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Implicit coupled Hilfer–Hadamard fractional differential systems under weak topologies

Saïd Abbas[1], Mouffak Benchohra[2], Naima Hamidi[2] and Yong Zhou[3,4*]

*Correspondence: yzhou@xtu.edu.cn
[3]Faculty of Mathematics and Computational Science, Xiangtan University, Hunan, P.R. China
[4]Nonlinear Analysis and Applied Mathematics (NAAM) Research Group, Faculty of Science, King Abdulaziz University, Jeddah, Saudi Arabia
Full list of author information is available at the end of the article

Abstract

In this article, we present some existence of weak solutions for a coupled system of implicit fractional differential equations of Hilfer–Hadamard type. Our approach is based on Mönch's fixed point theorem associated with the technique of measure of weak noncompactness.

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1 Introduction

In recent years, fractional calculus and fractional differential equations are emerging as a useful tool in modeling the dynamics of many physical systems and electrical phenomena, which has been demonstrated by many researchers in the fields of mathematics, science, and engineering; see [3, 4, 18, 19, 22, 23, 30, 31, 35–40]. Recently, considerable attention has been given to the existence of solutions of initial and boundary value problems for fractional differential equations with Hilfer fractional derivative [15, 16, 18, 20, 32, 34] and other problems with Hilfer–Hadamard fractional derivative [28, 29].

The measure of weak noncompactness was introduced by De Blasi [14]. The strong measure of noncompactness was developed first by Banaś and Goebel [8] and subsequently developed and used in many papers; see, for example, Akhmerov et al. [6], Alvárez [7], Benchohra et al. [12], Guo et al. [17], and the references therein. In [12, 26], the authors considered some existence results by the technique of measure of noncompactness. Recently, several researchers obtained other results by the technique of measure of weak noncompactness; see [2, 4, 10, 11] and the references therein.

Consider the following coupled system of implicit Hilfer–Hadamard fractional differential equations:

$$\begin{cases} ({}^H D_1^{\alpha,\beta} u_1)(t) = f_1(t, u_1(t), u_2(t), ({}^H D_1^{\alpha,\beta} u_1)(t), ({}^H D_1^{\alpha,\beta} u_2)(t)), \\ ({}^H D_1^{\alpha,\beta} u_2)(t) = f_2(t, u_1(t), u_2(t), ({}^H D_1^{\alpha,\beta} u_1)(t), ({}^H D_1^{\alpha,\beta} u_2)(t)), \end{cases} \quad t \in I, \tag{1}$$

with the initial conditions

$$\begin{cases} ({}^H I_1^{1-\gamma} u_i)(t)|_{t=1} = \phi_1, \\ ({}^H I_1^{1-\gamma} u_2)(t)|_{t=1} = \phi_2, \end{cases} \tag{2}$$

where $I := [1, T], T > 1, \alpha \in (0, 1), \beta \in [0, 1], \gamma = \alpha + \beta - \alpha\beta, \phi_i \in E, f_i : I \times E^4 \to E, i = 1, 2,$ are given continuous functions, E is a real (or complex) Banach space with norm $\| \cdot \|_E$ and dual E^*, such that E is the dual of a weakly compactly generated Banach space X, ${}^H I_1^{1-\gamma}$ is the left-sided mixed Hadamard integral of order $1 - \gamma$, and ${}^H D_1^{\alpha,\beta}$ is the Hilfer–Hadamard fractional derivative of order α and type β. In this paper, we prove the existence of weak solutions for a coupled system of implicit fractional differential equations of Hilfer–Hadamard type.

2 Preliminaries

Let C be the Banach space of all continuous functions v from I into E with the supremum (uniform) norm

$$\|v\|_\infty := \sup_{t \in I} \| v(t) \|_E.$$

As usual, AC(I) denotes the space of absolutely continuous functions from I into E. We define the space

$$\mathrm{AC}^1(I) := \{ w : I \to E : w' \in \mathrm{AC}(I) \},$$

where $w'(t) = \frac{\mathrm{d}}{\mathrm{d}t} w(t), t \in I$. Let

$$\delta = t \frac{\mathrm{d}}{\mathrm{d}t}, \qquad n = [q] + 1,$$

where $[q]$ is the integer part of $q > 0$. Define the space

$$\mathrm{AC}_\delta^n := \{ u : [1, T] \to E : \delta^{n-1}(u) \in \mathrm{AC}(I) \}.$$

Let $\gamma \in (0, 1]$. By $\mathrm{C}_\gamma(I), \mathrm{C}_\gamma^1(I)$, and $\mathrm{C}_{\gamma,\ln}(I)$ we denote the weighted spaces of continuous functions defined by

$$\mathrm{C}_\gamma(I) = \{ w : (1, T] \to E : \in \bar{w} \in \mathrm{C} \},$$

where $\bar{w}(t) = t^{1-\gamma} w(t), t \in (1, T]$, with the norm

$$\|w\|_{\mathrm{C}_\gamma} := \sup_{t \in I} \| \bar{w}(t) \|_E,$$

$$\mathrm{C}_\gamma^1(I) = \{ w \in \mathrm{C} : w' \in \mathrm{C}_\gamma \}$$

with the norm

$$\|w\|_{\mathrm{C}_\gamma^1} := \|w\|_\infty + \| w' \|_{\mathrm{C}_\gamma},$$

and

$$\mathrm{C}_{\gamma,\ln}(I) = \{w : I \to E : \widetilde{w} \in \mathrm{C}\},$$

where $\widetilde{w}(t) = (\ln t)^{1-\gamma} w(t), t \in I$, with the norm

$$\|w\|_{C_{\gamma,\ln}} := \sup_{t\in I} \big\| \widetilde{w}(t) \big\|_E.$$

We further denote $\|w\|_{C_{\gamma,\ln}}$ by $\|w\|_C$.

Define the weighted product space $\mathcal{C} := C_{\gamma,\ln}(I) \times C_{\gamma,\ln}(I)$ with the norm

$$\big\|(w_1, w_2)\big\|_{\mathcal{C}} := \|w_1\|_C + \|w_2\|_C.$$

In the same way, we can define the the weighted product space $\overline{C} := (C_{\gamma,\ln}(I))^n$ with the norm

$$\big\|(w_1, w_2, \ldots, w_n)\big\|_{\overline{C}} := \sum_{k=1}^{n} \|w_k\|_C.$$

Let $(E, w) = (E, \sigma(E, E^*))$ be the Banach space E with weak topology.

Definition 2.1 A Banach space X is said to be weakly compactly generated (WCG) if it contains a weakly compact set whose linear span is dense in X.

Definition 2.2 A function $h : E \to E$ is said to be weakly sequentially continuous if h takes each weakly convergent sequence in E to a weakly convergent sequence in E (i.e., for any (u_n) in E with $u_n \to u$ in (E, w), we have $h(u_n) \to h(u)$ in (E, w)).

Definition 2.3 ([27]) The function $u : I \to E$ is said to be Pettis integrable on I if and only if there is an element $u_J \in E$ corresponding to each $J \subset I$ such that $\phi(u_J) = \int_J \phi(u(s))\,\mathrm{d}s$ for all $\phi \in E^*$, where the integral on the right-hand side is assumed to exist in the Lebesgue sense (by definition $u_J = \int_J u(s)\,\mathrm{d}s$).

Let $\mathrm{P}(I, E)$ be the space of all E-valued Pettis-integrable functions on I, and let $L^1(I, E)$ be the Banach space of Bochner-integrable measurable functions $u : I \to E$. Define the class

$$\mathrm{P}_1(I, E) = \big\{u \in P(I, E) : \varphi(u) \in L^1(I, \mathbb{R}) \text{ for every } \varphi \in E^*\big\}.$$

The space $\mathrm{P}_1(I, E)$ is normed by

$$\|u\|_{\mathrm{P}_1} = \sup_{\varphi\in E^*, \|\varphi\|\le 1} \int_1^T \big|\varphi\big(u(x)\big)\big|\,\mathrm{d}\lambda x,$$

where λ is the Lebesgue measure on I.

The following result is due to Pettis [27, Thm. 3.4 and Cor. 3.41].

Proposition 2.4 ([27]) *If $u \in \mathrm{P}_1(I, E)$ and h is a measurable and essentially bounded E-valued function, then $uh \in \mathrm{P}_1(I, E)$.*

In what follows, the symbol "$\int$" denotes the Pettis integral.

Now, we give some results and properties of fractional calculus.

Definition 2.5 ([3, 22, 30]) The left-sided mixed Riemann–Liouville integral of order $r > 0$ of a function $w \in L^1(I)$ is defined by

$$\big(I_1^r w\big)(t) = \frac{1}{\Gamma(r)} \int_1^t (t-s)^{r-1} w(s)\,\mathrm{d}s \quad \text{for a.e. } t \in I,$$

where Γ is the (Euler) gamma function defined by

$$\Gamma(\xi) = \int_0^\infty t^{\xi-1} e^{-t}\,\mathrm{d}t, \quad \xi > 0.$$

Notice that, for all $r, r_1, r_2 > 0$ and $w \in \mathrm{C}$, we have $I_0^r w \in \mathrm{C}$ and

$$\big(I_1^{r_1} I_1^{r_2} w\big)(t) = \big(I_1^{r_1+r_2} w\big)(t); \quad \text{for a.e. } t \in I.$$

Definition 2.6 ([3, 22, 30]) The Riemann–Liouville fractional derivative of order $r > 0$ of a function $w \in L^1(I)$ is defined by

$$\begin{aligned}\big(D_1^r w\big)(t) &= \left(\frac{\mathrm{d}^n}{\mathrm{d}t^n} I_1^{n-r} w\right)(t) \\ &= \frac{1}{\Gamma(n-r)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_1^t (t-s)^{n-r-1} w(s)\,\mathrm{d}s \quad \text{for a.e. } t \in I,\end{aligned}$$

where $n = [r] + 1$, and $[r]$ is the integer part of r.

In particular, if $r \in (0,1]$, then

$$\begin{aligned}\big(D_1^r w\big)(t) &= \left(\frac{\mathrm{d}}{\mathrm{d}t} I_1^{1-r} w\right)(t) \\ &= \frac{1}{\Gamma(1-r)} \frac{\mathrm{d}}{\mathrm{d}t} \int_1^t (t-s)^{-r} w(s)\,\mathrm{d}s \quad \text{for a.e. } t \in I.\end{aligned}$$

Let $r \in (0,1], \gamma \in [0,1)$, and $w \in \mathrm{C}_{1-\gamma}(I)$. Then the following expression leads to the left inverse operator:

$$\big(D_1^r I_1^r w\big)(t) = w(t) \quad \text{for all } t \in (1,T].$$

Moreover, if $I_1^{1-r} w \in C_{1-\gamma}^1(I)$, then the following composition is proved in [30]:

$$\big(I_1^r D_1^r w\big)(t) = w(t) - \frac{(I_1^{1-r} w)(1^+)}{\Gamma(r)} t^{r-1} \quad \text{for all } t \in (1,T].$$

Definition 2.7 ([3, 22, 30]) The Caputo fractional derivative of order $r > 0$ of a function $w \in L^1(I)$ is defined by

$$\big({}^c D_1^r w\big)(t) = \left(I_1^{n-r} \frac{\mathrm{d}^n}{\mathrm{d}t^n} w\right)(t)$$

$$= \frac{1}{\Gamma(n-r)} \int_1^t (t-s)^{n-r-1} \frac{\mathrm{d}^n}{\mathrm{d}s^n} w(s)\,\mathrm{d}s \quad \text{for a.e. } t \in I.$$

In particular, if $r \in (0,1]$, then

$$\begin{aligned}({}^c D_1^r w)(t) &= \left(I_1^{1-r} \frac{\mathrm{d}}{\mathrm{d}t} w \right)(t) \\ &= \frac{1}{\Gamma(1-r)} \int_1^t (t-s)^{-r} \frac{d}{\mathrm{d}s} w(s)\,\mathrm{d}s \quad \text{for a.e. } t \in I.\end{aligned}$$

Let us recall some definitions and properties of Hadamard fractional integration and differentiation. We refer to [22] for more details.

Definition 2.8 ([22]) The Hadamard fractional integral of order $q > 0$ for a function $g \in L^1(I,E)$ is defined as

$$({}^H I_1^q g)(x) = \frac{1}{\Gamma(q)} \int_1^x \left(\ln \frac{x}{s} \right)^{q-1} \frac{g(s)}{s}\,\mathrm{d}s,$$

provided that the integral exists.

Example 2.9 Let $0 < q < 1$. Then

$${}^H I_1^q \ln t = \frac{1}{\Gamma(2+q)} (\ln t)^{1+q} \quad \text{for a.e. } t \in [0,e].$$

Remark 2.10 Let $g \in \mathrm{P}_1(I,E)$. For every $\varphi \in E^*$, we have

$$\varphi({}^H I_1^q g)(t) = ({}^H I_1^q \varphi g)(t) \quad \text{for a.e. } t \in I.$$

Similarly to the Riemann–Liouville fractional calculus, the Hadamard fractional derivative is defined in terms of the Hadamard fractional integral as follows.

Definition 2.11 ([22]) The Hadamard fractional derivative of order $q > 0$ applied to a function $w \in \mathrm{AC}_\delta^n$ is defined as

$$({}^H D_1^q w)(x) = \delta^n ({}^H I_1^{n-q} w)(x).$$

In particular, if $q \in (0,1]$, then

$$({}^H D_1^q w)(x) = \delta ({}^H I_1^{1-q} w)(x).$$

Example 2.12 Let $0 < q < 1$. Then

$${}^H D_1^q \ln t = \frac{1}{\Gamma(2-q)} (\ln t)^{1-q} \quad \text{for a.e. } t \in [0,e].$$

It has been proved (see, e.g., Kilbas [21, Thm. 4.8]) that, in the space $L^1(I,E)$, the Hadamard fractional derivative is the left-inverse operator to the Hadamard fractional integral, that is,

$$\left({}^{H}D_1^q\right)\left({}^{H}I_1^q w\right)(x) = w(x).$$

From [22, Thm. 2.3] we have

$$\left({}^{H}I_1^q\right)\left({}^{H}D_1^q w\right)(x) = w(x) - \frac{({}^{H}I_1^{1-q}w)(1)}{\Gamma(q)}(\ln x)^{q-1}.$$

Similarly to the Hadamard fractional calculus, the Caputo–Hadamard fractional derivative is defined as follows.

Definition 2.13 The Caputo–Hadamard fractional derivative of order $q > 0$ applied to a function $w \in AC_\delta^n$ is defined as

$$\left({}^{Hc}D_1^q w\right)(x) = \left({}^{H}I_1^{n-q}\delta^n w\right)(x).$$

In particular, if $q \in (0,1]$, then

$$\left({}^{Hc}D_1^q w\right)(x) = \left({}^{H}I_1^{1-q}\delta w\right)(x).$$

Hilfer [18] studied applications of the generalized fractional operator having the Riemann–Liouville and the Caputo derivatives as particular cases (see also [20, 32]).

Definition 2.14 Let $\alpha \in (0,1), \beta \in [0,1], w \in L^1(I)$ and $I_1^{(1-\alpha)(1-\beta)}w \in AC^1(I)$. The Hilfer fractional derivative of order α and type β of w is defined as

$$\left(D_1^{\alpha,\beta}w\right)(t) = \left(I_1^{\beta(1-\alpha)}\frac{d}{dt}I_1^{(1-\alpha)(1-\beta)}w\right)(t) \quad \text{for a.e. } t \in I. \tag{3}$$

Properties Let $\alpha \in (0,1), \beta \in [0,1], \gamma = \alpha + \beta - \alpha\beta$, and $w \in L^1(I)$.

1. The operator $(D_1^{\alpha,\beta}w)(t)$ can be written as

$$\left(D_1^{\alpha,\beta}w\right)(t) = \left(I_1^{\beta(1-\alpha)}\frac{d}{dt}I_1^{1-\gamma}w\right)(t) = \left(I_1^{\beta(1-\alpha)}D_1^{\gamma}w\right)(t) \quad \text{for a.e. } t \in I.$$

   Moreover, the parameter γ satisfies

$$\gamma \in (0,1], \qquad \gamma \geq \alpha, \qquad \gamma > \beta, \qquad 1-\gamma < 1-\beta(1-\alpha).$$

2. For $\beta = 0$, generalization (3) coincides with the Riemann–Liouville derivative and for $\beta = 1$, with the Caputo derivative:

$$D_1^{\alpha,0} = D_1^{\alpha} \quad \text{and} \quad D_1^{\alpha,1} = {}^{c}D_1^{\alpha}.$$

3. If $D_1^{\beta(1-\alpha)}w$ exists and is in $L^1(I)$, then

$$\left(D_1^{\alpha,\beta}I_1^{\alpha}w\right)(t) = \left(I_1^{\beta(1-\alpha)}D_1^{\beta(1-\alpha)}w\right)(t) \quad \text{for a.e. } t \in I.$$

Furthermore, if $w \in C_\gamma(I)$ and $I_1^{1-\beta(1-\alpha)} w \in C_\gamma^1(I)$, then

$$\left(D_1^{\alpha,\beta} I_1^\alpha w\right)(t) = w(t) \quad \text{for a.e. } t \in I.$$

4. If $D_1^\gamma w$ exists and is in $L^1(I)$, then

$$\left(I_1^\alpha D_1^{\alpha,\beta} w\right)(t) = \left(I_1^\gamma D_1^\gamma w\right)(t) = w(t) - \frac{I_1^{1-\gamma}(1^+)}{\Gamma(\gamma)} t^{\gamma-1} \quad \text{for a.e. } t \in I.$$

Based on the Hadamard fractional integral, the Hilfer–Hadamard fractional derivative (introduced for the first time in [28]) is defined as follows.

Definition 2.15 Let $\alpha \in (0,1), \beta \in [0,1]$, $\gamma = \alpha + \beta - \alpha\beta, w \in L^1(I)$, and ${}^H I_1^{(1-\alpha)(1-\beta)} w \in AC^1(I)$. The Hilfer–Hadamard fractional derivative of order α and type β applied to a function w is defined as

$$\begin{aligned}\left({}^H D_1^{\alpha,\beta} w\right)(t) &= \left({}^H I_1^{\beta(1-\alpha)}\left({}^H D_1^\gamma w\right)\right)(t) \\ &= \left({}^H I_1^{\beta(1-\alpha)} \delta\left({}^H I_1^{1-\gamma} w\right)\right)(t) \quad \text{for a.e. } t \in I.\end{aligned} \tag{4}$$

This new fractional derivative (4) may be viewed as interpolation of the Hadamard and Caputo–Hadamard fractional derivatives. Indeed, for $\beta = 0$, this derivative reduces to the Hadamard fractional derivative, and, for $\beta = 1$, we recover the Caputo–Hadamard fractional derivative:

$${}^H D_1^{\alpha,0} = {}^H D_1^\alpha \quad \text{and} \quad {}^H D_1^{\alpha,1} = {}^{Hc} D_1^\alpha.$$

From [29, Thm. 21] we have the following lemma.

Lemma 2.16 *Let $f_i : I \times E^4 \to E, i = 1,2$, be such that $f_i(\cdot, u, v, \bar{u}, \bar{v}) \in C_{\gamma,\ln}(I)$ for any $u, v, \bar{u}, \bar{v} \in C_{\gamma,\ln}(I)$. Then system (1)–(2) is equivalent to the problem of obtaining the solution of the coupled system*

$$\begin{cases} g_1(t) = f_1(t, \frac{\phi_1}{\Gamma(\gamma)}(\ln t)^{\gamma-1} + ({}^H I_1^\alpha g_1)(t), \frac{\phi_2}{\Gamma(\gamma)}(\ln t)^{\gamma-1} + ({}^H I_1^\alpha g_2)(t), g_1(t), g_2(t)), \\ g_2(t) = f_2(t, \frac{\phi_1}{\Gamma(\gamma)}(\ln t)^{\gamma-1} + ({}^H I_1^\alpha g_1)(t), \frac{\phi_2}{\Gamma(\gamma)}(\ln t)^{\gamma-1} + ({}^H I_1^\alpha g_2)(t), g_1(t), g_2(t)), \end{cases}$$

and if $g_i(\cdot) \in C_{\gamma,\ln}$ are the solutions of this system, then

$$\begin{cases} u_1(t) = \frac{\phi_1}{\Gamma(\gamma)}(\ln t)^{\gamma-1} + ({}^H I_1^\alpha g_1)(t), \\ u_2(t) = \frac{\phi_2}{\Gamma(\gamma)}(\ln t)^{\gamma-1} + ({}^H I_1^\alpha g_2)(t). \end{cases}$$

Definition 2.17 ([14]) Let E be a Banach space, let Ω_E be the set of bounded subsets of E, and let B_1 be the unit ball of E. The De Blasi measure of weak noncompactness is the map $\mu : \Omega_E \to [0,\infty)$ defined by

$$\mu(X) = \inf\{\varepsilon > 0 : \text{there exists a weakly compact set } \Omega \subset E \text{ such that } X \subset \varepsilon B_1 + \Omega\}.$$

The De Blasi measure of weak noncompactness satisfies the following properties:

(a) $A \subset B \Rightarrow \mu(A) \leq \mu(B)$,
(b) $\mu(A) = 0 \Leftrightarrow A$ is weakly relatively compact,
(c) $\mu(A \cup B) = \max\{\mu(A), \mu(B)\}$,
(d) $\mu(\overline{A}^{\omega}) = \mu(A)$, where $\overline{A}^{\omega}$ denotes the weak closure of A,
(e) $\mu(A + B) \leq \mu(A) + \mu(B)$,
(f) $\mu(\lambda A) = |\lambda|\mu(A)$,
(g) $\mu(\operatorname{conv}(A)) = \mu(A)$,
(h) $\mu(\bigcup_{|\lambda| \leq h} \lambda A) = h\mu(A)$.

The next result follows directly from the Hahn–Banach theorem.

Proposition 2.18 *If E is a normed space and $x_0 \in E - \{0\}$, then there exists $\varphi \in E^*$ with $\|\varphi\| = 1$ and $\varphi(x_0) = \|x_0\|$.*

For a given set V of functions $v : I \to E$, let us denote

$$V(t) = \{v(t) : v \in V\}; \quad t \in I \quad \textit{and} \quad V(I) = \{v(t) : v \in V, t \in I\}.$$

Lemma 2.19 ([17]) *Let $H \subset C$ be a bounded equicontinuous subset. Then the function $t \to \mu(H(t))$ is continuous on I,*

$$\mu_C(H) = \max_{t \in I} \mu(H(t)),$$

and

$$\mu\left(\int_I u(s)\,ds\right) \leq \int_I \mu(H(s))\,ds,$$

where $H(t) = \{u(t) : u \in H\}, t \in I$, and μ_C is the De Blasi measure of weak noncompactness defined on the bounded sets of C.

For our purpose, we will need the following fixed point theorem.

Theorem 2.20 ([25]) *Let Q be a nonempty, closed, convex, and equicontinuous subset of a metrizable locally convex vector space $C(I,E)$ such that $0 \in Q$. Suppose $T : Q \to Q$ is weakly sequentially continuous. If the implication*

$$\overline{V} = \overline{\operatorname{conv}}(\{0\} \cup T(V)) \Rightarrow V \quad \text{is relatively weakly compact} \tag{5}$$

holds for every subset $V \subset Q$, then the operator T has a fixed point.

3 Existence of weak solutions

Let us start by the definition of a weak solution of problem (1).

Definition 3.1 By a weak solution of the coupled system (1)–(2) we mean a coupled measurable functions $(u_1, u_2) \in \mathcal{C}$ such that $({}^H I_1^{1-\gamma} u_i)(1^+) = \phi_i, i = 1, 2$, and the equations $({}^H D_1^{\alpha,\beta} u_i)(t) = f_i(t, u_1(t), u_2(t), ({}^H D_1^{\alpha,\beta} u_1)(t), ({}^H D_1^{\alpha,\beta} u_2)(t))$ are satisfied on I.

We further will use the following hypotheses.

(H_1) The functions $v \to f_i(t,v,w,\bar{v},\bar{w}), w \to f_i(t,v,w,\bar{v},\bar{w}), \bar{v} \to f_i(t,v,w,\bar{v},\bar{w})$, and $\bar{w} \to f_i(t,v,w,\bar{v},\bar{w}), i = 1,2$, are weakly sequentially continuous for a.e. $t \in I$,

(H_2) For all $v,w,\bar{v},\bar{w} \in E$, the functions $t \to f_i(t,v,w,\bar{v},\bar{w}), i = 1,2$, are Pettis integrable a.e. on I,

(H_3) There exist $p_i, q_i \in C(I,[0,\infty))$ such that, for all $\varphi \in E^*$,

$$\left|\varphi\big(f_i(t,u,v,\bar{u},\bar{v})\big)\right| \le \frac{p_i(t)\|u\|_E + q_i(t)\|v\|_E}{1 + \|\varphi\| + \|u\|_E + \|v\|_E + \|\bar{u}\|_E + \|\bar{v}\|_E}$$

for a.e. $t \in I$ and all $u,v,\bar{u},\bar{v} \in E$,

(H_4) For all bounded measurable sets $B_i \subset E, i = 1,2$, and all $t \in I$, we have

$$\mu\big(f_1\big(t,B_1,B_2,{}^H D_1^{\alpha,\beta}B_1,{}^H D_1^{\alpha,\beta}B_2\big),0\big) \le p_1(t)\mu(B_1) + q_1(t)\mu(B_2)$$

and

$$\mu\big(0,f_2\big(t,B_1,B_2,{}^H D_1^{\alpha,\beta}B_1,{}^H D_1^{\alpha,\beta}B_2\big)\big) \le p_2(t)\mu(B_1) + q_2(t)\mu(B_2),$$

where ${}^H D_1^{\alpha,\beta}B_i = \{{}^H D_1^{\alpha,\beta}w : w \in B_i\}, i = 1,2$.

Set

$$p_i^* = \sup_{t\in I} p_i(t) \quad \text{and} \quad q_i^* = \sup_{t\in I} q_i(t), \quad i = 1,2.$$

Theorem 3.2 *Assume that the hypotheses* (H_1)–(H_4) *hold. If*

$$L := \frac{(p_1^* + p_2^* + q_1^* + q_2^*)(\ln T)^\alpha}{\Gamma(1+\alpha)} < 1, \tag{6}$$

then the coupled system (1)–(2) *has at least one weak solution defined on* I.

Proof Consider the operators $N_i : C_{\gamma,\ln} \to C_{\gamma,\ln}, i = 1,2$, defined by

$$(N_i u_i)(t) = \frac{\phi_i}{\Gamma(\gamma)}(\ln t)^{\gamma-1} + \big({}^H I_1^\alpha g_i\big)(t),$$

where $g_i \in C_{\gamma,\ln}, i = 1,2$, are defined as

$$g_i(t) = f_i\left(t, \frac{\phi_1}{\Gamma(\gamma)}(\ln t)^{\gamma-1} + \big({}^H I_1^\alpha g_1\big)(t), \frac{\phi_2}{\Gamma(\gamma)}(\ln t)^{\gamma-1} + \big({}^H I_1^\alpha g_2\big)(t), g_1(t), g_2(t)\right).$$

Consider the operator $N : \mathcal{C} \to \mathcal{C}$ such that, for any $(u_1,u_2) \in \mathcal{C}$,

$$\big(N(u_1,u_2)\big)(t) = \big((N_1u_1)(t),(N_2u_2)(t)\big). \tag{7}$$

First, notice that the hypotheses imply that, for each $g_i \in C_{\gamma,\ln}, i = 1,2$, the function

$$t \mapsto \left(\ln\frac{t}{s}\right)^{\alpha-1} g_i(s)$$

is Pettis integrable over I, and

$$t \mapsto f_i\left(t, \frac{\phi_1}{\Gamma(\gamma)}(\ln t)^{\gamma-1} + \left({}^H I_1^\alpha g_1\right)(t), \frac{\phi_2}{\Gamma(\gamma)}(\ln t)^{\gamma-1} + \left({}^H I_1^\alpha g_2\right)(t), g_1(t), g_2(t)\right)$$

for a.e. $t \in I$ is Pettis integrable. Thus, the operator N is well defined. Let $R > 0$ be such that $R > L_1 + L_2$, where

$$L_i := \frac{(p_i^* + q_i^*)(\ln T)^{1-\gamma+\alpha}}{\Gamma(1+\alpha)}, \quad i = 1, 2,$$

and consider the set

$$\begin{aligned} Q = \Bigg\{ &(u_1, u_2) \in \mathcal{C} : \|(u_1, u_2)\|_{\mathcal{C}} \le R \text{ and } \|(\ln t_2)^{1-\gamma} u_i(t_2) - (\ln t_1)^{1-\gamma} u_i(t_1)\|_E \\ &\le L_i \left(\ln \frac{t_2}{t_1}\right)^\alpha \\ &\quad + \frac{p_i^* + q_i^*}{\Gamma(\alpha)} \int_1^{t_1} \left| (\ln t_2)^{1-\gamma} \left(\ln \frac{t_2}{s}\right)^{\alpha-1} - (\ln t_1)^{1-\gamma} \left(\ln \frac{t_1}{s}\right)^{\alpha-1} \right| ds, i = 1, 2 \Bigg\}. \end{aligned}$$

Clearly, the subset Q is closed, convex, and equicontinuous. We will show that the operator N satisfies all the assumptions of Theorem 2.20. The proof will be given in several steps.

Step 1. *N maps Q into itself.* Let $(u_1, u_2) \in Q, t \in I$, and assume that $(N(u_1, u_2))(t) \neq (0.0)$. Then there exists $\varphi \in E^*$ such that $\|(\ln t)^{1-\gamma}(N_i u_i)(t)\|_E = |\varphi((\ln t)^{1-\gamma}(N_i u_i)(t))|, i = 1, 2$. Thus, for any $i \in \{1, 2\}$, we have

$$\|(\ln t)^{1-\gamma}(N_i u_i)(t)\|_E = \varphi\left(\frac{\phi_i}{\Gamma(\gamma)} + \frac{(\ln t)^{1-\gamma}}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} g_i(s) \frac{ds}{s}\right),$$

where $g_i \in C_{\gamma,\ln}$ are defined as

$$g_i(t) = f_i\left(t, \frac{\phi_1}{\Gamma(\gamma)}(\ln t)^{\gamma-1} + \left({}^H I_1^\alpha g_1\right)(t), \frac{\phi_2}{\Gamma(\gamma)}(\ln t)^{\gamma-1} + \left({}^H I_1^\alpha g_2\right)(t), g_1(t), g_2(t)\right).$$

Then from (H_3) we get

$$\left|\varphi\left(g_i(t)\right)\right| \le p_i^* + q_i^*.$$

Thus

$$\begin{aligned} &\|(\ln t)^{1-\gamma}(N_i u_i)(t)\|_E \\ &\quad \le \frac{(\ln t)^{1-\gamma}}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \left|\varphi\left(g_i(s)\right)\right| \frac{ds}{s} \\ &\quad \le \frac{(p_i^* + q_i^*)(\ln T)^{1-\gamma}}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{ds}{s} \\ &\quad \le \frac{(p_i^* + q_i^*)(\ln T)^{1-\gamma+\alpha}}{\Gamma(1+\alpha)} \\ &\quad = L_i. \end{aligned}$$

Hence we get

$$\left\|N(u_1,u_1)\right\|_{\mathcal{C}} \le L_1 + L_2 < R.$$

Next, let $t_1, t_2 \in I$ be such that $t_1 < t_2$, and let $u \in Q$ be such that

$$(\ln t_2)^{1-\gamma}(N_i u_i)(t_2) - (\ln t_1)^{1-\gamma}(N_i u_i)(t_1) \neq 0.$$

Then there exists $\varphi \in E^*$ such that

$$\begin{aligned}&\left\|(\ln t_2)^{1-\gamma}(N_i u_i)(t_2) - (\ln t_1)^{1-\gamma}(N_i u_i)(t_1)\right\|_E \\ &\quad = \left|\varphi\big((\ln t_2)^{1-\gamma}(N_i u_i)(t_2) - (\ln t_1)^{1-\gamma}(N_i u_i)(t_1)\big)\right|\end{aligned}$$

and $\|\varphi\| = 1$. Then, for any $i \in \{1,2\}$, we have

$$\begin{aligned}&\left\|(\ln t_2)^{1-\gamma}(N_i u_i)(t_2) - (\ln t_1)^{1-\gamma}(N_i u_i)(t_1)\right\|_E \\ &\quad = \left|\varphi\big((\ln t_2)^{1-\gamma}(N_i u_i)(t_2) - (\ln t_1)^{1-\gamma}(N_i u_i)(t_1)\big)\right| \\ &\quad \le \varphi\left((\ln t_2)^{1-\gamma}\int_1^{t_2}\left(\ln\frac{t_2}{s}\right)^{\alpha-1}\frac{g_i(s)}{s\Gamma(\alpha)}\,\mathrm{d}s - (\ln t_1)^{1-\gamma}\int_1^{t_1}\left(\ln\frac{t_1}{s}\right)^{\alpha-1}\frac{g_i(s)}{s\Gamma(\alpha)}\,\mathrm{d}s\right),\end{aligned}$$

where $g_i \in \mathrm{C}_{\gamma,\ln}$ are defined as

$$g_i(t) = f_i\left(t, \frac{\phi_1}{\Gamma(\gamma)}(\ln t)^{\gamma-1} + \left({}^H I_1^\alpha g_1\right)(t), \frac{\phi_1}{\Gamma(\gamma)}(\ln t)^{\gamma-1} + \left({}^H I_1^\alpha g_2\right)(t), g_1(t), g_2(t)\right).$$

Then

$$\begin{aligned}&\left\|(\ln t_2)^{1-\gamma}(N_i u_i)(t_2) - (\ln t_1)^{1-\gamma}(N_i u_i)(t_1)\right\|_E \\ &\quad \le (\ln t_2)^{1-\gamma}\int_{t_1}^{t_2}\left(\ln\frac{t_2}{s}\right)^{\alpha-1}\frac{|\varphi(g_i(s))|}{s\Gamma(\alpha)}\,\mathrm{d}s \\ &\qquad + \int_1^{t_1}\left|(\ln t_2)^{1-\gamma}\left(\ln\frac{t_2}{s}\right)^{\alpha-1} - (\ln t_1)^{1-\gamma}\left(\ln\frac{t_1}{s}\right)^{\alpha-1}\right|\frac{|\varphi(g_i(s))|}{s\Gamma(\alpha)}\,\mathrm{d}s \\ &\quad \le (\ln t_2)^{1-\gamma}\int_{t_1}^{t_2}\left(\ln\frac{t_2}{s}\right)^{\alpha-1}\frac{p_i(s)+q_i(s)}{s\Gamma(\alpha)}\,\mathrm{d}s \\ &\qquad + \int_1^{t_1}\left|(\ln t_2)^{1-\gamma}\left(\ln\frac{t_2}{s}\right)^{\alpha-1} - (\ln t_1)^{1-\gamma}\left(\ln\frac{t_1}{s}\right)^{\alpha-1}\right|\frac{p_i(s)+q_i(s)}{s\Gamma(\alpha)}\,\mathrm{d}s.\end{aligned}$$

Thus, we get

$$\begin{aligned}&\left\|(\ln t_2)^{1-\gamma}(N_i u_i)(t_2) - (\ln t_1)^{1-\gamma}(N_i u_i)(t_1)\right\|_E \\ &\quad \le L_i\left(\ln\frac{t_2}{t_1}\right)^{\alpha} \\ &\qquad + \frac{p_i^* + q_i^*}{\Gamma(\alpha)}\int_1^{t_1}\left|(\ln t_2)^{1-\gamma}\left(\ln\frac{t_2}{s}\right)^{\alpha-1} - (\ln t_1)^{1-\gamma}\left(\ln\frac{t_1}{s}\right)^{\alpha-1}\right|\mathrm{d}s.\end{aligned}$$

Hence $N(Q) \subset Q$.

Step 2. *N is weakly sequentially continuous.* Let $\{(u_n,v_n)\}_n$ be a sequence in Q, and let $(u_n(t),v_n(t)\to(u(t),v(t))$ in $(E,\omega)\times(E,\omega)$ for each $t\in I$. Fix $t\in I$. Since for any $i\in 1,2$, the function f_i satisfies assumption (H_1), we have that $f_i(t,u_n(t),v_n(t),({}^HD_1^{\alpha,\beta}u_n)(t),({}^HD_1^{\alpha,\beta}v_n)(t))$ converges weakly uniformly to $f_i(t,u(t),v(t),(D_0^{\alpha,\beta}u)(t),(D_0^{\alpha,\beta}v)(t))$. Hence the Lebesgue dominated convergence theorem for Pettis integral implies that $(N(u_n,v_n))(t)$ converges weakly uniformly to $(N(u,v))(t)$ in (E,ω) for each $t\in I$. Thus $N(u_n,v_n)\to N(u,v)$. Hence $N:Q\to Q$ is weakly sequentially continuous.

Step 3. *Implication (5) holds.* Let V be a subset of Q such that $\overline{V}=\overline{\operatorname{conv}}(N(V)\cup\{(0,0)\})$. Obviously,

$$V(t)\subset\overline{\operatorname{conv}}(NV)(t))\cup\bigl\{(0,0)\bigr\}),\quad t\in I.$$

Further, as V is bounded and equicontinuous, by [13, Lemma 3] the function $t\to\mu(V(t))$ is continuous on I. From (H_3), (H_4), Lemma 2.19, and the properties of the measure μ, for any $t\in I$, we have

$$\begin{aligned}
&\mu\bigl((\ln t)^{1-\gamma}V(t)\bigr)\\
&\quad\le\mu\bigl((\ln t)^{1-\gamma}(NV)(t)\cup\bigl\{(0,0)\bigr\}\bigr)\\
&\quad\le\mu\bigl((\ln t)^{1-\gamma}(NV)(t)\bigr)\\
&\quad\le\mu(\bigl\{\bigl((\ln t)^{1-\gamma}(N_1v_1)(t),(\ln t)^{1-\gamma}(N_2v_2)(t):(v_1,v_2)\in V\bigr\}\bigr)\\
&\quad\le\frac{1}{\Gamma(\alpha)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\mu\bigl(\bigl\{(\ln s)^{1-\gamma}\bigl(f_1\bigl(s,v_1(s),v_2(s),\\
&\qquad\bigl({}^HD_1^{\alpha,\beta}v_1\bigr)(t),\bigl({}^HD_1^{\alpha,\beta}v_2\bigr)(t)\bigr),0\bigr):(v_1,v_2)\in V\bigr\}\bigr)\frac{ds}{s}\\
&\qquad+\frac{1}{\Gamma(\alpha)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\mu\bigl(\bigl\{(\ln s)^{1-\gamma}\bigl(0,f_2\bigl(s,v_1(s),v_2(s),\\
&\qquad\bigl({}^HD_1^{\alpha,\beta}v_1\bigr)(t),\bigl({}^HD_1^{\alpha,\beta}v_2\bigr)(t)\bigr)\bigr):(v_1,v_2)\in V\bigr\}\bigr)\frac{ds}{s}\\
&\quad\le\frac{1}{\Gamma(\alpha)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\bigl[p_1(s)\mu\bigl(\bigl\{(\ln s)^{1-\gamma}\bigl(v_1(s),0\bigr):(v_1,0)\in V\bigr\}\bigr)\\
&\qquad+q_1(s)\mu\bigl(\bigl\{(\ln s)^{1-\gamma}\bigl(0,v_2(s)\bigr):(0,v_2)\in V\bigr\}\bigr)\bigr]\frac{ds}{s}\\
&\qquad+\frac{1}{\Gamma(\alpha)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\bigl[p_2(s)\mu\bigl(\bigl\{(\ln s)^{1-\gamma}\bigl(v_1(s),0\bigr):(v_1,0)\in V\bigr\}\bigr)\\
&\qquad+q_2(s)\mu\bigl(\bigl\{(\ln s)^{1-\gamma}\bigl(0,v_2(s)\bigr):(0,v_2)\in V\bigr\}\bigr)\bigr]\frac{ds}{s}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\mu\bigl((\ln t)^{1-\gamma}V(t)\bigr)\\
&\quad\le\frac{1}{\Gamma(\alpha)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\bigl(p_1(s)+q_1(s)+p_2(s)+q_2(s)\bigr)\\
&\qquad\times\mu\bigl((\ln s)^{1-\gamma}V(s)\bigr)\frac{ds}{s}
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{\Gamma(\alpha)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\big(p_1(s)+q_1(s)+p_2(s)+q_2(s)\big)\\
&\quad\times\sup_{s\in I}\mu\big((\ln s)^{1-\gamma}V(s)\big)\frac{ds}{s}\\
&\leq \frac{(p_1^*+p_2^*+q_1^*+q_2^*)(\ln T)^{\alpha}}{\Gamma(1+\alpha)}\sup_{t\in I}\mu\big((\ln t)^{1-\gamma}V(t)\big).
\end{aligned}$$

Hence

$$\sup_{t\in I}\mu\big((\ln t)^{1-\gamma}V(t)\big)\leq L\sup_{t\in I}\mu\big((\ln t)^{1-\gamma}V(t)\big).$$

From (6) we get $\sup_{t\in I}\mu((\ln t)^{1-\gamma}V(t))=0$, that is, $\mu(V(t))=0$ for each $t\in I$. Then by [24, Thm. 2] V is weakly relatively compact in $\mathcal{C}$. From Theorem 2.20 we conclude that N has a fixed point, which is a weak solution of the coupled system (1)–(2). □

As a consequence of the theorem, we get the following corollary.

Corollary 3.3 *Consider the following system of implicit Hilfer–Hadamard fractional differential equations*:

$$\begin{cases}
({}^HD_1^{\alpha,\beta}u_1)(t)\\
\quad = f_1(t,u_1(t),u_2(t),\dots,u_n(t),\\
\qquad ({}^HD_1^{\alpha,\beta}u_1)(t),({}^HD_1^{\alpha,\beta}u_2)(t),\dots,({}^HD_1^{\alpha,\beta}u_n)(t)),\\
({}^HD_1^{\alpha,\beta}u_2)(t)\\
\quad = f_2(t,u_1(t),u_2(t),\dots,u_n(t),\\
\qquad ({}^HD_1^{\alpha,\beta}u_1)(t),({}^HD_1^{\alpha,\beta}u_2)(t),\dots,({}^HD_1^{\alpha,\beta}u_n)(t)),\\
\vdots\\
({}^HD_1^{\alpha,\beta}u_n)(t)\\
\quad = f_n(t,u_1(t),u_2(t),\dots,u_n(t),\\
\qquad ({}^HD_1^{\alpha,\beta}u_1)(t),({}^HD_1^{\alpha,\beta}u_2)(t),\dots,({}^HD_1^{\alpha,\beta}u_n)(t)),
\end{cases}\quad t\in I, \tag{8}$$

$$\big({}^HI_1^{1-\gamma}u_i\big)(t)|_{t=1}=\phi_i,\quad i=1,2,\dots,n, \tag{9}$$

$I:=[1,T], T>1, \alpha\in(0,1), \beta\in[0,1], \gamma=\alpha+\beta-\alpha\beta, \phi_i\in E, f_i: I\times E^{2n}\to E, i=1,2,\dots,n$, are given continuous functions, E is a real (*or complex*) *Banach space with norm $\|\cdot\|_E$ and dual E^*, such that E is the dual of a weakly compactly generated Banach space X, ${}^HI_1^{1-\gamma}$ is the left-sided mixed Hadamard integral of order $1-\gamma$, and ${}^HD_1^{\alpha,\beta}$ is the Hilfer–Hadamard fractional derivative of order α and type β.*

Assume that the following hypotheses hold:

(H_{01}) *The functions $v_j\to f_i(t,v_1,v_2,\dots,v_j,\dots,v_{2n}), i=1,\dots,n, j=1,\dots,2n$, are weakly sequentially continuous* for a.e. $t\in I$,

(H_{02}) *For each $v_j\in E, j=1,\dots,2n$, the functions $t\to f_i(t,v_1,v_2,\dots,v_j,\dots,v_{2n}), i=1,2$, are Pettis integrable a.e. on I*,

(H_{03}) *There exist* $p_{ij} \in C(I,[0,\infty))$ *such that, for all* $\varphi \in E^*$, *we have*

$$\left|\varphi\big(f_i(t,v_1,v_2,\dots,v_{2n})\big)\right| \le \frac{\sum_{i=1}^{n}\sum_{j=1}^{n} p_{ij}(t)\|v_j\|_E}{1+\|\varphi\|+\sum_{j=1}^{n}\|v_i\|_E}$$

for a.e. $t \in I$ *and each* $v_i \in E, i = 1,2,\dots,n$,

(H_{04}) *For all bounded measurable sets* $B_i \subset E, i = 1,\dots,n$, *and for each* $t \in I$, *we have*

$$\begin{aligned}&\mu\big(0,\dots,f_j\big(t,B_1,B_2,\dots,B_n,{}^H D_1^{\alpha,\beta}B_1,{}^H D_1^{\alpha,\beta}B_2,\dots,{}^H D_1^{\alpha,\beta}B_n\big),\dots,0\big)\\&\quad\le \sum_{i=1}^{n} p_{ij}(t)\mu(B_i), \quad j=1,\dots,n,\end{aligned}$$

where ${}^H D_1^{\alpha,\beta}B_i = \{{}^H D_1^{\alpha,\beta}w : w \in B_i\}, i = 1,\dots,n$.

If

$$L^* := \frac{\sum_{i=1}^{n}\sum_{j=1}^{n} p_{ij}^*(\ln T)^\alpha}{\Gamma(1+\alpha)} < 1,$$

where

$$p_{ij}^* = \sup_{t\in I} p_{ij}(t), \quad i,j = 1,\dots,n,$$

then the coupled system (8)–(9) *has at least one weak solution defined on* I.

4 An example

Let

$$E = l^1 = \Big\{u = (u_1,u_2,\dots,u_n,\dots), \sum_{n=1}^{\infty}|u_n| < \infty\Big\}$$

be the Banach space with the norm

$$\|u\|_E = \sum_{n=1}^{\infty}|u_n|.$$

As an application of our results, we consider the coupled system of Hilfer–Hadamard fractional differential equations

$$\begin{cases}({}^H D_1^{\frac12,\frac12}u_n)(t) = f_n(t,u(t),v(t),({}^H D_1^{\frac12,\frac12}u_n)(t),({}^H D_1^{\frac12,\frac12}v_n)(t)),\\ ({}^H D_1^{\frac12,\frac12}v_n)(t) = g_n(t,u(t),v(t),({}^H D_1^{\frac12,\frac12}u_n)(t),({}^H D_1^{\frac12,\frac12}v_n)(t)),\end{cases} \quad t \in [1,e], \tag{10}$$

$$\big({}^H I_1^{\frac14}u\big)(t)|_{t=1} = \big({}^H I_1^{\frac14}v\big)(t)|_{t=1} = (0,0,\dots,0,\dots), \tag{11}$$

where

$$f_n\big(t,u(t),v(t)\big) = \frac{ct^2}{1+\|u(t)\|_E+\|v(t)\|_E+\|\bar u(t)\|_E+\|\bar v(t)\|_E}\,\frac{u_n(t)}{e^{t+4}}, \quad t \in [1,e],$$

and

$$g_n\big(t,u(t),v(t)\big) = \frac{ct^2}{1+\|v(t)\|_E+\|v(t)\|_E+\|\bar{u}(t)\|_E+\|\bar{v}(t)\|_E}\frac{u_n(t)}{e^{t+4}}, \quad t\in[1,e],$$

with

$$u=(u_1,u_2,\dots,u_n,\dots), \qquad v=(v_1,v_2,\dots,v_n,\dots) \quad \text{and} \quad c:=\frac{e^3}{16}\sqrt{\pi}.$$

Set

$$f=(f_1,f_2,\dots,f_n,\dots) \quad \text{and} \quad g=(g_1,g_2,\dots,g_n,\dots).$$

Clearly, the functions f and g are continuous.

For all $u,v,\bar{u},\bar{v}\in E$ and $t\in[1,e]$, we have

$$\big\|f\big(t,u(t),v(t),\bar{u}(t),\bar{v}(t)\big)\big\|_E \le ct^2\frac{1}{e^{t+4}} \quad \text{and} \quad \big\|g\big(t,u(t),v(t),\bar{u}(t),\bar{v}(t)\big)\big\|_E \le ct^2\frac{1}{e^{t+4}}.$$

Hence, hypothesis (H_3) is satisfied with $p_i^*=ce^{-3}$ and $q_i^*=0, i=1,2$. We will show that condition (6) holds with $T=e$. Indeed,

$$\frac{(p_1^*+q_1^*+p_2^*+q_2^*)(\ln T)^\alpha}{\Gamma(1+\alpha)} = \frac{4ce^{-3}}{\sqrt{\pi}} = \frac{1}{4} < 1.$$

Simple computations show that all conditions of Theorem 3.2 are satisfied. It follows that the coupled system (10)–(11) has at least one weak solution defined on $[1,e]$.

5 Conclusion

In the recent years, implicit functional differential equations have been considered by many authors [1, 5, 9, 33]. In this work, we give some existence results for coupled implicit Hilfer–Hadamard fractional differential systems. This paper initiates the application of the measure of weak noncompactness to such a class of problems.

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Author details
[1]Laboratory of Mathematics, Geometry, Analysis, Control and Applications, Tahar Moulay University of Saïda, Saïda, Algeria. [2]Laboratory of Mathematics, Djillali Liabes University of Sidi Bel-Abbès, Sidi Bel-Abbès, Algeria. [3]Faculty of Mathematics and Computational Science, Xiangtan University, Hunan, P.R. China. [4]Nonlinear Analysis and Applied Mathematics (NAAM) Research Group, Faculty of Science, King Abdulaziz University, Jeddah, Saudi Arabia.

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References

1. Abbas, S., Benchohra, M., Bohner, M.: Weak solutions for implicit differential equations of Hilfer–Hadamard fractional derivative. Adv. Dyn. Syst. Appl. **12**(1), 1–16 (2017)
2. Abbas, S., Benchohra, M., Graef, J.R., Henderson, J.: Implicit Fractional Differential and Integral Equations: Existence and Stability. De Gruyter, Berlin (2018)
3. Abbas, S., Benchohra, M., N'Guérékata, G.M.: Topics in Fractional Differential Equations. Springer, New York (2012)
4. Abbas, S., Benchohra, M., N'Guérékata, G.M.: Advanced Fractional Differential and Integral Equations. Nova Science Publishers, New York (2015)
5. Abbas, S., Benchohra, M., Vityuk, A.N.: On fractional order derivatives and Darboux problem for implicit differential equations. Fract. Calc. Appl. Anal. **15**(2), 168–182 (2012)
6. Akhmerov, R.R., Kamenskii, M.I., Patapov, A.S., Rodkina, A.E., Sadovskii, B.N.: Measures of Noncompactness and Condensing Operators. Birkhauser Verlag, Basel (1992)
7. Alvárez, J.C.: Measure of noncompactness and fixed points of nonexpansive condensing mappings in locally convex spaces. Rev. Real. Acad. Cienc. Exact. Fis. Natur. Madrid **79**, 53–66 (1985)
8. Banaś, J., Goebel, K.: Measures of Noncompactness in Banach Spaces. Marcel Dekker, New York (1980)
9. Benavides, T.D.: An existence theorem for implicit differential equations in a Banach space. Ann. Mat. Pura Appl. **4**, 119–130 (1978)
10. Benchohra, M., Graef, J., Mostefai, F.Z.: Weak solutions for boundary-value problems with nonlinear fractional differential inclusions. Nonlinear Dyn. Syst. Theory **11**(3), 227–237 (2011)
11. Benchohra, M., Henderson, J., Mostefai, F.Z.: Weak solutions for hyperbolic partial fractional differential inclusions in Banach spaces. Comput. Math. Appl. **64**, 3101–3107 (2012)
12. Benchohra, M., Henderson, J., Seba, D.: Measure of noncompactness and fractional differential equations in Banach spaces. Commun. Appl. Anal. **12**(4), 419–428 (2008)
13. Bugajewski, D., Szufla, S.: Kneser's theorem for weak solutions of the Darboux problem in a Banach space. Nonlinear Anal. **20**(2), 169–173 (1993)
14. De Blasi, F.S.: On the property of the unit sphere in a Banach space. Bull. Math. Soc. Sci. Math. Roum. **21**, 259–262 (1977)
15. Furati, K.M., Kassim, M.D.: Non-existence of global solutions for a differential equation involving Hilfer fractional derivative. Electron. J. Differ. Equ. **2013**, 235 (2013)
16. Furati, K.M., Kassim, M.D., Tatar, N.E.: Existence and uniqueness for a problem involving Hilfer fractional derivative. Comput. Math. Appl. **64**, 1616–1626 (2012)
17. Guo, D., Lakshmikantham, V., Liu, X.: Nonlinear Integral Equations in Abstract Spaces. Kluwer Academic, Dordrecht (1996)
18. Hilfer, R.: Applications of Fractional Calculus in Physics. World Scientific, Singapore (2000)
19. Jiao, F., Zhou, Y.: Existence results for fractional boundary value problem via critical point theory. Int. J. Bifurc. Chaos **22**(4), 1250086 (2012)
20. Kamocki, R., Obczyński, C.: On fractional Cauchy-type problems containing Hilfer's derivative. Electron. J. Qual. Theory Differ. Equ. **2016**, 50 (2016)
21. Kilbas, A.A.: Hadamard-type fractional calculus. J. Korean Math. Soc. **38**(6), 1191–1204 (2001)
22. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. Elsevier Science B.V., Amsterdam (2006)
23. Li, M., Wang, J.R.: Exploring delayed Mittag-Leffler type matrix functions to study finite time stability of fractional delay differential equations. Appl. Math. Comput. **324**, 254–265 (2018)
24. Mitchell, A.R., Smith, C.: Nonlinear equations in abstract spaces. In: Lakshmikantham, V. (ed.) An Existence Theorem for Weak Solutions of Differential Equations in Banach Spaces, pp. 387–403. Academic Press, New York (1978)
25. O'Regan, D.: Fixed point theory for weakly sequentially continuous mapping. Math. Comput. Model. **27**(5), 1–14 (1998)
26. O'Regan, D.: Weak solutions of ordinary differential equations in Banach spaces. Appl. Math. Lett. **12**, 101–105 (1999)
27. Pettis, B.J.: On integration in vector spaces. Trans. Am. Math. Soc. **44**, 277–304 (1938)
28. Qassim, M.D., Furati, K.M., Tatar, N.-E.: On a differential equation involving Hilfer–Hadamard fractional derivative. Abstr. Appl. Anal. **2012**, Article ID 391062 (2012)
29. Qassim, M.D., Tatar, N.E.: Well-posedness and stability for a differential problem with Hilfer–Hadamard fractional derivative. Abstr. Appl. Anal. **2013**, Article ID 605029 (2013)
30. Samko, S.G., Kilbas, A.A., Marichev, O.I.: Fractional Integrals and Derivatives. Theory and Applications. Gordon and Breach, Amsterdam (1987) Engl. Trans. from the Russian
31. Tarasov, V.E.: Fractional Dynamics: Application of Fractional Calculus to Dynamics of Particles, Fields and Media. Springer, Heidelberg (2010)
32. Tomovski, Ž., Hilfer, R., Srivastava, H.M.: Fractional and operational calculus with generalized fractional derivative operators and Mittag-Leffler type functions. Integral Transforms Spec. Funct. **21**(11), 797–814 (2010)
33. Vityuk, A.N., Mykhailenko, A.V.: The Darboux problem for an implicit fractional-order differential equation. J. Math. Sci. **175**(4), 391–401 (2011)
34. Wang, J.R., Zhang, Y.: Nonlocal initial value problems for differential equations with Hilfer fractional derivative. Appl. Math. Comput. **266**, 850–859 (2015)
35. Zhou, Y.: Attractivity for fractional evolution equations with almost sectorial operators. Fract. Calc. Appl. Anal. **21**(3), 786–800 (2018)
36. Zhou, Y., Ahmad, B., Alsaedi, A.: Existence of nonoscillatory solutions for fractional neutral differential equations. Appl. Math. Lett. **72**, 70–74 (2017)
37. Zhou, Y., Shangerganesh, L., Manimaran, J., Debbouche, A.: A class of time-fractional reaction–diffusion equation with nonlocal boundary condition. Math. Methods Appl. Sci. **41**, 2987–2999 (2018)
38. Zhou, Y., Vijayakumar, V., Murugesu, R.: Controllability for fractional evolution inclusions without compactness. Evol. Equ. Control Theory **4**, 507–524 (2015)

39. Zhou, Y., Zhang, L.: Existence and multiplicity results of homoclinic solutions for fractional Hamiltonian systems. Comput. Math. Appl. **73**, 1325–1345 (2017)
40. Zhou, Y., Zhang, L., Shen, X.H.: Existence of mild solutions for fractional evolution equations. J. Integral Equ. Appl. **25**, 557–586 (2013)