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Global Mittag-Leffler synchronization of delayed fractional-order memristive neural networks

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Abstract

This paper deals with the global Mittag-Leffler synchronization of fractional-order memristive neural networks (FMNNs) with time delay. Since the FMNNs are essentially a class of switched systems with irregular switching laws, it is more difficult to achieve synchronization than with the traditional neural networks. First, under the framework of fractional-order differential inclusions and set-valued maps, the FMNNs are transformed into a continuous system with uncertainties. Then a linear state feedback combined with switching control law is designed in order to achieve the Mittag-Leffler synchronization. In addition, several synchronization criteria are obtained by constructing appropriate Lyapunov functionals, together with the help of some inequality techniques. Finally, an example is given to demonstrate the effectiveness of the obtained results.

Keywords: Mittag-Leffler synchronization; Fractional-order systems; Memristive neural networks; State feedback control; Switching control

1 Introduction

The memristor was first postulated by Chua in 1971, as the fourth fundamental circuit element together with the resistor, inductor and capacitor [1]. In 2008, the Williams group announced a successful fabrication of a very compact and nonvolatile nano scale memory, called memristor [2], which describes the relationship between the magnetic flux and charge. If supplying current and voltage to the memristor, the resistance value of the memristor, (i.e., memristance) can be changed. When the voltage is turned off, the memristor remembers its most recent resistance value until the next time it is turned on. As shown in Fig. 1, it has been found that memristors exhibit the feature of pinched hysteresis, which is very similar to the behavior of neurons in the human brain. Owing to this special characteristic, memristor can be used in artificial neural networks to imitate the synapses. It inspires more and more researchers to replace resistors in traditional neural networks (NNs) by memristors to implement synaptic connection weights between neurons [3–6]. Thus, the so-called memristive neural networks (MNNs) have been established. Since then considerable efforts have been made devoted to the investigation of MNNs, such as stability [7], stabilization [8–10], nonlinear dynamics analysis [11], and synchronization [12–18].

On the other hand, fractional-order dynamical systems have attracted considerable research interests due to their widespread applications in many fields such as neural systems

Figure 1 Typical current–voltage characteristic of a memristor
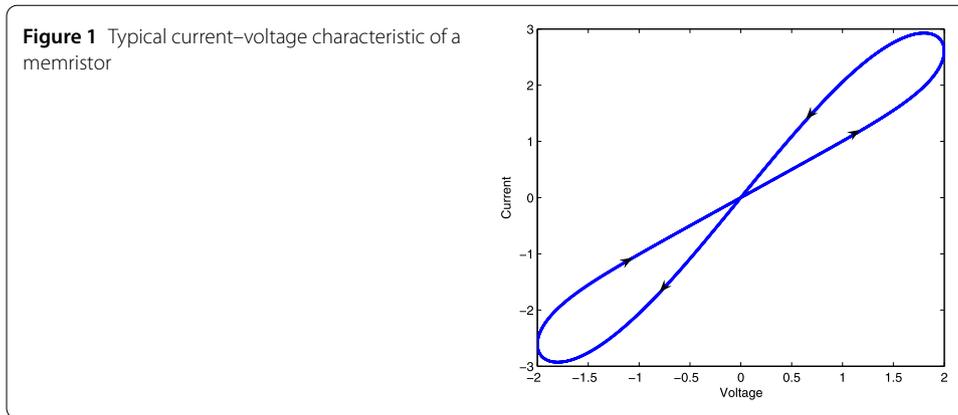
[19] and viscoelastic systems [20]. Compared with integer-order systems, a distinguishing feature of fractional-order systems is that they have long-term memory effects, hence it is suitable for describing various materials and processing more accurately [21]. In the field of electronics, the model of a fractional capacitor has been presented [22]. A fractional capacitor describes the fractional differentiation constitutive relationship $I(t) = C_0 D_t^\alpha V(t)$ between the voltage $V(t)$ and the current $I(t)$ passing through it [23, 24]. In the past few years, many researchers have incorporated fractional capacitors into conventional NNs to establish the fractional-order neural networks (FNNs) model and to improve the accuracy of NN models. In consequence, some excellent results on the dynamical analysis of FNNs have been developed in [25–28].

In recent years, FMNNs have been proposed by replacing the common resistors in circuit implementation of FNNs by memristors to emulate synapses in the human brain. It is evident that FMNN models provide a significant improvement on FNNs. Therefore, the research on FMNNs has become a topic of focus and attracted increasing attention of many researchers. However, many traditional analysis methods, which are applicable to integer-order chaotic systems, cannot be extended and applied directly to FMNNs for the reason that FMNNs are represented by fractional-order differential equations (FDEs). In recent years, more and more scholars devoted their efforts to the dynamical analysis of FMNNs.

At present, some remarkable results relating to stability and synchronization of FMNNs have been reported in [29–36]. For example, global Mittag-Leffler stability and synchronization have been investigated for FMNNs without time delay in [29]. Projective synchronization of FMNNs has been discussed by constructing appropriate Lyapunov functions in [30]. Adaptive synchronization of delayed FMNNs has been studied via delayed state feedback control in [31]. Finite-time synchronization has been addressed for fractional-order memristive BAM neural networks via linear feedback control in [32]. The synchronization issue for delayed FMNNs with parameter uncertainties or parameter mismatches has been discussed in [33–35] based on constructing suitable Lyapunov functionals together with the assistance of some fractional-order differential inequalities. Furthermore, synchronization of FMNNs with multiple delays has been discussed by using the maximum modulus principle [36]. To the best of our knowledge, there are few results on global Mittag-Leffler synchronization of FMNNs with time delay.

Motivated by the aforementioned discussions, the main objective of this paper is to discuss the global Mittag-Leffler synchronization of delayed FMNNs. The main contribu-

tions of this paper are highlighted as follows. First, considering that FMNNs are described by FDEs with discontinuous right-hand sides, under the framework of fractional-order differential inclusions and set-valued maps, the FMNNs can be transformed into a continuous system with uncertainties. Second, a linear feedback combined with switching control law is designed, which plays an important role in the achievement of Mittag-Leffler synchronization. Third, several global Mittag-Leffler synchronization criteria are obtained for FMNNs by constructing suitable Lyapunov functionals. The obtained results extend and improve some previously published work on synchronization of FMNNs.

The rest of this paper is organized as follows. In Sect. 2, some lemmas, definitions and assumptions are introduced and the synchronization control schemes for FMNNs are designed. In Sect. 3, several global Mittag-Leffler synchronization criteria are derived. In Sect. 4, a numerical example is provided to illustrate the feasibility of the proposed theoretical results. Finally, some conclusions are drawn in Sect. 5.

2 Preliminaries and problem formulation

Notations Throughout this paper, the notation $\| \cdot \|$ is used to denote the 1-norm for vectors or for matrices, whenever appropriate. Besides, solutions of all the systems are considered in Filippov's sense. C is the space of complex number, R is the space of real number, and R^n denotes the n-dimensional Euclidean space. $\text{sign}(\cdot)$ is the sign function. Let $\tau > 0$, $C([-\tau, 0], R^n)$ denotes the family of continuous functions from $[-\tau, 0]$ to R^n.

2.1 Definitions and lemmas

Definition 1 ([37]) The Caputo fractional derivative of order α for a function $f(t)$ is defined as

$$_{t_0}D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} \, d\tau,$$

where $t \geq t_0$, and n is the positive integer such that $n-1 < \alpha < n$. In particular, when $0 < \alpha < 1$, one has

$$_{t_0}D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} (t-\tau)^{-\alpha} f'(\tau) \, d\tau.$$

Definition 2 ([37]) The fractional integral of order α for a function $f(t)$ is defined as

$$_{t_0}I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} f(\tau) \, d\tau,$$

where $t \geq t_0$, $\alpha > 0$, and $\Gamma(\cdot)$ is the Gamma function, that is, $\Gamma(\alpha) = \int_0^{+\infty} e^{-t} t^{\alpha-1} \, dt$.

Definition 3 ([37]) The Mittag-Leffler function with one parameter is defined as

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + 1)},$$

where $\alpha > 0$, and $z \in C$.

The Mittag-Leffler function with two parameters is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + \beta)},$$

where $\alpha > 0$, $\beta > 0$. When $\beta = 1$, one has $E_{\alpha,1}(z) = E_\alpha(z)$, and when $\alpha = 1$, $\beta = 1$, one further has $E_{1,1}(z) = e^z$.

Lemma 1 ([38]) *For $\alpha \in (0,1)$, $t \in (-\infty, +\infty)$, we have*

$$E_\alpha(t) > 0, \qquad E_{\alpha,\alpha}(t) > 0.$$

Definition 4 ([37]) The Laplace transform of Mittag-Leffler function with two parameters is

$$\mathcal{L}\{t^{\beta-1}E_{\alpha,\beta}(-\lambda t^\alpha)\} = \frac{s^{\alpha-\beta}}{s^\alpha + \lambda}, \quad \mathrm{Re}(s) > |\lambda|^{1/\alpha},$$

where t and s denote the variables in the time domain and Laplace domain, respectively; $\mathrm{Re}(s)$ denotes the real part of s, $\lambda \in R$, and $\mathcal{L}$ stands for the Laplace transform.

Definition 5 ([39]) Suppose $E, Y \subset R^n$, then $F : E \to Y$ is called a set-valued map, if for each point $x \in E$, there corresponds a nonempty set $F(x) \subset Y$. A set-valued map F with nonempty values is said to be upper-semi-continuous at $x_0 \in E \subset R^n$ if, for any open set N containing $F(x_0)$, there exists a neighborhood M of x_0 such that $F(M) \subset N$. $F(x)$ is said to have a closed (convex, compact) image if, for each $x \in E$, F(x) is closed (convex, compact).

Definition 6 ([40]) Consider the system $\frac{dx}{dt} = g(x)$, $x \in R^n$, with discontinuous right-hand sides, a set-valued map is defined as

$$\phi(x) = \bigcap_{\delta>0} \bigcap_{\mu(N)=0} K[g(B(x,\delta) \setminus N)],$$

where $K[E]$ is the closure of the convex hull of set E, $B(x,\delta) = \{y : \|y - x\| \le \delta\}$, and $\mu(N)$ is the Lebesgue measure of the set N. A solution in Filippov's sense of the Cauchy problem for this system with initial condition $x(0) = x_0$ is an absolutely continuous function $x(t)$, $t \in [0, T]$, which satisfies $x(0) = x_0$ and differential inclusion:

$$\frac{dx}{dt} \in \phi(x)$$

for a.e. $t \in [0, T]$.

Lemma 2 ([41]) *If $x(t) \in C^1([0, +\infty), R)$ denotes a continuously differentiable function, for any $\alpha \in (0,1)$, the following inequality holds almost everywhere*:

$${}_0D_t^\alpha |x(t)| \le \mathrm{sign}(x(t)) \,{}_0D_t^\alpha x(t).$$

2.2 Problem formulation

In this paper, we consider a class of delayed FMNNs as drive system, which is described by

$$
{}_0D_t^\alpha x_i(t) = -c_i x_i(t) + \sum_{j=1}^n a_{ij}\big(x_j(t)\big) f_j\big(x_j(t)\big) + \sum_{j=1}^n b_{ij}\big(x_j(t)\big) f_j\big(x_j(t-\tau)\big), \tag{1}
$$

where $i = 1, 2, \ldots, n$, n is the number of neurons; $c_i > 0$ denotes the self-feedback connection weight; $0 < \alpha < 1$ is the fractional order; $x_i(t)$ corresponds to the state variable connected with the ith neuron; τ is the time delay; $f_j(\cdot)$ denotes the activation function satisfying $f_j(0) = 0$; $a_{ij}(x_j(t))$ and $b_{ij}(x_j(t))$ are memristive synaptic connection weights, which are defined by

$$
a_{ij}\big(x_j(t)\big) = \frac{W_{fij}}{C_i} \times \delta_{ij}, \qquad b_{ij}\big(x_j(t)\big) = \frac{W_{gij}}{C_i} \times \delta_{ij},
$$

in which $\delta_{ij} = 1$, if $i \neq j$ holds, otherwise, $\delta_{ij} = -1$. W_{fij} and W_{gij} denote the memductance of memristors R_{fij} and R_{gij}, respectively. In [42], it has been shown that, since digital computer applications require only two memory states, memristor needs to display only two sufficiently different equilibrium states. Hence, the memristive synaptic connection weights, in view of the pinched hysteresis curve, can be simply represented as

$$
a_{ij}\big(x_j(t)\big) = \begin{cases} a_{ij}^*, & |x_j(t)| \le T_j, \\ a_{ij}^{**}, & |x_j(t)| > T_j, \end{cases} \qquad b_{ij}\big(x_j(t)\big) = \begin{cases} b_{ij}^*, & |x_j(t)| \le T_j, \\ b_{ij}^{**}, & |x_j(t)| > T_j, \end{cases}
$$

for $i, j = 1, 2, \ldots, n$, where the switching jump $T_j > 0$, and $a_{ij}^*, a_{ij}^{**}, b_{ij}^*, b_{ij}^{**}$ are all constants. The initial condition of system (1) is assumed to be $x_i(s) = \vartheta_{xi}(s)$, $s \in [-\tau, 0]$, where $\vartheta_{xi}(s) \in C([-\tau, 0], R)$.

The response system is described by

$$
{}_0D_t^\alpha y_i(t) = -c_i y_i(t) + \sum_{j=1}^n a_{ij}\big(y_j(t)\big) f_j\big(y_j(t)\big) + \sum_{j=1}^n b_{ij}\big(y_j(t)\big) f_j\big(y_j(t-\tau)\big) + u_i(t), \tag{2}
$$

where $i = 1, 2, \ldots, n$, $c_i > 0$ denotes the self-feedback connection weight; $y_i(t)$ corresponds to the state variable connected with the ith neuron; $u_i(t)$ is the control input to be designed later.

Note that system (1) or (2) can be regarded as a class of discontinuous fractional-order differential equations because the memristive synaptic connection weights $a_{ij}(x_j(t))$, $b_{ij}(x_j(t))$, $a_{ij}(y_j(t))$, $b_{ij}(y_j(t))$ are all discontinuous. Hence, system (1) or (2) cannot be solved within the framework of classical Cauchy initial problem because the classical solutions to ordinary differential equations are not available. In this case, the solutions should be considered in Filippov's sense [40], which is a useful tool to deal with differential equations with discontinuous right-hand sides. Based on the theory of set-valued maps and differential inclusions, from (1) and (2), we have

$$
{}_0D_t^\alpha x_i(t) \in -c_i x_i(t) + \sum_{j=1}^n K\big[a_{ij}\big(x_j(t)\big)\big] f_j\big(x_j(t)\big) + \sum_{j=1}^n K\big[b_{ij}\big(x_j(t)\big)\big] f_j\big(x_j(t-\tau)\big) \tag{3}
$$

and

$$\,_0D_t^\alpha y_i(t) \in -c_i y_i(t) + \sum_{j=1}^n K\big[a_{ij}\big(y_j(t)\big)\big]f_j\big(y_j(t)\big)$$

$$+ \sum_{j=1}^n K\big[b_{ij}\big(y_j(t)\big)\big]f_j\big(y_j(t-\tau)\big) + u_i(t), \tag{4}$$

where

$$K\big[a_{ij}\big(x_j(t)\big)\big] = \begin{cases} a_{ij}^*, & |x_j(t)| < T_j, \\ \mathrm{co}\{a_{ij}^*, a_{ij}^{**}\}, & |x_j(t)| = T_j, \\ a_{ij}^{**}, & |x_j(t)| > T_j, \end{cases}$$

$$K\big[b_{ij}\big(x_j(t)\big)\big] = \begin{cases} b_{ij}^*, & |x_j(t)| < T_j, \\ \mathrm{co}\{b_{ij}^*, b_{ij}^{**}\}, & |x_j(t)| = T_j, \\ b_{ij}^{**}, & |x_j(t)| > T_j, \end{cases}$$

$$K\big[a_{ij}\big(y_j(t)\big)\big] = \begin{cases} a_{ij}^*, & |y_j(t)| < T_j, \\ \mathrm{co}\{a_{ij}^*, a_{ij}^{**}\}, & |y_j(t)| = T_j, \\ a_{ij}^{**}, & |y_j(t)| > T_j, \end{cases}$$

$$K\big[b_{ij}\big(y_j(t)\big)\big] = \begin{cases} b_{ij}^*, & |y_j(t)| < T_j, \\ \mathrm{co}\{b_{ij}^*, b_{ij}^{**}\}, & |y_j(t)| = T_j, \\ b_{ij}^{**}, & |y_j(t)| > T_j, \end{cases}$$

for $i,j = 1,2,\ldots,n$, and $\mathrm{co}\{a_{ij}^*, a_{ij}^{**}\} = [\underline{a}_{ij}, \overline{a}_{ij}]$, $\mathrm{co}\{b_{ij}^*, b_{ij}^{**}\} = [\underline{b}_{ij}, \overline{b}_{ij}]$, $\overline{a}_{ij} = \max\{a_{ij}^*, a_{ij}^{**}\}$, $\underline{a}_{ij} = \min\{a_{ij}^*, a_{ij}^{**}\}$, $\overline{b}_{ij} = \max\{b_{ij}^*, b_{ij}^{**}\}$, $\underline{b}_{ij} = \min\{b_{ij}^*, b_{ij}^{**}\}$.

Based on the measurable selection theorem in [43], there exist measurable functions $\gamma_{ij}(x_j(t)) \in K[a_{ij}(x_j(t))]$, $\chi_{ij}(x_j(t)) \in K[b_{ij}(x_j(t))]$, $\gamma_{ij}(y_j(t)) \in K[a_{ij}(y_j(t))]$, $\chi_{ij}(y_j(t)) \in K[b_{ij}(y_j(t))]$ such that

$$\,_0D_t^\alpha x_i(t) = -c_i x_i(t) + \sum_{j=1}^n \gamma_{ij}\big(x_j(t)\big)f_j\big(x_j(t)\big) + \sum_{j=1}^n \chi_{ij}\big(x_j(t)\big)f_j\big(x_j(t-\tau)\big) \tag{5}$$

and

$$\,_0D_t^\alpha y_i(t) = -c_i y_i(t) + \sum_{j=1}^n \gamma_{ij}\big(y_j(t)\big)f_j\big(y_j(t)\big) + \sum_{j=1}^n \chi_{ij}\big(y_j(t)\big)f_j\big(y_j(t-\tau)\big) + u_i(t). \tag{6}$$

To ensure the existence of the solutions of systems (5) and (6), respectively, throughout this paper, we assume that the activation function $f_j(\cdot)$ satisfies the following properties.

Assumption 1 The activation function $f_j(\cdot)$ is Lipschitz-continuous with Lipschitz constant $L_j > 0$, i.e.,

$$\big|f_j(u) - f_j(v)\big| \le L_j|u - v|,$$

for all $u, v \in R$, and $j = 1,2,\ldots,n$.

Assumption 2 The activation function $f_j(\cdot)$ is bounded on R, i.e., there exists a constant M_j such that

$$\left| f_j(u) \right| \le M_j, \quad u \in R, j = 1, 2, \ldots, n.$$

Let $e_i(t) = y_i(t) - x_i(t)$ be the synchronization error and the controller can be designed as

$$u_i(t) = -k_i e_i(t) - \text{sign}(e_i(t)) \epsilon_i \left| e_i(t - \tau) \right| - \rho_i \, \text{sign}(e_i(t)). \tag{7}$$

where k_i, ϵ_i, ρ_i denote the feedback gains.

Then the error system can be given by

$$\begin{aligned}
{}_0 D_t^\alpha e_i(t) = {}&-(c_i + k_i) e_i(t) - \text{sign}(e_i(t)) \epsilon_i \left| e_i(t - \tau) \right| \\
&+ \sum_{j=1}^n \gamma_{ij}(y_j(t)) \varphi_j(e_j(t)) + \sum_{j=1}^n \chi_{ij}(y_j(t)) \varphi_j(e_j(t - \tau)) \\
&+ \sum_{j=1}^n \big(\gamma_{ij}(y_j(t)) - \gamma_{ij}(x_j(t))\big) f_j(x_j(t)) \\
&+ \sum_{j=1}^n \big(\chi_{ij}(y_j(t)) - \chi_{ij}(x_j(t))\big) f_j(x_j(t - \tau)) - \rho_i \, \text{sign}(e_i(t)),
\end{aligned} \tag{8}$$

where $\varphi_j(e_j(t)) = f_j(y_j(t)) - f_j(x_j(t))$.

The initial condition of error system (8) is defined by

$$e_i(s) = \phi_i(s), \quad -\tau \le s \le 0,$$

where $\phi_i(s) = \vartheta_{yi}(s) - \vartheta_{xi}(s) \in C([-\tau, 0], R)$, $\phi(s) = [\phi_1(s), \phi_2(s), \ldots, \phi_n(s)]^T$.

In the next section, global Mittag-Leffler synchronization of drive–response systems (5)–(6) will be achieved. The definition of global Mittag-Leffler stability is as follows.

Definition 7 ([29]) The zero solution of the error system (8) is said to be globally Mittag-Leffler stable if there exist two positive constants N and λ such that, for any solutions $\phi(t)$ of error system (8) with different initial condition denoted by $\phi(0)$, one has

$$\left\| \phi(t) \right\| \le N \left\| \phi(0) \right\| E_\alpha(-\lambda t^\alpha), \quad t \ge 0.$$

3 Main result

Theorem 1 *Suppose Assumptions* 1–2 *hold and that the following algebraic conditions*:

(i) $\quad \min\limits_{1 \le i \le n} \left\{ c_i + k_i - \sum_{j=1}^n a_{ji}^u L_i \right\} > 0,$

(ii) $\quad \varepsilon_i - \sum_{j=1}^n b_{ji}^u L_i \ge 0, \quad i = 1, 2, \ldots, n,$

(iii) $\quad \rho_i - \sum_{j=1}^{n}\left(\left|a_{ij}^* - a_{ij}^{**}\right| + \left|b_{ij}^* - b_{ij}^{**}\right|\right)M_j \geq 0, \quad i = 1, 2, \dots, n,$

are satisfied, then error system (8) is globally Mittag-Leffler stable, where $a_{ji}^u = \max\{|a_{ji}^*|,$ $|a_{ji}^{**}|\}$, $b_{ji}^u = \max\{|b_{ji}^*|, |b_{ji}^{**}|\}$, $i, j = 1, 2, \dots, n$. That is, the response system (2) under the controller (7) can be globally Mittag-Leffler synchronized with the drive system (1).

Proof Construct the Lyapunov functional as follows:

$$V(t) = \sum_{i=1}^{n}\left|e_i(t)\right|.$$

From Lemma 2 and Assumptions 1–2, we obtain

$$
\begin{aligned}
{}_0D_t^\alpha V(t) &= {}_0D_t^\alpha \sum_{i=1}^{n}\left|e_i(t)\right| \leq \sum_{i=1}^{n}\text{sign}\left(e_i(t)\right){}_0D_t^\alpha e_i(t) \\
&= \sum_{i=1}^{n}\text{sign}\left(e_i(t)\right)\Bigg[-(c_i + k_i)e_i(t) - \text{sign}\left(e_i(t)\right)\epsilon_i\left|e_i(t - \tau)\right| \\
&\quad + \sum_{j=1}^{n}\gamma_{ij}\left(y_j(t)\right)\varphi_j\left(e_j(t)\right) + \sum_{j=1}^{n}\chi_{ij}\left(y_j(t)\right)\varphi_j\left(e_j(t - \tau)\right) \\
&\quad + \sum_{j=1}^{n}\left(\gamma_{ij}\left(y_j(t)\right) - \gamma_{ij}\left(x_j(t)\right)\right)f_j\left(x_j(t)\right) \\
&\quad + \sum_{j=1}^{n}\left(\chi_{ij}\left(y_j(t)\right) - \chi_{ij}\left(x_j(t)\right)\right)f_j\left(x_j(t - \tau)\right) - \rho_i\,\text{sign}\left(e_i(t)\right)\Bigg] \\
&\leq \sum_{i=1}^{n}\Bigg[-(c_i + k_i)\left|e_i(t)\right| - \epsilon_i\left|e_i(t - \tau)\right| - \rho_i + \sum_{j=1}^{n}a_{ij}^u L_j\left|e_j(t)\right| + \sum_{j=1}^{n}b_{ij}^u L_j\left|e_j(t - \tau)\right| \\
&\quad + \sum_{j=1}^{n}\left(\left|a_{ij}^* - a_{ij}^{**}\right| + \left|b_{ij}^* - b_{ij}^{**}\right|\right)M_j\Bigg] \\
&= -\sum_{i=1}^{n}\left(c_i + k_i - \sum_{j=1}^{n}a_{ji}^u L_i\right)\left|e_i(t)\right| - \sum_{i=1}^{n}\left(\epsilon_i - \sum_{j=1}^{n}b_{ji}^u L_i\right)\left|e_i(t - \tau)\right| \\
&\quad - \sum_{i=1}^{n}\Bigg[\rho_i - \sum_{j=1}^{n}\left(\left|a_{ij}^* - a_{ij}^{**}\right| + \left|b_{ij}^* - b_{ij}^{**}\right|\right)M_j\Bigg].
\end{aligned}
$$

Based on the conditions (i)–(iii) of Theorem 1, we have

$$
{}_0D_t^\alpha V(t) \leq -\min_{1 \leq i \leq n}\left(c_i + k_i - \sum_{j=1}^{n}a_{ji}^u L_i\right)\sum_{i=1}^{n}\left|e_i(t)\right|.
$$

Let $\lambda = \min_{1 \leq i \leq n}\{c_i + k_i - \sum_{j=1}^{n}a_{ji}^u L_i\}$, we have

$$
{}_0D_t^\alpha V(t) \leq -\lambda V(t).
$$

Thus, there exists a nonnegative function $F(t)$ satisfying

$$_0D_t^\alpha V(t) + F(t) = -\lambda V(t). \tag{9}$$

Taking the Laplace transform on both sides of (9), it follows that

$$V(s) = \frac{s^{\alpha-1}}{s^\alpha + \lambda} V(0) - \frac{F(s)}{s^\alpha + \lambda},$$

where $V(s) = \mathcal{L}\{V(t)\}$, $F(s) = \mathcal{L}\{F(t)\}$.

By the inverse Laplace transform, we obtain

$$V(t) = V(0)E_\alpha\left(-\lambda t^\alpha\right) - F(t) * t^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda t^\alpha\right), \quad t \geq 0,$$

where $*$ denotes the convolution operator. From Lemma 1, $F(t)$, $t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha)$ are all non-negative functions. Then we can obtain

$$V(t) \leq V(0)E_\alpha\left(-\lambda t^\alpha\right), \quad t \geq 0.$$

That is,

$$\|e(t)\| \leq \|e(0)\|E_\alpha\left(-\lambda t^\alpha\right), \quad t \geq 0.$$

By Definition 7, the error system (8) is globally Mittag-Leffler stable, which implies the response system (2) under the controller (7) can be globally Mittag-Leffler synchronized with the drive system (1). This completes the proof. □

If the drive–response systems are without time delay, then the drive–response systems (5)–(6) are reduced to the following two systems:

$$_0D_t^\alpha x_i(t) = -c_i x_i(t) + \sum_{j=1}^{n} \gamma_{ij}\left(x_j(t)\right)f_j\left(x_j(t)\right), \tag{10}$$

$$_0D_t^\alpha y_i(t) = -c_i y_i(t) + \sum_{j=1}^{n} \gamma_{ij}\left(y_j(t)\right)f_j\left(y_j(t)\right) + u_i(t). \tag{11}$$

Accordingly, the controller is reduced to

$$u_i(t) = -k_i e_i(t) - \rho_i \operatorname{sign}\left(e_i(t)\right). \tag{12}$$

Then the error system can be given by

$$_0D_t^\alpha e_i(t) = -(c_i + k_i)e_i(t) + \sum_{j=1}^{n} \gamma_{ij}\left(y_j(t)\right)\varphi_j\left(e_j(t)\right)$$

$$+ \sum_{j=1}^{n}\left(\gamma_{ij}\left(y_j(t)\right) - \gamma_{ij}\left(x_j(t)\right)\right)f_j\left(x_j(t)\right) - \rho_i \operatorname{sign}\left(e_i(t)\right), \tag{13}$$

where $\varphi_j(e_j(t)) = f_j(y_j(t)) - f_j(x_j(t))$.

Corollary 1 *Suppose Assumptions* 1–2 *hold and that the following algebraic conditions*:

(i) $\quad \min_{1 \le i \le n} \left\{ c_i + k_i - \sum_{j=1}^{n} a_{ji}^u L_i \right\} > 0,$

(ii) $\quad \rho_i - \sum_{j=1}^{n} \left(\left| a_{ij}^* - a_{ij}^{**} \right| \right) M_j \ge 0, \quad i = 1, 2, \ldots, n,$

are satisfied, then error system (13) *is globally Mittag-Leffler stable, where* $i, j = 1, 2, \ldots, n$, $a_{ji}^u = \max\{|a_{ji}^*|, |a_{ji}^{**}|\}$. *That is, the response system* (11) *under the controller* (12) *can be globally Mittag-Leffler synchronized with the drive system* (10).

Remark 1 In controller (7), the terms $-\rho_i \operatorname{sign}(e_i(t))$ play an important role in eliminating the undesired error caused by the memristive synaptic connection weights. In order to achieve the global Mittag-Leffler synchronization, the feedback gains ρ_i should be well designed such that $\rho_i - \sum_{j=1}^{n}(|a_{ij}^* - a_{ij}^{**}| + |b_{ij}^* - b_{ij}^{**}|)M_j \ge 0$ are ensured. It also implies that, besides the Lipschitz-continuous condition, boundedness constraint should be imposed on the activation functions. In our future work, we will discuss how to reduce the conservatism of the activation functions.

Remark 2 In this paper, by constructing appropriate Lyapunov functionals and designing suitable hybrid controllers, some sufficient criteria are established to achieve the globally Mittag-Leffler synchronization of delayed FMNNs. Notice that the memristive synaptic connection weights $a_{ij}(\cdot)$, $b_{ij}(\cdot)$ are both state-dependent, which makes it impossible to achieve the complete synchronization only via linear state feedback control. To this end, $-\operatorname{sign}(e_i(t))\epsilon_i |e_i(t-\tau)|$, and $-\rho_i \operatorname{sign}(e_i(t))$ are added to the linear state feedback to deal with the delay term and the undesirable error caused by the differences between the memristive synaptic connection weights. Actually, there are few results relevant to the global Mittag-Leffler synchronization of delayed FMNNs (1) in the current literature. Therefore, the obtained results in this paper can fill this gap.

Remark 3 FMNNs are a class of switched systems with 2^n FNN subsystems. It should be mentioned that the memristive synaptic connection weights of FMNNs, i.e., $a_{ij}(x_j(t))$, $a_{ij}(y_j(t))$, $b_{ij}(x_j(t))$, and $b_{ij}(y_j(t))$ are time-varying; they which are dependent on the corresponding system states $x_j(t)$ and $y_j(t)$. Hence, compared with the traditional FNNs [25–28], it is more difficult to realize the synchronization of FMNNs. In this paper, a linear state feedback combined with switching control law is designed to overcome this difficulty. It is believed that the obtained results are useful and effective for the designs and applications of FMNNs.

4 Numerical simulations

In this section, the Adams–Bashforth–Moulton predictor–corrector algorithm is applied in the numerical simulations [44, 45].

Consider the drive system (1) with $n = 3$, $\alpha = 0.9$, $f_j(x_j) = \tanh(x_j)$, $\tau = 0.8$, initial condition $\vartheta_x(s) = (0.3, -0.6, 0.2)^T$, $s \in [-0.8, 0]$. Set $c_1 = 2.2$, $c_2 = 1.2$, $c_3 = 1.8$,

$$a_{11}(x_1) = \begin{cases} 2.2, & |x_1| \le 1, \\ 2, & |x_1| > 1, \end{cases} \qquad a_{12}(x_2) = \begin{cases} -2, & |x_2| \le 1, \\ -2.1, & |x_2| > 1, \end{cases}$$

$$a_{13}(x_3) = \begin{cases} 2, & |x_3| \leq 1, \\ 1.8, & |x_3| > 1, \end{cases} \qquad a_{21}(x_1) = \begin{cases} -0.8, & |x_1| \leq 1, \\ -0.6, & |x_1| > 1, \end{cases}$$

$$a_{22}(x_2) = \begin{cases} 5.71, & |x_2| \leq 1, \\ 5.68, & |x_2| > 1, \end{cases} \qquad a_{23}(x_3) = \begin{cases} 1.15, & |x_3| \leq 1, \\ 1.1, & |x_3| > 1, \end{cases}$$

$$a_{31}(x_1) = \begin{cases} -4.75, & |x_1| \leq 1, \\ -4.5, & |x_1| > 1, \end{cases} \qquad a_{32}(x_2) = \begin{cases} -1, & |x_2| \leq 1, \\ -0.8, & |x_2| > 1, \end{cases}$$

$$a_{33}(x_3) = \begin{cases} 1.2, & |x_3| \leq 1, \\ 1.25, & |x_3| > 1, \end{cases} \qquad b_{11}(x_1) = \begin{cases} -4, & |x_1| \leq 1, \\ -3.8, & |x_1| > 1, \end{cases}$$

$$b_{12}(x_2) = \begin{cases} 2.5, & |x_2| \leq 1, \\ 2.3, & |x_2| > 1, \end{cases} \qquad b_{13}(x_3) = \begin{cases} -3.2, & |x_3| \leq 1, \\ -3.0, & |x_3| > 1, \end{cases}$$

$$b_{21}(x_1) = \begin{cases} -1.5, & |x_1| \leq 1, \\ -1.7, & |x_1| > 1, \end{cases} \qquad b_{22}(x_2) = \begin{cases} -3.6, & |x_2| \leq 1, \\ -3.8, & |x_2| > 1, \end{cases}$$

$$b_{23}(x_3) = \begin{cases} -2.3, & |x_3| \leq 1, \\ -2.5, & |x_3| > 1, \end{cases} \qquad b_{31}(x_1) = \begin{cases} 0.3, & |x_1| \leq 1, \\ 0.4, & |x_1| > 1, \end{cases}$$

$$b_{32}(x_2) = \begin{cases} 1.8, & |x_2| \leq 1, \\ 2, & |x_2| > 1, \end{cases} \qquad b_{33}(x_3) = \begin{cases} 1.2, & |x_3| \leq 1, \\ 1.5, & |x_3| > 1. \end{cases}$$

The response system (2) is assumed to have the same parameters as those for the drive system, except for the different initial condition $\vartheta_y(s) = (-2, -3, 2)^T$, $s \in [-0.8, 0]$.

Figure 2(a) and Fig. 2(b) depict the phase portraits of the x_2–x_1 plane and x_2–x_3 plane of the drive system, respectively. When the controller (7) is at rest, that is, $k_1 = k_2 = k_3 = 0$, $\epsilon_1 = \epsilon_2 = \epsilon_3 = 0$, $\rho_1 = \rho_2 = \rho_3 = 0$, the switching laws of the memristive synaptic connection weights $a_{11}(x_1)$, $a_{11}(y_1)$ are shown in Fig. 3(a) and Fig. 3(b), respectively. From Fig. 3, one can see that the switching laws of $a_{11}(x_1)$, $a_{11}(y_1)$ evoked by the nonidentical initial conditions are different from each other.
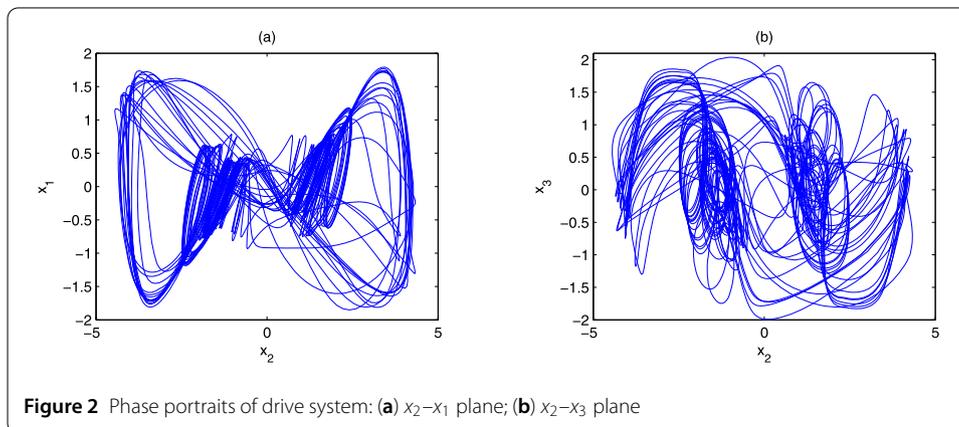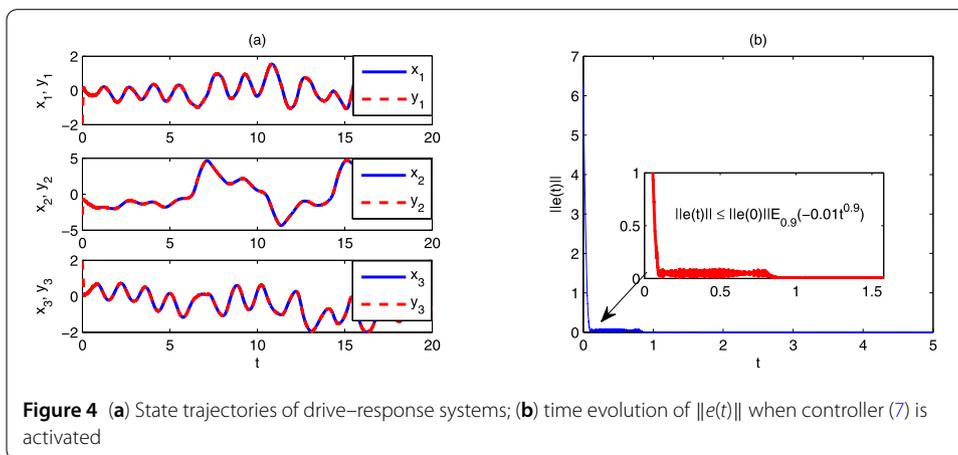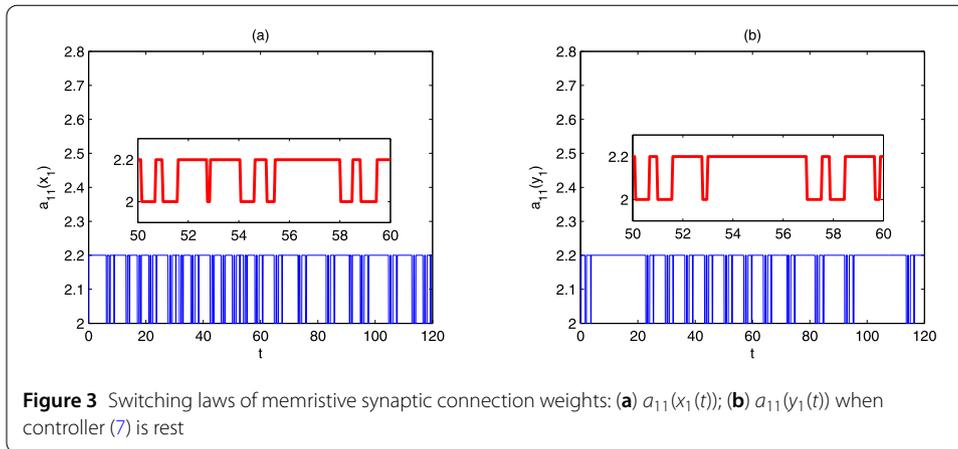

Figure 2 Phase portraits of drive system: (**a**) x_2–x_1 plane; (**b**) x_2–x_3 plane

Figure 3 Switching laws of memristive synaptic connection weights: (**a**) $a_{11}(x_1(t))$; (**b**) $a_{11}(y_1(t))$ when controller (7) is rest

Figure 4 (**a**) State trajectories of drive–response systems; (**b**) time evolution of $\|e(t)\|$ when controller (7) is activated
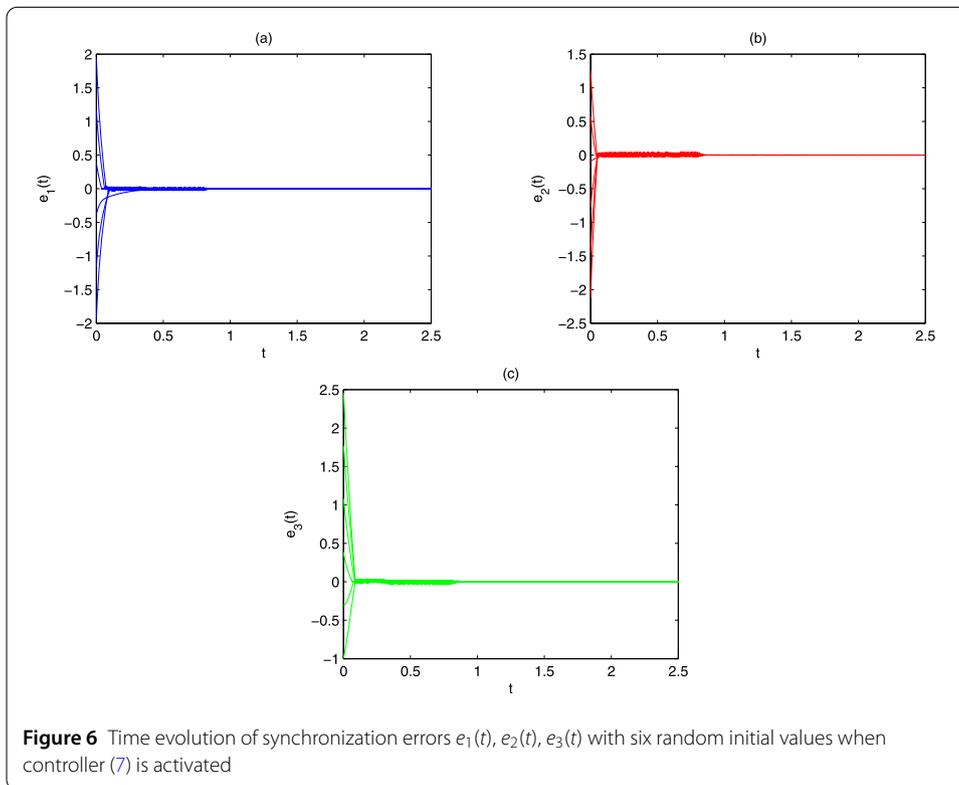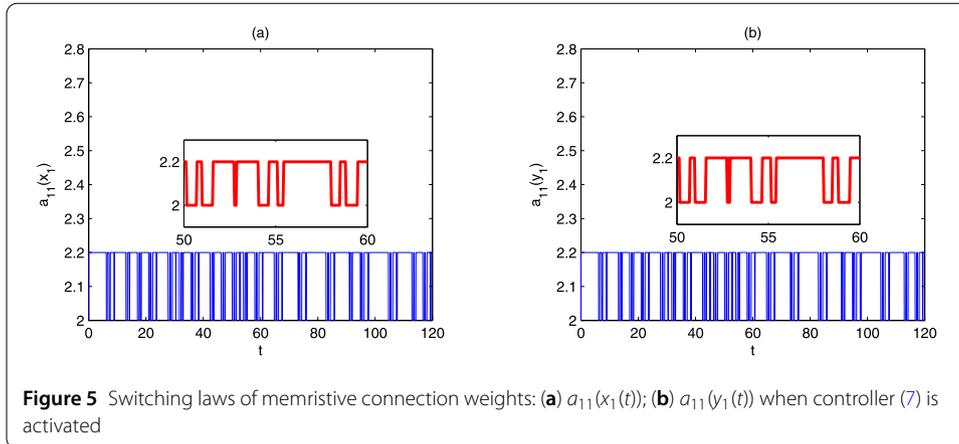
Taking $k_1 = 5.56$, $k_2 = 7.62$, $k_3 = 2.61$, $\epsilon_1 = 6.1$, $\epsilon_2 = 8.3$, $\epsilon_3 = 7.2$, $\rho_1 = 1.1$, $\rho_2 = 0.88$, $\rho_3 = 1.1$, by simple calculation, we can obtain $\lambda = \min_{1 \le i \le 3} \{c_i + k_i - \sum_{j=1}^{3} a_{ji}^u L_i\} = 0.01 > 0$, $\varepsilon_i - \sum_{j=1}^{3} b_{ji}^u L_i = 0$, $\rho_i - \sum_{j=1}^{3}(|a_{ij}^* - a_{ij}^{**}| + |b_{ij}^* - b_{ij}^{**}|)M_j = 0$. It means that the conditions of Theorem 1 are satisfied. According to Theorem 1, we can obtain the following inequality:

$$\|e(t)\| \le \|e(0)\| E_{0.9}(-0.01t^{0.9}), \quad t \ge 0.$$

Then the error system (8) is globally Mittag-Leffler stable, which is verified by Fig. 4. From Fig. 5, the switching laws of $a_{11}(x_1)$, $a_{11}(y_1)$, under the designed controller (7), become identical as time evolves. Choose six random initial values, Fig. 6 depicts the time evolution of synchronization error $e_1(t)$, $e_2(t)$, $e_3(t)$. From Figs. 4–6, one can find that the drive–response systems (1)–(2) can achieve globally Mittag-Leffler synchronization under controller (7). The numerical simulations verify the effectiveness of the proposed method.

5 Conclusions

In this paper, the global Mittag-Leffler synchronization of delayed FMNNs is investigated. A linear state feedback control combined with switching control law is designed, which is simple and easy to implement. Two Mittag-Leffler synchronization criteria are developed by constructing suitable Lyapunov functionals. It is the first time that the global Mittag-

Figure 5 Switching laws of memristive connection weights: (**a**) $a_{11}(x_1(t))$; (**b**) $a_{11}(y_1(t))$ when controller (7) is activated

Figure 6 Time evolution of synchronization errors $e_1(t)$, $e_2(t)$, $e_3(t)$ with six random initial values when controller (7) is activated

Leffler synchronization is discussed under the proposed controller. The effectiveness of the developed method has been demonstrated theoretically and numerically.

Availability of data and materials
Not applicable.

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