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Finite-time synchronization control relationship analysis for two classes of Markovian jump complex networks under feedback control

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Abstract

In this paper, finite-time synchronization problems for a class of nonlinear coupled Markovian jump delay time complex networks (NCMJDTCNs) with stochastic noises and a class of linear coupled Markovian jump delay time complex networks (LCMJDTCNs) with stochastic noises are investigated. Compared to the existing works [Li and Yang in IEEE Trans. Cybern. 46(1):171–180, 2016; Zhang et al. in Physica A 494:251–264, 2018; Liu and Chen in IEEE Trans. Neural Netw. Learn. Syst. 26(1):113–126, 2015; Jin et al. in IEEE Trans. Neural Netw. Learn. Syst. 23(9):1345–1355, 2012; Feng et al. in IEEE Access, 2018, https://doi.org/10.1109/ACCESS.2018.2836142; Tseng in Neural Netw. 86:18–31, 2017; Lei et al. in Neurocomputing 230:390–396, 2017; Cui et al. in J. Franklin Inst. 351:2543–2561, 2014; Wang et al. in Nonlinear Dyn. 79:47–61, 2015; Liu et al. in Neurocomputing 153:148–158, 2015], the main contribution of this paper consists of two new ideas which are applied to analyze a nonlinear coupling affecting synchronization dynamics of the NCMJDTCNs.

Keywords: Synchronization; Complex network; Time delay; Markovian jump; Control rule

1 Introduction

Complex networks have gained widespread interest because of the pioneering work of Watts and Strogatz [11]. In various fields, there are many applications of complex networks, such as communication networks, the World Trade Web, social networks, genetics regulatory networks, neuronal networks [12–14]. Synchronization is one of the most important collective behaviors of complex dynamical networks. There are many different kinds of synchronization, for instance, asymptotical, finite-time, and cluster synchronization [15–17].

It should be noted that nonlinear coupling is an important factor impacting synchronization dynamics of nonlinear coupled complex networks [1–7]. In [1], synchronization for a class of nonlinear coupled complex networks with adaptive distributed controller based on fuzzy logic systems was studied. In [2], Zhang et al. investigated robust outer synchronization for a class of nonlinear coupled complex networks with parametric disturbances and mixed time-varying delays. By analyzing the literature [1–7], one can conclude that although synchronization problems for some classes of nonlinear coupled complex networks were explored, except for [3], nobody considered nonlinear coupling to effect synchronization dynamics of the addressed networks. In [3], global synchronization for a class of nonlinear coupled complex networks was proposed. Furthermore, the above topic was analyzed from the simulation aspect. For a nonlinear coupling affecting synchronization dynamics of nonlinear coupled complex networks, it is very significant to adopt some new ideas to discuss the issue. The work can extend the already analyzed schemes of synchronization problems for nonlinear coupled complex networks. This is the first motivation of this paper.

In order to make a system attain a fast convergence speed, recently, finite-time synchronization of complex networks has been employed more and more [8, 18–20]. In [18], finite-time synchronization for a class of multi-weighted complex networks with and without coupling delay was investigated. In [19], Ren et al. studied finite-time synchronization for a class of Markovian jumping stochastic complex networks with mixed time delays. For finite-time synchronization and nonlinear effect problem of nonlinear coupled complex networks, until now, there is no literature on this issue.

Because Markovian jumps can model random abrupt variations which are often caused by random failures and repairs of the components, Markovian jump systems have attracted increasing attention [9, 10, 21–28]. In [21], Huang et al. investigated finite-time H_∞ synchronization for a class of Markovian jump complex networks with time-varying delays. In [22], almost sure cluster synchronization was used for a class of Markovian jump complex networks with stochastic noise via decentralized adaptive pinning control. Besides this, due to the limited travel speed of signals, processing speeds and the other environment elements, one experiences time delays and stochastic perturbations [29, 30].

Inspired by the above discussions, we observe that it is significant to study finite-time synchronization and nonlinear effects for NCTDMJCNs with stochastic noise. To the best of our knowledge, until now, for finite-time synchronization of NCTDMJCNs, there is no literature on this topic. In this paper, two ideas are adopted to analyze a nonlinear coupling which affects synchronization dynamics of the addressed NCTDMJCNs with stochastic noise. The first idea is as follows: (1) Assume that a nonlinear coupling function $g(x)$ satisfies the Lipschitz condition. That is to say, $\|g(x_1(t)) - g(x_2(t))\| \leq L\|x_1(t) - x_2(t)\|$, where $L > 0$. Thus, one has $\frac{\|g(x_1(t))-g(x_2(t))\|}{\|x_1(t)-x_2(t)\|} \leq L$. It is seen that if $\|x_1(t) - x_2(t)\|$ is fixed, and $\|g(x_1(t)) - g(x_2(t))\|$ becomes larger, L should become also larger. This means that if the nonlinearity of $g(x(t))$ is more serious, L will become larger. (2) Combining Lyapunov stability theory and some stochastic analysis techniques, sufficient conditions for finite-time synchronization of the addressed NCTDMJCNs with stochastic noise can be derived. Furthermore, in the sufficient conditions, because $g(x)$ satisfies the Lipschitz condition, parameter L must exist. Thus, the nonlinearity of $g(x)$ which affects synchronization dynamics of the addressed NCTDMJCNs with stochastic noise is analyzed by adjusting L. The detailed analysis process is given in Remarks 3 and 7–8. The second idea is to use a comparison analysis method as follows: Firstly, based on the addressed NCTDMJCNs with stochastic noise, a class of LCTDMJCNs with stochastic noise is considered. The addressed networks are the same except for coupling functions. Secondly, a sufficient condition for finite-time synchronization of the LCTDMJCNs with stochastic noise is obtained. Thirdly, according to the above sufficient conditions, finite-time synchronization control relationships of the addressed networks are built. By analyzing them, the nonlinear coupling function $g(x(t))$ affecting synchronization dynamics of the NCTDMJCNs with stochastic noise is analyzed.

The main contribution of this paper is that it extends exiting works [1–10]. From the second paragraph of the introduction, it is seen that although synchronization problems of nonlinear coupled complex networks were investigated in [1, 2, 4–7], the nonlinear coupling affecting synchronization dynamics was not analyzed. In [3], global synchronization and nonlinear coupling affecting nonlinear coupled complex networks were considered. Compared to this paper, there exist three differences, which include nonlinear effect analysis ideas, the addressed network models and the addressed synchronization problems. The detailed analysis is presented in Remarks 2–5 and 7–8. Besides these differences, in [8–10], the authors mainly focused on finite-time synchronization of linear coupled Markovian jump complex networks.

The paper is organized as follows. The model and preliminaries are presented in Sect. 2. The finite-time synchronization, relationships of synchronization control rules, and nonlinear effect analysis are given in Sect. 3. In order to illustrate the effectiveness of the derived results, Sect. 4 presents three examples. The conclusions are drawn in Sect. 5.

Notations: Throughout this paper, $\mathbb{R}^{n\times m}$ is the set of real matrices and $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. The superscript T denotes the matrix transposition; $I_n \in \mathbb{R}^{n\times n}$ means an n-dimensional identity matrix; $X \geq Y > 0$ (respectively, $X > Y > 0$), where $X, Y \in \mathbb{R}^{n\times n}$ are symmetric matrices, means that the matrix $X - Y$ is positive semidefinite (respectively, positive definite); $\|\cdot\|$ refers to the Euclidean vector norm in $\mathbb{R}^n$ and $\mathrm{diag}\{\cdots\}$ stands for a block diagonal matrix. The Kronecker product of matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{M\times N}$ is a matrix in $\mathbb{R}^{mM\times nN}$ which is denoted as $A \otimes B$. If A is a matrix, $\lambda_{\max}$ denotes the maximal eigenvalue. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., the filtration contains all P-null sets and is right continuous). Further, $E[x]$ means the expectation of the random variable x. If the dimensions of matrices are not explicitly indicated, they are assumed suitable for any algebraic operations.

2 Problem formulation and preliminaries

In this section, problem formulation and preliminaries are briefly introduced. In this paper, we consider a class of NCTDMJCNs with stochastic noises and a class of LCTDMJCNs with stochastic noises, respectively. They are as follows:

$$\begin{aligned} dx_i(t) = \Biggl[& f\bigl(x_i(t), x_i(t-\tau)\bigr) + c\bigl(r(t)\bigr) \sum_{j=1}^{N} a_{ij}\bigl(r(t)\bigr)\Gamma\bigl(r(t)\bigr)g\bigl(x_j(t)\bigr) \\ & + u_i\bigl(t, r(t)\bigr)\Biggr] dt + \sigma_i\bigl(x(t), t, r(t)\bigr)\, d\omega_i(t), \quad i = 1, 2, \ldots, N, \end{aligned} \tag{1}$$

$$\begin{aligned} dx_i(t) = \Biggl[& f\bigl(x_i(t), x_i(t-\tau)\bigr) + c\bigl(r(t)\bigr) \sum_{j=1}^{N} a_{ij}\bigl(r(t)\bigr)\Gamma\bigl(r(t)\bigr)x_j(t) \\ & + u_i\bigl(t, r(t)\bigr)\Biggr] dt + \sigma_i\bigl(x(t), t, r(t)\bigr)\, d\omega_i(t), \quad i = 1, 2, \ldots, N, \end{aligned} \tag{2}$$

where $\{r(t), t \geq 0\}$ is a right-continuous Markov chain on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ taking values in a finite state space $S = \{1, 2, \ldots, s\}$ with a generator $\Pi = (\delta_{ij})_{s\times s}$

$(i,j \in S)$ given by

$$P\{r(t+\Delta t)=j|r(t)=i\} = \begin{cases} \delta_{ij}\Delta t + o(\Delta t), & \text{if } (i \neq j), \\ 1+\delta_{ij}\Delta t + o(\Delta t), & \text{if } (i = j), \end{cases}$$

where $\Delta t > 0$, $\lim_{\Delta t \to 0}(o(\Delta t)/\Delta t) = 0$, $\delta_{ij} > 0$ $(\forall i \neq j)$ is the transition rate from state i to state j, and $\delta_{ii} = -\sum_{i \neq j}\delta_{ij} < 0$; $c(r(t))$ represents the coupling strength in state $r(t)$ and $c(r(t)) > 0$, $\Gamma(r(t)) \in \mathbb{R}^{n\times n}$ is an inner-coupling matrix in state $r(t)$, $A(r(t)) = (a_{ij}(r(t)))_{N\times N}$ is an outer-coupling matrix, and the diagonal elements of matrix $A(r(t))$ are defined by $a_{ii}(r(t)) = -\sum_{j=1,j\neq i}^{N} a_{ij}(r(t))$; $\tau > 0$ is node time delay, $f(\cdot,\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ describing for the activity of the ith node is a vector-valued function, $g(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ denotes a nonlinear coupling function and $u_i(t, r(t))$ denotes the control input of the ith node in state $r(t)$, $\sigma_i(x(t), t, r(t)) \in \mathbb{R}^{n\times n}$ is the noise intensity, $\omega_i = (\omega_{i1}, \omega_{i2}, \dots, \omega_{in})^T$ is a bounded vector-valued Brownian motion process that is independent of the Markov chain $r(t)$, satisfying

$$E[\omega_{ij}(t)] = 0, \qquad E[\omega_{ij}^2(t)] = 1, \qquad E[\omega_{ij}(t)\omega_{ij}(t+\Delta t)] = 0, \quad \Delta t \neq 0.$$

Remark 1 Comparing networks (1) and (2), it is not difficult to observe that except for the coupling functions, there is no difference. The coupling functions of networks (1) and (2) are a nonlinear coupling function $g(x_j(t))$ and a linear coupling function $x_j(t)$, where $j = 1, 2, \dots, N$. Why do we discuss them? The reason is closely related to the aim of this paper. From the introduction of this paper, it is clear that we mainly focus on the nonlinear coupling $g(\cdot)$ affecting synchronization dynamics of network (1). Furthermore, it is seen that two ideas are adopted to analyze the issue. In the second idea, by using finite-time synchronization conditions of networks (1) and (2), the finite-time synchronization control relationships are built. Based on the control relationships, the above issue is explored. Besides, for networks (1) and (2), the following two issues are also important: (i) First is the difference between the functions $f(\cdot,\cdot)$ and $g(\cdot)$. In network (1), the nonlinear function $g(\cdot)$ is a special case of a nonlinear function $f(\cdot,\cdot)$. For example, if $f(\cdot,\cdot) = f(\cdot, 0)$ and $f(\cdot, 0) = g(\cdot)$, then $f(\cdot,\cdot) = g(\cdot)$. (ii) Regarding the function $f(\cdot,\cdot)$, networks (1) and (2) are nonlinear systems. For the coupling functions $g(x_j(t))$ and $x_j(t)$ of networks (1) and (2), network (1) is a nonlinear coupled system, while network (2) is a linear coupled system.

Remark 2 From the introduction, we know the following facts: (i) Until now, although some results on Markovian jump complex networks have been obtained, they focused on linear Markovian jump complex networks instead of nonlinear Markovian jump complex networks, see, e.g., [8–10]. For finite-time synchronization and nonlinear coupling effect of NCTDMJCNs, there is no literature on the issue. (ii) In [1–7], the authors considered some classes of nonlinear coupled complex networks. But except for [3], there is no literature on nonlinear coupling affecting synchronization dynamics of the addressed complex networks. (iii) There are three differences between this paper and reference [3]. Firstly, the addressed network aspect. In the complex networks of [3], there were no Markovian jumps and stochastic noises. While in this paper a class of NCTDMJCNs with stochastic noises is considered. Secondly, the addressed synchronization problem aspect. In [3], the authors proposed global synchronization. This paper is devoted to finite-time synchronization. Thirdly, one aspect of the synchronization effect analysis idea of nonlinear coupling. In [3], this analysis was built on an assumption that the nonlinear coupling function

$\varphi(\cdot,\cdot)$ satisfied the following three conditions: (i) $\varphi(\cdot,\cdot)$ was a continuous mapping satisfying the local Lipschitz condition. (ii) There existed a positive constant $L > 0$ such that $(x-y)\varphi(y,x) \le -L(x-y)^2$, $\forall x \neq y$. (iii) $\varphi(y,x) = -\varphi(x,y)$. In this paper, our analysis is based on Assumption 1. It is clear that in [3] and this paper, the nonlinear coupling functions are handled by different methods. This constitutes a difference in synchronization effect analysis of [3] and this paper.

Let $s(t)$ be the synchronization state of the networks (1)–(2). Then, according to networks (1)–(2), $s(t)$ satisfies

$$ds(t) = \big[f\big(s(t), s(t-\tau)\big)\big]\,dt + \sigma\big(s(t), t, r(t)\big)\,d\omega(t). \tag{3}$$

Because linear feedback control scheme is simple and easily realized [31], in this paper, we use the following linear feedback controller:

$$u_i\big(t, r(t)\big) = -\varepsilon_i\big(r(t)\big)\Gamma\big(r(t)\big)e_i(t) - \frac{k_i(r(t))}{\varrho(r(t))}\operatorname{sign}\big(e_i(t)\big)\big|e_i(t)\big|^{\beta}, \tag{4}$$

where $\varepsilon_i(r(t)) > 0$, $k_i(r(t)) > 0$, $\varrho(r(t)) > 0$, $|e_i(t)|^{\beta} = (|e_{i1}(t)|^{\beta}, |e_{i2}(t)|^{\beta}, \ldots, |e_{in}(t)|^{\beta})^T$, $\operatorname{sign}(\cdot)$ is the sign function and $\operatorname{sign}(e_i(t)) = (\operatorname{sign}(e_{i1}(t)), \operatorname{sign}(e_{i2}(t)), \ldots, \operatorname{sign}(e_{in}(t)))^T$, β satisfies $0 < \beta < 1$ and $\beta \in \mathbb{R}$.

For notation simplicity, we denote $\Gamma(r(t))$, $c(r(t))$, $a_{ij}(r(t))$, $\varepsilon_i(r(t))$, $k_i(r(t))$ and $\varrho(r(t))$ as Γ_r, c_r, a_{ij}^r, ε_{ir}, k_{ir} and ϱ_r, respectively.

In order to obtain the main results, the following definition, assumptions and lemmas are needed.

Definition 1 Network (1) or (2) is said to achieve global synchronization in finite-time t^*, if there exists a constant $t^* > 0$, which depends on the initial state vector value $x(t)$, for any $t \ge t^*$, such that

$$E\big\|x_i(t) - s(t)\big\| = 0, \quad \text{as } t \to t^*$$

holds for any $i \in \{1, 2, \ldots, N\}$, where $x(t) = (x_1^T(0), \ldots, x_N^T(0))^T$, $s(t) = (s_1(t), \ldots, s_n(t))^T \in \mathbb{R}^n$ is the synchronization state of network (1) or (2).

Assumption 1 The functions $f(\cdot,\cdot)$ and $g(\cdot)$ satisfy Lipschitz conditions, that is, there exist constants L, $L_1 > 0$ and $L_2 > 0$ such that

$$\begin{aligned}
&\big\|g(x) - g(y)\big\| \le L\|x - y\|, \quad \forall x, y, \in \mathbb{R}^n, \\
&\big\|f(x_1, y_1) - f(x_2, y_2)\big\| \le L_1\|x_1 - x_2\| + L_2\|y_1 - y_2\|, \quad \forall x_1, x_2, y_1, y_2 \in \mathbb{R}^n.
\end{aligned}$$

Assumption 2 There exist nonnegative constants $\mu_{(r)ij}$ $(i, j = 1, 2, \ldots, N, r \in S)$, such that

$$\operatorname{trace}\big(\sigma_i^T\big(e(t), t, r(t)\big)\sigma_i\big(e(t), t, r(t)\big)\big) \le \sum_{j=1}^{N}\mu_{(r)ij}e_i^T(t)e_i(t).$$

Assumption 3 (Yin et al. [32]) Let $0 < \beta < 1$ and $\lambda > 0$. There exists a continuous function $g : [0,\infty) \to [0,\infty)$ with $g(0) > 0$, for any $0 \le u \le t$, such that

$$g(t) - g(u) \le -\lambda \int_u^t \bigl(g(s)\bigr)^\beta \, ds.$$

Assumption 4 Suppose that the initial conditions of networks (1) and (2) are given by $x_i(z) = \varphi_i(z) \in C([-\tau,0],\mathbb{R}^n)$, $i = 1,2,\ldots,N$, where $C([-\tau,0],\mathbb{R}^n)$ denotes the set of continuous functions mapping the interval $[-\tau,0]$ into $\mathbb{R}^n$.

Lemma 1 (Bhat et al. [33]) *Suppose that function* $V(t) : [0,\infty) \to [0,\infty)$ *is differentiable* (*the derivative of* $V(t)$ *at* 0 *is, in fact, its right derivative*) *and*

$$\frac{dV(t)}{dt} \le -\eta V^\alpha(t),$$

where $\eta > 0$ *and* $0 < \alpha < 1$. *Then* $V(t)$ *will reach zero in finite time* $t^* \le V^{1-\alpha}(t)/(\eta(1-\alpha))$ *and* $V(t) = 0$ *for all* $t \ge t^*$.

Lemma 2 (Boyd et al. [34]) *For any vectors* $x, y \in \mathbb{R}^n$ *and a positive-definite matrix* $Q > 0$, *the following inequality holds*:

$$2x^T y \le x^T Q^{-1} x + y^T Q y.$$

Lemma 3 (Mei et al. [35]) *Let* $x_1, x_2, \ldots, x_n \in \mathbb{R}^n$ *be any vectors and* $0 < q < 2$ *a real number. Then the following inequality holds*:

$$\|x_1\|^q + \cdots + \|x_n\|^q \ge \bigl(\|x_1\|^2 + \cdots + \|x_n\|^2\bigr)^{q/2}.$$

Lemma 4 (Wang and Xiao [36]) *If* $a_1, a_2, \ldots, a_n \ge 0$ *and* $0 < p \le 1$, *then*

$$\left(\sum_{i=1}^n a_i\right)^p \le \sum_{i=1}^n a_i^p.$$

3 Main results

In this section, the finite-time synchronization and nonlinear effect of networks (1)–(2) are studied. In order to show the advantages of the proposed method and the results in this paper, we give seven corollaries of Theorems 1–2.

3.1 Finite-time synchronization for a class of NCTDMJCNs with stochastic noises

Subtracting (3) from (1), we obtain the error system, which can be described as follows:

$$\begin{aligned} de_i(t) = {} & \Biggl[F\bigl(e_i(t), e_i(t-\tau)\bigr) + c_r \sum_{j=1}^N a_{ij}^r \Gamma_r G\bigl(e_j(t)\bigr) - \varepsilon_{ir} \Gamma_r e_i(t) \\ & - \frac{k_{ir}}{\varrho_r} \operatorname{sign}\bigl(e_i(t)\bigr) \bigl|e_i(t)\bigr|^\beta \Biggr] dt + \tilde{\sigma}_i\bigl(e(t), t, r\bigr) \, d\omega_i(t), \quad i = 1,2,\ldots,N, \end{aligned} \tag{5}$$

where $e_i(t) = x_i(t) - s(t)$, $F(e_i(t), e_i(t-\tau)) = f(x_i(t), x_i(t-\tau)) - f(s(t), s(t-\tau))$, $G(e_j(t)) = g(x_j(t)) - g(s(t))$, and $\tilde{\sigma}_i(e(t), t, r) = \sigma_i(x(t), t, r) - \sigma(s(t), t, r)$.

Theorem 1 *Let Assumptions 1–4 hold, then network (1) achieves global synchronization under the set of controller (4) in finite time t^* if the following conditions are satisfied*:

(i) *If $p \neq r$, $q_p\varrho_p - a_r \leq 0$, otherwise, if $p = r$, $q_p\varrho_p - a_r \geq 0$, where $r, p \in S$.*

(ii) *The following LMI holds*:

$$\begin{bmatrix} \Theta_r^{(1)} & 0 \\ \star & \Theta_r^{(2)} \end{bmatrix} \leq 0, \tag{6}$$

where $\Theta_r^{(1)} = \rho_r I_N \otimes I_n + L^2\|Q_{r(1)}\| I_N \otimes I_n + I_N \otimes Q^{-1} + \Psi_r \otimes I_n + c_r(A_r \otimes \Gamma_r)^T Q_{r(1)}^{-1}(A_r \otimes \Gamma_r) - 2\Xi_r \otimes \Gamma_r$, $\Theta_r^{(2)} = \eta_r I_N \otimes I_n$, $\rho_r = \varrho_r + 2L_1^2\|Q\| + \sum_{p=1}^{s} \frac{\delta_{rp}q_p}{q_r} - \upsilon$, $\eta_r = 2L_2^2\|Q\| + \upsilon - \varrho_r$, $\Xi_r = \operatorname{diag}\{\varepsilon_{1r}, \ldots, \varepsilon_{Nr}\}$, $\Psi_r = \operatorname{diag}\{\mu_{(r)1}, \ldots, \mu_{(r)N}\}$, $\mu_{(r)i} = \sum_{j=1}^{N} \mu_{(r)ij}$, $k = \min\{k_{ir}\}$, $i \in \{1, 2, \ldots, N\}$, $r \in S$.

(iii) *t^* is estimated by* $t^* \leq \tau + \frac{\varrho^{\frac{1+\beta}{2}} V(0, r(0))^{1-\frac{1+\beta}{2}}}{\nu(1-\frac{1+\beta}{2})}$, $\nu = \min\{\lambda\upsilon, 2k\}$, $k = \min_{i\in\{1,2,\ldots,N\}}^{r\in S}\{k_{ir}\}$, $\lambda > 0$, $0 < \beta < 1$, $V(0, r(0)) = q_{r(0)} \sum_{i=1}^{N} e_i^T(0)e_i(0)$, $e_i(0)$ $(i = 1, 2, \ldots, N)$ *is the initial condition.*

Proof Construct a Lyapunov–Krasovskii functional candidate as

$$V\bigl(e(t), t, r(t)\bigr) = q_r\left[\sum_{i=1}^{N} e_i^T(t)e_i(t) + \varrho_r \sum_{i=1}^{N} \int_{t-\tau}^{t} e_i^T(s)e_i(s)\,ds\right], \tag{7}$$

where $q_r \geq \varrho_r > 0$, $r \in S$.

Computing $\mathcal{L}V(e(t), t, r)$ along the trajectory of the error system (5), one can obtain

$$\begin{aligned}
\mathcal{L}V\bigl(e(t), t, r\bigr) &= V_t\bigl(e(t), t, r\bigr) + V_e\bigl(e(t), t, r\bigr)\left[F\bigl(e_i(t), e_i(t-\tau)\bigr) + c_r \sum_{j=1}^{N} a_{ij}^r \Gamma_r G\bigl(e_j(t)\bigr)\right. \\
&\quad \left. - \varepsilon_{ir}\Gamma_r e_i(t) - \frac{k_{ir}}{\varrho_r} \operatorname{sign}\bigl(e_i(t)\bigr)\bigl|e_i(t)\bigr|^{\beta}\right] + \sum_{p=1}^{s} \delta_{rp} V\bigl(e(t), t, p\bigr) \\
&\quad + q_r \sum_{i=1}^{N} \operatorname{trace}\bigl[\tilde{\sigma}_i^T\bigl(e(t), t, r\bigr)\tilde{\sigma}_i\bigl(e(t), t, r\bigr)\bigr] \\
&= q_r\varrho_r \sum_{i=1}^{N} \bigl[e_i^T(t)e_i(t) - e_i^T(t-\tau)e_i(t-\tau)\bigr] \\
&\quad + 2q_r \sum_{i=1}^{N} e_i^T(t)\left[F\bigl(e_i(t), e_i(t-\tau)\bigr) + c_r \sum_{j=1}^{N} a_{ij}^r \Gamma_r G\bigl(e_j(t)\bigr) - \varepsilon_{ir}\Gamma_r e_i(t)\right. \\
&\quad \left. - \frac{k_{ir}}{\varrho_r} \operatorname{sign}\bigl(e_i(t)\bigr)\bigl|e_i(t)\bigr|^{\beta}\right] + \sum_{p=1}^{s} \delta_{rp} V\bigl(e(t), t, p\bigr) \\
&\quad + q_r \sum_{i=1}^{N} \operatorname{trace}\bigl[\tilde{\sigma}_i^T\bigl(e(t), t, r\bigr)\tilde{\sigma}_i\bigl(e(t), t, r\bigr)\bigr].
\end{aligned} \tag{8}$$

Based on Assumption 1 and Lemma 2, we have

$$\begin{aligned} &e_i^T(t)F\big(e_i(t), e_i(t-\tau)\big) \\ &\quad\le \frac{1}{2}\big[e_i^T(t)Q^{-1}e_i(t) + F^T\big(e_i(t), e_i(t-\tau)\big)QF\big(e_i(t), e_i(t-\tau)\big)\big] \\ &\quad\le \frac{1}{2}\big[e_i^T(t)Q^{-1}e_i(t) + 2L_1^2\|Q\|e_i^T(t)e_i(t) + 2L_2^2\|Q\|e_i^T(t-\tau)e_i(t-\tau)\big], \end{aligned} \tag{9}$$

$$\begin{aligned} &2q_r\sum_{i=1}^{N} e_i^T(t)c_r\sum_{j=1}^{N} a_{ij}^r\Gamma_r G\big(e_j(t)\big) \\ &\quad= 2q_r c_r e^T(t)(A_r\otimes\Gamma_r)G\big(e(t)\big) \\ &\quad\le q_r c_r\big[e^T(t)(A_r\otimes\Gamma_r)^T Q_{r(1)}^{-1}(A_r\otimes\Gamma_r)e(t) + G^T\big(e(t)\big)Q_{r(1)}G\big(e(t)\big)\big] \\ &\quad\le q_r c_r\big[e^T(t)(A_r\otimes\Gamma_r)^T Q_{r(1)}^{-1}(A_r\otimes\Gamma_r)e(t) + L^2\|Q_{r(1)}\|e^T(t)e(t)\big]. \end{aligned} \tag{10}$$

Because of $\sum_{r,p\in S}\delta_{rp} = 0$, for $\forall a_r > 0$ $(r\in S)$, we get

$$\sum_{r,p\in S}\delta_{rp}a_r = 0. \tag{11}$$

Thus,

$$\begin{aligned} \sum_{p=1}^{s}\delta_{rp}V\big(e(t),t,p\big) &= \sum_{p=1}^{s}\delta_{rp}q_p\left[\sum_{i=1}^{N} e_i^T(t)e_i(t) + \varrho_p\sum_{i=1}^{N}\int_{t-\tau}^{t} e_i^T(s)e_i(s)\,ds\right] \\ &= \sum_{p=1}^{s}\delta_{rp}q_p\sum_{i=1}^{N} e_i^T(t)e_i(t) \\ &\quad + \sum_{r,p\in S}\delta_{rp}(q_p\varrho_p - a_r)\sum_{i=1}^{N}\int_{t-\tau}^{t} e_i^T(s)e_i(s)\,ds. \end{aligned} \tag{12}$$

From Assumption 2, we obtain

$$\operatorname{trace}\big[\tilde{\sigma}_i^T\big(e(t),t,r\big)\tilde{\sigma}_i\big(e(t),t,r\big)\big] \le \sum_{j=1}^{N}\mu_{(r)ij}e_i^T(t)e_i(t). \tag{13}$$

Denoting $\Xi_r = \operatorname{diag}\{\varepsilon_{1r},\dots,\varepsilon_{Nr}\}$, $\mu_{(r)i} = \sum_{j=1}^{N}\mu_{(r)ij}$, $\Psi_r = \operatorname{diag}\{\mu_{(r)1},\dots,\mu_{(r)N}\}$, $k = \min_{i\in\{1,2,\dots,N\}}^{r\in S}\{k_{ir}\}$, $e(t) = (e_1^T(t),\dots,e_N^T(t))^T$, $e_i(t) = (e_{i1}(t),\dots,e_{in}(t))^T$, $e(t-\tau(t)) = (e_1^T(t-\tau(t)),\dots,e_N^T(t-\tau)(t))^T$, $e_i(t-\tau) = (e_{i1}(t-\tau),\dots,e_{in}(t-\tau))^T$, and substituting (9)–(13) into (8), then taking the expectation on both sides of (8), according to condition (i) in Theorem 1, we have

$$\begin{aligned} E\big[\mathcal{L}V\big(e(t),t,r\big)\big] \le E\Big\{ & q_r e^T(t)\big[\rho_r I_N\otimes I_n + I_N\otimes Q^{-1} + \Psi_r\otimes I_n + L^2\|Q_{r(1)}\|I_N\otimes I_n \\ & + c_r(A_r\otimes\Gamma_r)^T Q_{r(1)}^{-1}(A_r\otimes\Gamma_r) - 2\Xi_r\otimes\Gamma_r\big]e(t) \\ & + q_r\eta_r e^T(t-\tau)e(t-\tau) \end{aligned}$$

$$+ q_r\upsilon\left[\sum_{i=1}^{N} e_i^T(t)e_i(t) - \sum_{i=1}^{N} e_i^T(t-\tau)e_i(t-\tau)\right]$$
$$- \frac{2q_r k}{\varrho_r}\sum_{i=1}^{N}\sum_{j=1}^{n}\left|e_{ij}(t)\right|^{1+\beta}\Bigg\}, \tag{14}$$

where $\rho_r = \varrho_r + 2L_1^2\|Q\| + \sum_{p=1}^{s}\frac{\delta_{rp}q_p}{q_r} - \upsilon$, $\eta_r = 2L_2^2\|Q\| + \upsilon - \varrho_r$.

By Lemma 4, we get

$$\begin{aligned} -\frac{2q_r k}{\varrho_r}\sum_{i=1}^{N}\sum_{j=1}^{n}\left|e_{ij}(t)\right|^{1+\beta} &= -\frac{2q_r k}{\varrho_r}\sum_{i=1}^{N}\left(e_i^T(t)e_i(t)\right)^{\frac{1+\beta}{2}} \\ &\le -2k\left(\frac{q_r}{\varrho_r}\sum_{i=1}^{N} e_i^T(t)e_i(t)\right)^{\frac{1+\beta}{2}}, \end{aligned} \tag{15}$$

where $q_r \ge \varrho_r > 0$.

Let $\lambda > 0$, then combining Assumption 4 and Lemma 4, we obtain

$$\begin{aligned} &q_r\upsilon\left[\sum_{i=1}^{N} e_i^T(t)e_i(t) - \sum_{i=1}^{N} e_i^T(t-\tau)e_i(t-\tau)\right] \\ &\quad\le -\lambda\upsilon\sum_{i=1}^{N}\int_{t-\tau}^{t}\left(q_r e_i^T(s)e_i(s)\right)^{\frac{1+\beta}{2}}\,ds \\ &\quad\le -\lambda\upsilon\left(q_r\sum_{i=1}^{N}\int_{t-\tau}^{t} e_i^T(s)e_i(s)\,ds\right)^{\frac{1+\beta}{2}}. \end{aligned} \tag{16}$$

Thus, substituting (15)–(16) into (14), we have

$$\begin{aligned} E\left[\mathcal{L}V\left(e(t),t,r(t)\right)\right] \le E\Bigg\{ & q_r e^T(t)\Theta_r^{(1)}e(t)e^T(t-\tau)\Theta_r^{(2)}e(t-\tau) \\ & - \left(2k\left(\frac{q_r}{\varrho_r}\sum_{i=1}^{N} e_i^T(t)e_i(t)\right)^{\frac{1+\beta}{2}}\right. \\ & \left. + \lambda\upsilon\left(q_r\sum_{i=1}^{N}\int_{t-\tau}^{t} e_i^T(s)e_i(s)\,ds\right)^{\frac{1+\beta}{2}}\right)\Bigg\}, \end{aligned} \tag{17}$$

where $\Theta_r^{(1)} = \rho_r I_N \otimes I_n + L^2\|Q_{r(1)}\| I_N \otimes I_n + I_N \otimes Q^{-1} + \Psi_r \otimes I_n + c_r(A_r \otimes \Gamma_r)^T Q_{r(1)}^{-1}(A_r \otimes \Gamma_r) - 2\Xi_r \otimes \Gamma_r$, $\Theta_r^{(2)} = \eta_r I_N \otimes I_n$.

Let $\nu = \min\{\lambda\upsilon, 2k\}$, then by Lemmas 3–4, and the condition (ii) in Theorem 1, we get

$$\begin{aligned} &E\left[\mathcal{L}V\left(e(t),t,r(t)\right)\right] \\ &\quad\le -\nu E\left\{\left(\left(\frac{q_r}{\varrho_r}\sum_{i=1}^{N} e_i^T(t)e_i(t)\right)^{\frac{1+\beta}{2}}\right.\right. \end{aligned}$$

$$+\left(q_r\sum_{i=1}^{N}\int_{t-\tau}^{t}e_i^T(s)e_i(s)\,ds\right)^{\frac{1+\beta}{2}}\Bigg)\Bigg\}$$
$$\le -\nu E\left\{\left(\frac{1}{\varrho_r}\left(q_r\sum_{i=1}^{N}e_i^T(t)e_i(t)+q_r\varrho_r\sum_{i=1}^{N}\int_{t-\tau}^{t}e_i^T(s)e_i(s)\,ds\right)\right)^{\frac{1+\beta}{2}}\right\}$$
$$\le -\frac{\nu}{\varrho^{\frac{1+\beta}{2}}}E\big[V^{\frac{1+\beta}{2}}\big(e(t),t,r(t)\big)\big], \tag{18}$$

where $\varrho = \max_{r\in S}\{\varrho_r\}$.

For any $t_0 \ge \tau > 0$, we have $E[V^{\frac{1+\beta}{2}}(t_0)] = (E[V(t_0)])^{\frac{1+\beta}{2}}$. Therefore, we can obtain

$$E\big[\mathcal{L}V\big(e(t),t,r\big)\big] \le -\frac{\nu}{\varrho^{\frac{1+\beta}{2}}}\big(E\big[V(t)\big]\big)^{\frac{1+\beta}{2}}. \tag{19}$$

According to Lemma 1, $E[V(t)]$ converges to zero in finite-time t^* and finite-time t^* is estimated by

$$t^* \le \tau + \frac{\varrho^{\frac{1+\beta}{2}}V(0,r(0))^{1-\frac{1+\beta}{2}}}{\nu(1-\frac{1+\beta}{2})}. \tag{20}$$

This shows that $V(e(t),t,r) = 0$ if $t \ge t^*$. Therefore, $e_i(t) = 0$ if $t \ge t^*$. By Definition 1, if $t \ge t^*$, we have $E\|x_i(t) - s(t)\| = 0$. Hence, network (1) will achieve finite-time global synchronization under the controller (4) within finite time t^*. This completes the proof. □

Remark 3 In Theorem 1, the nonlinear coupling function $G(\cdot)$ is handled using Assumption 1. In other words, we use inequality $\|G(x(t)) - G(y(t))\| \le L\|x(t) - y(t)\|$, where $G(x(t)) = g(x(t)) - g(s(t))$, $s(t)$ is synchronization state of network . Due to to $\|G(x(t)) - G(y(t))\| \le L\|x(t) - y(t)\|$, one has $\|g(x(t)) - g(y(t))\| \le L\|x(t) - y(t)\|$. From the introduction, we know that if the nonlinearity of $g(\cdot)$ is more serious, L becomes larger. Therefore, based on Theorem 1, the nonlinear coupling affecting synchronization dynamics of network (1) can be analyzed by adjusting L. The steps are as follows: (a) By Theorem 1, synchronization rule, which includes ε_{ir}, k_{ir}, ϱ_r and β of the controller (4), can be designed. The synchronization rule can make network (1) realize finite-time synchronization. (b) Synchronization simulation results of the network (1) with the synchronization rule are given. (c) The synchronization rule is fixed. This means ε_{ir}, k_{ir}, ϱ_r and β are kept fixed. (d) A new nonlinear coupling function $\tilde{g}(x)$ is chosen. According to Assumption 1, one obtains $\|\tilde{g}(x(t)) - \tilde{g}(y(t))\| \le \tilde{L}\|x(t) - y(t)\|$. Furthermore, let the nonlinearity of $\tilde{g}(x)$ be more serious than that of $g(\cdot)$. Thus, one can make $\tilde{L} > L$ hold. (e) The parameters of the above synchronization rule ε_{ir}, k_{ir}, ϱ_r, β and $\tilde{L}$ are substituted into Theorem 1, and if LMI (6) in Theorem 1 holds, one continues to the next step. Otherwise, one returns to step 4. (f) Under the synchronization rule, the synchronization trajectories of network (1) with $\tilde{g}(x)$ are simulated. (g) Comparing simulation results, the nonlinear coupling affecting synchronization dynamics of network (1) is analyzed.

3.2 Finite-time synchronization for a class of LCTDMJCNs with stochastic noises

Subtracting (3) from (2), we obtain the following error system of network (2):

$$de_i(t) = \left[F\big(e_i(t), e_i(t-\tau)\big) + c_r \sum_{j=1}^{N} a_{ij}^r \Gamma_r e_j(t-\tau) - \varepsilon_{ir}\Gamma_r e_i(t) - \frac{k_{ir}}{\varrho_r} \operatorname{sign}\big(e_i(t)\big)\big|e_i(t)\big|^{\beta} \right] dt + \tilde{\sigma}_i\big(e(t), t, r\big)\, d\omega_i(t), \quad i = 1, \ldots, N. \tag{21}$$

Theorem 2 *Let Assumptions 1–4 hold. Then network (2) achieves global synchronization under the set of controller (4) in finite time t^* if the following conditions are satisfied:*

(i) *If $p \neq r$, $q_p\varrho_p - a_r \leq 0$, otherwise, if $p = r$, $q_p\varrho_p - a_r \geq 0$, where $r, p \in S$.*

(ii) *The following LMI holds:*

$$\begin{bmatrix} \tilde{\Theta}_r^{(1)} & 0 \\ \star & \tilde{\Theta}_r^{(2)} \end{bmatrix} \leq 0, \tag{22}$$

where $\tilde{\Theta}_r^{(1)} = \rho_r I_N \otimes I_n + \|Q_{r(1)}\| I_N \otimes I_n + I_N \otimes Q^{-1} + \Psi_r \otimes I_n + c_r(A_r \otimes \Gamma_r)^T \times Q_{r(1)}^{-1}(A_r \otimes \Gamma_r) - 2\Xi_r \otimes \Gamma_r$, $\tilde{\Theta}_r^{(2)} = \eta_r I_N \otimes I_n$, $\rho_r = \varrho_r + 2L_1^2\|Q\| + \sum_{p=1}^{s} \frac{\delta_{rp}q_p}{q_r} - \upsilon$, $\eta_r = 2L_2^2\|Q\| + \upsilon - \varrho_r$, $\Xi_r = \operatorname{diag}\{\varepsilon_{1r}, \ldots, \varepsilon_{Nr}\}$, $\Psi_r = \operatorname{diag}\{\mu_{(r)1}, \ldots, \mu_{(r)N}\}$, $\mu_{(r)i} = \sum_{j=1}^{N} \mu_{(r)ij}$, $k = \min\{k_{ir}\}$, $i \in \{1, 2, \ldots, N\}$, $r \in S$.

(iii) *t^* is estimated by $t^* \leq \tau + \frac{\varrho^{\frac{1+\beta}{2}} V(0,r(0))^{1-\frac{1+\beta}{2}}}{\nu(1-\frac{1+\beta}{2})}$, $\nu = \min\{\lambda\upsilon, 2k\}$, $k = \min_{i\in\{1,2,\ldots,N\}}^{r\in S}\{k_{ir}\}$, $\lambda > 0$, $0 < \beta < 1$, $V(0, r(0)) = q_{r(0)} \sum_{i=1}^{N} e_i^T(0)e_i(0)$, $e_i(0)$ $(i = 1, 2, \ldots, N)$ is the initial condition.*

Proof Construct a Lyapunov–Krasovskii functional candidate as

$$V\big(e(t), t, r(t)\big) = q_r \left[\sum_{i=1}^{N} e_i^T(t)e_i(t) + \varrho_r \sum_{i=1}^{N} \int_{t-\tau}^{t} e_i^T(s)e_i(s)\, ds \right], \tag{23}$$

where $q_r \geq \varrho_r > 0$, $r \in S$.

The rest of the proof is similar to that of Theorem 1. □

3.3 Finite-time synchronization control relationship analysis of networks (1)–(2)

Corollary 1 *Under the conditions of Theorem 1, if $L > 1$, then network (1) with controller (4) must satisfy the conditions of Theorem 2.*

Proof Because the network (1) satisfies the conditions of Theorem 1, we have $\Theta_r^{(1)} \leq 0$. If $L > 1$, combining inequality (22) of Theorem 2, we have $\Theta_r^{(1)} - \tilde{\Theta}_r^{(1)} = c_r(L^2 - 1)\|Q_{r(1)}\| I_N \otimes I_n = \Delta\Theta > 0$. Therefore, we get $\tilde{\Theta}_r^{(1)} = \Theta_r^{(1)} - \Delta\Theta$. When $\Theta_r^{(1)} \leq 0$, one has $\tilde{\Theta}_r^{(1)} \leq -\Delta\Theta < 0$. This completes the proof. □

Corollary 2 *Under the conditions of Corollary 1, the synchronization control rule for network (1) will make network (2) achieve synchronization.*

Proof From Corollary 1, we know that if network (1) satisfies the conditions of Corollary 1, it must be true that $\tilde{\Theta}_r^{(1)} \leq -\Delta\Theta < 0$. Comparing Corollary 1 and Theorem 2, we observe that the differences between Corollary 1 and Theorem 2 are $\tilde{\Theta}_r^{(1)} \leq -\Delta\Theta < 0$ and $\tilde{\Theta}_r^{(1)} \leq 0$. Thus, we can obtain that synchronization control rule for network (1) in Corollary 1 is built on Theorem 2 and the principle of $\tilde{\Theta}_r^{(1)} \leq -\Delta\Theta < 0$. Therefore, under the conditions of Corollary 1, synchronization control rule for network (1) must make network (2) achieve synchronization. □

Remark 4 Under the conditions of Corollary 2, networks (1) and (2) can realize finite-time synchronization. Thus, comparing synchronization total errors of networks (1) and (2), the nonlinear coupling affecting synchronization dynamics of network (1) can be analyzed. The steps are as follows: (Step 1) Choose a nonlinear coupling function $g(\cdot)$, and let $\hat{L}_1\|x(t) - y(t)\| \leq \|g(x(t)) - g(y(t))\| \leq \hat{L}_2\|x(t) - y(t)\|$ hold, where $\hat{L}_2 \geq \hat{L}_1 > 1$. (Step 2) According to Corollary 2, one designs a synchronization control rule. The other steps are similar to those listed in Remark 3.

Corollary 3 *Under the conditions of Theorem 2, if $0 < L < 1$, then network (2) with controller (4) must satisfy the conditions of Theorem 1.*

Proof The proof is similar to that of Corollary 1. □

Corollary 4 *Under the conditions of Corollary 3, the synchronization control rule for network (2) will make network (1) achieve synchronization.*

Proof The proof is similar to that of Corollary 2. □

Remark 5 By Corollary 4, for networks (1) and (2), the synchronization control rule can be designed. It is clear that the synchronization control rule can make networks (1) and (2) achieve finite-time synchronization. Furthermore, similar to the steps of Remark 4, we address the issue of the nonlinear coupling affecting synchronization dynamics of network (1). One needs to emphasize that the nonlinear coupling function $g(\cdot)$ satisfies $\check{L}_1\|x(t) - y(t)\| \leq \|g(x(t)) - g(y(t))\| \leq \check{L}_2\|x(t) - y(t)\|$, where $0 < \check{L}_1 \leq \check{L}_2 < 1$.

3.4 Two novel design schemes of synchronization control rules for networks (1)–(2)

Corollary 5 *If $-\Delta\Theta < \tilde{\Theta}_r^{(1)T2} \leq 0$ and $\tilde{\Theta}_r^{(1)T2}$ is $\tilde{\Theta}_r^{(1)}$ in Theorem 2, $k_{ir}^{C1} = k_{ir}^{T2}$ and $\varrho_r^{C1} = \varrho_r^{T2}$ in Corollary 1 and Theorem 2, for network (2), the control rule under Corollary 1 is better than that of Theorem 2.*

Proof According to Corollary 1, we have $\tilde{\Theta}_r^{(1)} \leq -\Delta\Theta < 0$. By Theorem 2, one has $\tilde{\Theta}_r^{(1)} \leq 0$. If $-\Delta\Theta < \tilde{\Theta}_r^{(1)T2} \leq 0$, then $\tilde{\Theta}_r^{(1)} \leq -\Delta\Theta < \tilde{\Theta}_r^{(1)T2} \leq 0$. Thus, we can obtain two control rules to make network (2) achieve synchronization. If $k_{ir}^{C1} = k_{ir}^{T2}$ and $\varrho_r^{C1} = \varrho_r^{T2}$ in Corollary 1 and Theorem 2, the control rule built on $\tilde{\Theta}_r^{(1)} \leq -\Delta\Theta < 0$ is better than that of the control rule based on $-\Delta\Theta < \tilde{\Theta}_r^{(1)T2} \leq 0$. □

Corollary 6 *If $-\Delta\Theta < \Theta_r^{(1)T1} \leq 0$ and $\Theta_r^{(1)T1}$ is $\Theta_r^{(1)}$ in Theorem 1, $k_{ir}^{C3} = k_{ir}^{T1}$ and $\varrho_r^{C3} = \varrho_r^{T1}$ in Corollary 3 and Theorem 1, for network (1), the control rule under Corollary 3 is better than that of Theorem 1.*

Proof The proof is similar to that of Corollary 5. □

Corollary 7 *If* $L^2 = 1$, *Theorems 1–2 are equivalent.*

Proof The proof is similar to that of Corollary 1. □

Remark 6 From Theorem 1, it is observed that $\Theta_r^{(1)} \le 0$ and $\Theta_r^{(2)} \le 0$ are closely related to L and L_2. According to Remark 3, if synchronization control rule of network (1) is fixed, and nonlinearities of nonlinear coupling functions $g(\cdot)$ and $f(\cdot,\cdot)$ are increased, one must have $\Theta_r^{(1)} = 0$ and $\Theta_r^{(2)} = 0$. Because Theorem 1 gives a sufficient condition under which network (1) can realize finite-time synchronization, this means that, under the conditions of Theorem 1, network (1) must achieve finite-time synchronization. Otherwise, the result doesn't hold. That is to say, if network (1) can achieve synchronization within finite time, it perhaps doesn't satisfy the conditions of Theorem 1. If $\Theta_r^{(1)} = 0$ and $\Theta_r^{(2)} = 0$, Theorem 1 presents a stricter condition. From the above analysis, one can obtain that the stricter condition of Theorem 1 is a sufficient condition. It is clear that for network (1), stricter conditions of finite-time synchronization are not unique.

4 Numerical examples

This section gives three numerical examples to illustrate the effectiveness of the derived results. The initial conditions of the numerical simulations are taken as: $x_1(0) = (-1,-2)^T$, $x_2(0) = (-3,1)^T$, $x_3(0) = (2,3)^T$. The synchronization total error of networks (1) and (2) is defined as $e(t) = \sum_{i=1}^{3}\sum_{j=1}^{2}|e_{ij}(t)|$. A Markov chain with the following rate transition matrix is considered:

$$\Pi = \begin{bmatrix} -3 & 3 \\ 4 & -4 \end{bmatrix}. \tag{24}$$

Networks (1)–(2) are composed of three coupled nodes, and the main parameters are as follows:

$$\Gamma_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \Gamma_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -2 \end{bmatrix},$$

$$\tilde{\sigma}_i\big(e(t),t,r\big) = \begin{bmatrix} 0.1e_{i1}(t) & 0 \\ 0 & 0.1e_{i2}(t) \end{bmatrix}, \qquad c_1 = 1, \qquad c_2 = 1, \qquad \tau = 0.3,$$

$$f\big(x_i(t),x_i(t-0.3)\big) = \begin{bmatrix} -0.5x_{i1}(t) - \tanh(x_{i1}(t-0.3)) \\ \sin(x_{i2}(t)) - \tanh(x_{i2}(t-0.3)) \end{bmatrix}.$$

Example 1 According to Remark 3, nonlinear coupling function $g(\cdot)$ affecting the synchronization dynamics of network (1) is analyzed. Firstly, one chooses $g(\cdot)$ as

$$g\big(x_j(t)\big) = \begin{bmatrix} \tanh(-0.5x_{j1}(t)) \\ \tanh(-0.5x_{j2}(t)) \end{bmatrix}.$$

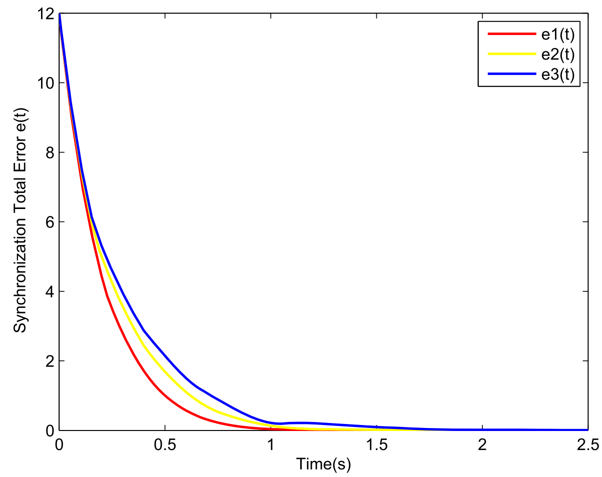

Figure 1 Synchronization total error trajectories for Example 1

According to Assumption 1, one has $L = 1$. By using Theorem 1, one gets $a_1 = a_2 = 1$, $\lambda = 2$, $\upsilon = 4$, $\beta = 0.5$, $q_1 = q_2 = 1$, $\varrho_1 = \varrho_2 = 1$, $\Psi_r = \mathrm{diag}\{0.2, 0.2, 0.2\}$, $L_1 = 0.5$, $L_2 = 1$, $\Xi_r = \mathrm{diag}\{5, 5, 5\}$, $V(0) = 28$, $k_{ir} = 2.5$, $r = 1, 2$, $i = 1, 2, 3$, $t^* \leq 2.14$. Secondly, a new nonlinear coupling function $h(\cdot)$ is chosen as

$$h\bigl(x_j(t)\bigr) = \begin{bmatrix} \tanh(-x_{j1}(t)) \\ \tanh(-x_{j2}(t)) \end{bmatrix}.$$

It is clear that nonlinearity of $h(\cdot)$ is more serious than that of $g(\cdot)$. From Assumption 1, we get $L = 1.2$. Fixing the above control rule, and substituting the above parameters into inequality (6) of Theorem 1, inequality (6) still holds. Similar to the above steps, the nonlinear coupling function $\ell(\cdot)$ is

$$\ell\bigl(x_j(t)\bigr) = \begin{bmatrix} \tanh(-1.5 * x_{j1}(t)) \\ \tanh(-1.5 * x_{j2}(t)) \end{bmatrix}.$$

By Assumption 1, one has $L = 1.6$. Simulation results are shown in Fig. 1. In this figure, $e1(t)$, $e2(t)$ and $e3(t)$ stand for the synchronization total error trajectories of network (1) with respect to the nonlinear coupling functions $g(\cdot)$, $h(\cdot)$ and $\ell(\cdot)$, respectively.

Example 2 This example shows the derived results of Remark 4.

Due to $\hat{L}_1\|x(t)-y(t)\| \leq \|g(x(t))-g(y(t))\| \leq \hat{L}_2\|x(t)-y(t)\|$, where $\hat{L}_2 \geq \hat{L}_1 > 1$, we choose the nonlinear coupling functions $\hat{g}(\cdot)$ and $\hat{h}(\cdot)$ as

$$\hat{g}\bigl(x_j(t)\bigr) = \begin{bmatrix} \tanh(-1.2 * x_{j1}(t)) \\ \tanh(-1.2 * x_{j2}(t)) \end{bmatrix}, \qquad \hat{h}\bigl(x_j(t)\bigr) = \begin{bmatrix} \tanh(-1.6 * x_{j1}(t)) \\ \tanh(-1.6 * x_{j2}(t)) \end{bmatrix}.$$

By Assumption 1, one gets $\hat{L} = 1.3$ and $\tilde{L} = 1.7$, where $\hat{L}$ and $\tilde{L}$ are the Lipschitz constants of $\hat{g}(\cdot)$ and $\hat{h}(\cdot)$, respectively. Similar to the steps of Example 1, one has $a_1 = a_2 = 1$, $\lambda = 1$, $\upsilon = 4$, $\beta = 0.5$, $q_1 = q_2 = 1$, $\varrho_1 = \varrho_2 = 1$, $\Psi_r = \mathrm{diag}\{0.2, 0.2, 0.2\}$, $L_1 = 0.5$, $L_2 = 1$,

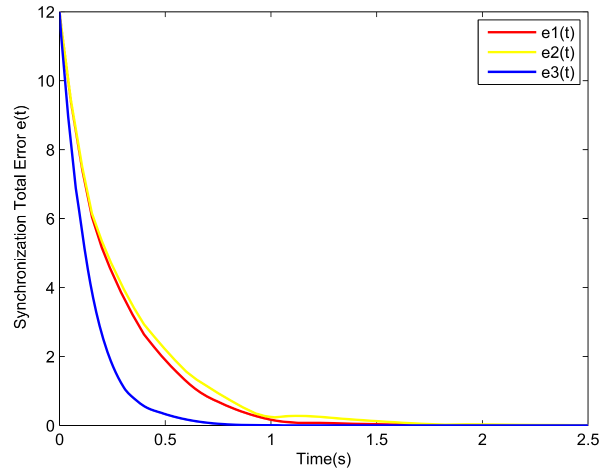

Figure 2 Synchronization total error trajectories for Example 2

$\Xi_r = \mathrm{diag}\{5.2, 5.2, 5.2\}$, $V(0) = 28$, $k_{ir} = 2.5$, $r = 1,2$, $i = 1,2,3$, $t^* \leq 2.6$. Figure 2 gives the simulation results. In this figure, $e1(t)$ and $e2(t)$ represent the synchronization total error trajectories of network (1) with respect to the nonlinear coupling functions $\hat{g}(\cdot)$ and $\hat{h}(\cdot)$, respectively, while $e3(t)$ is the total error trajectory of network (2).

Example 3 Based on Remark 5, the following example is given. In Remark 5, we had $\check{L}_1\|x(t) - y(t)\| \leq \|g(x(t)) - g(y(t))\| \leq \check{L}_2\|x(t) - y(t)\|$, where $0 < \check{L}_1 \leq \check{L}_2 < 1$. According to the same principle, one has

$$\vartheta\left(x_j(t)\right) = \begin{bmatrix} 0.5 * \tanh(-0.5 * x_{j1}(t)) \\ 0.5 * \tanh(-0.5 * x_{j2}(t)) \end{bmatrix}, \qquad \tilde{\vartheta}\left(x_j(t)\right) = \begin{bmatrix} 0.3 * \tanh(-0.1 * x_{j1}(t)) \\ 0.3 * \tanh(-0.1 * x_{j2}(t)) \end{bmatrix},$$

where $\vartheta(\cdot)$ and $\tilde{\vartheta}(\cdot)$ are nonlinear coupling functions of network (1). From the above principle, one obtains $\check{\iota}_1 = 0.6$ and $\check{\iota}_2 = 0.3$, where $\check{\iota}_1$ and $\check{\iota}_2$ are the constants corresponding to $\vartheta(\cdot)$ and $\tilde{\vartheta}(\cdot)$, respectively. Similar to Example 2, one gets $a_1 = a_2 = 1$, $\lambda = 2$, $\upsilon = 4$, $\beta = 0.5$, $q_1 = q_2 = 1$, $\varrho_1 = \varrho_2 = 1$, $\Psi_r = \mathrm{diag}\{0.2, 0.2, 0.2\}$, $L_1 = 0.5$, $L_2 = 1$, $\Xi_r = \mathrm{diag}\{2.5, 2.5, 2.5\}$, $V(0) = 28$, $k_{ir} = 3.5$, $r = 1,2$, $i = 1,2,3$, $t^* \leq 1.62$. In Fig. 3, $e1(t)$ is the total error trajectory of network (2) while $e2(t)$ and $e3(t)$ are the total error trajectories of network (1) with respect to the nonlinear coupling functions $\vartheta(\cdot)$ and $\tilde{\vartheta}(\cdot)$, respectively.

Remark 7 From Figs. 1–3, it is observed that if nonlinearity of the nonlinear coupling becomes more and more serious, its effect on synchronization dynamics of network (1) also becomes more and more serious. Furthermore, from Fig. 3, one can see that if the nonlinear coupling function satisfies $\check{L}_1\|x(t) - y(t)\| \leq \|g(x(t)) - g(y(t))\| \leq \check{L}_2\|x(t) - y(t)\|$, where $0 < \check{L}_1 \leq \check{L}_2 < 1$, under the conditions of Corollary 4 and with the same synchronization control rule, synchronization dynamics of network (1) is better than that of network (2). This further testifies that with decreasing nonlinearity of the nonlinear coupling, its effect on the synchronization dynamics of network (1) becomes smaller. Therefore, from Examples 1–3, one can obtain that the smaller the nonlinearity of function $g(\cdot)$ for network (1), the better its synchronization effect.

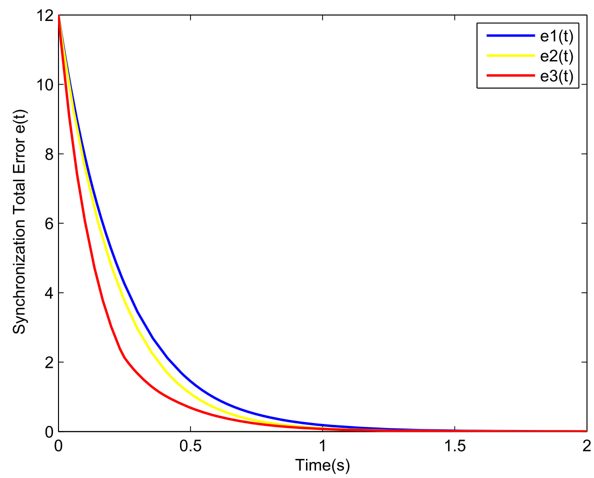

Figure 3 Synchronization total error trajectories for Example 3

Remark 8 In [3], in order to analyze the nonlinear coupling affecting synchronization dynamics of the addressed network, one uses $(x-y)\varphi(y,x) \le -L(x-y)^2$, where $\forall x \neq y$ and $L > 0$, and lets $(x-y)\varphi_1(y,x) < (x-y)\varphi_2(y,x) < (x-y)\varphi_3(y,x) < 0$. Thus, there must exist κ_1 and κ_2 such that $(x-y)\varphi_1(y,x) < -\kappa_1(x-y)^2$, $(x-y)\varphi_2(y,x) < -\kappa_2(x-y)^2$ and $(x-y)\varphi_3(y,x) \le -L(x-y)^2$ hold, where $\kappa_1 > \kappa_2 > L > 0$. Therefore, one has $-\kappa_1(x-y)^2 < -\kappa_2(x-y)^2 < -L(x-y)^2 < 0$. In [3], in the process of proving Theorems 1–3, the nonlinear term $-L\sum_{j=1}^{N}\sum_{k=1}^{N} a_{jk}[x_j(t)-x_k(t)]^2$ needed to be handled. Thus, if $-L[x_j(t)-x_k(t)]^2$ is decreased, $\dot{V}(t) \le 0$ will be decreased. Here, $V(t)$ is a Lyapunov function. This means that with decreasing $(x-y)\varphi(y,x) < 0$, a better synchronization effect of nonlinear coupling is achieved. This can be seen from simulation results of [3]. In this paper, in the process of proving Theorem 1, one can observe that if L is decreased, $E[\mathcal{L}V(e(t),t,r(t))] \le 0$ will be decreased. This means that with decreasing $L\|g(x)-g(y)\|$, the effect of nonlinear coupling function $g(\cdot)$ becomes smaller. The simulation results also reflect this result. So the results of [3] and Examples 1–3 of this paper are not contradictory.

Remark 9 In this paper, in order to analyze the nonlinear coupling affecting synchronization dynamics of network (1), some theoretical results have been obtained. Based on them, analysis schemes of the above addressed issue have been given. From the introduction and Remarks 2–8, it is seen that the motivation for wide practical use of the theoretical results includes: (a) From the derived results aspect, our results show that the nonlinear coupling affects synchronization dynamics of network (1). (b) From the motivation aspect, the analysis ideas of the addressed problem can be applied to analyze synchronization dynamics of other classes of complex networks. Here, we emphasize that this paper extends the existing analysis ideas for synchronization problems of complex networks, such as given in [1–10].

Remark 10 In network (1), the nonlinear coupling is given by $g(\cdot)$. If the nonlinear coupling becomes $g(\cdot,\cdot)$, based on the ideas of this paper, how can one analyze the addressed problem? Besides, for finite-time synchronization of network (1), Theorem 1 gives only a sufficient condition. This means that there exist other sufficient conditions which can make network (1) achieve finite-time synchronization. Among these sufficient conditions, there must exist some sufficient conditions which are less conservative than the condition of Theorem 1. How can one investigate this issue? In a future work, the above questions will be further explored.

5 Conclusions

This paper is concerned with a nonlinear coupling affecting synchronization dynamics of a class of NCMJDTCNs with stochastic noises. Firstly, finite-time synchronization problems of the NCMJDTCNs with stochastic noises and a class of LCMJDTCNs with stochastic noises are investigated. Secondly, comparing the existed works [1–10], two new ideas are applied to analyze the above problem. Combining simulation results, the conclusion of the presented analysis is that if the nonlinearity of the nonlinear coupling becomes smaller, the effect of the nonlinear coupling on the synchronization dynamics will also become smaller.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
XW wrote the main part of this manuscript. KG participated in the discussion and corrected the main theorem. All authors read and approved the final manuscript.

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