Advances in Difference Equations
a SpringerOpen Journal

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Multiplicity results for a class of fractional differential equations with impulse

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Abstract

In this paper, we apply Morse theory, local linking arguments and the Clark theorem to a study of the multiplicity of nontrivial solutions for a class of impulsive fractional differential equations with Dirichlet boundary conditions.

MSC: 26A33; 35A15; 34A08; 58E05

Keywords: Fractional differential equation; Nontrivial solution; Morse theory; Impulsive effects

1 Introduction

This paper focuses on the following fractional differential equations with impulsive effects:

$$
\begin{cases}
{}_tD_T^\alpha({}_0^cD_t^\alpha u(t)) + \kappa(t)u(t) = f(t, u(t)), & 0 < t < T, t \neq t_j, \\
\Delta({}_tD_T^{\alpha-1}({}_0^cD_t^\alpha u))(t_j) = I_j(u(t_j)), & j = 1, 2, \ldots, m, \\
u(0) = u(T) = 0,
\end{cases}
\tag{1.1}
$$

where $0 < \alpha \le 1$, ${}_0^cD_t^\alpha$ and ${}_tD_T^\alpha$ denote the left Caputo fractional derivative and the right Riemann–Liouville fractional derivative of order α, respectively, $0 = t_0 < t_1 < \cdots < t_{m+1} = T$, and $\Delta({}_tD_T^{\alpha-1}({}_0^cD_t^\alpha u))(t_j) = {}_tD_T^{\alpha-1}({}_0^cD_t^\alpha u)(t_j^+) - {}_tD_T^{\alpha-1}({}_0^cD_t^\alpha u)(t_j^-)$, where ${}_tD_T^{\alpha-1}({}_0^cD_t^\alpha u)(t_j^+) = \lim_{t \to t_j^+} {}_tD_T^{\alpha-1}({}_0^cD_t^\alpha u)(t), {}_tD_T^{\alpha-1}({}_0^cD_t^\alpha u)(t_j^-) = \lim_{t \to t_j^-} {}_tD_T^{\alpha-1}({}_0^cD_t^\alpha u)(t)$, and $f \in C([0, T] \times \mathbb{R})$, $I_j \in C(\mathbb{R}, \mathbb{R})$. For the function κ we assume the following condition:

(H0): $\kappa \in C[0, T]$ and $\text{essinf}_{t \in [0,T]} \kappa(t) = \kappa_0 > -\lambda_1$, where λ_1 is the first eigenvalue of the operator ${}_tD_T^\alpha({}_0^cD_t^\alpha u(t))$ (see (1.4)).

Fractional differential equations (FDEs for short) have been recently proved to be strong tools in the modeling of many phenomena in various areas of science as physics, biology, geology, polymer rheology, control theory, chemistry and engineering, etc. There have been important advances in the theory of fractional calculus and fractional ordinary and partial differential equations recently, for more details on fractional calculus theory, see the monographs of Podlubny [1], Kilbas *et al.* [2], Zhou [3] and the references therein. Based on variational methods and critical point theory, Jiao and Zhou [4] investigated the existence of at least one weak solution to the following nonlinear FDEs:

$$
\begin{cases}
{}_tD_T^\alpha({}_0D_t^\alpha u(t)) = \nabla F(t, u(t)), & \text{a.e. } t \in [0, T], \\
u(0) = u(T) = 0,
\end{cases}
$$

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where $\alpha \in (0, 1]$, $_0D_t^\alpha$ and $_tD_T^\alpha$ are the left and right Riemann–Liouville fractional derivatives respectively. $F : [0, T] \times \mathbf{R}^N \to \mathbf{R}$ (with $N \geq 1$) is an appropriate given function and $\nabla F(t, u)$ is the gradient of F at u. Since then the variational methods were used to deal with the existence of solutions for fractional differential equations by many authors. In addition, impulsive problems for differential equations appear in mathematical models with sudden and discontinuously changes of their states in pharmacology, population dynamics, optimal control, etc. [5]. Very recently, Bonanno *et al.* [6], Fečkan *et al.* [7] and D'Aguì *et al.* [8] have investigated the existence results of solutions for impulsive FDEs. For more related work on impulsive FDEs, we refer the interested reader to [9–18] and the references therein. However, the main techniques used in these references are the coincidence degree theory, fixed-point theorems and critical point theory in nonlinear analysis but not Morse theory.

On the other hand, in the last few years, Morse theory has been recently used to deal with the existence of solutions for impulsive differential equations [19–22] having the corresponding variational structure. In [21], Shi and Chen considered the following impulsive boundary value problems:

$$\begin{cases} -u''(t) = f(t, u), & t \in [0, T] \setminus \{t_1, t_2, \ldots, t_m\}, \\ \triangle u'(t_j) = I_j(u(t_j)), & j = 1, 2, \ldots, m, \\ u(0) = u(T) = 0. \end{cases} \tag{1.2}$$

Under some natural assumptions, the authors established some new results on the existence of three nontrivial solutions for problems (1.2) via Morse theory. However, to the best of our knowledge, with exception of [23], little work has been done on the existence and multiplicity of solutions for impulsive FDEs with Dirichlet boundary conditions by using Morse theory. More precisely, in [23], the authors studied problem (1.1) and obtained the existence of one nontrivial solution via computing the critical groups at infinity and at zero. Motivated by the above described work, we are interested in investigating the multiplicity of nontrivial solutions for problem (1.1) via Morse theory, local linking arguments and the Clark theorem.

A number $\lambda \in \mathbb{R}$ is said to be an eigenvalue of the following Dirichlet problem:

$$\begin{cases} _tD_T^\alpha(_0^cD_t^\alpha u(t)) = \lambda u(t), \\ u(0) = u(T) = 0, \end{cases} \tag{1.3}$$

if there exists a nontrivial solution $u \in E_0^\alpha$ (E_0^α be given in Definition 2.1) such that

$$\int_0^T {}_0^cD_t^\alpha u(t) {}_0^cD_t^\alpha v(t)\, dt - \lambda \int_0^T u(t) v(t)\, dt = 0$$

for any $v \in E_0^\alpha$. By Riesz–Schauder theory, we can show that problem (1.3) has a sequence of positive eigenvalues $\{\lambda_k\}_{k=1}^\infty$, and λ_1 can be characterized as

$$\lambda_1 = \inf_{u \in E_0^\alpha \setminus \{0\}} \frac{\int_0^T |{}_0^cD_t^\alpha u(t)|^2\, dt}{\int_0^T |u(t)|^2\, dt}. \tag{1.4}$$

Before stating our main results, we give the following hypotheses on the nonlinearity f and I_j ($j = 1, 2, \dots, m$):

(H1) For every j, there exist constants $a_j, b_j > 0$ and $\gamma_j \in [0, 1)$ such that $|I_j(u)| \leq a_j |u|^{\gamma_j} + b_j$, and I_j satisfy $\int_0^u I_j(s)\, ds \geq 0$ for all $u \in \mathbb{R}$.

(H2) There exist $c_0 > 0, q \geq 1$ such that $|f(t, u)| \leq c_0(1 + |u|^{q-1})$, a.e. $t \in [0, T], u \in \mathbb{R}$.

(H3) $\limsup_{|u| \to \infty} \frac{F(t,u)}{|u|^2} < \frac{1}{2}(\lambda_1 + \kappa_0)$ uniformly on $t \in [0, T]$, where $F(t, u) = \int_0^u f(t, s)\, ds$.

(H4) There exist $r > 0$ small and $c_1, c_2 > 0$ with $c_2 \xi_*^2 < 1, 0 < \mu < \min\{\gamma_j + 1\}$ such that

$$c_1 |u|^\mu \leq F(t, u) \leq \frac{c_2}{2} |u|^2, \quad |u| \leq r, t \in [0, T],$$

where ξ_* will be given in Remark A.

(H5) $f(t, u)$ and $I_j(u)$ ($j = 1, 2, \dots, m$) are odd with respect to u. Now we state the main results of this paper.

Theorem 1.1 *Assume conditions* (H0)–(H4) *hold, then the problem* (1.1) *has at least two nontrivial weak solutions in* E_0^α.

Theorem 1.2 *Assume conditions* (H0)–(H5) *hold, then the problem* (1.1) *possesses at least k distinct pairs of nontrivial weak solution in* E_0^α.

The paper is arranged as follows. The variational setting and some preliminaries are presented in Sect. 2 In Sect. 3, we complete the proof of Theorems 1.1 and 1.2.

2 Variational setting and preliminaries

For any fixed $t \in [0, T]$ and $1 \leq p < \infty$, let

$$\|x\|_\infty = \max_{[0,T]} |x(t)|, \qquad \|x\|_{L^p[0,t]} = \left(\int_0^t |x(s)|^p\, ds \right)^{1/p} \quad \text{and}$$

$$\|x\|_{L^p} = \left(\int_0^T |x(t)|^p\, dt \right)^{1/p}.$$

In order to establish a variational structure for problem (1.1), we need to construct suitable function spaces. Denote by $C_0^\infty([0, T], \mathbb{R})$ the set of all functions $u \in C^\infty([0, T], \mathbb{R})$ with $u(0) = u(T) = 0$. From Lemma 5.1 in [3], for any $u \in C_0^\infty([0, T], \mathbb{R})$ and $1 < p < \infty$, we have $u \in L^p([0, T], \mathbb{R})$ and ${}_0^c D_t^\alpha u \in L^p([0, T], \mathbb{R})$.

Definition 2.1 Let $0 < \alpha \leq 1$. The fractional derivative space E_0^α is defined by the closure of $C_0^\infty([0, T], \mathbb{R})$, that is $E_0^\alpha = \overline{C_0^\infty([0, T], \mathbb{R})}$ with respect to the norm

$$\|u\|_{\alpha,2} = \left(\int_0^T |{}_0^c D_t^\alpha u(t)|^2\, dt + \int_0^T |u(t)|^2\, dt \right)^{1/2}, \quad \forall u \in E_0^\alpha. \tag{2.1}$$

From Proposition 5.4 in [3], one finds that, for $0 < \alpha \leq 1$, the space E_0^α is a reflexive and separable Banach space. We present some properties of the fractional derivative space E_0^α.

Lemma 2.1 ([4]) *Let* $0 < \alpha \leq 1, \|{}_0^c D_t^\alpha u\|_{L^2} := (\int_0^T |{}_0^c D_t^\alpha u(t)|^2\, dt)^{1/2}$. *For any* $u \in E_0^\alpha$, *we have*

$$\|u\|_{L^2} \leq \frac{T^\alpha}{\Gamma(\alpha + 1)} \|{}_0^c D_t^\alpha u\|_{L^2}. \tag{2.2}$$

Moreover, if $\alpha > \frac{1}{2}$, then

$$\|u\|_\infty \le \frac{T^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)\sqrt{2\alpha-1}} \left\|{}_0^c D_t^\alpha u\right\|_{L^2}. \tag{2.3}$$

According to (2.2) we may consider the fractional space E_0^α with the norm $\|u\|_\alpha = \|{}_0^c D_t^\alpha u\|_{L^2}$, which is equivalent to (2.1).

Lemma 2.2 ([4]) *Let $\frac{1}{2} < \alpha \le 1$. Assume that the sequence $\{u_n\}$ converges weakly to u in E_0^α, i.e., $u_n \rightharpoonup u$. Then $\{u_n\}$ converges strongly to u in $C([0,T],\mathbb{R})$, i.e., $\|u_n - u\|_\infty \to 0$, as $n \to \infty$.*

Lemma 2.3 *Assume that $0 < \alpha \le 1$, If (H0) holds, then the norm*

$$\|u\| = \left(\int_0^T \left|{}_0^c D_t^\alpha u(t)\right|^2 dt + \int_0^T \kappa(t)\left|u(t)\right|^2 dt \right)^{1/2}, \quad \forall u \in E_0^\alpha, \tag{2.4}$$

and the norm $\|\cdot\|_\alpha$ are equivalent, namely, there exist two positive constants ξ, η such that

$$\xi\|u\|_\alpha \le \|u\| \le \eta\|u\|_\alpha, \quad \text{for any } u \in E_0^\alpha.$$

Proof Since $\operatorname{essinf}_{t\in[0,T]} \kappa(t) > -\lambda_1$, there exists $\varepsilon \in (0,1)$ such that $-\kappa_0 \le \lambda_1(1-\varepsilon)$. By virtue of (1.4), we get

$$(1-\varepsilon)\int_0^T \left|{}_0^c D_t^\alpha u(t)\right|^2 dt \ge \lambda_1(1-\varepsilon)\int_0^T \left|u(t)\right|^2 dt \ge -\kappa_0\int_0^T \left|u(t)\right|^2 dt$$

for any $u \in E_0^\alpha$. Therefore, for all $u \in E_0^\alpha$, we have

$$\|u\|^2 \ge \int_0^T \left(\left|{}_0^c D_t^\alpha u(t)\right|^2 + \kappa_0\left|u(t)\right|^2\right) dt \ge \varepsilon\int_0^T \left|{}_0^c D_t^\alpha u(t)\right|^2 dt = \varepsilon\|u\|_\alpha^2.$$

On the other hand, from (1.4) and (2.4) we deduce that

$$\|u\|^2 \le \int_0^T \left(\left|{}_0^c D_t^\alpha u(t)\right|^2 + \|\kappa\|_\infty\left|u(t)\right|^2\right) dt \le \left(1 + \frac{\|\kappa\|_\infty}{\lambda_1}\right)\int_0^T \left|{}_0^c D_t^\alpha u(t)\right|^2 dt$$

$$= \left(1 + \frac{\|\kappa\|_\infty}{\lambda_1}\right)\|u\|_\alpha^2.$$

Thus the norm $\|\cdot\|$ and the norm $\|\cdot\|_\alpha$ are equivalent and we may choose, for instance, $\xi = \sqrt{\varepsilon}$ and $\eta = \sqrt{1 + \frac{\|\kappa\|_\infty}{\lambda_1}}$. In the sequel, we will consider E_0^α with the norm $\|\cdot\|$ defined in (2.4). □

Remark A From Lemmas 2.1 and 2.3, we have

(i) for $0 < \alpha \le 1$, $\|u\|_{L^2} \le \frac{T^\alpha}{\Gamma(\alpha+1)}\|u\|_\alpha \le \xi_*\|u\|$, where $\xi_* = \frac{T^\alpha}{\xi\Gamma(\alpha+1)}$;

(ii) for $\frac{1}{2} < \alpha \le 1$, $\|u\|_\infty \le \frac{T^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)\sqrt{2\alpha-1}}\|u\|_\alpha \le \eta_*\|u\|$, where $\eta_* = \frac{T^{\alpha-\frac{1}{2}}}{\xi\Gamma(\alpha)\sqrt{2\alpha-1}}$.

Definition 2.2 We say that $u \in E_0^\alpha$ is a weak solution of problem (1.1) if the identity

$$\int_0^T \left({}_0^c D_t^\alpha u(t) {}_0^c D_t^\alpha v(t) + \kappa(t) u(t) v(t) \right) dt + \sum_{j=1}^m I_j \left(u(t_j) \right) v(t_j) = \int_0^T f \left(t, u(t) \right) v(t) \, dt$$

holds for any $v \in E_0^\alpha$.

Consider the functional $\varphi : E_0^\alpha \to \mathbb{R}$ defined by

$$\varphi(u) = \frac{1}{2} \|u\|^2 + \sum_{j=1}^m \int_0^{u(t_j)} I_j(s) \, ds - \int_0^T F \left(t, u(t) \right) dt. \tag{2.5}$$

Since f and I_j $(j = 1, \ldots, m)$ are continuous, we can infer that φ is continuous, differentiable and for all $u, v \in E_0^\alpha$

$$\langle \varphi'(u), v \rangle = \int_0^T \left({}_0^c D_t^\alpha u(t) {}_0^c D_t^\alpha v(t) + \kappa(t) u(t) v(t) \right) dt + \sum_{j=1}^m I_j \left(u(t_j) \right) v(t_j)$$

$$- \int_0^T f \left(t, u(t) \right) v(t) \, dt.$$

Then it is clear that the critical points of φ are weak solutions of the problem (1.1).

Definition 2.3 We say that φ satisfies the (PS) condition in E_0^α, if any (PS) sequence $\{u_n\}_{n \in \mathbb{N}} \subset E_0^\alpha$, i.e., $\{\varphi(u_n)\}_{n \in \mathbb{N}}$ is bounded and $\varphi'(u_n) \to 0$ *as* $n \to \infty$, has a strongly convergent subsequence in E_0^α.

Definition 2.4 Let E be a real Banach space and $\varphi \in C^1(E, \mathbb{R})$. *Set* u be an isolated critical point of φ with $\varphi(u) = c$, for $c \in \mathbb{R}$, and let U be a neighborhood of u such that φ has only u as a critical point in U. We call

$$C_k(\varphi, u) := H_k \left(\varphi^c \cap U, \varphi^c \cap U \setminus \{u\} \right), \quad \left(k \in \mathbb{N} := \{0, 1, 2, \ldots\} \right)$$

the kth critical group of φ at u, where $\varphi^c := \{u \in E : \varphi(u) \le c\}$ is the c-sublevel set, and H_k is the singular relative homology group with coefficients in an Abelian group G. For more details, we refer the reader to [24].

One calls u a homological nontrivial critical point of the functional φ if at least one of the critical groups of φ is nontrivial.

Lemma 2.4 ([25], Proposition 2.1) *Let* 0 *be a critical point of* φ *with* $\varphi(0) = 0$. *Suppose that* φ *has a local linking at* 0 *with respect to* $E = V \oplus Y$, $k = \dim V < \infty$, *that is, there exists* $\rho > 0$ *small enough such that*

$$\varphi(u) \le 0, \quad \forall u \in V, \|u\| \le \rho \quad and \quad \varphi(u) > 0, \quad \forall u \in Y, 0 < \|u\| \le \rho.$$

Then $C_k(\varphi, 0) \ncong 0$, *hence* 0 *is a homological nontrivial critical point of* φ.

Lemma 2.5 ([25], Theorem 2.1) *Let E be a real Banach space and let $\varphi \in C^1(E, \mathbb{R})$ satisfy the (PS)-condition and be bounded from below. If φ has a critical point that is homological nontrivial and is not a minimizer of φ, then φ has at least three critical points.*

Lemma 2.6 ([26], Theorem 9.1) *Let E be a real Banach space, $\varphi \in C^1(E, \mathbb{R})$ with φ even, bounded from below, and satisfying (PS)-condition. Suppose $\varphi(0) = 0$, there is a set $\Omega \subset E$ such that Ω is homeomorphic to S^{k-1} by an odd map, and $\sup_\Omega \varphi < 0$. Then φ possesses at least k distinct pairs of critical points.*

3 Proofs of main results

To obtain multiple nontrivial solutions of the problem (1.1), we shall present the following lemmas and compute the critical groups of the functional φ.

Lemma 3.1 *Assume conditions* (H0)–(H2) *hold, then any bounded sequence $\{u_n\} \subset E_0^\alpha$ such that $\varphi'(u_n) \to 0$ as $n \to \infty$ admits a strongly convergent subsequence in E_0^α.*

Proof Let $\{u_n\}$ be a bounded sequence in E_0^α. Going to a subsequence if necessary, we can assume that $u_n \rightharpoonup u_0$ weakly in E_0^α. Hence,

$$u_n \to u_0 \quad \text{in } L^p([0, T]), \qquad u_n \to u_0 \quad \text{a.e. } t \in [0, T]. \tag{3.1}$$

We will show that $\{u_n\}$ converges strongly to u_0 in E_0^α. Since E_0^α is reflexive space and $u_n \rightharpoonup u_0$ in E_0^α, $\varphi'(u_n) \to 0$ in $(E_0^\alpha)^*$, then from (2.5), we can obtain

$$\langle \varphi'(u_n) - \varphi'(u_0), u_n - u_0 \rangle \le \|\varphi'(u_n)\| \cdot \|u_n - u_0\| - \langle \varphi'(u_0), u_n - u_0 \rangle \to 0 \tag{3.2}$$

as $n \to \infty$. Moreover, it follows from (2.4) and Lemma 2.2 that u_n is bounded in $C[0, T]$ and $\|u_n - u_0\|_\infty \to 0$ as $n \to \infty$. Hence, by (H1), (H2) and the Hölder inequality, we have

$$\begin{cases} \lim_{n\to\infty} \int_0^T (f(t, u_n(t)) - f(t, u_0(t)))(u_n(t) - u_0(t)) \, dt = 0, \\ \lim_{n\to\infty} \sum_{j=1}^m (I_j(u_n(t_j)) - I_j(t, u_0(t_j)))(u_n(t_j) - u_0(t_j)) = 0. \end{cases} \tag{3.3}$$

Observe that

$$\begin{aligned}
0 \leftarrow \langle \varphi'(u_n) &- \varphi'(u_0), u_n - u_0 \rangle \\
&= \|u_n - u_0\|^2 - \int_0^T \big[f(t, u_n) - f(t, u_0) \big](u_n - u_0) \, dt \\
&\quad + \sum_{j=1}^m \big[I_j\big(u_n(t_j)\big) - I_j\big(u_0(t_j)\big) \big]\big(u_n(t_j) - u_0(t_j)\big) \\
&\ge \|u_n - u_0\|^2 - \left| \int_0^T \big[f(t, u_n) - f(t, u_0) \big] \, dt \right| \cdot \|u_n - u_0\|_\infty \\
&\quad - \left| \sum_{j=1}^m \big[I_j\big(u_n(t_j)\big) - I_j\big(u_0(t_j)\big) \big] \right| \cdot \|u_n - u_0\|_\infty.
\end{aligned}$$

Combining (3.2) and (3.3), we deduce that $\|u_n - u_0\|^2 \to 0$ as $n \to \infty$, and we have $u_n \to u_0$ in E_0^α. Thus, $\{u_n\}$ admits a convergent subsequence. This completes the proof. □

Lemma 3.2 *Assume that* (H0)–(H3) *hold. Then* φ *satisfies the* (PS) *condition.*

Proof According to Lemma 3.1, it suffices to verify the boundedness of (PS) sequences. It follows from (H3) that, for some $\epsilon > 0$ small, there exists a constant $C_\epsilon > 0$ such that

$$\left| F(t,u) \right| \leq \frac{1}{2}(\lambda_1 + \kappa_0 - \epsilon)|u|^2 + C_\epsilon, \quad \text{for all } u \in \mathbb{R}, t \in [0,T]. \tag{3.4}$$

By assumption (H1) we get

$$\int_0^{u(t_j)} I_j(s)\, ds \leq C_j |u|^{\gamma_j + 1}, \quad \text{for some } C_j > 0 \ (j = 1, 2, \ldots, m). \tag{3.5}$$

Thus, by means of (1.3), (3.4) and (3.5), for $u \in E_0^\alpha, u \neq 0$, we have

$$\begin{aligned}
\varphi(u) &\geq \frac{1}{2} \int_0^T \left| {}_0^c D_t^\alpha u(t) \right|^2 dt + \frac{\kappa_0}{2} \int_0^T |u(t)|^2 dt \\
&\quad - \sum_{j=1}^m C_j |u|^{\gamma_j + 1} - \frac{1}{2}(\lambda_1 + \kappa_0 - \epsilon) \int_0^T |u(t)|^2 dt - C_\epsilon T \\
&= \frac{1}{2}\|u\|_\alpha^2 - \frac{1}{2}(\lambda_1 - \epsilon)\|u\|_{L^2}^2 - \sum_{j=1}^m C_j |u|^{\gamma_j + 1} - C_\epsilon T \\
&\geq \frac{1}{2}\|u\|_\alpha^2 - \frac{1}{2\lambda_1}(\lambda_1 - \epsilon)\|u\|_\alpha^2 - \sum_{j=1}^m C_j (\eta_*)^{\gamma_j + 1}\|u\|^{\gamma_j + 1} - C_\epsilon T \\
&\geq \frac{\epsilon}{2\eta^2}\|u\|^2 - \sum_{j=1}^m C_j^* \|u\|^{\gamma_j + 1} - C_\epsilon T \to +\infty
\end{aligned}$$

as $\|u\| \to \infty$, since $\eta > 0, 1 \leq \max\{\gamma_j + 1\} < 2$ and $C_j^* > 0$ $(j = 1, 2, \ldots, m)$ are positive constants. That is to say, φ is coercive on E_0^α. Therefore, we get the desired assertion.

Since E_0^α is a reflexive and separable Banach space, we may choose an orthogonal basis $\{e_i\}$ of E_0^α with $E_0^\alpha = \overline{\text{span}\{e_i : i = 1, 2, \ldots\}}$. For $k = 1, 2, \ldots$, denote $X_i := \text{span}\{e_i\}, V_k = \bigoplus_{i=1}^k X_i, Y_k = \overline{\bigoplus_{i=k}^\infty X_i}$, then $E_0^\alpha = V_k \oplus Y_k$. $\square$

Lemma 3.3 *Assume that* (H0)–(H2) *and* (H4) *are satisfied. Then there exists* $k_0 \in \mathbb{N}$ *such that* $C_{k_0}(\varphi, 0) \neq 0$.

Proof According to Lemma 2.4, it suffices to show that the functional φ has a local linking at zero with respect to $E_0^\alpha = V_k \oplus Y_k$.

Step 1: Set $u \in V_k$. Since V is finite dimensional, from Remark A(ii) one sees that, for given r, there is $\rho \in (0, 1]$ small such that, for $u \in V_k, \|u\| \leq \rho$ implies $|u(t)| \leq r$ for $t \in [0, T]$. Denote $F^*(t, u) = F(t, u) - c_1 |u|^\mu$. Using (H2) and (H4), there exists $c_3 > 0$ such that, for some $q > 2$, we have $F^*(t, u) \geq -c_3 |u|^q$ for all $t \in [0, T]$ and $|u| > 0$. Therefore, for $u \in V_k$ with $\|u\| \leq \rho$, we get

$$\varphi(u) = \frac{1}{2}\|u\|^2 + \sum_{j=1}^m \int_0^{u(t_j)} I_j(s)\, ds - c_1 \int_0^T |u|^\mu\, dt - \left(\int_{\{|u(t)| \leq r\}} + \int_{\{|u(t)| > r\}} \right) F^*(t, u(t))\, dt$$

$$\leq \frac{1}{2}\|u\|^2 + \sum_{j=1}^{m} C_j |u(t_j)|^{\gamma_j+1} - c_1 \int_0^T |u|^\mu \, dt + c_3 \int_{\{|u(t)|>r\}} |u|^q \, dt$$

$$\leq \frac{1}{2}\|u\|^2 + \sum_{j=1}^{m} C_j^* \|u\|^{\gamma_j+1} - c_1 \|u\|_{L^\mu}^\mu + c_4 \|u\|^q,$$

where c_4 and C_j^* $(j = 1, 2, \ldots, m)$ are positive constants. Notice that the norms on V_k are equivalent to each other, thus $\|u\|_{L^\mu}$ is equivalent to $\|u\|$. Since $0 < \mu < \min\{\gamma_j + 1\} \leq \max\{\gamma_j + 1\} < 2 < q$, we have $\varphi(u) \leq 0$, for any $u \in V_k$ with $\|u\| \leq \rho$.

Step 2: Choose $u \in Y_k$, Due to the continuous embedding $E_0^\alpha \to C_0^\infty([0, T], \mathbb{R})$ is compact. Then, for given $r > 0$, by (H2), (H4) and Young's inequality, there is a constant $c_5 > 0$ such that, for some $q > 2$, one has

$$|F(t, u)| \leq \frac{c_2}{2}|u|^2 + c_5|u|^q, \quad \text{for all } t \in [0, T] \text{ and } |u| \geq r. \tag{3.6}$$

Denoting $H(t, u) = F(t, u) - \frac{c_2}{2}|u|^2$, from (H1), Remark A(i) and (3.6) we obtain

$$\varphi(u) = \frac{1}{2}\|u\|^2 + \sum_{j=1}^{m} \int_0^{u(t_j)} I_j(s) \, ds - \frac{c_2}{2}\int_0^T |u|^2 \, dt - \left(\int_{\{|u(t)|\leq r\}} + \int_{\{|u(t)|>r\}}\right) H(t, u(t)) \, dt$$

$$\geq \frac{1}{2}(1 - c_2 \xi_*^2)\|u\|^2 - c_5 \int_{\{|u(t)>r\}} |u(t)|^q \, dt$$

$$\geq \frac{1}{2}(1 - c_2 \xi_*^2)\|u\|^2 - c_6 \|u\|^q,$$

where c_6 is positive constant. Since $c_2 \xi_*^2 < 1, q \in (2, +\infty)$, we can show that $\varphi(u) > 0$ as $u \in Y_k$ and $0 < \|u\| < \rho$ with $\rho > 0$ small enough. This completes the proof. □

Proof of Theorem 1.1 By Lemma 3.2, φ satisfies the (PS) condition and is bounded from below. According Lemma 3.3, the trivial solution $u = 0$ is homological nontrivial and is not a minimizer. Then the conclusion follows immediately by Lemma 2.5. □

Proof of Theorem 1.2 From (H5), we can infer that φ is even. Lemma 3.2 shows that φ satisfies the (PS) condition and is bounded from below. For given $\rho > 0$, set $\Omega = S_\rho = \{u \in V_k : \|u\| = \rho\}$. Therefore, just as shown in the proof of Lemma 3.3, if ρ is small enough, we show that $\sup_\Omega \varphi(u) < 0$. Obviously, $\dim V_k = k$, then it follows from Lemma 2.6 that φ has at least k distinct pairs of critical points. Thus, problem (1.1) has at least k distinct pairs of nontrivial solutions. □

4 Conclusion

In this paper, we have studied the existence and multiplicity of the solutions for an impulsive fractional differential equations with Dirichlet boundary conditions. Our approach is based on Morse theory, local linking arguments and the Clark theorem. In the future, we will consider the existence of solutions for the impulsive fractional differential equation with p-Laplacian via Morse theory.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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