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Exact controllability of fractional order evolution equations in Banach spaces

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Abstract

In this paper, we deal with the exact controllability of a class of fractional evolution equations with time-varying delay. Under the nonlocal condition, the exact controllability of this system is established by applying a Leray–Schauder alternative theorem and the theory of propagation families in a Banach space. As an application, the controllability of a fractional partial differential equation is examined to show the effectiveness of our result.

Keywords: Fractional order evolution equation; Caputo fractional derivative; Banach space

1 Introduction and main results

It is well known that the fractional differential equations (FDE) are regarded as a more precise description of real life phenomena. There are many papers in the literature investigating various fractional dynamical systems; see, for example, [1–5]. Controllability of linear and nonlinear fractional dynamical systems, which plays a vital role in various areas of science and engineering, was established in finite dimensional spaces, see [6–15]. In [16] the authors consider the Cauchy problem of a class of semilinear fractional evolution systems in a Banach space. It is noteworthy that a nonlocal Cauchy condition initiated in [17] is related to the diffusion phenomenon of a little amount of gas in a transparent container. Moreover, this nonlocal condition usually covers four classical cases: the initial valued problem, the periodic and antiperiodic problem, and mean valued problems, see [18]. However very little is known about nonlocal problems of fractional control systems. For research on nonlocal problems of evolution equations, we refer the reader to the papers of [19–24] and the references therein.

Fractional differential equations with delay features are present in areas such as physical and medical ones with non-constant delay. Recently, several researchers have been increasingly interested in the issues of controllability results of mild solutions for these problems. Subsequently, a few papers on the existence of fractional order integro-differential equations and impulsive differential equations with delay have been published, see [25–31]. For research on approximate controllability of fractional order systems with delay, we refer the reader to the papers of [32–36] and the references therein. In [37] the authors prove the controllability of fractional functional evolution equations of Sobolev type with constant delay. Some of the works on fractional partial differential equations (PDEs) and on the controllability of fractional PDEs are investigated in [38–42]. As we shall see, the fractional derivative may bring some memory effect which has a great impact on the control properties of these systems. In [41], Lü and Zuazua defined a new notion of controllability which does not only control the value of the state at the final time, but also the memory accumulated by the long-tail effects that the fractional derivative introduces. In this setting, they proved the controllability properties for fractional in time ODEs and PDEs, the analysis of the problem of controllability of fractional (in time) ordinary and partial differential equations is failure. In this sense, for fractional in time derivatives, due to memory effects induced by the integral term, the fact that the solution reaches the final state at time $t = T$ does not guarantee that the solution stays at rest for $t \geq T$ when the control action stops. Consequently, in order to gain the controllability properties for a fractional in time system, we only consider the controllability in the classical control sense, which means that we do not ask for the state of the system to remain for $t \geq T$ without control. Furthermore, the fractional order control system with time-varying delay and the nonlocal condition considered in this paper is different from these problems mentioned above.

Let X and Y be two real Banach spaces with norms $\|\cdot\|$ and $\|\cdot\|_Y$, respectively. The aim of this paper is to discuss the controllability of the following fractional order control system with time-varying delay:

$$\begin{cases} \mathcal{D}_t^{\alpha}(Lx(t)) + Ax(t) = f(t, x(t-\nu(t))) + Bu(t), & t \in I := [0, T], \\ Lx(t) = \phi(t) + b(x)(t), & t \in [-\tau, 0]. \end{cases} \tag{1.1}$$

Here $\mathcal{D}_t^{\alpha}$, $0 < \alpha < 1$, is the regularized Caputo fractional derivative of order α (see Definition 1.2); $A : D(A) \subset X \to Y$ is a linear closed operator; $L : D(L) \subset X \to Y$ is a linear operator where the domain $D(L)$ of L becomes a Banach space with norm $\|x\|_{D(L)} := \|Lx\|$, $x \in D(L)$. Also, the control function $u(\cdot)$ is given in a Banach space $L^{\infty}(I, U)$ where U is also a Banach space; $B : U \to Y$ is a bounded linear operator; the functions $\nu : I \to (0, \tau]$ and $\phi : [-\tau, 0] \to Y$ are continuous; $b : C([-\tau, T], X) \to C([-\tau, 0], Y)$ is a continuous function and $f : I \times C([-\tau, 0], X) \to Y$ is a given function to be specified later. In this paper, we will not assume that the linear operator A generates a compact continuous semigroup (see [43]) and the Lipschitz continuity on f and b which is essential in [44].

Let $J \subset R$ be a compact set, we denote by $C(J, X)$ a Banach space with norm given by $\|y\|_C = \sup_{t \in J} \|y(t)\|$ for $y \in C(J, X)$. Let $\Lambda(X, Y)$ be a Banach space of all bounded linear operators from X into Y endowed with the norm $\|\cdot\|_{\Lambda(X,Y)}$ in the uniform operator topology. The aim of this paper is to establish sufficient conditions for the controllability of system (1.1) via a Leray–Schauder alternative theorem and the theory of propagation families. Also, an example of a fractional feedback control system is discussed to illustrate our theory.

Let us recall some basic definitions and facts which are essential throughout the work. In particular, we introduce some properties of fractional calculus [45], some facts in semigroup theory [2, 4, 46].

Definition 1.1 The fractional integral of order β with the lower limit zero for a function $x:[0,\infty)\to R$ is defined as

$$D_t^{-\beta}x(t)=\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}x(s)\,ds,\quad t>0,\beta>0$$

provided the right-hand side is point-wise defined on $[0,\infty)$, where Γ is the gamma function.

Definition 1.2 The Caputo fractional derivative of order $\beta\in(0,1)$ of a function $x(t)$ is defined by

$$D_t^{\beta}x(t)=\frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}x'(s)\,ds.$$

Lemma 1.1 *If* $x(t)\in C^n[0,T]$, *then*

$$D_t^{-\beta}D_t^{\beta}x(t)=x(t)-\sum_{k=0}^{n-1}\frac{x^k(0)}{k!}t^k,$$

where $\beta\in[n-1,n)$, $n\in N$. *In particular, if* $x(t)\in C^1[0,T]$ *and* $\beta\in(0,1)$, *then* $D_t^{-\beta}D_t^{\beta}x(t)=x(t)-x(0)$.

We need the following assumptions on the operators A and L.

(H1) A is a closed linear operator, L is also a linear operator and bijective, and $D(L)\subset D(A)$.

(H2) The linear operator $L^{-1}:Y\to X$ is compact (which implies that L^{-1} is bounded).

Remark 1.1 From (H1), we know that L is closed due to the fact that L^{-1} is injective and closed. It follows from (H1)–(H2) and the closed graph theorem that the linear operator $-AL^{-1}:Y\to Y$ is bounded. Thus, $-AL^{-1}$ generates a semigroup $\{W_L(t),t\geq 0\}$, $W_L(t):=e^{-AL^{-1}t}$.

Now, let

$$\psi_\alpha(s)=\frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}s^{-\alpha n-1}\frac{\Gamma(1+\alpha n)}{n!}\sin(n\pi\alpha),\quad s\in[0,\infty),$$

be one-sided stable probability density whose Laplace transform is provided by

$$\int_0^\infty e^{-\lambda\theta}\psi_\alpha(s)\,d\theta=e^{\lambda^\alpha},\tag{1.2}$$

where $\alpha\in(0,1)$. We state families of operators $P(t)$, $t\geq 0$, on Y as follows:

$$P(t)=\int_0^\infty \alpha s\eta_\alpha(s)W_L\bigl(st^\alpha\bigr)\,ds,$$

where $\eta_\alpha(s):=\frac{1}{\alpha}s^{-(1+\frac{1}{\alpha})}\psi_\alpha(s^{-\frac{1}{\alpha}})$ is the function of Wright type where $\eta_\alpha(s)\geq 0$, $s\in(0,\infty)$, and $\int_0^\infty \eta_\alpha(s)\,ds=1$. Note that $P(t)$ is a bounded linear operator on Y and continuous for

$t > 0$ in the means of uniform operator topology. Moreover, for all $x \in Y$, then

$$\left\|P(t)x\right\|_Y \leq \frac{\alpha M_1 \|x\|_Y}{\Gamma(1+\alpha)}, \tag{1.3}$$

where $\|W\|_{\Lambda(Y)} := \sup_{t\geq 0}\|W_L(t)\| \leq M_1$ for $t > 0$ (see Remark 2.1.3, [47]). For details, we refer the reader to [2, 4].

In view of Lemma 1.1, we can rewrite system (1.1) in the equivalent fractional integral equation

$$\begin{cases} Lx(t) = \phi(0) + b(x)(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s)\,ds, & t \in I, \\ Lx(t) = \phi(t) + b(x)(t), & t \in [-\tau, 0], \end{cases} \tag{1.4}$$

provided that the integral in (1.4) exists, where $g(t) := -Ax(t) + f(t, x(t-\nu(t))) + Bu(t)$ for $t \in I$.

For $x \in Y$, we define two families $\mathcal{A}_L(t)$, $t \geq 0$, and $\mathcal{B}_L(t)$, $t \geq 0$, of operators given by

$$\mathcal{A}_L(t)x = \int_0^\infty L^{-1}\eta_\alpha(s) W_L\big(st^\alpha\big)x\,ds,$$

$$\mathcal{B}_L(t)x = \int_0^\infty L^{-1}\alpha s\eta_\alpha(s) W_L\big(st^\alpha\big)x\,ds.$$

Lemma 1.2 *If*

$$Lx(t) = Lx(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\big(-Ax(s) + f\big(s, x\big(s-\nu(s)\big)\big) + Bu(s)\big)\,ds \tag{1.5}$$

for $t \in I$ holds, then we obtain

$$Lx(t) = \mathcal{A}_L(t)\big(Lx(0)\big) + \int_0^t (t-s)^{\alpha-1}\mathcal{B}_L(t-s)\big(f\big(s, x\big(s-\nu(s)\big)\big) + Bu(s)\big)\,ds, \quad t \in I.$$

Proof Using the Laplace transforms with respect to t on (1.5), one has that

$$\begin{aligned} Ly(\lambda) &= \frac{1}{\lambda}Lx(0) - \frac{1}{\lambda^\alpha}AL^{-1}Ly(\lambda) + \frac{1}{\lambda^\alpha}w(\lambda) \\ &= \lambda^{\alpha-1}\big(\lambda^\alpha\mathcal{I} + AL^{-1}\big)^{-1}\big(Lx(0)\big) + \big(\lambda^\alpha\mathcal{I} + AL^{-1}\big)^{-1}w(\lambda) \\ &= \lambda^{\alpha-1}\int_0^\infty e^{-\lambda^\alpha s}W_L(s)\big(Lx(0)\big)\,ds + \int_0^\infty e^{-\lambda^\alpha s}W_L(s)\big(Lx(0)\big)\,ds, \end{aligned} \tag{1.6}$$

provided that the integrals in (1.5) exist, where $\mathcal{I}$ is the identity operator defined on Y, the Laplace transform of $x, f(t,x)$, and u are defined by

$$y(\lambda) = \int_0^\infty e^{-\lambda s}x(s)\,ds$$

and

$$w(\lambda) = \int_0^\infty e^{-\lambda s}\big(f\big(s, x_\nu(s)\big) + Bu(s)\big)\,ds, \quad \lambda > 0.$$

From (1.2) and (1.6), we get

$$\begin{aligned}
&\lambda^{\alpha-1}\int_0^\infty e^{-\lambda^\alpha s}W_L(s)\big(Lx(0)\big)\,ds \\
&\quad=\int_0^\infty \alpha(\lambda t)^{\alpha-1}e^{-(\lambda t)^\alpha}W_L\big(t^\alpha\big)\big(Lx(0)\big)\,dt \\
&\quad=-\int_0^\infty \frac{1}{\lambda}\frac{d}{dt}\big[e^{-(\lambda t)^\alpha}\big]W_L\big(t^\alpha\big)\big(Lx(0)\big)\,dt \\
&\quad=\int_0^\infty\int_0^\infty \theta\psi_\alpha(\theta)e^{-\lambda t\theta}W_L\big(t^\alpha\big)\big(Lx(0)\big)\,d\theta\,dt \\
&\quad=\int_0^\infty e^{-\lambda t}\bigg[\int_0^\infty \psi_\alpha(\theta)W_L\bigg(\frac{t^\alpha}{\theta^\alpha}\bigg)\big(Lx(0)\big)\,d\theta\bigg]\,dt,
\end{aligned}\tag{1.7}$$

and

$$\begin{aligned}
&\int_0^\infty e^{-\lambda^\alpha s}W_L(s)w(\lambda)\,ds \\
&\quad=\int_0^\infty\int_0^\infty \alpha t^{\alpha-1}e^{-(\lambda t)^\alpha}W_L\big(t^\alpha\big)e^{-\lambda s}\big(f\big(s,x_\nu(s)\big)+Bu(s)\big)\,ds\,dt \\
&\quad=\int_0^\infty\int_0^\infty\int_0^\infty \alpha\psi_\alpha(\theta)e^{-\lambda t\theta}t^{\alpha-1}W_L\big(t^\alpha\big)e^{-\lambda s}\big(f\big(s,x_\nu(s)\big)+Bu(s)\big)\,d\theta\,ds\,dt \\
&\quad=\int_0^\infty\int_0^\infty\int_0^\infty \alpha\psi_\alpha(\theta)e^{-\lambda(t+s)}\frac{t^{\alpha-1}}{\theta^\alpha}W_L\bigg(\frac{t^\alpha}{\theta^\alpha}\bigg)e^{-\lambda s}\big(f\big(s,x_\nu(s)\big)+Bu(s)\big)\,d\theta\,ds\,dt \\
&\quad=\int_0^\infty e^{-\lambda t}\bigg[\alpha\int_0^t\int_0^\infty \psi_\alpha(\theta)W_L\bigg(\frac{(t-s)^\alpha}{\theta^\alpha}\bigg) \\
&\qquad\times\big(f\big(s,x_\nu(s)\big)+Bu(s)\big)\frac{(t-s)^{\alpha-1}}{\theta^\alpha}\,d\theta\,ds\bigg]\,dt.
\end{aligned}\tag{1.8}$$

In view of (1.7) and (1.8), we have

$$\begin{aligned}
Ly(\lambda)={}&\int_0^\infty e^{-\lambda t}\bigg[\int_0^\infty \psi_\alpha(\theta)W_L\bigg(\frac{t^\alpha}{\theta^\alpha}\bigg)\big(Lx(0)\big)\,d\theta \\
&+\alpha\int_0^t\int_0^\infty \psi_\alpha(\theta)W_L\bigg(\frac{(t-s)^\alpha}{\theta^\alpha}\bigg)\big(f\big(s,x_\nu(s)\big)+Bu(s)\big)\frac{(t-s)^{\alpha-1}}{\theta^\alpha}\,d\theta\,ds\bigg]\,dt.
\end{aligned}\tag{1.9}$$

Now we take the invert Laplace transform on (1.9) to get

$$\begin{aligned}
x(t)={}&\int_0^\infty L^{-1}\psi_\alpha(\theta)W_L\bigg(\frac{t^\alpha}{\theta^\alpha}\bigg)\big(Lx(0)\big)\,d\theta \\
&+\alpha\int_0^t\int_0^\infty L^{-1}\psi_\alpha(\theta)W_L\bigg(\frac{(t-s)^\alpha}{\theta^\alpha}\bigg)\frac{(t-s)^{\alpha-1}}{\theta^\alpha}\big(f\big(s,x_\nu(s)\big)+Bu(s)\big)\,d\theta\,ds \\
={}&\int_0^\infty L^{-1}\eta_\alpha(\theta)W_L\big(t^\alpha\theta\big)\big(Lx(0)\big)\,d\theta \\
&+\int_0^t\int_0^\infty L^{-1}\alpha\theta\eta_\alpha(\theta)W_L\big((t-s)^\alpha\theta\big)(t-s)^{\alpha-1}\big(f\big(s,x_\nu(s)\big)+Bu(s)\big)\,d\theta\,ds \\
={}&\mathcal{A}_L(t)\big(Lx(0)\big)+\int_0^t (t-s)^{\alpha-1}\mathcal{B}_L(t-s)\big(f\big(s,x_\nu(s)\big)+Bu(s)\big)\,ds,\quad t\in I.
\end{aligned}\tag{1.10}$$

The proof is completed. □

The following lemma comes from the results with minor modifications in [37].

Lemma 1.3 *The following properties on $\mathcal{A}_L(t)$ and $\mathcal{B}_L(t)$ are valid.*

(1) *For every $t \geq 0$, $\mathcal{A}_L(t)$ and $\mathcal{B}_L(t)$ are linear and bounded operators, i.e., $\|\mathcal{A}_L(t)x\| \leq M_1\|L^{-1}\|_{\Lambda(Y,X)}\|x\|_Y$, $\|\mathcal{B}_L(t)x\| \leq \frac{\|L^{-1}\|_{\Lambda(Y,X)}M_1}{\Gamma(\alpha)}\|x\|_Y$ for all $x \in Y$, and $t \in (0,\infty)$.*

(2) *For every $x \in Y$, $t \to \mathcal{A}_L(t)x$, $t \to \mathcal{B}_L(t)x$ are continuous functions from $[0,\infty)$ into Y.*

(3) *$\mathcal{A}_L(t)$ and $\mathcal{B}_L(t)$ are compact operators on X for $t > 0$.*

Definition 1.3 For given $u \in L^\infty(I,U)$, $x \in C([-\tau,T],X)$ will be said to be a mild solution of the fractional system (1.1) if and only if it satisfies the following integral equation:

$$\begin{cases} x(t) = \mathcal{A}_L(t)(\phi(0) + b(x)(0)) + \int_0^t (t-s)^{\alpha-1}\mathcal{B}_L(t-s)f(s,x(s-\nu(s)))\,ds \\ \qquad + \int_0^t (t-s)^{\alpha-1}\mathcal{B}_L(t-s)Bu(s)\,ds, & t \in I, \\ x(t) = L^{-1}(\phi(t) + b(x)(t)), & t \in [-\tau,0]. \end{cases} \tag{1.11}$$

Definition 1.4 The fractional system (1.1) is said to be controllable on the interval I if and only if, for every $x_1 \in X$, there exists a control $u \in L^\infty(I,U)$ such that the mild solution x of system (1.1) satisfies $x(T) = x_1$.

Remark 1.2 In order to gain the excellent controllability properties, Lü and Zuazua [41] give a new definition of controllability that system is null controllable at time T if, for any $x_0 \in H$, there is a control $u \in L^2(I,U)$ such that the corresponding solution $x(\cdot)$ satisfies that $x(t) = 0$ for all $t \geq T$. In this setting, they are interested in the problem of controllability. More precisely, they resolve the problem of null controllability in which the objective is to drive the solution to rest, in other words, to the trivial null state, in finite time. Nevertheless, in the work of Lü and Zuazua this controllability property cannot be achieved in this case for the fractional in time differential system. This negative result holds even for finite-dimensional systems in which the control is of full dimension. Consequently, the same negative results hold also for fractional in time PDE and memory PDEs, regardless of whether they are of hyperbolic or parabolic nature. This negative result exhibits a completely opposite behavior with respect to the existing literature on classical ODE and PDE control.

Remark 1.3 When $L = I$, $I : Y \to Y$ is the identity operator, we have $\mathcal{A}_I(t)x = \int_0^\infty \eta_\alpha(s)W_L(st^\alpha)x\,ds$, $\mathcal{B}_I(t)x = \int_0^\infty \alpha s\eta_\alpha(s)W_L(st^\alpha)x\,ds$.

Let $\Omega_r := \{y \in C([-\tau,T],X) : \|y\|_C < r\}$ for $r \in R_+$. Obviously, Ω_r is an open, bounded, convex subset of $C([-\tau,T],X)$. In addition to (H1) and (H2), we still need the following hypotheses:

(H3) The function $f : I \times C([-\tau,0],X) \to Y$ satisfies the following two conditions:
  - (i) for every $y \in C([-\tau,0],X)$, $t \to f(t,y)$ is strongly measurable, and for each $t \in I$, $y \to f(t,y)$ is continuous;
  - (ii) there are a measurable function $h \in L_+^\infty(I)$ and a nondecreasing continuous function ψ such that $\|f(t,y)\|_Y \leq h(t)\psi(\|y\|_C)$ for almost all $t \in I$, every $y \in C([-\tau,0],X)$.

(H4) The operator $b : C([-\tau, T], X) \to C([-\tau, 0], Y)$ is continuous, and there exists a nondecreasing function $g : R_+ \to R_+$ such that $\|b(y)\|_C \le g(r)$, $\forall y \in \Omega_r$.

(H5) $B : U \to Y$ is a linear bounded operator and the linear operator $K : L^\infty(I, U) \to D(L)$ defined by

$$Ku = \int_0^T (T-s)^{\alpha-1} \mathcal{B}_L(T-s)Bu(s)\,ds \tag{1.12}$$

has an invertible operator K^{-1} which takes values in $L^\infty(I, U) \setminus \mathrm{Ker}K$, and there exist positive constants M_3, M_4 such that $\|K^{-1}\|_{\Lambda(D(L),L^\infty(I,U))} \le M_3$ and $\|B\|_{\Lambda(U,Y)} \le M_4$.

(H6) Let $M_2 = \frac{T^\alpha \|L^{-1}\|_{\Lambda(Y,X)} M_1}{\Gamma(\alpha)}$, $x_1 \in X$, there exists $r > 0$ such that

$$D_1 g(r) + D_2 \psi(r) + D_3 < r,$$

where $D_1 = (1 + M_2M_3M_4)M_1\|L^{-1}\|_{\Lambda(Y,X)}$, $D_2 = (1 + M_2M_3M_4)M_2\|h\|_\infty$,

$$D_3 = M_1 \|L^{-1}\|_{\Lambda(Y,X)}\|\phi\|_C + M_2M_3M_4\bigl(\|x_1\| + M_1\|L^{-1}\|_{\Lambda(Y,X)}\|\phi\|_C\bigr),$$

and M_1, M_3, M_4 are constants stated in (1.3) and (H5).

It is evident that K is well defined on $D(L)$, since

$$\begin{aligned} \|LKu\|_Y &= \left\| \int_0^T (T-s)^{\alpha-1} L\mathcal{B}_L(T-s)Bu(s)\,ds \right\|_Y \\ &\le \frac{\alpha M_1 T^\alpha M_4}{\Gamma(1+\alpha)} \|u\|_{L^\infty(I,U)}. \end{aligned} \tag{1.13}$$

Now we state a Leray–Schauder alternative theorem and our main result.

Lemma 1.4 (see [44]) *Let C be a bounded convex subset of a Banach space X, Q be an open subset of C and $0 \in Q$. Let $\Phi : \overline{Q} \to C$ be a continuous, compact (that is, $\Phi(\overline{Q})$ is a precompact subset of C) map. Then either there exist $x \in \partial Q$ (the boundary of Q in C) and $\lambda \in (0, 1)$ with $x = \lambda\Phi(x)$ or Φ has a fixed point $x \in \overline{Q}$.*

Theorem 1.1 *Assume that* (H1)–(H6) *are satisfied, then system* (1.1) *is exactly controllable on I.*

2 Proof of Theorem 1.1

The controllability of system (1.1) is equivalent to showing that for $x_1 \in X$, there exists $u \in L^\infty(I, U)$ such that the solution x of system (1.1) satisfies $x(T) = x_1$. We observe that for arbitrary $x \in C([-\tau, T], X)$ the control u_x can be defined by

$$\begin{aligned} u_x(t) = K^{-1}\biggl(& x_1 - \mathcal{A}_L(T)\bigl(\phi(0) + b(x)(0)\bigr) \\ & - \int_0^T (T-s)^{\alpha-1}\mathcal{B}_L(t-s) f\bigl(s, x\bigl(s - \nu(s)\bigr)\bigr)\,ds \biggr) \end{aligned} \tag{2.1}$$

$t \in I$. Using this control, we show that the operator $\Phi : C([-\tau, T], X) \to C([-\tau, T], X)$, defined by

$$\begin{cases} \Phi(x)(t) = \mathcal{A}_L(t)(\phi(0) + b(x)) + \int_0^t (t-s)^{\alpha-1}\mathcal{B}_L(t-s)f(s, x(s-\nu(s)))\,ds \\ \qquad + \int_0^t (t-s)^{\alpha-1}\mathcal{B}_L(t-s)Bu_x(s)\,ds, & t \in I, \\ \Phi(x)(t) = L^{-1}(\phi(t) + b(x)(t)), & t \in [-\tau, 0], \end{cases} \tag{2.2}$$

has a fixed point x, which is a solution of system (1.1). Putting (2.1) into (2.2), we note that $x_1 = \Phi(x)(T)$, which means that u_x operates system (1.1) from x_0 to x_1 in time T, i.e., (1.1) is exactly controllable on I. In view of Lemma 1.3 and (H3)–(H5), we derive that for every $x \in \Omega_r$,

$$\begin{aligned} &\|u_x\|_{L^\infty(I,U)} \\ &\quad \le M_3\bigl(\|x_1\| + \bigl\|\mathcal{A}_L(T)\bigr\|_{\Lambda(X)}\bigl(\|\phi\|_C + \bigl\|b(x)\bigr\|_C\bigr)\bigr) \\ &\qquad + M_3\int_0^T (T-s)^{\alpha-1}\|\mathcal{B}_L\|_{\Lambda(X)}\bigl\|f\bigl(s, x\bigl(s-\nu(s)\bigr)\bigr)\bigr\|_Y\,ds \\ &\quad \le M_3\biggl(\|x_1\| + M_1\bigl\|L^{-1}\bigr\|\bigl(\|\phi\|_C + g(r)\bigr) + \frac{\|h\|_\infty\psi(r)M_1T^\alpha}{\Gamma(\alpha)}\biggr). \end{aligned} \tag{2.3}$$

We proceed in the following four steps.

Step 1. We show that the set $\{\Phi(x)(t) : x \in \overline{\Omega}_r, t \in [-\tau, T]\}$ is relatively compact in X.

Fix $t \in [-\tau, T]$. First, we define

$$\begin{aligned} \Phi_1(x)(t) &= \mathcal{A}_I(t)\bigl(\phi(0) + b(x)(0)\bigr) + \int_0^t (t-s)^{\alpha-1}\mathcal{B}_I(t-s)f\bigl(s, x\bigl(s-\nu(s)\bigr)\bigr)\,ds \\ &\quad + \int_0^t (t-s)^{\alpha-1}\mathcal{B}_I(t-s)Bu_x(s)\,ds, \quad t \in I, \\ \Phi_1(x)(t) &= \phi(t) + b(x)(t), \quad t \in [-\tau, 0]. \end{aligned} \tag{2.4}$$

Then $\Phi(x)(t) = L^{-1}\Phi_1(x)(t)$, $x \in \overline{\Omega}_r$. Next, for $x \in \overline{\Omega}_r$, one has that

$$\bigl\|\Phi_1(x)\bigr\|_Y \le \|\phi\|_Y + \bigl|g(r)\bigr|$$

if $t \in [-\tau, 0]$ and

$$\begin{aligned} \bigl\|\Phi_1(x)\bigr\|_Y &= \bigl\|\mathcal{A}_I(t)\bigl(\phi(0) + b(x)(0)\bigr)\bigr\|_Y + \biggl\|\int_0^t (t-s)^{\alpha-1}\mathcal{B}_I(t-s)f\bigl(s, x\bigl(s-\nu(s)\bigr)\bigr)\,ds\biggr\|_Y \\ &\quad + \biggl\|\int_0^t (t-s)^{\alpha-1}\mathcal{B}_I(t-s)Bu_x(s)\,ds\biggr\|_Y \\ &\le \|\mathcal{A}_I\|_{\Lambda(X,Y)}\bigl(\bigl\|\phi(0)\bigr\|_Y + \bigl|g(r)\bigr|\bigr) + \frac{T^\alpha}{\Gamma(\alpha)}\|\mathcal{B}_I\|_{\Lambda(X,Y)}M_1\|h\|_\infty\psi\bigl(\|x\|_C\bigr) \\ &\quad + \frac{T^\alpha}{\Gamma(\alpha)}\|\mathcal{B}_I\|_{\Lambda(X,Y)}M_1\|B\|_{\Lambda(U,Y)}\|u\|_{L^\infty(I,U)} \end{aligned}$$

if $t \in I$, and so $\{\Phi_1(x)(t) : x \in \overline{\Omega}_r, t \in [-\tau, T]\}$ is bounded in Y by (2.3). Since $L^{-1} : Y \to X$ is compact, then $\{\Phi(x)(t) : x \in \overline{\Omega}_r, t \in [-\tau, T]\} = L^{-1}\{\Phi_1(x)(t) : x \in \overline{\Omega}_r, t \in [-\tau, T]\}$ is relatively compact in X.

Step 2. We show that the set $\{\Phi(x)(t) : x \in C([-\tau, T], X)\}$ is equicontinuous on $[-\tau, T]$.

From the strong continuity of $(\mathcal{A}_L(t))_{t\geq 0}$ and $(\mathcal{B}_L(t))_{t\geq 0}$, for any fixed $t \in I$, we can choose $-\tau < \delta < T - t$ such that

$$\begin{aligned} &\|\mathcal{A}_L(t+\delta)x - \mathcal{A}_L(t)x\| \to 0 \quad \text{for } x \in X \text{ as } \delta \to 0; \\ &\|\mathcal{B}_L(t+\delta)x - \mathcal{B}_L(t)x\| \to 0 \quad \text{for } x \in X \text{ as } \delta \to 0. \end{aligned}$$

For every $t \in [-\tau, 0]$, from (H4) it is clear that $\Phi(x)(t) = L^{-1}(\phi(t) + b(x)(t))$ is continuous with respect to t. For every $t \in I$, $\delta \in (0, T - t)$ and $x \in C([-\tau, T], X)$, from (H3) and (2.2), it follows that

$$\begin{aligned} &\|\Phi(x)(t+\delta) - \Phi(x)(t)\| \\ &\quad\leq \|\mathcal{A}_L(t+\delta)(\phi(0)+b(x)) - \mathcal{A}_L(t)(\phi(0)+b(x))\| \\ &\qquad + \left\| \int_t^{t+\delta} (t+\delta-s)^{\alpha-1}\mathcal{B}_L(t+\delta-s)\big[f\big(s,x(s-\nu(s))\big) + Bu_x(s)\big]\,ds \right\| \\ &\qquad + \left\| \int_0^t \big[(t+\delta-s)^{\alpha-1}\mathcal{B}_L(t+\delta-s) - (t-s)^{\alpha-1}\mathcal{B}_L(t-s)\big] \right. \\ &\qquad \left. \times \big(f\big(s,x(s-\nu(s))\big) + Bu_x(s)\big)\,ds \right\| \\ &\quad\leq \|\mathcal{A}_L(t+\delta)(\phi(0)+b(x)) - \mathcal{A}_L(t)(\phi(0)+b(x))\| \\ &\qquad + \int_t^{t+\delta} (t+\delta-s)^{\alpha-1}\|\mathcal{B}_L\|_{\Lambda(X)}\big[\|f(s,x(s))\|_Y + \|B\|_{\Lambda(U,Y)}\|u_x\|_{L^\infty(I,U)}\big]\,ds \\ &\qquad + \left\| \int_0^t \big[(t+\delta-s)^{\alpha-1}\mathcal{B}_L(t+\delta-s) - (t-s)^{\alpha-1}\mathcal{B}_L(t+\delta-s) \right. \\ &\qquad \left. + (t-s)^{\alpha-1}\mathcal{B}_L(t+\delta-s) - (t-s)^{\alpha-1}\mathcal{B}_L(t-s)\big]\big(f(s,x(s)) + Bu_x(s)\big)\,ds \right\| \\ &\quad\leq q_1 + q_2 + q_3 + q_4, \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} q_1 &= \|\mathcal{A}_L(t+\delta)(\phi(0)+b(x)) - \mathcal{A}_L(t)(\phi(0)+b(x))\|, \\ q_2 &= \int_t^{t+\delta} (t+\delta-s)^{\alpha-1}\|\mathcal{B}_L\|_{\Lambda(X)}\big[\psi(\|x\|_C)\|h\|_\infty + \|B\|_{\Lambda(U,Y)}\|u_x\|_{L^\infty(I,U)}\big]\,ds, \\ q_3 &= \left| \int_0^t \big[(t+\delta-s)^{\alpha-1} - (t-s)^{\alpha-1}\big]\|\mathcal{B}_L\|_{\Lambda(X)}\big[\|f(s,x(s))\| \right. \\ &\qquad \left. + \|B\|_{\Lambda(U,Y)}\|u_x\|_{L^\infty(I,U)}\big]\,ds \right|, \\ q_4 &= \int_0^t (t-s)^{\alpha-1}\|\mathcal{B}_L(t+\delta-s) - \mathcal{B}_L(t-s)\|\big[\|f(s,x(s))\| + \|Bu_x(s)\|\big]\,ds. \end{aligned}$$

Obviously, $q_1 \to 0$ as $\delta \to 0$. From Lemma 1.3 and (H1), $\mathcal{B}_L(t)$ is continuous in the uniform operator topology for $t \geq 0$ and u is bounded by (2.3). From (H5) and Lemma 1.3, we can derive that $q_2, q_3 \to 0$ as $\delta \to 0$. Moreover, by applying Lebesgue's dominated convergence theorem, we have $q_4 \to 0$ as $\delta \to 0$. Therefore, $\|\Phi(x)(t+\delta) - \Phi(x)(t)\| \to 0$ for each $t \in I$ as $\delta \to 0$. Thus, the operator Φ maps $C([-\tau, T], X)$ into an equicontinuous family of functions.

Step 3. We show that $\Phi : C([-\tau, T], X) \to C([-\tau, T], X)$ is continuous.

Let $x_n \to x$ on $C([-\tau, T], X)$ as $n \to \infty$. From (H3) and (H4), it follows that for almost all $t \in [-\tau, T]$,

$$\left\|f(t,x_n) - f(t,x)\right\|_Y \to 0 \tag{2.6}$$

and

$$\left\|b(x_n) - b(x)\right\|_C \to 0, \tag{2.7}$$

as $n \to 0$, which, together with (2.1) and (2.2), yields

$$\begin{aligned}
&\left\|\Phi(x_n)(t) - \Phi(x)(t)\right\|_C \\
&\quad\leq \left\|\mathcal{A}_L b(x_n) - \mathcal{A}_L b(x)\right\|_C \\
&\qquad + \int_0^t (t-s)^{\alpha-1}\mathcal{B}_L(t-s)\left[f\left(s, x_n\left(s - \nu(s)\right)\right) - f\left(s, x\left(s - \nu(s)\right)\right)\right] ds\|_C \\
&\qquad + \int_0^t (t-s)^{\alpha-1}\mathcal{B}_L(t-s)B\left[u_{x_n}(s) - u_x(s)\right] ds\|_C \\
&\quad\leq p_1 + p_2 + p_3,
\end{aligned} \tag{2.8}$$

where

$$\begin{aligned}
p_1 &= \|\mathcal{A}_L\|_{\Lambda(X)}\left\|b(x_n) - b(x)\right\|_C \\
p_2 &= \int_0^T (T-s)^{\alpha-1}\|\mathcal{B}_L\|_{\Lambda(X)}\left\|f\left(s, x_n\left(s-\nu(s)\right)\right) - f\left(s, x\left(s-\nu(s)\right)\right)\right\|_Y ds \\
p_3 &= \int_0^T (T-s)^{\alpha-1}\|\mathcal{B}_L\|_{\Lambda(X)}\|B\|_{\Lambda(U,Y)}\left\|K^{-1}\right\|_{\Lambda(D(L),L^\infty(I,U))} \\
&\quad \times \left[\|\mathcal{A}_L\|_{\Lambda(X)}\left\|b(x_n) - b(x)\right\|_C \right.\\
&\quad \left. + \int_0^T (T-s)^{\alpha-1}\|\mathcal{B}_L\|_{\Lambda(X)}\left\|f\left(s, x_n\left(s-\nu(s)\right)\right) - f\left(s, x(s)\right)\right\|_Y ds\right] ds.
\end{aligned} \tag{2.9}$$

Clearly, $p_1 \to 0$ as $x_n \to x$ in $C([-\tau, T], X)$. From (2.6), (2.7) and applying Lebesgue's dominated convergence theorem, we have $p_2, p_3 \to 0$ as $x_n \to x$ in $C([-\tau, T], X)$. Hence, the mapping Φ is continuous. As a consequence of the Arzela–Ascoli theorem, we can conclude that the operator $\Phi : \overline{\Omega}_r \to C(I, X)$ is continuous and compact.

Step 4. We prove that the operator Φ has a fixed point $x \in \overline{\Omega}_r$.

Suppose $x \in \partial\Omega_r$, $\lambda \in (0,1)$ such that $x = \lambda\Phi(x)$; from (H3)–(H5) and (2.1) we have

$$\begin{aligned}
r &= \|x\|_C = \lambda \big\|\Phi(x)\big\|_C \\
&\le \lambda\|\mathcal{A}_L\|_{\Lambda(X)}\big(\|\phi\|_C + \big\|b(x)\big\|_C\big) + \lambda\left\|\int_0^t (t-s)^{\alpha-1}\mathcal{B}_L(t-s)f\big(s,x\big(s-\nu(s)\big)\big)\,ds\right\|_C \\
&\quad + \lambda\left\|\int_0^t (t-s)^{\alpha-1}\mathcal{B}_L(t-s)Bu_x(s)\,ds\right\|_C \\
&\le \lambda\|\mathcal{A}_L\|_{\Lambda(X)}\big(\|\phi\|_C + g\big(\|x\|_C\big)\big) + \lambda\int_0^T (T-s)^{\alpha-1}\big\|\mathcal{B}_L(t-s)\big\|_{\Lambda(X)}\big(\psi\big(\|x\|_C\big)\|h\|_\infty\big)\,ds \\
&\quad + \lambda\int_0^T (T-s)^{\alpha-1}\big\|\mathcal{B}_L(t-s)\big\|_{\Lambda(X)}\|B\|_{\Lambda(U,Y)}\big\|u_x(s)\big\|\,ds \\
&\le \lambda\|\mathcal{A}_L\|_{\Lambda(X)}\big(\|\phi\|_C + g\big(\|x\|_C\big)\big) + \lambda M_2\|h\|_\infty\psi\big(\|x\|_C\big) + \lambda M_2\|B\|_{\Lambda(U,Y)}\|u_x\|_{L^\infty(I,U)} \\
&\le \lambda\big(1 + M_2\|B\|_{\Lambda(U,Y)}\big\|K^{-1}\big\|_{\Lambda(D(L),L^\infty(I,U))}\big)\|\mathcal{A}_L\|_{\Lambda(X)}g(r) \\
&\quad + \lambda\big(1 + M_2\|B\|_{\Lambda(U,Y)}\big\|K^{-1}\big\|_{\Lambda(D(L),L^\infty(I,U))}\big)M_2\|h\|_\infty\psi(r) \\
&\quad + \lambda\|\mathcal{A}_L\|_{\Lambda(X)}\|\phi\|_C + \lambda M_2\|B\|_{\Lambda(U,Y)}\big\|K^{-1}\big\|_{\Lambda(D(L),L^\infty(I,U))}\big(\|x_1\| + \|\mathcal{A}_L\|_{\Lambda(X)}\|\phi\|_C\big) \\
&\le \lambda\big(D_1 g(r) + D_2\psi(r) + D_3\big),
\end{aligned} \tag{2.10}$$

where

$$\begin{aligned}
D_1 &= (1 + M_2M_3M_4)M_1\big\|L^{-1}\big\|_{\Lambda(Y,X)}, \\
D_2 &= (1 + M_2M_3M_4)M_2\|h\|_\infty, \\
D_3 &= M_1\big\|L^{-1}\big\|_{\Lambda(Y,X)}\|\phi\|_C + M_2M_3M_4\big(\|x_1\| + M_1\big\|L^{-1}\big\|_{\Lambda(Y,X)}\|\phi\|_C\big), \\
M_2 &= \frac{T^\alpha\|L^{-1}\|_{\Lambda(Y,X)}M_1}{\Gamma(\alpha)}.
\end{aligned}$$

Thus, by (H6) and (2.10),

$$r \le \lambda\big(D_1 g(r) + D_2\psi(r) + D_3\big) < D_1 g(r) + D_2\psi(r) + D_3 < r,$$

a contradiction. Applying Lemma 1.4, the operator Φ has a fixed point $x \in \overline{\Omega}_r$, i.e., x is the solution of (1.1). Therefore system (1.1) is exactly controllable on I.

3 Example

In this section, we present an example of a fractional feedback control system as an application of our theory. Let $X = Y = U = L^2[0,\pi]$, $I = [0,1]$, and $J = [0,\pi]$, consider a fractional control system represented by the following fractional partial differential equation:

$$\begin{cases}
\mathcal{D}_t^\alpha\big(x(t,\eta) - \lambda_1 x_{\eta\eta}(t,\eta)\big) & \\
\quad = \lambda_2 x_{\eta\eta} + f\big(t, x(t-\nu(t),\eta)\big) + Bu(t,\eta), & t \in I, \eta \in J, \\
x(t,0) = x(t,\pi) = 0, & t \in I, \\
x(t,\eta) = \lambda_1 x_{\eta\eta}(t,\eta) + \phi(t,\eta) + \int_0^1 \hat{K}(s,t)\sin x(s,\eta)\,ds, & -\tau \le t \le 0, 0 \le \eta \le \pi,
\end{cases} \tag{3.1}$$

where $\mathcal{D}_t^\alpha$, $0 < \alpha < 1$ is the regularized Caputo fractional derivative of order α; $\lambda_1, \lambda_2 \in R_+$; $\nu : [0,1] \to (0,\tau]$ is a continuous function; the function f and map B are given below; $\phi \in C(I \times J, R)$ and $\hat{K} \in C(I \times [-\tau, 0], R)$.

Define the operator $A : D(A) \subset X \to X$ by $Ax = -\lambda_2 x_{\eta\eta}$, and $L : D(L) \subset X \to X$ by $Lx = x - \lambda_1 x_{\eta\eta}$, where each domain $D(A)$, $D(L)$ is given by $\{x \in X, x, x_\eta,$ are absolutely continuous, $x_{\eta\eta} \in X, x(0) = x(\pi) = 0\}$. From [48], A and L can be written as

$$Ax := \sum_{n=1}^{\infty} \lambda_2 n^2 \langle x, e_n \rangle e_n, \quad x \in D(A),$$

$$Lx := \sum_{n=1}^{\infty} \bigl(1 + \lambda_1 n^2\bigr) \langle x, e_n \rangle e_n, \quad x \in D(L),$$

where $e_n(\eta) = \sqrt{\frac{2}{\pi}} \sin n\eta$, $n = 1, 2, \ldots$, is the orthogonal eigenfunctions set of A. Moreover, for any $x \in X$, we obtain

$$L^{-1}x = \sum_{n=1}^{\infty} \frac{1}{1 + \lambda_1 n^2} \langle x, e_n \rangle e_n.$$

$$-AL^{-1}x = \sum_{n=1}^{\infty} \frac{-\lambda_2 n^2}{1 + \lambda_1 n^2} \langle x, e_n \rangle e_n,$$

and a semigroup $W_L(t)$ is given by

$$W_L(t)x = \sum_{n=1}^{\infty} e^{\frac{-\lambda_2 n^2}{1+\lambda_1 n^2} t} \langle x, e_n \rangle e_n.$$

It is easy to see that L^{-1} is compact and $\|L^{-1}\| \leq 1$, and $-AL^{-1}$ generates a strongly continuous semigroup $W_L(t)$ on X and $\|W_L(t)\| \leq 1$ for each $t > 0$. Then, the characterized operators $\mathcal{A}_L(t)x$ and $\mathcal{B}_L(t)x$ can be written as

$$\mathcal{A}_L(t)x = \int_0^\infty L^{-1} \delta_\alpha(\varepsilon) \sum_{n=1}^{\infty} e^{-\frac{\lambda_2 n^2}{1+\lambda_1 n^2} t^\alpha \varepsilon} \langle x, x_n \rangle x_n \, d\varepsilon,$$

$$\mathcal{B}_L(t)x = \int_0^\infty L^{-1} \alpha \varepsilon \delta_\alpha(\varepsilon) \sum_{n=1}^{\infty} e^{-\frac{\lambda_2 n^2}{1+\lambda_1 n^2} t^\alpha \varepsilon} \langle x, x_n \rangle x_n \, d\varepsilon,$$

where

$$\delta_\alpha(\varepsilon) = \frac{1}{\pi\alpha} \sum_{n=1}^{\infty} (-\varepsilon)^{n-1} \frac{\Gamma(1 + \alpha n)}{n!} \sin(n\pi\alpha), \quad \varepsilon \in (0, \infty).$$

Note $\|\mathcal{A}_L(t)\| \leq 1$, $\|\mathcal{B}_L(t)\| \leq \frac{1}{\Gamma(\alpha)}$ for $t \geq 0$. Now, $B : U \to U$ is defined by $B := \mu\mathcal{I}$, $\mu > 0$, where $\mathcal{I}$ denotes the identity operator, and $K : L^\infty[I, U] \to X$ is defined by

$$Ku := \mu \int_0^1 (1 - z)^{\alpha-1} \mathcal{B}_L(1 - z) u(z, \eta) \, dz.$$

It is easy to show that K is surjective. Further, if $u(t,\eta) := x(\eta) \in X$, then

$$\begin{aligned}
Ku &= \mu \int_0^1 (1-z)^{\alpha-1} \int_0^\infty L^{-1}\alpha\varepsilon\delta_\alpha(\varepsilon) \sum_{n=1}^\infty e^{-\frac{\lambda_2 n^2}{1+\lambda_1 n^2}(1-z)^\alpha \varepsilon} \langle x, e_n\rangle e_n \, d\varepsilon \, dz \\
&= \mu \int_0^\infty L^{-1}\delta_\alpha(\varepsilon) \sum_{n=1}^\infty \int_0^1 \alpha\varepsilon(1-z)^{\alpha-1} e^{-\frac{\lambda_2 n^2}{1+\lambda_1 n^2}(1-z)^\alpha \varepsilon} \, dz \langle x, e_n\rangle e_n \, d\varepsilon \\
&= \mu \int_0^\infty \delta_\alpha(\varepsilon) \sum_{n=1}^\infty \int_0^1 \frac{1}{1+\lambda_1 n^2} \alpha\varepsilon(1-z)^{\alpha-1} e^{-\frac{\lambda_2 n^2}{1+\lambda_1 n^2}(1-z)^\alpha \varepsilon} \, dz \langle x, e_n\rangle e_n \, d\varepsilon \\
&= \mu \int_0^\infty \delta_\alpha(\varepsilon) \sum_{n=1}^\infty \int_0^1 \frac{1}{\lambda_2 n^2} \frac{d}{dz}\big(e^{-\frac{\lambda_2 n^2}{1+\lambda_1 n^2}(1-z)^\alpha \varepsilon}\big) \langle x, e_n\rangle e_n \, dz \, d\varepsilon \\
&= \mu \sum_{n=1}^\infty \frac{1}{\lambda_2 n^2}\left(1 - \int_0^\infty \delta_\alpha(\varepsilon) e^{-\frac{\lambda_2 n^2}{1+\lambda_1 n^2}\varepsilon} \, d\varepsilon\right) \langle x, x_n\rangle x_n \\
&= \mu \sum_{n=1}^\infty \frac{[1 - E_\alpha(-\frac{\lambda_2 n^2}{1+\lambda_1 n^2})]}{\lambda_2 n^2} \langle x, x_n\rangle x_n,
\end{aligned} \tag{3.2}$$

where

$$E_\alpha\left(-\frac{\lambda_2 n^2}{1+\lambda_1 n^2}\right) := \int_0^\infty e^{-\frac{\lambda_2 n^2}{1+\lambda_1 n^2}\varepsilon} \delta_\alpha(\varepsilon) \, d\varepsilon$$

is a Mittag-Leffler function (for details, see formulas (24)–(27) in [49]). Note that $0 < 1 - e^{-\frac{\lambda_2 n^2}{1+\lambda_1 n^2}\varepsilon} < 1 - e^{-\frac{\lambda_2}{\lambda_1}\varepsilon} < 1$ for any $\varepsilon > 0$. So we have

$$0 < 1 - E_\alpha\left(-\frac{\lambda_2 n^2}{1+\lambda_1 n^2}\right) < 1 - E_\alpha\left(-\frac{\lambda_2}{\lambda_1}\right) < 1.$$

Thus, an inverse operator $K^{-1} : X \to L^\infty[I, U]$ can be defined by

$$\big(K^{-1}x\big)(t,z) := \frac{1}{\mu} \sum_{n=1}^\infty \frac{\lambda_2 n^2}{[1 - E_\alpha(-\frac{\lambda_2 n^2}{1+\lambda_1 n^2})]} \langle x, x_n\rangle x_n$$

for $x = \sum_{n=1}^\infty \langle x, x_n\rangle x_n$. Since

$$\|x\|_{D(L)} := \sqrt{\sum_{n=1}^\infty \big(1+\lambda_1 n^2\big)^2 \langle x, x_n\rangle^2},$$

for $x \in X$, we can deduce

$$\big\|\big(K^{-1}x\big)(t,\eta)\big\| \le \frac{1}{\mu} \sqrt{\sum_{n=1}^\infty \frac{{\lambda_2}^2 n^4}{[1 - E_\alpha(-\frac{\lambda_2 n^2}{1+\lambda_1 n^2})]^2} \langle x, x_n\rangle^2}$$

$$\le \frac{\lambda_2}{\mu\lambda_1[1-E_\alpha(-\frac{\lambda_2}{1+\lambda_1})]}\sqrt{\sum_{n=1}^{\infty}\bigl(1+\lambda_1 n^2\bigr)^2\langle x,x_n\rangle^2}$$
$$= \frac{\lambda_2}{\mu\lambda_1[1-E_\alpha(-\frac{\lambda_2}{1+\lambda_1})]}\|x\|_{D(L)}. \tag{3.3}$$

Consequently, we obtain $\|K^{-1}\| \le M := \frac{\lambda_2}{\lambda_1\mu[1-E_\alpha(-\frac{\lambda_2}{1+\lambda_1})]}$. So, (H1),(H2), and (H5) hold.

We also need the following assumptions:

(i) $f: I\times R \to R$ is a Carathéodory function, i.e., for each $x\in R$, $t\to f(t,x)$ is measurable and for each $\in I$, $x\to f(t,x)$ is continuous. Moreover, for every $t\in I$ and $x\in R$, there exists a function $h\in L_+^\infty(I)$ such that $|f(t,x)|\le h(t)|x|$.

(ii) The function $\hat{K}(s,t): I\times[-\tau,0]\to R$ is continuous, and let $M^* := \sup_{s\in I,t\in[-\tau,0]}|\hat{K}(s,t)|$.

(iii) The following inequality

$$\frac{\|h\|_\infty(\Gamma(\alpha)+M\mu)}{\Gamma^2(\alpha)} < 1$$

is satisfied.

Define $x(t)(\eta) = x(t,\eta)$, $f(t,x(t))(\eta)$, $\phi(t)(\eta) = \phi(t,\eta)$, and $b(x)(t)(\eta) = \int_0^1 \hat{K}(s,t)\sin x(s,\eta)\,ds$. Therefore, the above control system (3.1) driven by nonlocal fractional partial differential equations with time-varying delay can be written as the abstract form (1.1). Note $\|b(x)\|_C \le M^*$, by calculation, we can get that there exists a constant

$$r > \frac{M\mu(M^*+\|x_1\|)+(\Gamma(\alpha)+M\mu)\|\phi\|_C+M^*\Gamma(\alpha)}{\Gamma(\alpha)-(1+\frac{M\mu}{\Gamma(\alpha)})\|h\|_\infty}$$

such that

$$\left(1+\frac{M\mu}{\Gamma(\alpha)}\right)M^* + \frac{\|h\|_\infty(\Gamma(\alpha)+\mu M)}{\Gamma^2(\alpha)}r + \|\phi\|_C + \frac{M\mu(\|x_1\|+\|\phi\|_C)}{\Gamma(\alpha)} < r,$$

where $x_1\in X$ and $\|\phi\|_C = \sup_{t\in I,\eta\in[0,\pi]}|\phi(t,\eta)|$. Thus, (H6) holds. We can also easily see that the functions $f(t,\cdot): C([-\tau,1],X)\to X$ for each $t\in I$ and $b: C([-\tau,1],X)\to C([-\tau,0],X)$ are continuous and satisfy assumptions (H3) and (H4), respectively.

If the above conditions are given, it is easy to see that all the assumptions of Theorem 1.1 are satisfied. Then system (3.1) is controllable on I, *that is*, for given $x_1\in X$, we can find suitable $u\in L^\infty(I,U)$ which steers the solution x of system (3.1) to satisfy $x(1)=x_1$.

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