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Single upper-solution or lower-solution method for Langevin equations with two fractional orders

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Abstract

The purpose of this paper is to investigate the existence and uniqueness of nonnegative solutions for Langevin equations with two fractional orders:

$$\begin{cases} {}_0^cD_t^\beta({}_0^cD_t^\alpha - \gamma)x(t) = f(t,x(t)), & 0 < t < 1, \\ x^{(k)}(0) = \mu_k, & 0 \le k < l, \\ x^{(\alpha+k)}(0) = \nu_k, & 0 \le k < n, \end{cases}$$

where ${}_0^cD_t^\alpha$ and ${}_0^cD_t^\beta$ denote the Caputo fractional derivatives, $f : [0,1] \times \mathbf{R} \to \mathbf{R}$ is a continuous function, and $m-1 < \alpha \le m$, $n-1 < \beta \le n$, $l = \max\{m,n\}$, $n,m \in \mathbf{N}$, $\gamma > 0$, $\mu_j \ge 0$, $\forall j \in \{0,\dots,m-1\}$, $\nu_i - \gamma\mu_i \ge 0$, $\forall i \in \{0,\dots,n-1\}$. By using a single upper-solution or lower-solution method and monotone iterative approach, several existence and uniqueness results of nonnegative solutions are obtained. Moreover, an example is given to illustrate the main results.

MSC: 26A33; 34A08; 34A12; 34B18; 34B15

Keywords: Single upper-solution or lower-solution method; Monotone iterative; Initial value problem; Fractional Langevin equation; Nonnegative solutions

1 Introduction

In 1908, Paul Langevin gave an elaborate description of Brownian motion, and thus Langevin equations were proposed, see [9, 12]. Langevin equations can also describe many stochastic problems in fluctuating environments. In 1966, Kube gave a generalized Langevin equation for modeling anomalous diffusive processes in a complex and viscoelastic environment [10, 11]. An important extension of the topic is fractional Langevin equation, which was introduced by Mainardi and collaborators [16, 17] in the early 1990s. A lot of fractional Langevin equations have been established, e.g., fractional Langevin equations for modeling of single-file diffusion [6] and for a free particle driven by power law type of noise [18]. So fractional Langevin equations have been studied widely, see [1–9, 13, 15, 16, 19–26] for example. Recently, there have been many papers considering fractional Langevin equations involving two fractional orders, see [1–6, 8, 13, 21, 24–26]. Most of these articles studied the existence and uniqueness of solutions for Langevin equations, and some good results have been given by using the Banach contraction principle, Krasnoselskii's fixed point theorem, Schauder's fixed point theorem, Leray–Schauder nonlinear alternative, Leray–Schauder degree, and so on.

In [26], by using the Leray–Schauder nonlinear alternative, the authors studied the following initial value problem of Langevin equations with two fractional orders:

$$\begin{cases} {}_0^cD_t^\beta({}_0^cD_t^\alpha + \gamma)x(t) = f(t,x(t)), \quad 0 < t < 1, \\ x^{(k)}(0) = \mu_k, \quad 0 \le k < l, \\ x^{(\alpha+k)}(0) = \nu_k, \quad 0 \le k < n, \end{cases}$$

where ${}_0^cD_t^\alpha$ and ${}_0^cD_t^\beta$ denote the Caputo fractional derivatives, $f : [0,1] \times \mathbf{R} \to R$ is a continuous function, $\gamma \in \mathbf{R}$, $n, m \in \mathbf{N}^+$, $m-1 < \alpha \le m$, $n-1 < \beta \le n$, $l = \max\{m,n\}$. The existence of solutions was given. Further, the uniqueness of solutions was also obtained by means of the Banach contraction principle. Recently, the author [4] studied this problem by introducing a new norm

$$\|f\|_{p,\alpha} = \sup_{t\in[0,1]} \left(\int_0^t \frac{|f(s)|^p}{(t-s)^\alpha} \, ds \right)^{\frac{1}{p}}, \quad \alpha \in (0,1), p \ge 1$$

for a measurable function $f : [0,1] \to \mathbf{R}$ and got the existence, uniqueness of solutions for this problem via the Banach contraction principle.

We can find that there are few papers devoted to the study of nonnegative solutions for Langevin equations involving two fractional orders. In this paper we use a single upper-solution or lower-solution method and a monotone iterative approach to consider the following initial value problem of Langevin equations involving two fractional orders:

$$\begin{cases} {}_0^cD_t^\beta({}_0^cD_t^\alpha - \gamma)x(t) = f(t,x(t)), \quad 0 < t < 1, \\ x^{(k)}(0) = \mu_k, \quad 0 \le k < l, \\ x^{(\alpha+k)}(0) = \nu_k, \quad 0 \le k < n, \end{cases} \tag{1.1}$$

where ${}_0^cD_t^\alpha$ and ${}_0^cD_t^\beta$ denote the Caputo fractional derivatives, $f : [0,1] \times \mathbf{R} \to R$ is a continuous function, and some initial conditions are given: $m-1 < \alpha \le m$, $n-1 < \beta \le n$, $l = \max\{m,n\}$, $n, m \in \mathbf{N}^+$, $\gamma > 0$, $\mu_j \ge 0$, $\forall j \in \{0,\ldots,m-1\}$, $\nu_i - \gamma\mu_i \ge 0$, $\forall i \in \{0,\ldots,n-1\}$. In [14], by using e-positive operators and Altman fixed point theory, we gave some existence and uniqueness results of solutions for (1.1). Different from the above-mentioned results, in this paper we establish the existence and uniqueness of nonnegative solutions for problem (1.1), which are new results on initial value problems for Langevin equations. It should be pointed out that we only use single lower-solution or single upper-solution to get the existence and uniqueness of nonnegative solutions for problem (1.1). This method is novel and our results are new.

In this paper, we always assume that the function f satisfies the following two conditions:

(H_1) $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous;

(H_2) $f(t,x)$ is increasing in $x \in [0,\infty)$ for each $t \in [0,1]$.

2 Preliminaries

In order to obtain our results, we first list necessary definitions, lemmas, and basic results.

Definition 2.1 ([4, 26]) For a function $x(t)$, the Riemann–Liouville fractional integral of order $\alpha > 0$ is

$$ {}_aI_t^\alpha x(t) = \int_a^t \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)} x(u)\, du. \tag{2.1}$$

Definition 2.2 ([4, 26]) For a continuous function $x(t)$, the Caputo fractional derivative of order $\alpha > 0$ is

$$ {}_a^cD_t^\alpha x(t) = \int_a^t \frac{(t-u)^{n-\alpha-1}}{\Gamma(n-\alpha)} x^{(n)}(u)\, du, \quad n = [\alpha] + 1, \tag{2.2}$$

where $[\alpha]$ denotes the integer part of the real number α.

Lemma 2.1 ([26]) *$x(t)$ is a solution of problem (1.1) if and only if $x(t)$ is a solution of the integral equation*

$$ x(t) = \int_0^t \frac{(t-u)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} f\bigl(u, x(u)\bigr)\, du + \gamma \int_0^t \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)} x(u)\, du + \phi(t), \tag{2.3}$$

where

$$ \phi(t) = \sum_{i=0}^{n-1} \frac{\nu_i - \gamma\mu_i}{\Gamma(\alpha+i+1)} t^{\alpha+i} + \sum_{j=0}^{m-1} \frac{\mu_j}{\Gamma(j+1)} t^j. \tag{2.4}$$

Define an operator $T : C[0,1] \to C[0,1]$ by

$$ Tx(t) = \int_0^t \frac{(t-u)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} f\bigl(u, x(u)\bigr)\, du + \gamma \int_0^t \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)} x(u)\, du + \phi(t). \tag{2.5}$$

From Lemma 2.1, we can see that $x(t)$ is a solution of problem (1.1) if and only if x is a fixed point of T.

3 Main results

In this section, we apply a single upper-solution or lower-solution method and a monotone iterative approach to study problem (1.1), and we obtain some new results on the existence results of unique nonnegative solutions.

Let $E = C[0,1]$ be the Banach space with the norm $\|x\| = \max_{t\in[0,1]} |x(t)|$, and θ denotes the zero element in E. Given the usual normal cone $P = \{x \in E : x(t) \geq 0, \forall t \in [0,1]\}$. Then $x \leq y$ if and only if $x(t) \leq y(t)$, $\forall t \in [0,1]$. For $x, y \in P$ with $x \leq y$, we have $\|x\| \leq \|y\|$. Define

$$ D_1 = \bigl\{x \in E : x(t) \geq \phi(t), t \in [0,1]\bigr\}.$$

Theorem 3.1 *Assume* (H_1), (H_2), *and*

(H_3) *there exists a constant $\sigma > 0$ such that, for $x, y \in [0,\infty)$ with $y \geq x$,*

$$ f(t,y) - f(t,x) \leq \sigma(y - x), \quad \forall t \in [0,1].$$

If

$$\tau := \frac{\sigma}{\Gamma(\alpha+\beta+1)} + \frac{\gamma}{\Gamma(\alpha+1)} < 1,$$

then problem (1.1) *has a unique nonnegative solution in* D_1.

Proof Since $\mu_j \geq 0$ for $j \in \{0,\dots,m-1\}$ and $\nu_i - \gamma\mu_i \geq 0$ for $i \in \{0,\dots,n-1\}$, we obtain $\phi(t) \geq 0$, $t \in [0,1]$. From (H_1), $T\phi(t) \geq \phi(t)$, $\forall t \in [0,1]$. For $x,y \in D_1$ with $x \leq y$, we have $y(t) \geq x(t) \geq 0$, $t \in [0,1]$, from (H_2),

$$\begin{aligned} Ty(t) &= \int_0^t \frac{(t-u)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} f\bigl(u,y(u)\bigr)\,du + \gamma \int_0^t \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)} y(u)\,du + \phi(t) \\ &\geq \int_0^t \frac{(t-u)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} f\bigl(u,x(u)\bigr)\,du + \gamma \int_0^t \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)} x(u)\,du + \phi(t) \\ &= Tx(t), \end{aligned}$$

which means that T is increasing in D_1. Hence, for $t \in [0,1]$, by (H_3),

$$\begin{aligned} 0 &\leq Ty(t) - Tx(t) \\ &= \int_0^t \frac{(t-u)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \bigl(f\bigl(u,y(u)\bigr) - f\bigl(u,x(u)\bigr)\bigr)\,du \\ &\quad + \gamma \int_0^t \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)} \bigl(y(u)-x(u)\bigr)\,du \\ &\leq \int_0^t \sigma \frac{(t-u)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \bigl(y(u)-x(u)\bigr)\,du \\ &\quad + \gamma \int_0^t \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)} \bigl(y(u)-x(u)\bigr)\,du \\ &= \int_0^t \left(\sigma \frac{(t-u)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} + \gamma \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)}\right) \bigl(y(u)-x(u)\bigr)\,du \\ &= L\bigl(y(t)-x(t)\bigr), \end{aligned}$$

where

$$Lx(t) := \int_0^t \left(\sigma \frac{(t-u)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} + \gamma \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)}\right) x(u)\,du, \quad \forall t \in [0,1]. \tag{3.1}$$

We know that L is a positive linear bounded operator, and its norm

$$\begin{aligned} \|L\| &\leq \max_{t\in[0,1]} \int_0^t \left(\sigma \frac{(t-u)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} + \gamma \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)}\right) du \\ &= \max_{t\in[0,1]} \left(\sigma \frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \gamma \frac{t^{\alpha}}{\Gamma(\alpha+1)}\right) \\ &= \frac{\sigma}{\Gamma(\alpha+\beta+1)} + \frac{\gamma}{\Gamma(\alpha+1)} = \tau < 1. \end{aligned}$$

Therefore,

$$Ty - Tx \leq L(y - x) \quad \text{for } x, y \in D_1 \text{ with } x \leq y. \tag{3.2}$$

Let

$$x_0 = \phi, \qquad x_n = Tx_{n-1} \quad (n = 1, 2, \ldots).$$

Because T is increasing in D_1, we get

$$\phi = x_0 \leq x_1 \leq \cdots \leq x_n \leq \cdots. \tag{3.3}$$

Further, by (3.2),

$$\begin{aligned}\theta \leq x_{n+p} - x_n &= Tx_{n+p-1} - Tx_{n-1}\\ &\leq L(x_{n+p-1} - x_{n-1})\\ &\leq L^2(x_{n+p-2} - x_{n-2})\\ &\leq \cdots \leq L^n(x_p - x_0),\end{aligned}$$

and thus,

$$\begin{aligned}\|x_{n+p} - x_n\| &\leq \left\|L^n(x_p - x_0)\right\|\\ &\leq \left\|L^n\right\| \cdot \|x_p - x_0\|\\ &\leq \tau^n \|x_p - x_0\| \quad (n, p = 1, 2, \ldots).\end{aligned}$$

Since $\tau \in (0,1)$, $\{x_n\}$ is a Cauchy sequence in D_1. Because D_1 is a close set in E, so it is complete. Hence, there exists $x^* \in D_1$ such that $x_n \to x^*$ as $n \to \infty$. By (3.3), $x_n \leq x^*$ and then $Tx_n \leq Tx^*$, that is, $x_{n+1} \leq Tx^*$, which implies

$$\theta \leq Tx^* - Tx_n \leq L\bigl(x^* - x_n\bigr).$$

Also,

$$\left\|Tx^* - x_{n+1}\right\| = \left\|Tx^* - Tx_n\right\| \leq \left\|L\bigl(x^* - x_n\bigr)\right\| \leq \|L\| \cdot \left\|x^* - x_n\right\| \leq \tau \left\|x^* - x_n\right\|.$$

Because $x_n \to x^*$ as $n \to \infty$, we get $\|Tx^* - x^*\| = 0$, and thus $Tx^* = x^*$. That is, x^* is a fixed point of T. Therefore, x^* is a nonnegative solution of problem (1.1).

In the following, we show that the solution x^* of problem (1.1) is a unique solution in D_1.

Suppose that $\overline{x} \in D_1$ is the other solution of problem (1.1). Then $\overline{x}$ is a fixed point of T in D_1. Since $\overline{x} \geq x_0$, we get $T\overline{x} \geq Tx_0$, that is, $\overline{x} \geq x_1$. In general, $\overline{x} \geq x_n$ $(n = 0, 1, 2, \ldots)$. Let $n \to \infty$, we have $\overline{x} \geq x^*$, and by (3.2),

$$\theta \leq \overline{x} - x_{n+1} = T\overline{x} - Tx_n \leq L(\overline{x} - x_n) \leq \cdots \leq L^n(\overline{x} - x_0).$$

Also,

$$\|\overline{x}-x_{n+1}\| \leq \|L^n(\overline{x}-x_0)\| \leq \tau^n\|\overline{x}-x_0\| \to 0 \quad \text{as } n\to\infty.$$

So $\|\overline{x}-x^*\| = 0$, and thus $\overline{x} = x^*$. □

Let $L_1 = \max\{f(t,0) : t \in [0,1]\}$ and

$$C_1 = \frac{L_1}{\Gamma(\alpha+\beta+1)}, \qquad C = \sum_{i=0}^{n-1} \frac{\nu_i - \gamma\mu_i}{\Gamma(\alpha+i+1)} + \sum_{j=0}^{m-1} \frac{\mu_j}{\Gamma(j+1)}.$$

Then $L_1, C_1, C \geq 0$. Take

$$R \geq \frac{C+C_1}{1-\tau}. \tag{3.4}$$

Define

$$D_2 = \{x \in E : x(t) \leq R, t \in [0,1]\}.$$

Theorem 3.2 *Assume that* (H_1)–(H_3) *hold. If*

$$\tau := \frac{\sigma}{\Gamma(\alpha+\beta+1)} + \frac{\gamma}{\Gamma(\alpha+1)} < 1,$$

then problem (1.1) *has a unique nonnegative solution in* D_2.

Proof Like the proof of Theorem 3.1, T is increasing in D_2 and (3.2) holds. By (H_3), we obtain

$$f(t,x) \leq \sigma x + f(t,0) \quad \text{for } x \geq 0, t \in [0,1]. \tag{3.5}$$

Let

$$v_0(t) = R, \qquad v_n(t) = Tv_{n-1}(t), \quad n = 1,2,\ldots.$$

From (3.4), (3.5), we have

$$\begin{aligned}
Tv_0(t) &= \int_0^t \frac{(t-u)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} f\big(u,v_0(u)\big)\,du + \gamma \int_0^t \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)} v_0(u)\,du + \phi(t) \\
&\leq \int_0^t \frac{(t-u)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \sigma R\,du + \int_0^t \frac{(t-u)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} f(u,0)\,du + \gamma \int_0^t \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)} R\,du + C \\
&\leq \int_0^t \frac{(t-u)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \sigma R\,du + \int_0^t \frac{(t-u)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} L_1\,du + \gamma \int_0^t \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)} R\,du + C \\
&= R\left(\sigma \frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \gamma \frac{t^{\alpha}}{\Gamma(\alpha+1)}\right) + L_1 \frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + C \\
&\leq R\left(\sigma \frac{1}{\Gamma(\alpha+\beta+1)} + \gamma \frac{1}{\Gamma(\alpha+1)}\right) + C_1 + C \\
&= R\tau + C_1 + C \leq R = v_0(t),
\end{aligned}$$

so from the fact that T is increasing on D_2, we can easily get

$$\cdots \le v_n \le \cdots \le v_1 \le v_0 = R. \tag{3.6}$$

Further, by (3.2),

$$\begin{aligned}\theta \le v_n - v_{n+p} &= Tv_{n-1} - Tv_{n+p-1} \\ &\le L(v_{n-1} - v_{n+p-1}) \\ &\le L^2(v_{n-2} - v_{n+p-2}) \\ &\le \cdots \le L^n(v_0 - v_p),\end{aligned}$$

and thus

$$\begin{aligned}\|v_n - v_{n+p}\| &\le \big\|L^n(v_0 - v_p)\big\| \\ &\le \big\|L^n\big\| \cdot \big\|v_0 - v_p\big\| \\ &\le \tau^n\|v_0 - v_p\|, \quad n, p = 1, 2, \ldots.\end{aligned}$$

Note that $\tau \in (0,1)$, $\{v_n\}$ is a Cauchy sequence in D_2. Because D_2 is a close set in E, so it is complete. Hence, there exists $v^* \in D_2$ such that $v_n \to v^*$ as $n \to \infty$. From (3.6), $v_n \ge v^*$ and then $Tv_n \ge Tv^*$, that is, $v_{n+1} \ge Tv^*$. So we have

$$\theta \le Tv_n - Tv^* \le L\big(x^* - x_n\big),$$

then we can obtain

$$\big\|Tv_n - Tv^*\big\| = \big\|v_{n+1} - Tv^*\big\| \le \big\|L\big(v_n - v^*\big)\big\| \le \|L\| \cdot \big\|v_n - v^*\big\| \le \tau\big\|v_n - v^*\big\|.$$

Hence, from $v_n \to v^*$ as $n \to \infty$, we have $\|Tv^* - v^*\| = 0$, and thus $Tv^* = v^*$. That is, v^* is a fixed point of T. Therefore, v^* is a nonnegative solution of problem (1.1).

In the following, we show that v^* is a unique solution of problem (1.1) in D_2.

Suppose that $\overline{v}$ is the other solution of problem (1.1) in D_2. Then $\overline{v}$ is a fixed point of T in D_2. Since $\overline{v} \ge v_0$, we get $T\overline{v} \ge Tv_0$, that is, $\overline{v} \ge v_1$. In general, $\overline{v} \ge v_n$ ($n = 0, 1, 2, \ldots$). Let $n \to \infty$, we have $\overline{v} \ge v^*$, and by (3.2),

$$\theta \le \overline{v} - v_{n+1} = T\overline{v} - Tv_n \le L(\overline{v} - v_n) \le \cdots \le L^n(\overline{v} - v_0).$$

Therefore,

$$\|\overline{v} - v_{n+1}\| \le \big\|L^n(\overline{v} - v_0)\big\| \le \tau^n\|\overline{v} - v_0\| \to 0 \quad \text{as } n \to \infty.$$

So $\|\overline{v} - v^*\| = 0$, and thus $\overline{v} = v^*$. □

Define

$$D = \big\{x \in E : \phi(t) \le x(t) \le R, t \in [0,1]\big\},$$

where ϕ, R are given as in (2.4) and (3.4), respectively. From Theorems 3.1 and 3.2, we can obtain the following.

Theorem 3.3 *Assume that* (H_1)–(H_3) *hold. If*

$$\tau := \frac{\sigma}{\Gamma(\alpha+\beta+1)} + \frac{\gamma}{\Gamma(\alpha+1)} < 1,$$

then problem (1.1) *has a unique nonnegative solution in* D.

Remark 3.1 In Theorem 3.1, $T\phi \geq \phi$, we call ϕ a lower solution of operator T. In Theorem 3.2, $TR \leq R$, we call R an upper solution of T. From these results, we see that we get the existence and uniqueness of nonnegative solutions for Langevin equations with boundary conditions only via using single lower solution or single upper solution. So we can call the method to be a *single lower-solution* or *single upper-solution method*.

Corollary 3.4 *Assume* (H_1)–(H_3) *and there exists* $j_0 \in \{0,\dots,m-1\}$ *such that* $\mu_{j_0} \neq 0$. *If* $\tau < 1$, *then problem* (1.1) *has a unique positive solution in* D_i $(i = 1,2)$.

Proof By Theorems 3.1 and 3.2, problem (1.1) has a unique nonnegative solution x^* in D_i $(i = 1,2)$. Further,

$$x^*(t) = \int_0^t \frac{(t-u)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} f\bigl(u, x^*(u)\bigr)\,du + \gamma \int_0^t \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)} x^*(u)\,du + \phi(t).$$

Since $\mu_{j_0} \neq 0$, we can know that $\mu_{j_0} > 0$ and thus $\phi(t) > 0$, $\forall t \in (0,1]$. Further, the solution $x^*(t) \geq \phi(t) > 0$ for $t \in (0,1]$. So problem (1.1) has a unique positive solution in D_i $(i = 1,2)$. □

Corollary 3.5 *Assume* (H_1)–(H_3) *and there exists* $i_0 \in \{0,\dots,n-1\}$ *such that* $\nu_{i_0} - \gamma\mu_{i_0} \neq 0$. *If* $\tau < 1$, *then problem* (1.1) *has a unique positive solution in* D_i $(i = 1,2)$.

Proof The proof of this theorem is similar to the proof of Corollary 3.4. □

Corollary 3.6 *Assume* (H_1)–(H_3) *and there exists* $t_0 \in [0,1]$ *such that* $f(t_0,0) \neq 0$. *If* $\tau < 1$, *then problem* (1.1) *has a unique positive solution in* D_i $(i = 1,2)$.

Proof By Theorem 3.1, problem (1.1) has a nonnegative solution

$$x^*(t) = \int_0^t \frac{(t-u)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} f\bigl(u, x^*(u)\bigr)\,du + \gamma \int_0^t \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)} x^*(u)\,du + \phi(t).$$

If $\phi(t) \not\equiv 0$ for $t \in [0,1]$, then x^* is a positive solution. If $\phi(t) \equiv 0$ for $t \in [0,1]$, then

$$x^*(t) = \int_0^t \frac{(t-u)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} f\bigl(u, x^*(u)\bigr)\,du + \gamma \int_0^t \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)} x^*(u)\,du.$$

Suppose $x^*(t) \equiv 0$ for $t \in [0,1]$, then

$$\int_0^t \frac{(t-u)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} f(u,0)\,du = 0,$$

and thus

$$(t-u)^{\alpha+\beta-1}f(u,0)=0, \quad \text{a.e. } (u).$$

Since $(t-u)^{\alpha+\beta-1}\not\equiv 0$, a.e. (u), we obtain $f(u,0)=0$, a.e. (u). On the other hand, as $f(t_0,0)\neq 0$, for certain $t_0\in[0,1]$ and $f(t_0,x)\geq 0$, we have $f(t_0,0)>0$. The continuity of f implies that there is a set $B\subset[0,1]$ with $t_0\in B$ and $\mu(B)>0$, where μ is the Lebesgue measure such that $f(t,0)>0$, $\forall t\in B$. This leads to a contradiction. Therefore, $x^*(t)>0$ for $t\in(0,1)$. That is, problem (1.1) has a unique positive solution. □

4 An example

We present an example to better illustrate our main results.

Example 4.1 Consider the following initial value problem:

$$\begin{cases} {}_0^cD_t^{\frac{1}{2}}({}_0^cD_t^{\frac{3}{2}}-\Gamma(\frac{3}{2}))x(t)=\frac{\arctan x(t)+t^3}{(t+2)^2}, & 0<t<1,\\ x^{(0)}(0)=x^{(1)}(0)=\frac{1}{2}, \\ x^{(\frac{3}{2})}(0)=\Gamma(\frac{3}{2}). \end{cases} \tag{4.1}$$

In this example, $\alpha=\frac{3}{2}$, $\beta=\frac{1}{2}$, $\mu_0=\mu_1=\frac{1}{2}$, $\nu_0=\Gamma(\frac{3}{2})$, $\gamma=\Gamma(\frac{3}{2})$, $m=2$, $n=1$, $l=2$, and $f(t,x)=\frac{\arctan x+t^3}{(t+2)^2}$. It is not difficult to see that $f(t,x)$ satisfies the conditions (H_1), (H_2) and for $x,y\in[0,\infty)$, with $y\geq x$, then, by the mean value theorem for function $\arctan z$ in $[x,y]$, there exists $\varepsilon\in(x,y)$ such that

$$f(t,y)-f(t,x)=\frac{\arctan y-\arctan x}{(t+2)^2}=\frac{1}{1+\varepsilon^2}\cdot\frac{1}{(t+2)^2}(y-x)\leq\frac{1}{4}(y-x), \quad \forall t\in[0,1].$$

Choosing $\sigma=\frac{1}{4}$. Further,

$$\tau=\frac{\frac{1}{4}}{\Gamma(\frac{3}{2}+\frac{1}{2}+1)}+\frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{3}{2}+1)}=\frac{1}{8}+\frac{2}{3}=\frac{19}{24}<1.$$

Since $f(t,0)=\frac{t^3}{(t+2)^2}$, we get $L_1=\max\{f(t,0):t\in[0,1]\}=\frac{1}{9}$, and thus

$$C_1=\frac{\frac{1}{9}}{\Gamma(\frac{3}{2}+\frac{1}{2}+1)}=\frac{1}{18}, \qquad C=\frac{\Gamma(\frac{3}{2})-\frac{1}{2}\Gamma(\frac{3}{2})}{\Gamma(\frac{3}{2}+1)}+\frac{1}{2}+\frac{1}{2}=\frac{4}{3}.$$

We also know $\phi(t)=\frac{1}{3}t^{\frac{3}{2}}+\frac{1}{2}t^{\frac{1}{2}}+\frac{1}{2}$. Therefore, $R\geq\frac{\frac{4}{3}+\frac{1}{18}}{1-\frac{1}{18}}=\frac{20}{3}>\phi(t)$. In addition, $\mu_0=\mu_1=\frac{1}{2}>0$, $\nu_0-\gamma\mu_0=\Gamma(\frac{3}{2})-\frac{1}{2}\Gamma(\frac{3}{2})=\frac{1}{2}\Gamma(\frac{3}{2})=\frac{\sqrt{\pi}}{4}>0$, and $f(t,0)\neq 0$, $\forall t\in(0,1]$.

Thus, from Corollaries 3.4, 3.5, and 3.6, we know that problem (4.1) has a unique positive solution x^* in D_1, D_2, and D. Moreover,

$$\begin{aligned} x^*(t)={}&\int_0^t\frac{(t-u)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\cdot\frac{\arctan x^*(u)+u^3}{(u+2)^2}\,du\\ &+\gamma\int_0^t\frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)}x^*(u)\,du+\phi(t), \quad t\in[0,1]. \end{aligned}$$

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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