\}$,

$$
l(x) = \begin{cases} 4(c - a)x + (9a - 8c), & x \in [2, 2.25), \\ c, & x \in [2.25, 2.75), \\ -4cx + 12c & x \in [2.75, 3), \end{cases} \tag{36}
$$

and

$$
g(x) = \begin{cases} a, & x \in [0, 2), \\ l(x), & x \in [2, 3), \\ -ax + 3a, & x \in [3, 3 + \frac{1}{a}], \\ \frac{a}{a-1}x - \frac{4a}{a-1}, & x \in (3 + \frac{1}{a}, 4), \\ x - 4 & x \in [4, +\infty). \end{cases} \tag{37}
$$

Then $g(x)$ is a piecewise linear continuous function which is negative just in $(3, 4)$. From the fundamental theorem of calculus, we know the function $f(t) = \int_0^t g(x)\,dx$ satisfies $f'(0+) = g(0)$ and $f'(t) = g(t)$. By the definitions of $g(x)$ and $f(t)$, $f(t)$ is monotonous increasing in $[0, 3]$ and $[4, +\infty)$ but monotonous decreasing in $(3, 4)$, which means that $f(t)$ is not monotone in $[0, +\infty)$.

Step 10. Now we prove

$$
D^\alpha(f)(t) = \int_0^t \frac{g(\tau)}{(t - \tau)^\alpha}\,d\tau \geq 0. \tag{38}
$$

If $t \in [0, 3)$, then $(t - \tau)^\alpha > 0$ ($\tau \in [0, t)$), and thus $\frac{g(\tau)}{(t-\tau)^\alpha} \geq 0$ ($\tau \in [0, t)$) by the definition of $g(x)$, which means $D^\alpha(f)(t) \geq 0$. If $t \in [3, 4]$, then $\varphi'(t) \leq 0$ (i.e., $\varphi(t) = t^{1-\alpha} - (t - 2)^{1-\alpha}$ is monotonous decreasing) by Step 3 and the minimum of $\varphi(t)$ in $[3, 4]$ is $\varphi(4) = 4^{1-\alpha} - 2^{1-\alpha}$. As $\psi(t) = (t - 3)^{1-\alpha}$ is monotonous increasing, its maximum on $[3, 4]$ is $\psi(4) = 1$. By Step 1,

$$
\begin{aligned}
D^\alpha(f)(t) &= \int_0^2 \frac{a}{(t - \tau)^\alpha}\,d\tau + \int_2^3 \frac{l(\tau)}{(t - \tau)^\alpha}\,d\tau + \int_3^t \frac{g(\tau)}{(t - \tau)^\alpha}\,d\tau \\
&\geq \int_0^2 \frac{a}{(t - \tau)^\alpha}\,d\tau + \int_3^t \frac{g(\tau)}{(t - \tau)^\alpha}\,d\tau \\
&\geq a\int_0^2 \frac{1}{(t - \tau)^\alpha}\,d\tau + \int_3^t \frac{-1}{(t - \tau)^\alpha}\,d\tau \\
&= \frac{a}{1 - \alpha}\left[t^{1-\alpha} - (t - 2)^{1-\alpha}\right] + \frac{1}{1 - \alpha}\left[(t - t)^{1-\alpha} - (t - 3)^{1-\alpha}\right] \\
&\geq \frac{1}{1 - \alpha}\frac{t^{1-\alpha} - (t - 2)^{1-\alpha}}{4^{1-\alpha} - 2^{1-\alpha}} - \frac{1}{1 - \alpha} \geq 0.
\end{aligned} \tag{39}
$$

If $t \in (4, +\infty)$, then

$$
\int_0^t \frac{g(\tau)}{(t - \tau)^\alpha}\,d\tau \geq \int_0^2 \frac{a}{(t - \tau)^\alpha}\,d\tau + \int_2^3 \frac{l(\tau)}{(t - \tau)^\alpha}\,d\tau
$$

$$
+ \int_3^4 \frac{-1}{(t - \tau)^\alpha}\,d\tau + \int_4^t \frac{g(\tau)}{(t - \tau)^\alpha}\,d\tau
$$

$$\geq \int_0^2 \frac{1}{(t-\tau)^\alpha}\, d\tau + \int_3^4 \frac{-1}{(t-\tau)^\alpha}\, d\tau$$

$$+ \int_2^3 \frac{l(\tau)}{(t-\tau)^\alpha}\, d\tau + \int_4^t \frac{g(\tau)}{(t-\tau)^\alpha}\, d\tau. \tag{40}$$

If $t \in (4, t_0]$, then

$$D^\alpha(f)(t) = \int_0^t \frac{g(\tau)}{(t-\tau)^\alpha}\, d\tau \geq I(t) + \int_2^3 \frac{l(\tau)}{(t-\tau)^\alpha}\, d\tau + \int_4^t \frac{g(\tau)}{(t-\tau)^\alpha}\, d\tau$$

$$\geq -b + \int_{2.25}^{2.75} \frac{c}{(t-\tau)^\alpha}\, d\tau$$

$$= -b + b\left[\int_{2.25}^{2.75} \frac{d\tau}{(t_0-\tau)^\alpha}\right]^{-1} \int_{2.25}^{2.75} \frac{d\tau}{(t-\tau)^\alpha} = 0. \tag{41}$$

If $t \in (t_0, +\infty)$, then

$$D^\alpha(f)(t) = \int_0^t \frac{g(\tau)}{(t-\tau)^\alpha}\, d\tau \geq I(t) + \int_2^3 \frac{l(\tau)}{(t-\tau)^\alpha}\, d\tau + \int_4^t \frac{g(\tau)}{(t-\tau)^\alpha}\, d\tau \geq 0. \tag{42}$$

Step 11. From Steps 1–10 we know $h(t) = -f(t)$ is differentiable in $[0, +\infty)$, h' is continuous, and $D^\alpha(h)(t) \leq 0$ ($t \in [0, +\infty)$), but $h(t)$ is not monotone in $[0, +\infty)$.

Remark 2 This example shows a difference between fractional and integer order calculus. And the results in [32] are still right because their proofs are valid as long as we replace Lemma 5 in [32] by the following Lemma 4. In many closed-loop systems, it is difficult to determine that all signals are bounded. With the following lemma, we can manage to do this. For example, the signals $\hat{\gamma}$ and $\hat{\tilde{d}}$ in system (13).

Lemma 4 *Assume that f' is continuous and bounded in $[0, +\infty)$. Then $f(t) \geq f(0)$ if $D^\alpha(f)(t) \geq 0$ ($t \in [0, +\infty)$), and $f(t) \leq f(0)$ if $D^\alpha(f)(t) \leq 0$.*

Proof We just prove the first part of this lemma.

Step 1.On $[0, +\infty)$, if f is continuous $S_t = \{(u,v) \in R^2 \mid 0 \leq u \leq v \leq t\}$ ($t > 0$) and $\alpha, \beta \in (0, 1)$, by Fubini's theorem [54], the following holds:

$$\iint_{S_t} \frac{f(u)\, du\, dv}{(t-v)^{1-\alpha}(v-u)^{1-\beta}} = \int_0^t \left[\frac{1}{(t-v)^{1-\alpha}} \int_0^v \frac{f(u)\, du}{(v-u)^{1-\beta}}\right] dv$$

$$= \int_0^t \left[f(u) \int_u^t \frac{dv}{(t-v)^{1-\alpha}(v-u)^{1-\beta}}\right] du. \tag{43}$$

Step 2. If f is continuous on $[0, +\infty)$, we have

$$D^\alpha \circ D^\beta(f)(t) = D^{\alpha+\beta}(f)(t) = D^\beta \circ D^\alpha(f)(t) \quad (\alpha, \beta \in (0, 1)). \tag{44}$$

In fact, let

$$\eta = u, \qquad \xi = \frac{v-u}{t-u}, \tag{45}$$

the partial derivatives of (45) are continuous and their Jacobian matrix is not 0 almost everywhere in the domain. By the definition of Caputo fractional-order integral and Step 1, we have (where B is the beta function)

$$
\begin{aligned}
I^{\alpha} \circ I^{\beta}(f)(t) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \left[\frac{1}{(t-v)^{1-\alpha}} \int_0^v \frac{f(u)\,du}{(v-u)^{1-\beta}} \right] dv \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \left[f(u) \int_u^t \frac{(t-v)^{\alpha-1}\,dv}{(v-u)^{1-\beta}} \right] du \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \left[f(\eta)(t-\eta)^{\alpha+\beta-1} \int_0^1 \frac{\xi^{\beta-1}\,d\xi}{(1-\xi)^{1-\alpha}} \right] d\eta \\
&= \frac{B(\beta,\alpha)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \frac{f(\eta)\,d\eta}{(t-\eta)^{1-\alpha-\beta}} = I^{\alpha+\beta}(f)(t).
\end{aligned}
\tag{46}
$$

Step 3. If f' is bounded and continuous in $[0,+\infty)$, then

$$
I^{\alpha} \circ D^{\alpha}(f)(t) = f(t) - f(0).
\tag{47}
$$

In fact, from the definitions of Caputo derivative and integral, we know $D^{\alpha}(f) = I^{1-\alpha}(f')$. By Step 2,

$$
I^{\alpha} \circ D^{\alpha}(f)(t) = I^{\alpha} \circ I^{1-\alpha}(f')(t) = I^1(f').
\tag{48}
$$

By the fundamental theorem of calculus, $I^1(f') = f(t) - f(0)$.

Step 4. Since $D^{\alpha}(f)(t) \geq 0$ $(t \in [0,+\infty))$, $f(t) - f(0) = I^{\alpha} \circ D^{\alpha}(f)(t) \geq 0$ $(\forall t \in [0,\infty))$ by Step 3 and the definition of I^{α}, which means $f(t) \geq f(0)$ $(\forall t \in [0,\infty))$. □

And we show the following lemma.

Lemma 5 *Let* $V_1(t) = \frac{1}{2}x^2(t) + \frac{1}{2}y^2(t)$, *where* $x, y : [0,+\infty) \longrightarrow R$ *is a continuous function. If*

$$
D^{\alpha}(V_1)(t) \leq -kx^2(t),
\tag{49}
$$

where k *is a positive constant, then we have*

$$
x^2(t) \leq 2V_1(0)E_{\alpha}\left(-2kt^{\alpha}\right).
\tag{50}
$$

Proof Using the fractional integral operator I^{α} to both sides of (49), it follows from Lemma 2 that

$$
V_1(t) - V_1(0) \leq -kI^{\alpha}x^2(t).
\tag{51}
$$

It follows from (51) that

$$
x^2(t) \leq 2V_1(0) - 2kI^{\alpha}x^2(t).
\tag{52}
$$

There exists a nonnegative function $m(t)$ such that

$$
x^2(t) + m(t) = 2V_1(0) - 2kI^{\alpha}x^2(t).
\tag{53}
$$

Taking the Laplace transform ($\mathscr{L}\{\cdot\}$) on (53) by (3) gives

$$_2X(s) = 2V_1(0)\frac{s^{\alpha-1}}{s^\alpha + 2k} - \frac{s^\alpha}{s^\alpha + 2k}M(s), \tag{54}$$

where $_2X(s)$ and $M(s)$ are Laplace transforms of $x^2(t)$ and $m(t)$, respectively. Using (5), the solution of (54) will be given as

$$x^2(t) = 2V_1(0)E_\alpha\left(-2kt^\alpha\right) - m(t) * \left[t^{-1}E_{\alpha,0}\left(-2kt^\alpha\right)\right], \tag{55}$$

where $*$ represents the convolution operator. Noting that $m(t)$, $E_{\alpha,0}(-2kt^\alpha)$ and t^{-1} are nonnegative functions, it follows from (55) that (50) holds. Then this ends the proof for Lemma 5. $\square$

Based on the above discussions, now we are ready to give the following results.

Theorem 1 *Consider the fractional-order chaotic system* (13). *Under Assumptions* 1 *and* 2, *let the control input be* (19). *If $\hat{\gamma}_i(t)$ and $\hat{\bar{d}}_i(t)$ are updated by*

$$D^\alpha\hat{\gamma}_i(t) = h_i|e_i(t)|\|\mathbf{x}(t)\| \tag{56}$$

and

$$D^\alpha\hat{\bar{d}}_i(t) = m_i|e_i(t)|, \tag{57}$$

respectively, where h_i and m_i are positive design parameters, then the tracking error $e_i(t)$ will tend to the origin asymptotically, and all signals in the closed-loop system will remain bounded.

Proof Let us consider the following Lyapunov function candidate:

$$V_i(t) = \frac{1}{2}e_i^2(t) + \frac{1}{2h_i}\tilde{\gamma}_i^2(t) + \frac{1}{2m_i}\tilde{\bar{d}}_i^{\,2}(t). \tag{58}$$

Then, by using Lemma 3, we have

$$D^\alpha V_i(t) \leq e_i(t)D^\alpha e_i(t) + \frac{1}{h_i}\tilde{\gamma}_i(t)D^\alpha\tilde{\gamma}_i(t) + \frac{1}{m_i}\tilde{\bar{d}}_i(t)D^\alpha\tilde{\bar{d}}_i(t). \tag{59}$$

Noting that the fractional-order derivative of a constant is zero, from (21) and (22) we have

$$D^\alpha\tilde{\gamma}_i(t) = D^\alpha\hat{\gamma}_i(t) \tag{60}$$

and

$$D^\alpha\tilde{\bar{d}}_i(t) = D^\alpha\hat{\bar{d}}_i(t). \tag{61}$$

Substituting (20), (60), and (61) into (59), we have

$$D^\alpha V_i(t) \le -k_i|e_i(t)|^2 - |e_i(t)||\tilde{\gamma}_i(t)|\|\mathbf{x}(t)\| - |e_i(t)||\tilde{\bar{d}}_i|$$
$$+ \frac{1}{h_i}\tilde{\gamma}_i(t)D^\alpha\hat{\gamma}_i(t) + \frac{1}{m_i}\tilde{\bar{d}}_i(t)D^\alpha\hat{\bar{d}}_i(t). \tag{62}$$

Then substituting (56) and (57) into (62) gives

$$D^\alpha V_i(t) \le -k_i|e_i(t)|^2. \tag{63}$$

Thus, we have $D^\alpha V_i(t) \le -k_i|e_i(t)|^2 \le 0$. According to Lemma 4, $V_i(t) \le V_i(0)$. In addition, for $V_i(t) = \frac{1}{2}e_i^2(t) + \frac{1}{2h_i}\tilde{\gamma}_i^2(t) + \frac{1}{2m_i}\tilde{\bar{d}}_i^2(t)$, we can know that $e_i(t)$, $\tilde{\gamma}_i(t)$, and $\tilde{\bar{d}}_i(t)$ are all bounded. From Lemma 5 and (63), we can conclude that $e_i(t)$ will asymptotically converge to the origin. This ends the proof for Theorem 1. □

Remark 3 In the stability analysis of fractional-order nonlinear systems, the Lyapunov function candidate $V(t) = 2e^T(t)e(t)$ is often used. The αth-order of $V(t)$ can be given as

$$D^\alpha V(t) = \left(D^\alpha e(t)\right)^T e(t) + e^T(t)D^\alpha e(t) + 2\Lambda, \tag{64}$$

where

$$\Lambda = \sum_{i=1}^{\infty} \frac{\Gamma(1+\alpha)}{\Gamma(1+i)\Gamma(1-i+\alpha)} D^i e(t)D^{\alpha-i}e(t). \tag{65}$$

We can see that it is very hard to use the above complicated infinite series to analyze the stability of fractional-order systems. However, in this paper, by using Lemma 3 and the proposed Lemma 5, we need not tackle the above complicated infinite series.

4 Simulation studies

Consider the following fractional-order system with Caputo fractional derivative:

$$\begin{cases} D^\alpha(x_1) = 0.8x_1(t) - x_2(t)x_3(t), \\ D^\alpha(x_2) = -x_2(t) + x_1(t)x_3(t), \\ D^\alpha(x_3) = -4x_3(t) + x_1(t)x_2(t). \end{cases} \tag{66}$$

Let $\alpha = 0.95$, and the equilibrium point is $(2, 3.2^{0.5}, 0.8^{0.5})^T$.

The system uncertainties are chosen as:

$$\begin{cases} \Delta f_1(\mathbf{x}(t)) = 0.1x_1(t), \\ \Delta f_2(\mathbf{x}(t)) = 0.1x_2(t) + 0.1x_3(t), \\ \Delta f_3(\mathbf{x}(t)) = 0.1\sin(x(t)) + 0.1x_2(t). \end{cases} \tag{67}$$

Figure 1 Chaotic behavior of fractional-order system

Figure 2 Stability of fractional-order system
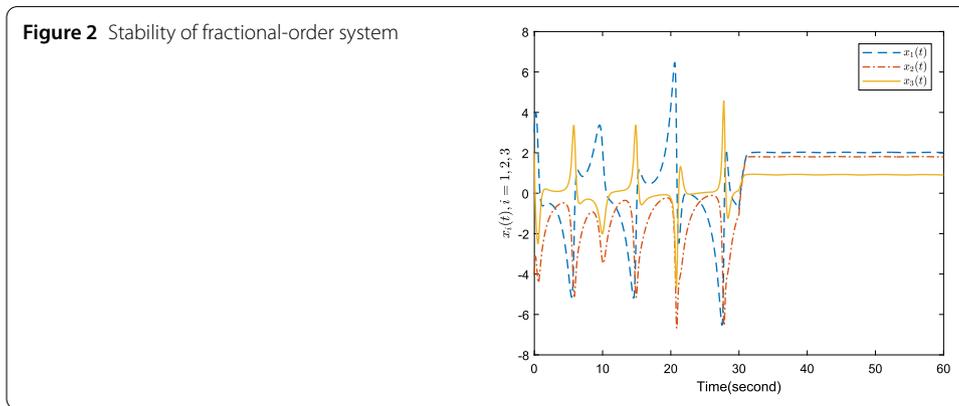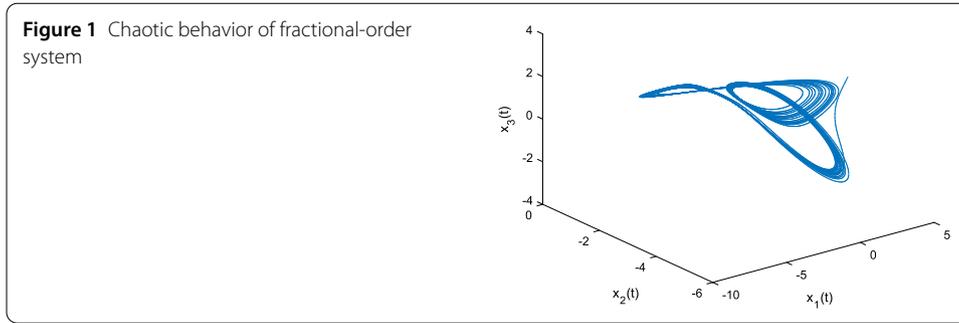
We can easily conclude that Assumption 1 is satisfied. Let the external disturbances be

$$\Delta d_1(t) = 0.1 \sin t,$$

$$\Delta d_2(t) = 0.1 \cos t, \tag{68}$$

$$\Delta d_3(t) = -0.1 \sin t + 0.1 \cos t.$$

Assumption 2 is satisfied, too. The controller design parameters are chosen as $k_1 = k_2 = k_3 = 1$, $h_1 = h_2 = h_3 = 0.1$, $m_1 = m_2 = m_3 = 0.1$. The initial conditions of the fractional-order adaptation law are chosen as $\hat{\gamma}_1(0) = 0.1$, $\hat{\gamma}_2(0) = 0.2$, $\hat{\gamma}_3(0) = 0.2$, $\hat{\bar{d}}_1(0) = 0.1$, $\hat{\bar{d}}_2(0) = 0.1$, $\hat{\bar{d}}_3(0) = 0.2$. To eliminate the chattering phenomenon, the discontinuous term $\text{sign}(\cdot)$ is replaced by $\arctan(10\cdot)$.

Figure 1 shows the behavior of the chaotic system. Figure 2 shows how the fractional-order system has become stable when the controller is activated since $t = 30$. Figure 3 shows the action of controllers. In this figure, the control inputs do not converge to zero because $f_i(t)$ do not converge to zero. And after $t = 30$, the error states of the fractional-order system which converge to the origin are shown in Fig. 4. From the simulation results, we can see that the good control performance has been obtained, and the system variables converge to the origin rapidly when the controller is activated at $t = 30$.

5 Conclusions

Chaotic systems can be used in many fields. Controlling fractional-order chaotic systems by using an effective control method is an interesting yet challenging work. In this paper, an example is built to show that being less than zero for a fractional-order derivative does not mean that the function monotonically decreases. And the robust control of

Figure 3 Controllers of fractional-order system

Figure 4 Error states of fractional-order system
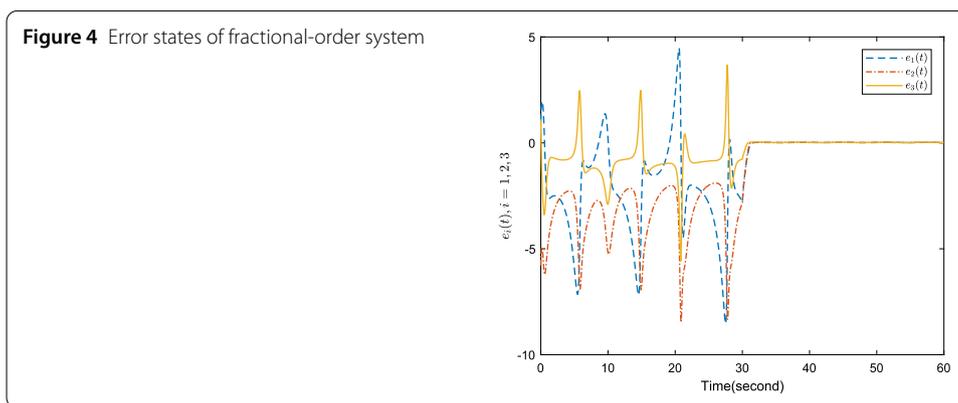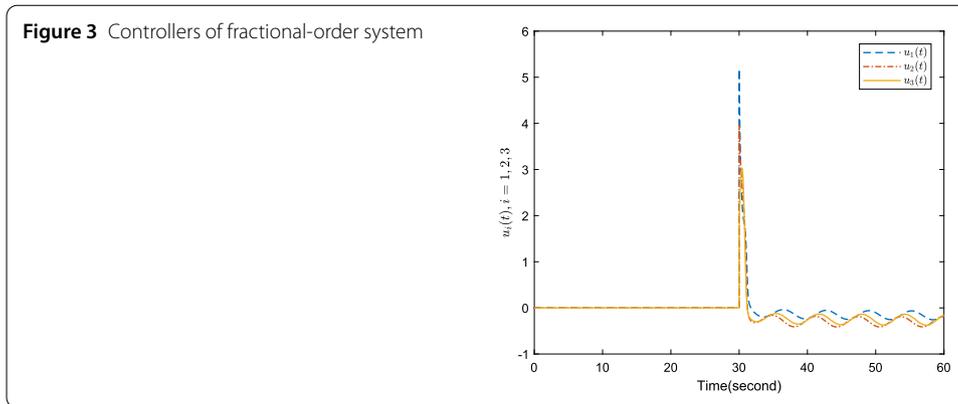
fractional-order chaotic systems by means of adaptive control is primarily discussed. The proposed method can be divided into three aspects: (1) Based on the fractional-order Lyapunov second method, analyzing the stability of general fractional-order chaotic systems; (2) Designing the fractional-order adaptation laws and uploading the controller; (3) Using the quadratic Lyapunov functions in the stability analysis of fractional-order systems. The control theorem of fractional-order systems may be enriched by our results, and the proposed control method can also be extended to other fractional-order systems.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors conceived of the study, participated in its design and coordination, read and approved the final manuscript.

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