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Singularity problems to fourth-order Rayleigh equation with time-dependent deviating argument

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Abstract

The paper is devoted to an investigation of the existence of a positive periodic solution for a kind of fourth-order singular Rayleigh equation with time-dependent deviating argument, where the nonlinear term g satisfies singularities of attractive and repulsive type at the origin and has time-dependent deviating argument. By applications of coincidence degree theory, we prove that this equation has at least one positive periodic solution. At last, two examples are given to show applications of theorems.

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Keywords: Positive periodic solution; Fourth-order Rayleigh equation; Singularities of attractive and repulsive type; Time-dependent deviating argument

1 Introduction

In this paper, we consider the following fourth-order p -Laplacian singular Rayleigh equation with time-dependent deviating argument:

$$(\phi_p(x''(t)))'' + f(t, x'(t)) + g(t, x(t - \delta(t))) = e(t), \quad (1.1)$$

where $\phi_p : \mathbb{R} \rightarrow \mathbb{R}$ is given by $\phi_p(s) = |s|^{p-2}s$, and $p > 1$ is a constant; $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous T -periodic function about t and $f(t, 0) = 0$; $e : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous periodic functions with $e(t + T) \equiv e(t)$; $\delta \in C^1(\mathbb{R}, \mathbb{R})$ is a T -periodic function; $g : \mathbb{R} \times (0, +\infty) \rightarrow \mathbb{R}$ is a L^2 -Carathéodory function defined on \mathbb{R}^2 , and $g(t, \cdot) = g(t + T, \cdot)$; it is said that Eq. (1.1) is singularity of attractive type (resp. repulsive type) if $g(t, x) \rightarrow +\infty$ (resp. $g(t, x) \rightarrow -\infty$) as $x \rightarrow 0^+$ for $t \in \mathbb{R}$.

As is well known, the Rayleigh equation can be derived from many fields, such as engineering technique, physics and mechanics fields, and an important question is whether we have periodic solutions to the Rayleigh equation. Gaines and Mawhin [1] in 1977 introduced continuation theorems and applied this theorem to prove the existence of a periodic solutions for the Rayleigh equation ([1], p. 99)

$$x''(t) + f(x'(t)) + g(t, x(t)) = 0. \quad (1.2)$$

Gaines and Mawhin’s work has attracted the attention of many scholars to the Rayleigh equation. More recently, the method of lower and upper solutions [2], Mawhin’s continuous theorem [3–5], topological degree and time maps [6, 7], and the Manásevich–Mawhin continuation theorem [8–10] have been employed to investigate the existence of a periodic solution of Rayleigh equations.

On the other hand, the existence of a periodic solutions for a singular differential equation was extensively studied (see [11–20]). Among these, there are some results on the Liénard equation with a singularity of repulsive type [14, 15, 19, 20]. Zhang [19] in 1996 introduced the problem of periodic solutions of the following Liénard equation with a repulsive singularity:

$$x''(t) + f(x(t))x'(t) + g(t, x(t)) = 0, \tag{1.3}$$

where the nonlinear term g has a singularity of repulsive type at the origin and satisfies semilinear condition at $x = \infty$, the author proved that Eq. (1.3) has at least one positive T -periodic solution by using Mawhin’s continuous theorem. Wang [20] in 2014 improved Eq. (1.3) and proved the existence of a positive periodic solution of the following Liénard equation with a singularity of repulsive type and a deviating argument:

$$x''(t) + f(x(t))x'(t) + g(t, x(t - \delta)) = 0, \tag{1.4}$$

where δ is a constant and $\delta \in [0, T]$.

Nowadays, a good deal of work has been performed on the existence of a positive periodic solution of the Rayleigh equation with a singularity [21–24]. Wang and Ma [24] in 2015 discussed a kind of singular Rayleigh equation as follows:

$$x''(t) + f(t, x'(t)) + g(x(t)) = p(t), \tag{1.5}$$

where the nonlinear term g has a singularity of repulsive type at the origin and satisfies semilinear condition. The authors obtained the existence of a positive periodic solution for Eq. (1.5) by applications of the limit properties of time map. Lu and Chen [21] in 2017 studied the following Rayleigh equation with singularity of repulsive type:

$$x'' + f(t, x'(t)) + \varphi(t)x(t) - \frac{1}{x^\gamma(t)} = p(t), \tag{1.6}$$

where $\gamma \geq 1$. By using topological degree theory, the authors proved Eq. (1.6) has at least one positive T -periodic solution.

Inspired by the above paper [1, 19–21, 24], in this paper, we further consider the existence of a positive T -periodic solution for Eq. (1.1) with singularities of attractive and repulsive type. By applications of coincidence degree theory, we obtain the following conclusions.

Theorem 1.1 *Assume that the following conditions hold:*

(H_1) *There exists a positive constant N such that*

$$|f(t, u)| \leq N, \quad \text{for } (t, u) \in [0, T] \times \mathbb{R}.$$

(H₂) There exist two positive constants D_1, D_2 with $D_1 < D_2$ such that $g(t, x) - e(t) < -N$ for all $(t, x) \in [0, T] \times (0, D_1)$, and $g(t, x) - e(t) > N$ for all $(t, x) \in [0, T] \times (D_2, +\infty)$.

(H₃) There exist positive constants a, b such that

$$g(t, x) \leq ax^{p-1} + b, \quad \text{for all } (t, x) \in [0, T] \times (0, +\infty).$$

(H₄) $g(t, x) = g_0(x) + g_1(t, x)$, where $g_0 \in C((0, \infty); \mathbb{R})$ and $g_1 : [0, T] \times [0, \infty) \rightarrow \mathbb{R}$ is an L^2 -Carathéodory function.

(H₅) (Singularity of repulsive type)

$$\int_0^1 g_0(x) dx = -\infty.$$

Then Eq. (1.1) has at least one positive T -periodic solution if

$$0 < \frac{aT}{1 - \delta'} \left(\frac{T}{\pi_p} \right)^{2p-1} < 1,$$

where $\delta' =: \max_{t \in [0, T]} |\delta(t)|$, and $1 \leq p < \infty, \pi_p = 2 \int_0^{(p-1)/p} \frac{ds}{(1 - \frac{s^p}{p-1})^{1/p}} = \frac{2\pi(p-1)^{1/p}}{p \sin(\pi/p)}$.

Remark 1.2 The friction term $f(x(t))x'(t)$ in Eqs. (1.3) and (1.4) satisfy $\int_0^T f(x(t))x'(t) dt = 0$, which is crucial to estimate *a priori bounds* of a positive T -periodic solution for these equations. However, in this paper, the friction term $f(t, x')$ may not satisfy $\int_0^T f(t, x'(t)) dt = 0$. For example, let

$$f(t, x') = (\cos^2(2t) + 100) \sin x'(t).$$

Obviously, $\int_0^T (\cos^2(2t) + 100) \sin x'(t) dt \neq 0$. This implies that our methods to estimate *a priori bounds* of a positive T -periodic solution for Eq. (1.1) is more complex than Eqs. (1.3) and (1.4).

Remark 1.3 From Eqs. (1.3), (1.4), (1.5), (1.6) in [19–21, 24], the nonlinear term g has a deviating argument (i.e., δ is a positive constant and $0 \leq \delta < T$). But, in this paper, the nonlinear term g satisfies a time-dependent deviating argument. For example, let

$$\delta(t) = \frac{1}{4} \sin 2t.$$

Obviously, the work on estimating *a lower bounds* of a positive T -periodic solution for Eq. (1.1) is more difficult than the corresponding work on Eqs. (1.3), (1.4), (1.5), (1.6). Therefore, we have to find another way to get over the difficulty.

Remark 1.4 If Eq. (1.1) satisfies singularity of attractive type, i.e., $\int_0^1 g_0(x) dx = +\infty$. Obviously, attractive condition and (H₂), (H₃), (H₅) are contradiction. Therefore, the above method and conditions are no long applicable to the proof of existence of a positive periodic solution for Eq. (1.1) with singularity of attractive type. Next, we give other conditions to prove the existence of a positive T -periodic solution for Eq. (1.1) with singularity of attractive type.

Theorem 1.5 Assume that conditions (H_1) and (H_4) hold. Suppose the following conditions are satisfied:

- (H_6) There exist two positive constants D_3, D_4 with $D_3 < D_4$ such that $g(t, x) - e(t) > N$ for all $(t, x) \in [0, T] \times (0, D_3)$, and $g(t, x) - e(t) < -N$ for all $(t, x) \in [0, T] \times (D_4, +\infty)$.
- (H_7) There exist positive constants a', b' such that

$$-g(t, x) \leq a'x^{p-1} + b', \quad \text{for all } (t, x) \in [0, T] \times (0, +\infty).$$

(H_8) (Singularity of attractive type)

$$\int_0^1 g_0(x) dx = +\infty.$$

Then Eq. (1.1) has at least one positive T -periodic solution if

$$0 < \frac{a'T}{1 - \delta'} \left(\frac{T}{\pi_p}\right)^{2p-1} < 1.$$

2 Preparation

Lemma 2.1 (Gaines and Mawhin [1]) Suppose that X and Y are two Banach spaces, and $L : D(L) \subset X \rightarrow Y$ is a Fredholm operator with index zero. Let $\Omega \subset X$ be an open bounded set and $N : \overline{\Omega} \rightarrow Y$ be L -compact on $\overline{\Omega}$. Assume that the following conditions hold:

- (1) $Lx \neq \lambda Nx, \forall x \in \partial\Omega \cap D(L), \lambda \in (0, 1)$;
- (2) $Nx \notin \text{Im } L, \forall x \in \partial\Omega \cap \text{Ker } L$;
- (3) $\text{deg}\{JQN, \Omega \cap \text{Ker } L, 0\} \neq 0$, where $J : \text{Im } Q \rightarrow \text{Ker } L$ is an isomorphism.

Then the equation $Lx = Nx$ has a solution in $\overline{\Omega} \cap D(L)$.

Lemma 2.2 ([25]) If $\omega \in C^1(\mathbb{R}, \mathbb{R})$ and $\omega(0) = \omega(T) = 0$, then

$$\left(\int_0^T |\omega(t)|^p dt\right)^{\frac{1}{p}} \leq \left(\frac{T}{\pi_p}\right) \left(\int_0^T |\omega'(t)|^p dt\right)^{\frac{1}{p}}.$$

In order to apply topological degree theorem to study the existence of a positive periodic solution for Eq. (1.1), we rewrite Eq. (1.1) in the form:

$$\begin{cases} x_1''(t) = (\phi_q(x_2(t))), \\ x_2''(t) = -f(t, x_1'(t)) - g(t, x_1(t - \delta(t))) + e(t), \end{cases} \tag{2.1}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Clearly, if $x(t) = (x_1(t), x_2(t))^T$ is a T -periodic solution to Eq. (2.1), then $x_1(t)$ must be a T -periodic solution to Eq. (1.1). Thus, the problem of finding a T -periodic solution for Eq. (1.1) reduces to finding one for Eq. (2.1).

Let

$$X := \{x = (x_1(t), x_2(t)) \in C^2(\mathbb{R}, \mathbb{R}^2) : x(t + T) - x(t) \equiv 0\}$$

with the norm $\|x\| := \max\{\|x_1\|, \|x_2\|\}$;

$$Y := \{x = (x_1(t), x_2(t)) \in C^1(\mathbb{R}, \mathbb{R}^2) : x(t + T) - x(t) \equiv 0\}$$

with the norm $\|x\|_\infty := \max\{\|x\|, \|x'\|\}$. Clearly, X and Y are both Banach spaces. Meanwhile, define

$$L : D(L) \subset X \rightarrow Y, \quad \text{by } (Lx)(t) = \begin{pmatrix} x_1''(t) \\ x_2''(t) \end{pmatrix}, \tag{2.2}$$

where $D(L) = \{x = (x_1, x_2)^\top \in C^2(\mathbb{R}, \mathbb{R}^2) : x(t + T) - x(t) \equiv 0, t \in \mathbb{R}\}$. Define a nonlinear operator $N : X \rightarrow Y$ as follows:

$$(Nx)(t) = \begin{pmatrix} \phi_q(x_2(t)) \\ -f(t, x_1'(t)) - g(t, x_1(t - \delta(t))) + e(t) \end{pmatrix}. \tag{2.3}$$

Then Eq. (2.1) can be converted to the abstract equation $Lx = Nx$.

From the definition of L , one can easily see that

$$\text{Ker } L \cong \mathbb{R}^2, \quad \text{Im } L = \left\{ y \in Y : \int_0^T \begin{pmatrix} y_1(s) \\ y_2(s) \end{pmatrix} ds = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.$$

So L is a Fredholm operator with index zero. Let $P : X \rightarrow \text{Ker } L$ and $Q : Y \rightarrow \text{Im } Q \subset \mathbb{R}^2$ be defined by

$$Px := \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}; \quad Qy := \frac{1}{T} \int_0^T \begin{pmatrix} y_1(s) \\ y_2(s) \end{pmatrix} ds,$$

then $\text{Im } P = \text{Ker } L$, $\text{Ker } Q = \text{Im } L$. let K denote the inverse of $L|_{\text{Ker } P \cap D(L)}$. It is easy to see that $\text{Ker } L = \text{Im } Q = \mathbb{R}^2$ and

$$[Ky](t) = \text{col} \left(\int_0^T G_1(t, s) y_1(s) ds, \int_0^T G_2(t, s) y_2(s) ds \right),$$

where

$$G_i(t, s) = \begin{cases} \frac{-s(T-t)}{T}, & 0 \leq s \leq t \leq T, \\ \frac{-t(T-s)}{T}, & 0 \leq t < s \leq T, \end{cases} \quad i = 1, 2. \tag{2.4}$$

3 Main results

In the section, we first consider the existence of a positive T -periodic solution for Eq. (1.1) with singularity of repulsive type.

Proof of Theorem 1.1 Consider the operator equation

$$Lx = \lambda Nx, \quad \lambda \in (0, 1),$$

where L and N are defined by Eqs. (2.2) and (2.4). Set

$$\Omega_1 = \{x : Lx = \lambda Nx, \lambda \in (0, 1)\}.$$

If $x(t) = (x_1(t), x_2(t))^T \in \Omega_1$, then

$$\begin{cases} x_1''(t) = \lambda \phi_q(x_2(t)), \\ x_2''(t) = -\lambda f(t, x_1'(t)) - \lambda g(t, x_1(t - \delta(t))) + \lambda e(t). \end{cases} \tag{3.1}$$

Substituting $x_2(t) = \frac{1}{\lambda^{p-1}} (\phi_p(x_1)''(t))$ into the second equation of (3.1)

$$\phi_p(x_1''(t))''(t) + \lambda^p f(t, x_1'(t)) + \lambda^p g(t, x_1(t - \delta(t))) = \lambda^p e(t). \tag{3.2}$$

Integrating both side of Eq. (3.2) over $[0, T]$, we have

$$\int_0^T (f(t, x_1'(t)) + g(t, x_1(t - \delta(t))) - e(t)) dt = 0, \tag{3.3}$$

since $\int_0^T (\phi_p(Ax_1)''(t))'' = 0$. From Eq. (3.3) and condition (H_1) , we deduce

$$-NT \leq \int_0^T (g(t, x_1(t - \delta(t))) - e(t)) dt \leq NT.$$

Then, by condition (H_2) , we know that there are two points $\xi, \eta \in (0, T)$ such that

$$x_1(\xi - \delta(\xi)) \geq D_1, \quad x_1(\eta) \leq D_2. \tag{3.4}$$

Then, from Eq. (3.4), we have

$$\begin{aligned} x(t) &= \frac{1}{2} (x(t) + x(t - T)) \\ &= \frac{1}{2} \left(x(\eta) + \int_{\eta}^t x'(s) ds + x(\eta) - \int_{t-T}^{\eta} x'(s) ds \right) \\ &= x(\eta) + \frac{1}{2} \int_{t-T}^t x'(s) ds \\ &\leq D_2 + \frac{1}{2} \int_0^T |x'(t)| dt. \end{aligned} \tag{3.5}$$

Multiplying both sides of Eq. (3.2) by $x_1(t)$ and integrating over the interval $[0, T]$, we get

$$\begin{aligned} &\int_0^T \phi_p(x_1''(t))'' x_1(t) dt + \lambda^p \int_0^T f(t, x_1'(t)) x_1(t) dt + \lambda^p \int_0^T g(t, x_1(t - \delta(t))) x_1(t) dt \\ &= \lambda^p \int_0^T e(t) x_1(t) dt. \end{aligned} \tag{3.6}$$

Substituting $\int_0^T \phi_p(x_1''(t))'' x_1(t) dt = \int_0^T |x_1''(t)|^p dt$ into Eq. (3.6), we arrive at

$$\begin{aligned} \int_0^T |x_1''(t)|^p dt &= -\lambda^p \int_0^T f(t, x_1'(t)) x_1(t) dt - \lambda^p \int_0^T g(t, x_1(t - \delta(t))) x_1(t) dt \\ &\quad + \lambda^p \int_0^T e(t) x_1(t) dt. \end{aligned}$$

Therefore, from condition (H_1) , we deduce

$$\begin{aligned} \int_0^T |x_1''(t)|^p dt &\leq \int_0^T |f(t, x_1'(t))||x_1(t)| dt + \int_0^T |g(t, x_1(t - \delta(t))||x_1(t)| dt \\ &\quad + \int_0^T |e(t)||x_1(t)| dt \\ &\leq \|x_1\|NT + \|x_1\| \int_0^T |g(t, x_1(t - \delta(t)))| dt + \|e\|T\|x_1\|, \end{aligned} \tag{3.7}$$

where $\|e\| := \max_{t \in [0, T]} |e(t)|$. From conditions (H_1) , (H_2) and Eq. (3.3), we obtain

$$\begin{aligned} &\int_0^T |g(t, x_1(t - \delta(t)))| dt \\ &= \int_{g(t, x_1) \geq 0} g(t, x_1(t - \delta(t))) dt - \int_{g(t, x_1) \leq 0} g(t, x_1(t - \delta(t))) dt \\ &= 2 \int_{g(t, x_1) \geq 0} g(t, x_1(t - \delta(t))) dt + \int_0^T f(t, x_1'(t)) dt - \int_0^T e(t) dt \\ &\leq 2a \int_0^T x_1^{p-1}(t - \delta(t)) dt + 2bT + \int_0^T |f(t, x_1'(t))| dt + \int_0^T |e(t)| dt \\ &= 2a \int_0^T |x_1(t - \delta(t))|^{p-1} dt + 2bT + NT + \|e\|T \\ &\leq \frac{2a}{1 - \delta'} \int_0^T |x_1(t)|^{p-1} dt + 2bT + NT + \|e\|T, \end{aligned} \tag{3.8}$$

from $\frac{aT}{1 - \delta'} (\frac{T}{\pi p})^{2p-1} > 0$, we know that $\delta' < 1$. Substituting Eqs. (3.5) and (3.8) into (3.7), and by using the Hölder inequality, we see that

$$\begin{aligned} \int_0^T |x_1''(t)|^p dt &\leq \frac{2a}{1 - \delta'} \|x_1\| \int_0^T |x_1(t)|^p dt + 2T(N + b + \|e\|)\|x_1\| \\ &\leq \frac{2a}{1 - \delta'} \left(D_2 + \frac{1}{2} \int_0^T |x_1'(t)| dt \right) \int_0^T |x_1(t)|^{p-1} dt \\ &\quad + F_1 \left(D_2 + \frac{1}{2} \int_0^T |x_1'(t)| dt \right) \\ &\leq \frac{aT}{1 - \delta'} \left(\int_0^T |x_1'(t)|^p dt \right)^{\frac{1}{p}} \left(\int_0^T |x_1(t)|^p dt \right)^{\frac{p-1}{p}} \\ &\quad + \frac{2aD_2 T^{\frac{1}{p}}}{1 - \delta'} \left(\int_0^T |x_1(t)|^p dt \right)^{\frac{p-1}{p}} \\ &\quad + \frac{F_1 T^{\frac{1}{q}}}{2} \left(\int_0^T |x_1'(t)|^p dt \right)^{\frac{1}{p}} + F_1 D_2, \end{aligned} \tag{3.9}$$

where $F_1 := 2(N + b + \|e\|)T$.

Let $\omega(t) = x_1(t + \eta) - x_1(\eta)$, here $x_1(\eta) \leq D_2$, and then $\omega(0) = \omega(T) = 0$. By Lemma 2.2 and the Minkowski inequality [26], it is clear that

$$\begin{aligned} \left(\int_0^T |x_1(t)|^p dt\right)^{\frac{1}{p}} &= \left(\int_0^T |\omega(t) + x_1(\xi)|^p dt\right)^{\frac{1}{p}} \\ &\leq \left(\int_0^T |\omega(t)|^p dt\right)^{\frac{1}{p}} + \left(\int_0^T |x_1(\eta)|^p dt\right)^{\frac{1}{p}} \\ &\leq \left(\frac{T}{\pi_p}\right) \left(\int_0^T |\omega'(t)|^p dt\right)^{\frac{1}{p}} + D_2 T^{\frac{1}{p}} \\ &= \left(\frac{T}{\pi_p}\right) \left(\int_0^T |x_1'(t)|^p dt\right)^{\frac{1}{p}} + D_2 T^{\frac{1}{p}}. \end{aligned} \tag{3.10}$$

On the other hand, in view of $x_1(0) = x_1(T)$, there exists a point $t_1 \in (0, T)$ such that $x_1'(t_1) = 0$. Let $\omega_*(t) = x_1'(t + t_1)$, it is easy to see that $\omega_*(0) = \omega_*(T) = 0$ for $\omega_* \in C^1(\mathbb{R}, \mathbb{R})$. By Lemma 2.2,

$$\begin{aligned} \left(\int_0^T |x_1'(t)|^p dt\right)^{\frac{1}{p}} &= \left(\int_0^T |\omega_*(t)|^p dt\right)^{\frac{1}{p}} \leq \left(\frac{T}{\pi_p}\right) \left(\int_0^T |\omega_*'(t)|^p dt\right)^{\frac{1}{p}} \\ &= \left(\frac{T}{\pi_p}\right) \left(\int_0^T |x_1''(t)|^p dt\right)^{\frac{1}{p}}. \end{aligned} \tag{3.11}$$

Substituting Eqs. (3.11) into (3.10), we obtain

$$\left(\int_0^T |x_1(t)|^p dt\right)^{\frac{1}{p}} \leq \left(\frac{T}{\pi_p}\right)^2 \left(\int_0^T |x_1''(t)|^p dt\right)^{\frac{1}{p}} + D_2 T^{\frac{1}{p}}. \tag{3.12}$$

Furthermore, substituting Eqs. (3.11) and (3.12) into (3.9), we see that

$$\begin{aligned} &\int_0^T |(x_1)''(t)|^p dt \\ &\leq \frac{aT}{1 - \delta'} \left(\frac{T}{\pi_p}\right) \left(\int_0^T |x_1''(t)|^p dt\right)^{\frac{1}{p}} \left(\left(\frac{T}{\pi_p}\right)^2 \left(\int_0^T |x_1''(t)|^p dt\right)^{\frac{1}{p}} + D_2 T^{\frac{1}{p}}\right)^{p-1} \\ &\quad + \frac{2aD_2 T^{\frac{1}{p}}}{1 - \delta'} \left(\left(\frac{T}{\pi_p}\right)^2 \left(\int_0^T |x_1''(t)|^p dt\right)^{\frac{1}{p}} + D_2 T^{\frac{1}{p}}\right)^{p-1} \\ &\quad + \frac{F_1 T^{\frac{1}{q}}}{2} \left(\frac{T}{\pi_p}\right) \left(\int_0^T |x_1''(t)|^p dt\right)^{\frac{1}{p}} + F_1 D_2 \\ &= \frac{aT}{1 - \delta'} \left(\frac{T}{\pi_p}\right)^{2p-1} \left(1 + \frac{D_2 T^{\frac{1}{p}}}{\left(\frac{T}{\pi_p}\right)^2 \left(\int_0^T |x_1''(t)|^p dt\right)^{\frac{1}{p}}}\right)^{p-1} \int_0^T |x_1''(t)|^p dt \\ &\quad + \frac{2aD_2 T^{\frac{1}{p}}}{1 - \delta'} \left(\frac{T}{\pi_p}\right)^{2p-2} \left(1 + \frac{D_2 T^{\frac{1}{p}}}{\left(\frac{T}{\pi_p}\right)^2 \left(\int_0^T |x_1''(t)|^p dt\right)^{\frac{1}{p}}}\right)^{p-1} \left(\int_0^T |x_1''(t)|^p dt\right)^{\frac{p-1}{p}} \\ &\quad + \frac{F_1 T^{\frac{1}{q}}}{2} \left(\frac{T}{\pi_p}\right) \left(\int_0^T |x_1''(t)|^p dt\right)^{\frac{1}{p}} + F_1 D_2. \end{aligned} \tag{3.13}$$

Next, we introduce a classical inequality, there exists a $k(p) > 0$ which is dependent on p only,

$$(1 + x)^p \leq 1 + (1 + p)x, \quad \text{for } x \in [0, k(p)]. \tag{3.14}$$

Then, we consider the following two cases:

Case 1. If $\frac{D_2 T^{\frac{1}{p}}}{(\frac{T}{\pi_p})^2 (\int_0^T |x_1''(t)|^p dt)^{\frac{1}{p}}} > k(p)$, then it is obvious that

$$\left(\int_0^T |x_1''(t)|^p dt \right)^{\frac{1}{p}} < \frac{D_2 T^{\frac{1}{p}}}{k(p)} \left(\frac{T}{\pi_p} \right)^{-2}.$$

From Eqs. (3.5) and (3.11), by using the Hölder inequality, we deduce

$$\begin{aligned} x(t) &\leq D_2 + \frac{1}{2} T^{\frac{1}{q}} \left(\int_0^T |x'(t)|^p dt \right)^{\frac{1}{p}} \\ &\leq D_2 + \frac{1}{2} T^{\frac{1}{q}} \left(\frac{T}{\pi_p} \right) \left(\int_0^T |x''(t)|^p dt \right)^{\frac{1}{p}} \\ &\leq D_2 + \frac{TD_2}{2k(p)} \left(\frac{T}{\pi_p} \right)^{-1} := M_1. \end{aligned} \tag{3.15}$$

Case 2. If $\frac{D_2 T^{\frac{1}{p}}}{(\frac{T}{\pi_p})^2 (\int_0^T |x_1''(t)|^p dt)^{\frac{1}{p}}} < k(p)$, from Eqs. (3.13) and (3.14), we obtain

$$\begin{aligned} &\int_0^T |(x_1)''(t)|^p dt \\ &\leq \frac{aT}{1 - \delta'} \left(\frac{T}{\pi_p} \right)^{2p-1} \left(1 + \frac{D_2 T^{\frac{1}{p}} p}{(\frac{T}{\pi_p})^2 (\int_0^T |x_1''(t)|^p dt)^{\frac{1}{p}}} \right) \int_0^T |x_1''(t)|^p dt \\ &\quad + \frac{2aD_2 T^{\frac{1}{p}}}{1 - \delta'} \left(\frac{T}{\pi_p} \right)^{2p-2} \left(1 + \frac{D_2 T^{\frac{1}{p}} p}{(\frac{T}{\pi_p})^2 (\int_0^T |x_1''(t)|^p dt)^{\frac{1}{p}}} \right) \left(\int_0^T |x_1''(t)|^p dt \right)^{\frac{p-1}{p}} \\ &\quad + \frac{F_1 T^{\frac{1}{q}}}{2} \left(\frac{T}{\pi_p} \right) \left(\int_0^T |x_1''(t)|^p dt \right)^{\frac{1}{p}} + F_1 D_2 \\ &= \frac{aT}{1 - \delta'} \left(\frac{T}{\pi_p} \right)^{2p-1} \int_0^T |x_1''(t)|^p dt + \frac{aD_2 T^{1+\frac{1}{p}}}{1 - \delta'} \left(Tp + 2 \left(\frac{T}{\pi} \right) \right) \left(\frac{T}{\pi_p} \right)^{2p-2} \\ &\quad \cdot \left(\int_0^T |x_1''(t)|^p dt \right)^{\frac{p-1}{p}} + \frac{2aD_2^2 T^{\frac{2}{p}} p}{1 - \delta'} \left(\frac{T}{\pi_p} \right)^{2p-4} \left(\int_0^T |x_1''(t)|^p dt \right)^{\frac{p-2}{p}} \\ &\quad + \frac{F_1 T^{\frac{1}{q}}}{2} \left(\frac{T}{\pi_p} \right) \left(\int_0^T |x_1''(t)|^p dt \right)^{\frac{1}{p}} + F_1 D_2. \end{aligned} \tag{3.16}$$

Since

$$\frac{aT}{1 - \delta'} \left(\frac{T}{\pi_p} \right)^{2p-1} < 1,$$

it is easy to see that there exists a positive constant M'_1 such that

$$\int_0^T |x''_1(t)|^p dt \leq M'_1.$$

From Eq. (3.5) and Lemma 2.2, we have

$$\begin{aligned} x_1(t) &\leq D_2 + \frac{1}{2} \int_0^T |x'_1(t)| dt \\ &\leq D_2 + \frac{1}{2} T^{\frac{1}{q}} \left(\frac{T}{\pi_p} \right) \left(\int_0^T |x''_1(t)|^p dt \right)^{\frac{1}{p}} \\ &\leq D_2 + \frac{1}{2} T^{\frac{1}{q}} \left(\frac{T}{\pi_p} \right) (M'_1)^{\frac{1}{p}} := M_1. \end{aligned} \tag{3.17}$$

From Eq. (3.5), we have

$$\begin{aligned} \|x'_1\| &\leq x'_1(t_1) + \frac{1}{2} \int_0^T |x''_1(t)| dt \\ &\leq \frac{1}{2} T^{\frac{1}{q}} \left(\int_0^T |x''_1(t)| dt \right)^{\frac{1}{p}} \\ &\leq \frac{1}{2} T^{\frac{1}{q}} M_1^{\frac{1}{p}} := M_2, \end{aligned} \tag{3.18}$$

since $x'_1(t_1) = 0$. From $x_2(0) = x_2(T)$, we know that there exists a point $t_2 \in (0, T)$ such that $x'_2(t_2) = 0$. From the second equation of (3.1), Eqs. (3.8), (3.17), (3.18) and condition (H_1) , we obtain

$$\begin{aligned} \|x'_2\| &\leq \frac{1}{2} \int_0^T |x''_2(t)| dt \\ &\leq \frac{\lambda}{2} \left(\int_0^T |f(t, x_1(t))| dt + \int_0^T |g(t, x_1(t))| dt + \int_0^T |e(t)| dt \right) \\ &\leq \lambda (NT + aTM_1^{p-1} + bT + T\|e\|) := \lambda M_3. \end{aligned} \tag{3.19}$$

Integrating the first equation of (3.1) over $[0, T]$, we have $\int_0^T x_2(t) dt = \int_0^T \phi_p(x'_1(t)) dt = 0$, which implies there is a point $t_3 \in (0, T)$ such that $x_2(t_3) = 0$, so

$$\|x_2\| \leq \frac{1}{2} \int_0^T |x'_2(t)| dt \leq \lambda TM_3 := \lambda M_4. \tag{3.20}$$

On the other hand, it follows from Eq. (3.2) and condition (H_4) that

$$\begin{aligned} &(\phi_p(x'_1(t)))'' + \lambda^p f(t, x_1(t))x'_1(t) + \lambda^p (g_0(x_1(t - \delta(t))) + g_1(t, x_1(t - \delta(t)))) \\ &= \lambda^p e(t). \end{aligned} \tag{3.21}$$

Let $\xi \in [0, T]$ be as in Eq. (3.4), for any $t \in [\xi, T]$. Multiplying both sides of Eq. (3.21) by $x'_1(t - \delta(t))(1 - \delta'(t))$ and integrating on $[\xi, t]$ we get

$$\begin{aligned} & \lambda^p \int_{x_1(\xi - \delta(\xi))}^{x_1(t - \delta(t))} g_0(u) \, du \\ &= \lambda^p \int_{\xi}^t g_0(x_1(s - \delta(s))) x'_1(s - \delta(s)) (1 - \delta'(s)) \, ds \\ &= - \int_{\xi}^t (\phi_p(x''_1(s)))'' x'_1(s - \delta(s)) (1 - \delta'(s)) \, ds \\ &\quad - \lambda^p \int_{\xi}^t f(s, x'_1(s)) x'_1(s - \delta(s)) (1 - \delta'(s)) \, ds \\ &\quad - \lambda^p \int_{\xi}^t g_1(s, x_1(s - \delta(s))) x'_1(s - \delta(s)) (1 - \delta'(s)) \, ds \\ &\quad + \lambda^p \int_{\xi}^t e(s) x'_1(s - \delta(s)) (1 - \delta'(s)) \, ds. \end{aligned} \tag{3.22}$$

Furthermore, by Eqs. (3.16), (3.17), (3.18) and (3.19), we have

$$\begin{aligned} & \lambda^p \left| \int_{x_1(\xi - \delta(\xi))}^{x_1(t - \delta(t))} g_0(u) \, du \right| \\ &= \left| - \int_{\xi}^t (\phi_p(x''_1(s)))'' x'_1(s - \delta(s)) (1 - \delta'(s)) \, ds \right. \\ &\quad - \lambda^p \int_{\xi}^t f(s, x'_1(s)) x'_1(s - \delta(s)) (1 - \delta'(s)) \, ds \\ &\quad - \lambda^p \int_{\xi}^t g_1(s, x_1(s - \delta(s))) x'_1(s - \delta(s)) (1 - \delta'(s)) \, ds \\ &\quad \left. + \lambda^p \int_{\xi}^t e(s) x'_1(s - \delta(s)) (1 - \delta'(s)) \, ds \right| \\ &\leq (1 + \delta') \|x'_1\| \int_0^T |\phi_p(x''_1(s))| \, ds + \lambda^p (1 + \delta') \|x'_1\| \int_0^T |f(s, x'_1(s))| \, ds \\ &\quad + \lambda^p (1 + \delta') \|x'_1\| \int_0^T |g_1(s, x_1(s - \delta(s)))| \, ds + \lambda^p (1 + \delta') \|x'_1\| \int_0^T |e(s)| \, ds \\ &\leq \lambda^p (1 + \delta') M_2 \left(\int_0^T |f(s, x'(s))| \, ds + \int_0^T |g(s, x_1(s))| \, ds + \int_0^T |e(s)| \, ds \right) \\ &\quad + \lambda^p (1 + \delta') (M_2 N T + M_2 \|g_{M_1}\| T + M_2 \|e\| T) \\ &\leq \lambda^p (1 + \delta') M_2 (M_3 + N T + \|g_{M_1}\| T + \|e\| T), \end{aligned} \tag{3.23}$$

where $g_{M_1} = \max_{0 \leq x \leq M_1} |g_1(t, x)| \in L^2(0, T)$ are as in condition (H_4) . According to the singular condition (H_5) , we know that there exists a positive constant M_5 such that

$$x_1(t - \delta(t)) \geq M_5, \quad \forall t \in [\xi, T]. \tag{3.24}$$

The case $t \in [0, \xi]$ (i.e., $x_1(t - \delta(t)) \in [-\delta(0), \xi - \delta(\xi)]$) can be treated similarly.

From Eqs. (3.16), (3.17), (3.18), (3.19) and (3.24), we let

$$\Omega = \{x = (x_1, x_2)^\top : E_1 \leq x_1(t) \leq E_2, \|x'_1\| \leq E_3, \|x_2\| \leq E_4 \text{ and } \|x'_2\| \leq E_5, \forall t \in [0, T]\},$$

where $0 < E_1 < \min(M_5, D_1), E_2 > \max(M_1, D_2), E_3 > M_2, E_4 > M_4$ and $E_5 > M_3$. $\Omega_2 = \{x : x \in \partial\Omega \cap \text{Ker } L\}$ then $\forall x \in \partial\Omega \cap \text{Ker } L$

$$QNx = \frac{1}{T} \int_0^T \begin{pmatrix} \phi_q(x_2(t)) \\ -f(t, x'_1) - g(t, x_1) + e(t) \end{pmatrix} dt.$$

If $QNx = 0$, then $x_2(t) = 0, x_1 = E_2$ or E_1 . But if $x_1(t) = E_2$, we know

$$0 = \int_0^T (g(t, E_2) - e(t)) dt.$$

From condition (H_2) , we have $x_1(t) \leq D_2 \leq E_2$, which yields a contradiction. Similarly if $x_1 = E_1$. We also have $QNx \neq 0$, i.e., $\forall x \in \partial\Omega \cap \text{Ker } L, x \notin \text{Im } L$, so assumptions (1) and (2) of Lemma 2.1 are both satisfied. Define the isomorphism $J : \text{Im } Q \rightarrow \text{Ker } L$ as follows:

$$J(x_1, x_2)^\top = (x_2, -x_1)^\top.$$

Let $H(\mu, x) = -\mu x + (1 - \mu)JQNx, (\mu, x) \in [0, 1] \times \Omega$, then $\forall (\mu, x) \in (0, 1) \times (\partial\Omega \cap \text{Ker } L)$,

$$H(\mu, x) = \begin{pmatrix} -\mu x_1 - \frac{1-\mu}{T} \int_0^T (g(t, x_1) - e(t)) dt \\ -\mu x_2 - (1 - \mu)\phi_q(x_2) \end{pmatrix}.$$

From condition (H_2) , we get $x^\top H(\mu, x) \neq 0, \forall (\mu, x) \in (0, 1) \times (\partial\Omega \cap \text{Ker } L)$. Hence

$$\begin{aligned} \deg\{JQN, \Omega \cap \text{Ker } L, 0\} &= \deg\{H(0, x), \Omega \cap \text{Ker } L, 0\} \\ &= \deg\{H(1, x), \Omega \cap \text{Ker } L, 0\} \\ &= \deg\{I, \Omega \cap \text{Ker } L, 0\} \neq 0. \end{aligned}$$

So assumption (3) of Lemma 2.1 is satisfied. By applying Lemma 2.1, we conclude that the equation $Lx = Nx$ has a solution $x = (x_1, x_2)^\top$ on $\bar{\Omega} \cap D(L)$, i.e., Eq. (2.1) has a T -periodic solution $x_1(t)$. □

Next, we investigate the existence of a positive T -periodic solution for Eq. (1.1) with a singularity of attractive type.

Proof of Theorem 1.5 We follow the same strategy and notation as in the proof of Theorem 1.1. From Eq. (3.3) and condition (H_6) , we know that there are points $\tau, \nu \in (0, T)$ such that

$$x_1(\tau - \delta(\tau)) \geq D_3, \quad x_1(\nu) \leq D_4. \tag{3.25}$$

Next, we consider $\int_0^T |g(t, x_1(t - \delta(t)))| dt$. From Eq. (3.11) and conditions (H_1) , (H_7) , we obtain

$$\begin{aligned} \int_0^T |g(t, x_1(t - \delta(t)))| dt &= \int_{g(t, x_1) \geq 0} g(t, x_1(t - \delta(t))) dt \\ &\quad - \int_{g(t, x_1) \leq 0} g(t, x_1(t - \delta(t))) dt \\ &= -2 \int_{g(t, x_1) \leq 0} g(t, x_1(t - \delta(t))) dt \\ &\quad - \int_0^T f(t, x_1'(t)) dt + \int_0^T e(t) dt \\ &\leq 2a' \int_0^T x_1^{p-1}(t - \delta(t)) dt + 2b'T \\ &\quad + \int_0^T |f(t, x_1'(t))| dt + \int_0^T |e(t)| dt \\ &= 2a' \int_0^T |x_1(t - \delta(t))|^{p-1} dt + 2b'T + NT + \|e\|T \\ &\leq \frac{2a'}{1 - \delta'} \int_0^T |x_1(t)|^{p-1} dt + 2b'T + NT + \|e\|T. \end{aligned}$$

The proof further is the same as Theorem 1.1. □

4 Examples

In this section, we present two examples to illustrate Theorems 1.1 and 1.5.

Example 4.1 Consider the fourth-order Rayleigh equation with singularity of repulsive type and time-dependent deviating argument:

$$\begin{aligned} &(\phi_p(x''(t)))'' + (\sin^2 t + 5) \cos x'(t) + \frac{1}{20\pi} (\sin 2t + 3)x \left(t - \frac{1}{4} \sin 2t \right) - \frac{1}{x^\mu (t - \frac{1}{4} \sin 2t)} \\ &= e^{\cos 2t}, \end{aligned} \tag{4.1}$$

where $\mu \geq 1$ and $p = 4$.

It is clear that $T = \pi$, $f(t, x') = (\sin^2 t + 5) \cos x'(t)$, $g(t, x) = \frac{1}{20\pi} (\sin 2t + 3)x - \frac{1}{x^\mu}$, $\delta(t) = \frac{1}{4} \sin 2t$, $a = \frac{1}{5\pi}$, $\delta' = \frac{1}{2} < 1$, $\pi_4 = \frac{2\pi(p-1)^{\frac{1}{p}}}{p \sin(\pi/p)} = \frac{2\pi(4-1)^{\frac{1}{4}}}{4 \cdot \frac{\sqrt{2}}{2}} = \pi \times (\frac{3}{4})^{\frac{1}{4}}$. It is obviously that (H_1) – (H_5) hold. Now we consider

$$\frac{aT}{1 - \delta'} \left(\frac{T}{\pi_p} \right)^{2p-1} = \frac{2}{5\pi} \times \pi \left(\frac{\pi}{\pi \times (\frac{3}{4})^{\frac{1}{4}}} \right)^7 = \frac{2}{5} \times \left(\frac{4}{3} \right)^{\frac{7}{4}} \approx 0.6618 < 1.$$

Therefore, by Theorem 1.1, we know that Eq. (4.1) has at least one positive π -periodic solution.

Example 4.2 Consider the fourth-order Rayleigh equation with singularity of attractive type:

$$\begin{aligned}
 x^{(4)}(t) - (\cos^2(2t) + 100) \sin x'(t) - \frac{1}{3\pi}(\sin 4t + 5)x\left(t - \frac{1}{8} \cos 4t\right) + \frac{5}{x^{\kappa'}\left(t - \frac{1}{8} \cos 4t\right)} \\
 = e^{\sin^2(2t)},
 \end{aligned}
 \tag{4.2}$$

where $\kappa' \geq 1$ and $p = 2$.

It is clear that $T = \frac{\pi}{2}$, $f(t, x') = -(\cos^2(2t) + 100) \sin x'$, $g(t, x) = -\frac{1}{3\pi}(\sin 4t + 5)x + \frac{5}{x^{\kappa'}}$, $\delta = \frac{1}{8} \cos 4t$, $\delta' = \frac{1}{2}$, $\pi_2 = \frac{2\pi(p-1)^{\frac{1}{p}}}{p \sin(\pi/p)} = \frac{2\pi(2-1)^{\frac{1}{2}}}{4 \times \frac{1}{2}} = \pi$. Take $N = 6$, $a' = \frac{2}{\pi}$, $b' = 1$. It is obvious that (H_1) , (H_4) , (H_6) – (H_8) hold. Now we consider

$$\frac{a'T}{1 - \delta'} \left(\frac{T}{\pi_p}\right)^{2p-1} = \frac{4}{\pi} \times \frac{\pi}{2} \times \left(\frac{1}{2}\right)^3 = \frac{1}{4} < 1.$$

Therefore, applying Theorem 1.5, we know that Eq. (4.2) has at least one positive $\frac{\pi}{2}$ -periodic solution.

5 Conclusions

In this paper, we introduce the existence of a positive T -periodic solution for the fourth-order p -Laplacian singular Rayleigh equation with time-dependent deviating argument. The nonlinear function has a time-dependent deviating argument. This implies that the work on estimating *lower bounds* of periodic solutions for Eq. (1.1) is more difficult than the corresponding work on Eq. (1.4) in [21]. Secondly, attractive conditions (H_6) , (H_7) and (H_8) are in contradiction with the repulsive conditions (H_2) , (H_3) and (H_5) , and the methods of singularity of repulsive type are no longer applicable to the proof of a periodic solution for Eq. (1.1) with singularity of attractive type. In this paper, by using coincidence degree theory and conditions (H_1) – (H_5) , we prove the existence of a positive T -periodic solution for Eq. (1.1) with a singularity of repulsive type; applying conditions (H_1) , (H_4) , (H_6) – (H_8) , we find that Eq. (1.1) with singularity of attractive type has at least one positive T -periodic solution.

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Consent for publication

YX and HML read and approved the final version of the manuscript.

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