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New exact solutions for Kudryashov–Sinelshchikov equation

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Abstract

In this paper, we firstly change the auxiliary second order ordinary differential equation in the $\frac{G'}{G}$-polynomial expansion method to the Riccati equation. By solving the Riccati equation, we obtain more exact solutions to the auxiliary equation and thus obtain more new exact solutions to the Kudryashov–Sinelshchikov equation, which mainly include three types of solutions with parameters: hyperbolic function traveling wave solutions, trigonometric function traveling wave solutions, and rational function traveling wave solutions. At last, some examples and figures are given to demonstrate the solutions.

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1 Introduction

A mixture of liquid and gas bubbles of the same size may be considered as an example of a classic nonlinear medium. The analysis of propagation of the pressure waves in a liquid with gas bubbles is an important problem in mathematics and/or physics fields. Indeed, there are solitary and periodic waves in such mixtures and they can be described by nonlinear partial differential equations like the Burgers, Korteweg–de Vries, and the Korteweg–de Vries–Burgers ones. In 2010, Kudryashov and Sinelshchikov [1, 2] obtained a more general nonlinear partial differential equation to describe the pressure waves in a liquid and gas bubbles mixture taking into consideration the viscosity of liquid and the heat transfer. They introduced the equation

$$u_t + \gamma u u_x + u_{xxx} - \epsilon (u u_{xx})_x - \kappa u_x u_{xx} - \nu u_{xx} - \delta (u u_x)_x = 0, \tag{1.1}$$

where u is a density and models heat transfer and viscosity; γ, ϵ, κ, ν and δ are real parameters, which describes pressure waves in the liquid with gas bubbles taking into account the heat transfer and viscosity. Equation (1.1) is called the Kudryashov–Sinelshchikov (KS) equation. Clearly, when $\epsilon = \kappa = \nu = \delta = 0$, Eq. (1.1) reads

$$u_t + \gamma u u_x + u_{xxx} = 0, \tag{1.2}$$

which is known as the Korteweg–de Vries (KdV) equation [3]; while, when $\epsilon = \kappa = \delta = 0$, Eq. (1.1) reads

$$u_t + \gamma u u_x + u_{xxx} - \nu u_{xx} = 0, \tag{1.3}$$

which is the Korteweg–de Vries–Burgers (KdVB) equation [4]. So, Eq. (1.1) is a generalization of the KdV equation and the KdVB equation and it is similar but not identical to the Camassa–Holm (CH) equation (see [5] and the references therein). It is well known that pressure waves in a gas–liquid mixture is characterized by the KdVB equation and KdV equation [3, 6].

Equations (1.1), (1.2), (1.3) are called nonlinear evolution equations. Undistorted waves are governed by a corresponding ordinary differential equation which is solved analytically in [1] for special values of some integration constant. In [2], the authors derived partial cases of nonlinear evolution equations of the fourth order for describing nonlinear pressure waves in a mixture liquid and gas bubbles. They obtained some exact solutions and discussed properties of nonlinear waves in a liquid with gas bubbles. In recent decades, to find the exact solutions of nonlinear evolution equations arising in mathematical physics plays an important role in the study of nonlinear physical phenomena. A class of important solutions to nonlinear evolution equations, called traveling wave solutions, attracts the interest of many mathematicians and physicists. The traveling wave solutions reduce the two variables, namely, the space variable x and the time variable t, of a partial differential equation (PDE) to an ordinary differential equation (ODE) with one independent variable $\xi = x - ct$ where $c \in (\mathbb{R} - \{0\})$ is the wave speed with which the wave travels either to the right or to the left. There are many classical methods proposed to find exact traveling wave solutions of PDE. For example, under conditions $\gamma = \epsilon = 1$, $\nu = \delta = 0$, Eq. (1.1) becomes

$$u_t + u u_x + u_{xxx} - (u u_{xx})_x - \kappa u_x u_{xx} = 0. \tag{1.4}$$

In [7], the author found four families of solitary wave solutions of (1.4) when $\kappa = -3$, or $\kappa = -4$ using a modification of the truncated expansion method [8, 9]. In [10], the authors discussed the existence of different kinds of traveling wave solutions by using the approach of dynamical systems, according to different phase orbits of the traveling wave system (1.4), 26 kinds of exact traveling wave solutions are obtained under the parameter choices $\kappa = -3, -4, 1, 2$. In [11], the author Randrüüt studied Eq. (1.4) under the conditions $\kappa > -2$, $\kappa = -2$, and $\kappa < -2$. He obtained some exact solitary wave solutions and discussed their dynamical behaviors. Some interesting phenomena of the solitary waves are successfully explained. Particularly, a kind of new periodic wave solutions, called meandering solution type, was obtained. In [12], the authors obtained the most complete family of evolutionary equations for describing nonlinear wave processes in a liquid containing gas bubbles and they classified the effects of physical properties of the gas-bubble system on the evolution of nonlinear waves. At the same time, the authors also obtained a peakon solution of Eq. (1.1). In [13], the authors discussed the cases β ($\neq -1$) is an odd number and $\alpha \neq 0$ of Eq. (1.1) by the bifurcation theory and the method of phase portraits analysis and they gave some new exact traveling wave solutions. In [14], the author obtained some soliton solutions to the nonlinear (3+1)-dimensional variable-coefficient

Kudryashov–Sinelshchikov model by using an auto-Bäcklund transformation. In [15], the authors obtained all of the geometric vector fields of the equation and some new exact explicit solutions to the 3-dimensional Kudryashov–Sinelshchikov equation by using the Lie symmetry analysis. In [16], the authors applied the Lie group method to derive the symmetries of the Kudryashov–Sinelshchikov equation. Then, by using the optimal system of 1-dimensional subalgebras, they reduced the equation to ordinary differential equations. Finally, some exact wave solutions were obtained by applying the simplest equation method. In [17], the authors, based on the power series theory, obtained a kind of explicit power series solutions to the Kudryashov–Sinelshchikov equation. In [18], the authors solved numerically the nonlinear time-fractional Kudryashov–Sinelshchikov equation by using radial basis function (RBF) method. In [19], the authors considered the Kudryashov–Sinelshchikov equation, which contains nonlinear dispersive effects, and proved that as the diffusion parameter tends to zero, the solutions of the dispersive equation converge to the entropy ones of the Burgers equation. In [20], the authors used Hermite transform for transforming the Wick-type stochastic Kudryashov–Sinelshchikov equation to deterministic partial differential equation and obtained exact solutions of Wick-type stochastic Kudryashov–Sinelshchikov equation by using improved Sub-equation method.

Recently, more and more methods to find traveling wave solutions are made. For example, the homogeneous balance method [21], the tanh method [22], the Jacobi elliptic function expansion [23–26], the truncated Painlevé expansion [27], differential quadrature method [28], Hirota bilinear method[29], Darboux transformations [30], the trial equation method [31]. Seadawy et al. [32] proposed the sech-tanh method to solve the Olver equation and the fifth-order KdV equation and obtained traveling wave solutions; in [33–38] was introduced a method called the $\frac{G'}{G}$-expansion method and one obtained a traveling solution for the four well established nonlinear evolution equations. In [5], the authors obtained traveling wave solutions for the generalized Camassa–Holm equation by polynomial expansion methods. Those methods are very efficient, reliable, simple in solving many PDEs.

In this paper, on the basis of the $\frac{G'}{G}$-expansion method [5, 33–38], we use the solutions of the Riccati equation [39] to extend the auxiliary equation $G'' + \lambda G' + \mu G = 0$ and obtain more exact solutions of the auxiliary equation [40], thus we derive more new exact solutions of Eq. (1.1).

This paper is organized as follows. In Sect. 1, an introduction is presented. In Sect. 2, we give a brief description of the modification of the $\frac{G'}{G}$-polynomial expansion method. In Sect. 3, the exact solutions of the KS equation are obtained. Finally, the paper ends with a conclusion and remark in Sect. 4.

2 Preliminaries

In this section we describe the modified $\frac{G'}{G}$-polynomial expansion method for finding the exact solutions of nonlinear evolution equation. Suppose a nonlinear equation which has independent space variable x and time variable t is given by

$$P(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \ldots) = 0, \tag{2.1}$$

where $u = u(x,t)$ is an unknown function, P is a polynomial of u and its partial derivatives and the polynomial P includes the highest order derivatives and the nonlinear terms. In the following, we will describe the modified $\frac{G'}{G}$-polynomial expansion method.

Suppose that $u(x,t) = \phi(x - ct) = \phi(\xi)$, where c is the wave speed and $\xi = x - ct$. Equation (2.1) can be reduced to an ODE with variable $\phi(\xi)$

$$P\bigl(\phi, \phi', \phi'', \ldots\bigr) = 0, \tag{2.2}$$

where the prime is the derivative with respect to ξ.

2.1 Algorithm of the modified $\frac{G'}{G}$-polynomial expansion method

The aim of this subsection is to present the algorithm of the modified $\frac{G'}{G}$-polynomial expansion method for finding exact solutions of the KS equation.

Step 1. Determination of the dominant terms. To find dominant terms we substitute

$$\phi = z^{-P}$$

into all terms of Eq. (3.4), then we compare degrees of all terms in Eq. (3.4) and choose two or more with the smallest degree. The minimum value of p defines the pole of the solution of Eq. (3.4) and we denote it N.

Step 2. Suppose the solution of Eq. (2.2) can be expressed by a polynomial in $\frac{G'}{G}$ as follows:

$$\phi(\xi) = \sum_{i=-N}^{N} a_i \left(\frac{G'}{G}\right)^i, \tag{2.3}$$

where a_i are real constants to be determined, and at least one of a_N and a_{-N} is not equal to zero. The function $G(\xi)$ is the solutions of the auxiliary linear ODE

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0, \tag{2.4}$$

where λ and μ are real constants to be determined.

Step 3. Changing the auxiliary linear equation (2.4). From auxiliary equation (2.4), we can obtain

$$\left(\frac{G'}{G}\right)' = -\mu - \lambda\left(\frac{G'}{G}\right) - \left(\frac{G'}{G}\right)^2. \tag{2.5}$$

Letting $\omega = \frac{G'}{G}$, then Eq. (2.4) is equivalent to

$$\omega' = -\mu - \lambda\omega - \omega^2, \tag{2.6}$$

which is the Riccati equation. By solving the Riccati equation (2.6), we can get all the solutions to Eq. (2.4).

Step 4. Substitution of derivatives for function $\varphi(\xi)$ with resect to ξ and the expression for $\varphi(\xi)$ into Eq. (2.2).

Step 5. Substituting (2.3) into (2.2), collecting all terms with the same powers of the function $\frac{G'}{G}$ and finding the algebraic system of equations for coefficients a_i and for parameters λ, μ. Solving this system we get the values of the unknown parameters.

Step 6. Substituting the solutions of Eq. (2.6) into (2.3), we obtain the exact traveling wave solutions to PDE (1.1).

2.2 The solutions to Riccati equation (2.6)

Considering differential equation

$$\frac{dy}{dx} = r(x) + q(x)y + p(x)y^2, \tag{2.7}$$

where the functions $p(x)$, $q(x)$ and $r(x)$ are continuous and $p(x) \not\equiv 0$. Equation (2.7) is called the Riccati equation.

At first, we give some properties of the Riccati equation (2.7) and its elementary integration method.

Lemma 2.1 *Assume $r(x) \equiv 0$, then Eq. (2.7) can be solved by elementary integration method.*

Proof Since $r(x) \equiv 0$, then Eq. (2.7) is

$$\frac{dy}{dx} = q(x)y + p(x)y^2,$$

which is a Bernoulli equation with $n = 2$, and it can be solved by elementary integration method to get solutions to Eq. (2.7). □

Lemma 2.2 *Assume $y = \varphi(x)$ is a special solution to Eq. (2.7), then Eq. (2.7) can be solved by elementary integration method.*

Proof Suppose $y = u(x) + \varphi(x)$, substituting it into Eq. (2.7), we have

$$\frac{du}{dx} + \frac{d\varphi}{dx} = r(x) + q(x)\big(u(x) + \varphi(x)\big) + p(x)\big(u(x) + \varphi(x)\big)^2. \tag{2.8}$$

Because $y = \varphi(x)$ is a special solution to Eq. (2.7), Eq. (2.8) can be simplified to

$$\frac{du}{dx} = q(x)u + p(x)u^2,$$

which is a Bernoulli equation with $n = 2$, and it can be solved by elementary integration method, so we can obtain solutions to Eq. (2.7). □

Lemma 2.3 *If $r(x)$, $q(x)$ and $p(x)$ all are constant numbers, then Eq. (2.7) can be solved by elementary integration method.*

Proof If $r(x)$, $q(x)$ and $p(x)$ all are constant numbers, then Eq. (2.7) is an independent variable equation, which can be solved by elementary integration method. □

Next, we solve Eq. (2.6), which is a Riccati equation with $r(x) = -\mu$, $q(x) = -\lambda$ and $p(x) = -1$. According to Lemma 2.3, Eq. (2.6) can be solved by an elementary integration method.

Let $q = 4\mu - \lambda^2$, the solutions of the equation (2.6) have the following three cases.

Case i: If $q > 0$, then Eq. (2.6) has the general solutions

$$\omega_1 = -\frac{\sqrt{q}}{2} \tan \frac{\sqrt{q}(\xi + k_1)}{2} - \frac{\lambda}{2},$$

where k_1 is an arbitrary constant. According to Lemma 2.2, suppose $k_1 = 0$, then

$$\omega_1 = -\frac{\sqrt{q}}{2}\tan\frac{\sqrt{q}\xi}{2} - \frac{\lambda}{2}$$

is a special solution to Eq. (2.6). So we assume that $\omega_{11} = u + \omega_1$ is a solution to Eq. (2.6), then

$$\frac{du}{dx} = \sqrt{q}\tan\frac{\sqrt{q}\xi}{2}u - u^2. \tag{2.9}$$

Solving Eq. (2.9), we obtain

$$u_1 = \frac{\sqrt{q}}{\sin\sqrt{q}\xi - k_{11}\sqrt{q}\cos^2\frac{\sqrt{q}\xi}{2}},$$

where k_{11} is an arbitrary constant. Thus, the general solutions to Eq. (2.6) are

$$\omega_{11} = \frac{\sqrt{q}}{\sin\sqrt{q}\xi - k_{11}\sqrt{q}\cos^2\frac{\sqrt{q}\xi}{2}} - \frac{\sqrt{q}}{2}\tan\frac{\sqrt{q}(\xi + k_1)}{2} - \frac{\lambda}{2},$$

where k_1, k_{11} are arbitrary constants.

Case ii: If $q < 0$, then Eq. (2.6) has a general solutions

$$\omega_2 = \frac{\sqrt{-q}}{2}\tanh\frac{\sqrt{-q}(\xi + k_2)}{2} - \frac{\lambda}{2},$$

where k_2 is an arbitrary constant. According to Lemma 2.2, similarly to Case i, we obtain the general solutions to Eq. (2.6),

$$\omega_{21} = \frac{\sqrt{-q}}{2\sinh\frac{\sqrt{-q}\xi}{2} + k_{21}\sqrt{-q}\cosh^2\frac{\sqrt{-q}\xi}{2}} + \frac{\sqrt{-q}}{2}\tanh\frac{\sqrt{-q}(\xi + k_2)}{2} - \frac{\lambda}{2},$$

where k_2, k_{21} are arbitrary constants.

Specially, if $\mu = 0$,

$$\omega_{22} = \frac{k_{22}\lambda}{e^{\lambda\xi} - k_{22}},$$

where k_{22} is an arbitrary constant.

Case iii: If $q = 0$, i.e., $\mu = \frac{\lambda^2}{4}$, then Eq. (2.6) can be changed as

$$\omega' = -\left(\omega + \frac{\lambda}{2}\right)^2, \tag{2.10}$$

Solving Eq. (2.10), we obtain the solutions

$$\omega_3 = \frac{1}{\xi + k_3} - \frac{\lambda}{2},$$

where k_3 is an arbitrary constant.

3 Main results

In this section, we obtain the exact traveling wave solutions to Eq. (3.1).

3.1 Analysis of the modified $\frac{G'}{G}$-polynomial expansion method

In this section, we will employ the proposed $\frac{G'}{G}$-polynomial expansion methods to solve the KS equation (1.1). At first, we simplify Eq. (1.1). Let $u = \frac{\tilde{u}}{\epsilon}$, and substitute it into Eq. (1.1), we obtain

$$\frac{1}{\epsilon}\tilde{u}_t + \frac{\gamma}{\epsilon^2}\tilde{u}\tilde{u}_x + \frac{1}{\epsilon}\tilde{u}_{xxx} - \frac{1}{\epsilon}(\tilde{u}\tilde{u}_{xx})_x - \frac{\kappa}{\epsilon^2}\tilde{u}_x\tilde{u}_{xx} - \frac{\nu}{\epsilon}\tilde{u}\tilde{u}_{xx} - \frac{\delta}{\epsilon^2}(\tilde{u}\tilde{u}_x)_x = 0,$$

multiplying ϵ to the above equation and let $\alpha = \frac{\gamma}{\epsilon}$, $\beta = \frac{\kappa}{\epsilon}$, $\eta = \frac{\delta}{\epsilon}$, then Eq. (1.1) can be written as

$$\tilde{u}_t + \alpha\tilde{u}\tilde{u}_x + \tilde{u}_{xxx} - (\tilde{u}\tilde{u}_{xx})_x - \beta\tilde{u}_x\tilde{u}_{xx} - \nu\tilde{u}\tilde{u}_{xx} - \eta(\tilde{u}\tilde{u}_x)_x = 0.$$

For simplifying, we write $\tilde{u}$ as u, so Eq. (1.1) can be written as

$$u_t + \alpha uu_x + u_{xxx} - (uu_{xx})_x - \beta u_x u_{xx} - \nu u_{xx} - \eta(uu_x)_x = 0. \tag{3.1}$$

Substituting $u(x,t) = \phi(x - ct) = \phi(\xi)$ into (3.1) and integrating it once, we have

$$-c\phi + \frac{\alpha}{2}\phi^2 + \phi'' - \phi\phi'' - \frac{\beta}{2}(\phi')^2\nu\phi' - \eta\phi\phi' = 0, \tag{3.2}$$

where primes denote the derivatives with respect to ξ and the integration constant is taken to zero.

From step 1, the pole order of Eq. (3.2) is $N = 1$. Therefore, we can write the solutions of Eq. (3.2) in the form

$$\phi(\xi) = a_{-1}\left(\frac{G'}{G}\right)^{-1} + a_0 + a_1\left(\frac{G'}{G}\right) = a_{-1}\omega^{-1} + a_0 + a_1\omega, \tag{3.3}$$

where $a_{-1}^2 + a_1^2 \neq 0$ and $G = G(\xi)$ satisfies the auxiliary LODE (2.4).

From Eqs. (2.5) and (3.3), we obtain

$$\begin{aligned}\phi'(\xi) &= \mu a_{-1}\left(\frac{G'}{G}\right)^{-2} - \lambda a_{-1}\left(\frac{G'}{G}\right) + a_{-1} \\ &\quad - \mu a_1 - \lambda a_1\left(\frac{G'}{G}\right) - a_1\left(\frac{G'}{G}\right)^2 \\ &= \mu a_{-1}\omega^{-2} - \lambda a_{-1}\omega + a_{-1} - \mu a_1 - \lambda a_1\omega - a_1\omega^2 \\ &= \mu a_{-1}\omega^{-2} - \lambda(a_{-1} + a_1)\omega + (a_{-1} - \mu a_1) - a_1\omega^2,\end{aligned} \tag{3.4}$$

$$\begin{aligned}\phi''(\xi) &= 2\mu^2 a_{-1}\omega^{-3} + 2\mu\lambda a_{-1}\omega^{-2} + 2\mu a_{-1}\omega^{-1} + \mu\lambda(a_{-1} + a_1) \\ &\quad + \lambda^2(a_{-1} + a_1 + 2\mu a_1)\omega + \lambda(a_{-1} + a_1 + 2\lambda a_1)\omega^2 + 2a_1\omega^3.\end{aligned} \tag{3.5}$$

Substituting (2.6), (3.3), (3.4), and (3.5) into Eq. (3.2), we obtain the polynomial of ω. Collecting all terms with the same power of ω and equate this expressions to zero, thus we obtain the coefficients of ω^i ($i = -4, -3, -2, -1, 0, 1, 2, 3, 4$) be zero, and get the algebraic equation system for a_{-1}, a_0, a_1, c, α, β, λ and μ as follows:

$$\begin{aligned}
\omega^4:&\quad -\frac{1}{2}a_1^2(\beta+4)=0;\\
\omega^3:&\quad -\big(\big((\beta+3)\lambda-\eta\big)a_1+2a_0-2\big)a_1=0;\\
\omega^2:&\quad \left(\left(\left(-\frac{1}{2}\beta-1\right)\lambda^2+\lambda\eta-(\beta+2)\mu+\frac{1}{2}\alpha\right)a_1+(\beta-2)a_{-1}\right.\\
&\qquad \left.-3(a_0-1)\lambda+a_0\eta+\nu\right)a_1=0;\\
\omega^1:&\quad \big(-\mu\big((\beta+1)\lambda-\eta\big)a_1+2\lambda(\beta-2)a_{-1}-(a_0-1)\lambda^2+(a_0\eta+\nu)\lambda\\
&\qquad -\big(2(a_0-1)\big)\mu-a_0\alpha+c\big)a_1=0;\\
\omega^0:&\quad -\frac{1}{2}\mu^2\beta a_1^2+\big(\big((\beta-2)\lambda^2+(2\beta-4)\mu+\alpha\big)a_{-1}-\big((a_0-1)\lambda-a_0\eta\\
&\qquad -\nu\big)\mu\big)a_1-a_{-1}^2\beta-\left(\frac{1}{2}\big((a_0-1)\lambda-a_0\eta-\nu\big)\right)a_{-1}+\frac{1}{2}a_0^2\alpha-ca_0=0;\\
\omega^{-1}:&\quad \big(2\lambda\mu(\beta-2)a_1-\big((\beta+1)\lambda+\eta\big)a_{-1}-(a_0-1)\lambda^2-(a_0\eta+\nu)\lambda\\
&\qquad -2(a_0-1)\mu+a_0\alpha-c\big)a_{-1}=0;\\
\omega^{-2}:&\quad \left(\mu^2(\beta-2)a_1-\left(\left(\frac{1}{2}(\beta+2)\right)\lambda^2+\lambda\eta+(\beta+2)\mu-\alpha\right)a_{-1}\right.\\
&\qquad \left.-\big(\big(3(a_0-1)\big)\lambda+a_0\eta+\nu\big)\mu\right)a_{-1}=0;\\
\omega^{-3}:&\quad -\big(\big((\beta+3)\lambda+\eta\big)a_{-1}+2\mu(a_0-1)\big)\mu a_{-1}=0;\\
\omega^{-4}:&\quad -\frac{1}{2}(\beta+4)\mu^2a_{-1}=0.
\end{aligned}$$

Solving the algebraic equation system by Maple we obtained ten types of solutions:

$$\begin{aligned}
\text{I}:&\quad \beta=-4,\qquad a_1=0,\qquad a_0=\frac{\lambda(\alpha-c)+c\eta}{\alpha\eta},\qquad a_{-1}=\frac{\lambda^2(\alpha-c)^2-c^2\eta^2}{2\eta\alpha(\alpha-c)},\\
&\quad \mu=\lambda^2-\frac{c^2\eta^2}{4(\alpha-c)^2},\qquad \nu=\frac{(\alpha-c)^2-c\eta^2}{\eta(\alpha-c)},
\end{aligned}\tag{3.6}$$

where α, λ, η and c are arbitrary constants.

$$\begin{aligned}
\text{II}:&\quad \beta=-4,\qquad a_{-1}=0,\qquad a_0=-\frac{\lambda(\alpha-c)-c\eta}{\alpha\eta},\qquad a_1=\frac{2(\alpha-c)}{\alpha\eta},\\
&\quad \mu=\frac{(\alpha-c)^2-c\eta^2}{\eta(\alpha-c)},\qquad \nu=\frac{(\alpha-c)^2-c\eta^2}{\eta(\alpha-c)},
\end{aligned}\tag{3.7}$$

where α, λ, η and c are arbitrary constants.

$$\begin{aligned}
\text{III}:&\quad \beta=-4,\qquad \mu=0,\qquad a_0=\frac{8\lambda}{4\lambda^2-\alpha},\qquad a_1=\frac{4\lambda}{4\lambda^2-\alpha},\\
&\quad a_{-1}=\frac{4\lambda^3}{4\lambda^2-\alpha},\qquad c=-6\lambda^2,\qquad \eta=\frac{2\lambda^2\alpha}{2\lambda},\nu=5\lambda,
\end{aligned}\tag{3.8}$$

where α and λ are arbitrary constants.

$$\begin{aligned}&\mathrm{IV}: \quad \beta=-4, \qquad \mu=0, \qquad a_0=\frac{4\lambda^2(\alpha-12\lambda^2)}{(\alpha^2-4\lambda^2)(\alpha+12\lambda^2)},\\ &\quad a_1=-\frac{4\alpha\lambda}{(\alpha^2-4\lambda^2)(\alpha+12\lambda^2)}, \qquad a_{-1}=-\frac{4\alpha\lambda^3}{(\alpha^2-4\lambda^2)(\alpha+12\lambda^2)},\\ &\quad c=\frac{6\alpha\lambda^2}{\alpha+12\lambda^2}, \qquad \eta=\frac{\alpha+2\lambda^2}{2\lambda}, \qquad \nu=-\lambda,\end{aligned} \tag{3.9}$$

where α and λ are arbitrary constants.

$$\begin{aligned}&\mathrm{V}: \quad \beta=-4, \qquad \lambda=\eta=0, \qquad a_0=1, \qquad \alpha=-\frac{2(a_1\nu-2)}{a_1^2},\\ &\quad a_{-1}=\frac{1}{4a_1}, \qquad c=-\frac{2(a_1\nu-2)}{a_1^2}, \qquad \mu=-\frac{1}{4a_1^2},\end{aligned} \tag{3.10}$$

where a_1 and ν are arbitrary constants.

$$\begin{aligned}&\mathrm{VI}: \quad \beta=-4, \qquad \lambda=\eta=0, \qquad a_0=1, \qquad a_1=\frac{6}{\nu}, \qquad a_{-1}=0,\\ &\quad \alpha=-\frac{2}{9}\nu^2, \qquad \mu=-\frac{1}{36}\nu^2, \qquad c=-\frac{2}{9}\nu^2,\end{aligned} \tag{3.11}$$

where a_1 and ν are arbitrary constants.

$$\begin{aligned}&\mathrm{VII}: \quad \beta=-4, \qquad \lambda=\eta=0, \qquad a_0=1, \qquad a_{-1}=-\frac{1}{36}a_1\nu^2+\frac{1}{6}\nu,\\ &\quad c=-\frac{2}{9}\nu^2, \qquad \mu=-\frac{1}{36}\nu^2,\end{aligned} \tag{3.12}$$

where a_1 and ν are arbitrary constants.

$$\begin{aligned}&\mathrm{VIII}: \quad \beta=-4, \qquad \lambda=\eta=-\frac{1}{4}\nu, \qquad a_0=1, \qquad a_1=0, \qquad \alpha=-\frac{1}{2}\nu^2,\\ &\quad a_{-1}=\frac{1}{8}\nu, \qquad c=-\frac{5}{16}\nu^2, \qquad \mu=-\frac{1}{16}\nu^2,\end{aligned} \tag{3.13}$$

where ν is an arbitrary constant.

$$\begin{aligned}&\mathrm{IX}: \quad \beta=-4, \qquad \alpha=\mu=0, \qquad \lambda=\eta=-\frac{1}{4}\nu, \qquad a_0=5, \qquad a_1=-\frac{16}{\nu},\\ &\quad a_{-1}=\frac{1}{8}\nu, \qquad c=\frac{3}{16}\nu^2,\end{aligned} \tag{3.14}$$

where ν is an arbitrary constant.

$$\begin{aligned}&\mathrm{X}: \quad a_1=\mu=0, \qquad a_{-1}=\frac{2(\lambda^2-\lambda\nu-\nu)((\beta c+\alpha)\lambda^2-\nu\alpha\lambda+\alpha(c-\nu))\lambda}{(\beta\lambda^2+\alpha)^2(c-\nu)},\\ &\quad a_0=\frac{2(\lambda^2-\lambda\nu-\nu)((\beta\nu+\alpha)\lambda-\nu\alpha)\lambda}{(\beta\lambda^2+\alpha)^2(c-\nu)}, \qquad \eta=-\frac{(\beta+2)\lambda^2+\alpha}{2\lambda},\end{aligned} \tag{3.15}$$

where α, β, λ, ν and c are arbitrary constants.

Remark: these ten types of solutions to the algebraic equation system are different from each other.

3.2 The exact traveling wave solutions of the KS equation

Now, we use the solution sets from I to X and the solutions of (2.6) to obtain the solutions of (3.1).

For I, substituting the solution set (3.6) and the corresponding solutions of (2.6) into (3.1), we obtain the corresponding traveling wave solutions of (3.1) as follows, respectively.

If $q > 0$, the solutions of Eq. (3.1) are

$$\phi_1^{\mathrm{I}}(\xi) = -\frac{\lambda^2(\alpha - c)^2 - c^2\eta^2}{2\alpha\eta(\alpha - c)}\left(\frac{\sqrt{q}}{2}\tan\frac{\sqrt{q}(\xi + k_1)}{2} + \frac{\lambda}{2}\right)^{-1} + \frac{\lambda(\alpha - c) + c\eta}{\alpha\eta}, \tag{3.16}$$

or

$$\begin{aligned}\phi_{11}^{\mathrm{I}}(\xi) = {}& \frac{\lambda^2(\alpha - c)^2 - c^2\eta^2}{2\alpha\eta(\alpha - c)}\left(\frac{\sqrt{q}}{\sin\sqrt{q}\xi - k_{11}\sqrt{q}\cos^2\frac{\sqrt{q}\xi}{2}}\right.\\ & \left. - \frac{\sqrt{q}}{2}\tan\frac{\sqrt{q}(\xi + k_1)}{2} - \frac{\lambda}{2}\right)^{-1} + \frac{\lambda(\alpha - c) + c\eta}{\alpha\eta},\end{aligned} \tag{3.17}$$

where $\xi = x - ct$ and α, η, c, λ, k_1 and k_{11} are arbitrary constants.

If $q < 0$, the solutions of Eq. (3.1) are

$$\phi_2^{\mathrm{I}}(\xi) = \frac{\lambda^2(\alpha - c)^2 - c^2\eta^2}{2\alpha\eta(\alpha - c)}\left(\frac{\sqrt{-q}}{2}\tanh\frac{\sqrt{-q}(\xi + k_2)}{2} - \frac{\lambda}{2}\right)^{-1} + \frac{\lambda(\alpha - c) + c\eta}{\alpha\eta}, \tag{3.18}$$

or

$$\begin{aligned}\phi_{21}^{\mathrm{I}}(\xi) = {}& \frac{\lambda^2(\alpha - c)^2 - c^2\eta^2}{2\alpha\eta(\alpha - c)}\left(\frac{\sqrt{-q}}{\sinh\sqrt{-q}\xi - k_{21}\sqrt{-q}\cosh^2\frac{\sqrt{-q}\xi}{2}}\right.\\ & \left. + \frac{\sqrt{-q}}{2}\tanh\frac{\sqrt{-q}(\xi + k_2)}{2} - \frac{\lambda}{2}\right)^{-1} + \frac{\lambda(\alpha - c) + c\eta}{\alpha\eta},\end{aligned} \tag{3.19}$$

where $\xi = x - ct$ and α, η, c, λ, k_2 and k_{21} are arbitrary constants.

Specially, if $\mu = 0$, then $q < 0$, the solutions of Eq. (3.1) are

$$\phi_{22}^{\mathrm{I}}(\xi) = \frac{\lambda^2(\alpha - c)^2 - c^2\eta^2}{2\alpha\eta(\alpha - c)}\left(\frac{k_{22}\lambda}{e^{\lambda\xi} - k_{22}}\right)^{-1} + \frac{\lambda(\alpha - c) + c\eta}{\alpha\eta}, \tag{3.20}$$

where $\xi = x - ct$ and α, η, c, λ and k_{22} are arbitrary constants.

If $q = 0$, the solutions of Eq. (3.1) are

$$\phi_3^{\mathrm{I}}(\xi) = \frac{\lambda^2(\alpha - c)^2 - c^2\eta^2}{2\alpha\eta(\alpha - c)}\left(\frac{1}{\xi + k_3} - \frac{\lambda}{2}\right)^{-1} + \frac{\lambda(\alpha - c) + c\eta}{\alpha\eta}, \tag{3.21}$$

where $\xi = x - ct$ and α, η, c, λ and k_3 are arbitrary constants.

Similarly, according to the sign of q, we can obtain all the solutions of Eq. (3.1) and all the figures of the solutions.

For II, the solutions of Eq. (3.1) are as follows: if $q > 0$,

$$\phi_1^{\mathrm{II}}(\xi) = -\frac{\lambda(\alpha - c) - c\eta}{\alpha\eta} - \frac{2(\alpha - c)}{\alpha\eta}\left(\frac{\sqrt{q}}{2}\tan\frac{\sqrt{q}(\xi + k_1)}{2} + \frac{\lambda}{2}\right), \tag{3.22}$$

or

$$\begin{aligned}\phi_{11}^{\mathrm{II}}(\xi) = {} & -\frac{\lambda(\alpha - c) - c\eta}{\alpha\eta} + \frac{2(\alpha - c)}{\alpha\eta}\left(\frac{\sqrt{q}}{\sin\sqrt{q}\xi - k_{11}\sqrt{q}\cos^2\frac{\sqrt{q}\xi}{2}}\right.\\ & \left. - \frac{\sqrt{q}}{2}\tan\frac{\sqrt{q}(\xi + k_1)}{2} - \frac{\lambda}{2}\right).\end{aligned} \tag{3.23}$$

If $q < 0$,

$$\phi_2^{\mathrm{II}}(\xi) = -\frac{\lambda(\alpha - c) - c\eta}{\alpha\eta} + \frac{2(\alpha - c)}{\alpha\eta}\left(\frac{\sqrt{-q}}{2}\tanh\frac{\sqrt{-q}(\xi + k_2)}{2} - \frac{\lambda}{2}\right), \tag{3.24}$$

or

$$\begin{aligned}\phi_{21}^{\mathrm{II}}(\xi) = {} & -\frac{\lambda(\alpha - c) - c\eta}{\alpha\eta} + \frac{2(\alpha - c)}{\alpha\eta}\left(\frac{\sqrt{-q}}{\sinh\sqrt{-q}\xi - k_{21}\sqrt{-q}\cosh^2\frac{\sqrt{-q}\xi}{2}}\right.\\ & \left. + \frac{\sqrt{-q}}{2}\tanh\frac{\sqrt{-q}(\xi + k_2)}{2} - \frac{\lambda}{2}\right).\end{aligned} \tag{3.25}$$

If $\mu = 0$,

$$\phi_{22}^{\mathrm{II}}(\xi) = -\frac{\lambda(\alpha - c) - c\eta}{\alpha\eta} + \frac{2(\alpha - c)}{\alpha\eta}\left(\frac{k_{22}\lambda}{e^{\lambda\xi} - k_{22}}\right). \tag{3.26}$$

If $q = 0$,

$$\phi_3^{\mathrm{II}}(\xi) = -\frac{\lambda(\alpha - c) - c\eta}{\alpha\eta} + \frac{2(\alpha - c)}{\alpha\eta}\left(\frac{1}{\xi + k_3} - \frac{\lambda}{2}\right), \tag{3.27}$$

where $\xi = x - ct$ and α, η, c, λ, k_1, k_{11}, k_2, k_{21}, and k_3 are arbitrary constants.

For III, because $\mu = 0$, the solutions to Eq. (3.1) are

$$\phi_1^{\mathrm{III}}(\xi) = \frac{4\lambda^3}{4\lambda^2 - \alpha}\left(\frac{k_{22}\lambda}{e^{\lambda\xi} - k_{22}}\right)^{-1} + \frac{4\lambda}{8\lambda^2 - \alpha} + \frac{4\lambda}{8\lambda^2 - \alpha}\frac{k_{22}\lambda}{e^{\lambda\xi} - k_{22}}, \tag{3.28}$$

where $\xi = x + 6\lambda^2 t$ and α, λ, k_{22} are arbitrary constants.

For IV, the solutions to Eq. (3.1) are

$$\begin{aligned}\phi_1^{\mathrm{IV}}(\xi) = {} & -\frac{4\alpha\lambda^3}{(\alpha - 4\lambda^2)(\alpha + 12\lambda^2)}\left(\frac{k_{22}\lambda}{e^{\lambda\xi} - k_{22}}\right)^{-1} + \frac{4\lambda^2(\alpha - 12\lambda^2)}{(\alpha - 4\lambda^2)(\alpha + 12\lambda^2)}\\ & - \frac{4\alpha\lambda}{(\alpha - 4\lambda^2)(\alpha + 12\lambda^2)}\frac{k_{22}\lambda}{e^{\lambda\xi} - k_{22}},\end{aligned} \tag{3.29}$$

where $\xi = x - \frac{6\alpha\lambda^2}{\alpha + 12\lambda^2}t$ and α, λ, k_{22} are arbitrary constants.

For V, because $\mu = -\frac{1}{4a_1^2}$, thus $q < 0$, then

$$\phi_2^{\mathrm{V}}(\xi) = -\frac{1}{4a_1}\left(\frac{1}{2|a_1|}\tanh\frac{\xi + k_2}{2|a_1|} - \frac{\lambda}{2}\right)^{-1} + 1 + a_1\left(\frac{1}{2|a_1|}\tanh\frac{\xi + k_2}{2|a_1|} - \frac{\lambda}{2}\right), \tag{3.30}$$

or

$$\phi_{21}^{\mathrm{V}}(\xi) = \frac{1}{4a_1}\left(\frac{1}{2|a_1|\sinh\frac{\xi}{2|a_1|} + k_{21}\cosh^2\frac{\xi}{2|a_1|}} + \frac{1}{2|a_1|}\tanh\frac{\xi + k_2}{2|a_1|} - \frac{\lambda}{2}\right)^{-1} + 1 + a_1\left(\frac{1}{2|a_1|\sinh\frac{\xi}{2|a_1|} + k_{21}\cosh^2\frac{\xi}{2|a_1|}} + \frac{1}{2|a_1|}\tanh\frac{\xi + k_2}{2|a_1|} - \frac{\lambda}{2}\right), \tag{3.31}$$

where $\xi = x + \frac{2(a_1\nu - 2)}{a_1^2}t$ and a_1, ν, k_2, k_{21} are arbitrary constants.

For VI, because $\mu = -\frac{1}{36}\nu^2$, thus $q < 0$, then

$$\phi_2^{\mathrm{VI}}(\xi) = 1 + \frac{6}{\nu}\left(\frac{|\nu|}{6}\tanh\frac{|\nu|(\xi + k_2)}{6} - \frac{\lambda}{2}\right), \tag{3.32}$$

or

$$\phi_{21}^{\mathrm{VI}}(\xi) = 1 + \frac{6}{\nu}\left(\frac{|\nu|}{3\sinh\frac{|\nu|\xi}{3} + k_{21}|\nu|\cosh^2\frac{|\nu|\xi}{6}} + \frac{|\nu|}{6}\tanh\frac{|\nu|(\xi + k_2)}{6} - \frac{\lambda}{2}\right), \tag{3.33}$$

where $\xi = x + \frac{2\nu^2}{9}t$ and ν, k_2, k_{21} are arbitrary constants.

For VII, because $\mu = -\frac{1}{36}\nu^2$, thus $q < 0$, then

$$\phi_2^{\mathrm{VII}}(\xi) = \left(-\frac{1}{36}a_1\nu^2 + \frac{1}{6}\nu\right)\left(\frac{|\nu|}{6}\tanh\frac{|\nu|(\xi + k_2)}{6} - \frac{\lambda}{2}\right)^{-1} + 1 + \frac{6}{\nu}\left(\frac{|\nu|}{6}\tanh\frac{|\nu|(\xi + k_2)}{6} - \frac{\lambda}{2}\right), \tag{3.34}$$

or

$$\phi_{21}^{\mathrm{VII}}(\xi) = \left(-\frac{1}{36}a_1\nu^2 + \frac{1}{6}\nu\right)\left(\frac{|\nu|}{3\sinh\frac{|\nu|\xi}{3} + k_{21}|\nu|\cosh^2\frac{|\nu|\xi}{6}} + \frac{|\nu|}{6}\tanh\frac{|\nu|(\xi + k_2)}{6} - \frac{\lambda}{2}\right)^{-1} + 1 + \frac{6}{\nu}\left(\frac{|\nu|}{3\sinh\frac{|\nu|\xi}{3} + k_{21}|\nu|\cosh^2\frac{|\nu|\xi}{6}} + \frac{|\nu|}{6}\tanh\frac{|\nu|(\xi + k_2)}{6} - \frac{\lambda}{2}\right), \tag{3.35}$$

where $\xi = x + \frac{2\nu^2}{9}t$ and a_1, ν, k_2, k_{21} are arbitrary constants.

For VIII, because $\mu = -\frac{1}{16}\nu^2$, $\lambda = -\frac{1}{4}\nu$, thus $q = 4\nu - \lambda^2 = -\frac{5}{16}\nu^2 < 0$, then

$$\phi_2^{\mathrm{VIII}}(\xi) = \frac{\nu}{8}\left(\frac{\sqrt{5}|\nu|}{8}\tanh\frac{\sqrt{5}|\nu|(\xi + k_2)}{8} - \frac{\lambda}{2}\right)^{-1} + 1, \tag{3.36}$$

or

$$\begin{aligned}\phi_{21}^{\mathrm{VIII}}(\xi) = \frac{\nu}{8}\Bigg(&\frac{\sqrt{5}|\nu|}{4\sinh\frac{\sqrt{5}|\nu|\xi}{4} + \sqrt{5}k_{21}|\nu|\cosh^2\frac{\sqrt{5}|\nu|\xi}{8}} \\ &+ \frac{\sqrt{5}|\nu|}{8}\tanh\frac{\sqrt{5}|\nu|(\xi + k_2)}{8} - \frac{\lambda}{2}\Bigg)^{-1} + 1,\end{aligned} \tag{3.37}$$

where $\xi = x + \frac{5\nu^2}{16}t$ and ν, k_2, k_{21} are arbitrary constants.

For IX, because $\mu = 0$, $\lambda = -\frac{1}{4}\nu$, thus $q = 4\nu - \lambda^2 = -\frac{1}{16}\nu^2 < 0$, then

$$\begin{aligned}\phi_2^{\mathrm{IX}}(\xi) = {}&\frac{\nu}{8}\left(\frac{|\nu|}{8}\tanh\frac{|\nu|(\xi + k_2)}{8} - \frac{\lambda}{2}\right)^{-1} + 5 \\ &- \frac{16}{\nu}\left(\frac{|\nu|}{8}\tanh\frac{|\nu|(\xi + k_2)}{8} - \frac{\lambda}{2}\right)^{-1},\end{aligned} \tag{3.38}$$

or

$$\begin{aligned}\phi_{21}^{\mathrm{IX}}(\xi) = {}&\frac{\nu}{8}\left(\frac{|\nu|}{4\sinh\frac{|\nu|\xi}{4} + k_{21}|\nu|\cosh^2\frac{|\nu|\xi}{8}} + \frac{|\nu|}{8}\tanh\frac{|\nu|(\xi + k_2)}{8} - \frac{\lambda}{2}\right)^{-1}, \\ &+ 5 - \frac{16}{\nu}\left(\frac{|\nu|}{4\sinh\frac{|\nu|\xi}{4} + k_{21}|\nu|\cosh^2\frac{|\nu|\xi}{8}} + \frac{|\nu|}{8}\tanh\frac{|\nu|(\xi + k_2)}{8} - \frac{\lambda}{2}\right),\end{aligned} \tag{3.39}$$

where $\xi = x - \frac{3\nu}{16}t$ and ν, k_2, k_{21} are arbitrary constants.

For X, because $\mu = 0$, then $q = 4\nu - \lambda^2 = -\lambda^2 < 0$, thus

$$\begin{aligned}\phi_2^{\mathrm{X}}(\xi) = {}&\frac{2(\lambda^2 - \lambda\nu - \nu)((\beta c + \alpha)\lambda^2 - \nu\alpha\lambda + \alpha(c - \nu))\lambda}{(\beta\lambda^2 + \alpha)^2(c - \nu)} \\ &\cdot\left(\frac{|\lambda|}{2}\tanh\frac{|\lambda|(\xi + k_2)}{2} - \frac{\lambda}{2}\right)^{-1} + \frac{2(\lambda^2 - \lambda\nu - \nu)((\beta\nu + \alpha)\lambda - \nu\alpha)\lambda}{(\beta\lambda^2 + \alpha)^2(c - \nu)},\end{aligned} \tag{3.40}$$

or

$$\begin{aligned}\phi_{21}^{\mathrm{X}}(\xi) = {}&\frac{2(\lambda^2 - \lambda\nu - \nu)((\beta c + \alpha)\lambda^2 - \nu\alpha\lambda + \alpha(c - \nu))\lambda}{(\beta\lambda^2 + \alpha)^2(c - \nu)} \\ &\cdot\left(\frac{|\lambda|}{2\sinh\frac{|\lambda|\xi}{2} + k_{21}|\lambda|\cosh^2\frac{|\lambda|\xi}{2}} + \frac{|\lambda|}{2}\tanh\frac{|\lambda|(\xi + k_2)}{2} - \frac{\lambda}{2}\right)^{-1} \\ &+ \frac{2(\lambda^2 - \lambda\nu - \nu)((\beta\nu + \alpha)\lambda - \nu\alpha)\lambda}{(\beta\lambda^2 + \alpha)^2(c - \nu)},\end{aligned} \tag{3.41}$$

where $\xi = x - ct$ and α, β, λ, ν, c, k_2, k_{21} are arbitrary constants.

4 Illustrative examples and the corresponding figures

Here we provide simple numerical examples to confirm our main results and demonstrate the system (3.1) as follows.

Example 1 In this example, we assume the following parameters:

$$\begin{aligned} &\mu = 2, \qquad \lambda = 2, \qquad \alpha = 2, \qquad c = \frac{1}{2}, \\ &\eta = 2, \qquad k_1 = 0, \qquad k_{11} = 1, \end{aligned}$$

which satisfy set (3.6) and $q = 4 > 0$, so the solutions to the system (3.1) are $\phi_1^{\mathrm{I}}(\xi)$ and $\phi_{11}^{\mathrm{I}}(\xi)$. The figures of $\phi_1^{\mathrm{I}}(\xi)$ and $\phi_{11}^{\mathrm{I}}(\xi)$ are like Fig. 1.

Example 2 In this example, we assume the following parameters:

$$\begin{aligned} &\mu = 1, \qquad \lambda = 3, \qquad \alpha = 2, \qquad c = \frac{1}{2}, \\ &\eta = 2, \qquad k_2 = 0, \qquad k_{21} = 1, \end{aligned}$$

which satisfy set (3.6) and $q = -5 < 0$, so the solutions to the system (3.1) are $\phi_2^{\mathrm{I}}(\xi)$ and $\phi_{21}^{\mathrm{I}}(\xi)$. The figures of $\phi_2^{\mathrm{I}}(\xi)$ and $\phi_{21}^{\mathrm{I}}(\xi)$ are like Fig. 2.

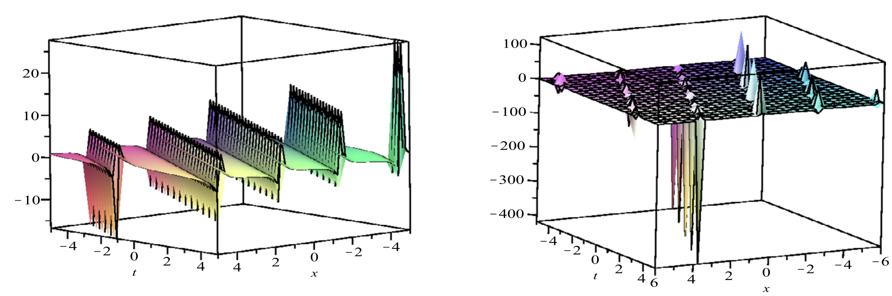


Figure 1 The figure of $\phi_1^{\mathrm{I}}(\xi)$ as shown in the left figure and the figure of $\phi_{11}^{\mathrm{I}}(\xi)$ as shown in the right figure

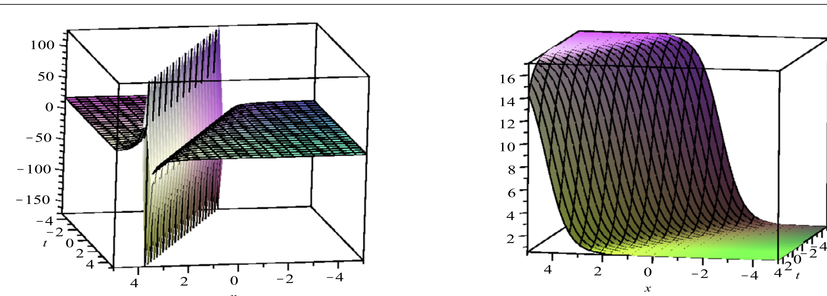


Figure 2 The figure of $\phi_2^{\mathrm{I}}(\xi)$ as shown in the left figure and the figure of $\phi_{21}^{\mathrm{I}}(\xi)$ as shown in the right figure

Figure 3 The figure of $\phi_{22}^{\mathrm{I}}(\xi)$

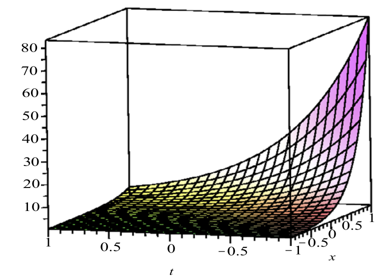
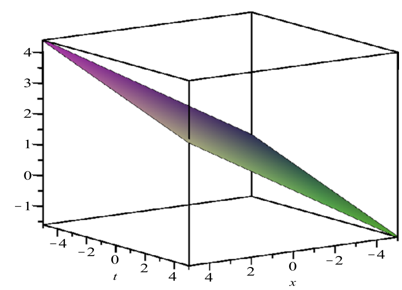


Figure 4 The figure of $\phi_{3}^{\mathrm{I}}(\xi)$



Example 3 In this example, we assume the following parameters:

$$\mu = 0, \qquad \lambda = 3, \qquad \alpha = 2, \qquad c = \frac{1}{2},$$
$$\eta = 2, \qquad k_{22} = 1,$$

which satisfy set (3.6) and $q = -9 < 0$, so the solutions to the system (3.1) are $\phi_{22}^{\mathrm{I}}(\xi)$ and the figures of $\phi_{22}^{\mathrm{I}}(\xi)$ are like Fig. 3.

Example 4 In this example, we assume the following parameters:

$$\mu = 1, \qquad \lambda = 2, \qquad \alpha = 2, \qquad c = \frac{1}{2},$$
$$\eta = 2, \qquad k_{3} = 1,$$

which satisfy set (3.6) and $q = 0$, so the solutions to the system (3.1) are $\phi_{3}^{\mathrm{I}}(\xi)$ and the figures of $\phi_{3}^{\mathrm{I}}(\xi)$ are like Fig. 4.

Unfortunately, it does not seem mathematically tractable to determine the figures of the other nine types solutions to Eq. (3.1), thus, we omit the examples and the figures about them.

5 Conclusions and remarks

We proposed the efficient modified polynomial expansion method and obtain more new exact traveling wave solutions to the Kudryashov–Sinelshchikov equation. By the modified polynomial expansion method we obtain hyperbolic function traveling wave solutions, trigonometric function traveling wave solutions, and rational function traveling wave solutions. On comparing with the modified polynomial expansion method and other methods to find the traveling wave for PDEs, the modified polynomial expansion method is more effective, more powerful and more convenient. Moreover, the modified polynomial expansion method can be used to solve any high-order degree PDEs.

Remark: In [5, 33–37], the solutions "G" of the auxiliary equation are directly obtained according to the method of solving the second ordinary differential equation, and by the expression of G, the authors obtained the expression of $\frac{G'}{G}$. In [33–37], there are only two kinds of solutions G corresponding to the discriminant of the second ordinary differential equation which is larger than zero and less than zero, respectively; and in [5], the authors obtain three kinds of solutions corresponding to the discriminant of the second ordinary differential equation which is larger than zero, equal to zero and less than zero, respectively. However, it is difficult to solve the second order ordinary differential equation, sometimes it cannot get the simple expression of its solution or cannot obtain its solutions. In this paper, the auxiliary equation is transformed into a Riccati equation for $\frac{G'}{G}$, the corresponding solutions $\frac{G'}{G}$ are directly easily obtained by the Riccati equation. Moreover, the solutions of the auxiliary equation by the Riccati obtained include the results in [5, 33–37].

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Authors' contributions
The main idea of this paper was proposed by the author and the author completed the final manuscript alone. All authors read and approved the final manuscript.

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