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# Dynamic behaviors of a stage-structured commensalism system

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## Abstract

A two species stage-structured commensalism model is proposed and studied in this paper. Local and global stability property of the boundary equilibrium and the positive equilibrium are investigated, respectively. If the stage-structured species is extinct, then depending on the intensity of cooperation, the species may still be extinct or become persistent. If the stage-structured species is permanent, then the final system is always globally asymptotically stable, which means the species is always permanent. Our study shows that increasing the intensity of the cooperation between the species is one of very useful methods to avoid extinction of the endangered species. Such a finding may be useful in protecting the endangered species. An example together with its numeric simulations is presented to verify our main results.

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**Keywords:** Stage structure; Commensalism; Lyapunov function; Global stability

## 1 Introduction

The aim of this paper is to investigate the dynamic behaviors of the following stage-structured commensalism system:

$$\begin{aligned}\frac{dx_1}{dt} &= \alpha x_2 - \beta x_1 - \delta_1 x_1, \\ \frac{dx_2}{dt} &= \beta x_1 - \delta_2 x_2 - \gamma x_2^2 + dx_2 y, \\ \frac{dy}{dt} &= y(b_2 - a_2 y),\end{aligned}\tag{1.1}$$

with  $x_1(0) > 0$ ,  $x_2(0) > 0$ , and  $y(0) > 0$ , where  $\alpha$ ,  $\beta$ ,  $\delta_1$ ,  $\delta_2$ ,  $d$ ,  $b_2$ ,  $a_2$ , and  $\gamma$  are all positive constants,  $x_1(t)$  and  $x_2(t)$  are the densities of the immature and mature first species at time  $t$ ,  $y$  is the density of the second species at time  $t$ . For the simplicity of our model, we only consider the stage structure of immaturity and maturity of the first species and do not consider the stage structure of the second species. The following assumptions are made in formulating model (1.1):

1. For the first species, the per capita birth rate of the immature population is  $\alpha > 0$ ;  
The per capita death rate of the immature population is  $\delta_1 > 0$ ; The per capita death rate of the mature population is proportional to the current mature population with

- a proportionality constant  $\delta_2 > 0$ ;  $\beta > 0$  denotes the surviving rate of immaturity to reach maturity; The mature species is density-dependent with the parameter  $\gamma > 0$ .
2. The second species is only beneficial to the mature first species, and their relationship is bilinear ( $dx_2y$ ).
  3. The second species satisfies the logistic model, where  $b_2$  is the intrinsic growth rate of the second species, and  $\frac{a_2}{b_2}$  is the catching capacity of the second species.

During the last decade, many scholars investigated the dynamic behaviors of the mutualism model or commensalism model [1–25]. Han and Chen [15] proposed the following commensalism model with feedback controls:

$$\begin{aligned} \frac{dx_1(t)}{dt} &= x_1(t)(b_1 - a_{11}x_1(t) + a_{12}x_2(t) - \alpha_1u_1(t)), \\ \frac{dx_2(t)}{dt} &= x_2(t)(b_2 - a_{22}x_2(t) - \alpha_2u_2(t)), \\ \frac{du_1(t)}{dt} &= -\eta_1u_1(t) + a_1x_1(t), \\ \frac{du_2(t)}{dt} &= -\eta_2u_2(t) + a_2x_2(t). \end{aligned} \tag{1.2}$$

They first showed that the subsystem, i.e., the two species commensalism model

$$\begin{aligned} \frac{dx_1(t)}{dt} &= x_1(t)(b_1 - a_{11}x_1(t) + a_{12}x_2(t)), \\ \frac{dx_2(t)}{dt} &= x_2(t)(b_2 - a_{22}x_2(t)), \end{aligned} \tag{1.3}$$

admits a unique globally asymptotically stable positive equilibrium. After that, they further showed that system (1.2) admits a unique globally stable positive equilibrium, which means that feedback control variables have no influence on the stability property of system (1.2). Several scholars argued that discrete models are more suitable if the species have non-overlapping generations. Xie et al. [16] proposed the following discrete commensal symbiosis model:

$$\begin{aligned} x_1(k + 1) &= x_1(k) \exp\{a_1(k) - b_1(k)x_1(k) + c_1(k)x_2(k)\}, \\ x_2(k + 1) &= x_2(k) \exp\{a_2(k) - b_2(k)x_2(k)\}. \end{aligned} \tag{1.4}$$

They obtained a set of sufficient conditions which ensure the existence of the positive periodic solution of system (1.4). Xue et al. [17] further proposed a discrete commensalism model with delays, they investigated the almost periodic solution of the system. Li et al. [21] studied the positive periodic solution of a discrete commensalism model with Holling II functional response. Wu [13] argued that it may be more suitable to assume that the relationship between two species is of nonlinear type instead of linear one, and she established the following two species commensal symbiosis model:

$$\begin{aligned} \frac{dx}{dt} &= x \left( a_1 - b_1x + \frac{c_1y^p}{1 + y^p} \right), \\ \frac{dy}{dt} &= y(a_2 - b_2y), \end{aligned} \tag{1.5}$$

where  $a_i, b_i, i = 1, 2, p$  and  $c_1$  are all positive constants,  $p \geq 1$ . The results of [13] were then generalized by Wu et al. [12] to the following commensalism model with Allee effect:

$$\begin{aligned} \frac{dx}{dt} &= x \left( a_1 - b_1x + \frac{c_1y^p}{1 + y^p} \right), \\ \frac{dy}{dt} &= y(a_2 - b_2y) \frac{y}{u + y}, \end{aligned} \tag{1.6}$$

where  $a_i, b_i, i = 1, 2, p, u$ , and  $c_1$  are all positive constants,  $p \geq 1$ .

Recently, several scholars studied the influence of partial closure on the non-selective harvesting commensalism model. Deng and Huang [21] studied the dynamic behaviors of the following system:

$$\begin{aligned} \frac{dx}{dt} &= r_1x \left( 1 - \frac{x}{K_1} + \alpha \frac{y}{K_1} \right) - q_1Emx, \\ \frac{dy}{dt} &= r_2y \left( 1 - \frac{y}{K_2} \right) - q_2Emy, \end{aligned} \tag{1.7}$$

where  $r_1, r_2, K_1, K_2, \alpha$  are all positive constants.  $E$  is the combined fishing effort used to harvest, and  $m$  ( $0 < m < 1$ ) is the fraction of the stock available for harvesting. They showed that depending on the fraction of the stock available for harvesting, the system may undergo extinction, partial survival, or two species may coexist in a stable state. The dynamic behaviors of the system become complicated compared with the non-harvesting system. Lin [10] further studied the dynamic behaviors of a commensal symbiosis model with non-monotonic functional response and non-selective harvesting in a partial closure.

On the other hand, many scholars investigated the dynamic behaviors of the stage-structured species, see [26–45], and the references cited therein. In constructing a stage-structured model, two different ideas were applied. Firstly, assume that the species needs time to grow up, and this leads to the delayed model. Aiello and Freedman [42] for the first time proposed the following stage-structured single species model:

$$\begin{aligned} \frac{dx_1(t)}{dt} &= \alpha x_2(t) - \gamma x_1(t) - \alpha e^{-\gamma\tau} x_2(t - \tau), \\ \frac{dx_2(t)}{dt} &= \alpha e^{-\gamma\tau} x_2(t - \tau) - \beta x_2^2(t). \end{aligned} \tag{1.8}$$

They showed that system (1.8) admits a unique positive equilibrium which is globally asymptotically stable. Many scholars [26–39] used the idea of Aiello and Freedman to establish stage-structured ecological models. For example, Chen et al. [28] studied the persistence property of the following stage-structured predator–prey model:

$$\begin{aligned} \frac{dx_1(t)}{dt} &= r_1x_2(t) - d_{11}x_1(t) - r_1e^{-d_{11}\tau_1}x_2(t - \tau_1), \\ \frac{dx_2(t)}{dt} &= r_1e^{-d_{11}\tau_1}x_2(t - \tau_1) - d_{12}x_2(t) - b_1x_2^2(t) - c_1x_2(t)y_2(t), \\ \frac{dy_1(t)}{dt} &= r_2y_2(t) - d_{22}y_1(t) - r_2e^{-d_{22}\tau_2}y_2(t - \tau_2), \\ \frac{dy_2(t)}{dt} &= r_2e^{-d_{22}\tau_2}y_2(t - \tau_2) - d_{21}y_2(t) - b_2y_2^2(t) + c_2y_2(t)x_2(t). \end{aligned} \tag{1.9}$$

They obtained a set of sufficient conditions which ensure the global asymptotic stability of the positive equilibrium. Chen et al. [34] proposed and studied the following May type stage-structured cooperation model:

$$\begin{aligned}
 \dot{x}_1(t) &= b_1 e^{-d_{11}\tau_1} x_1(t - \tau_1) - d_{12}x_1(t) - \frac{a_{11}x_1^2(t)}{c_1 + f_1x_2(t)} - a_{12}x_1^2(t), \\
 \dot{y}_1(t) &= b_1x_1(t) - d_{11}y_1(t) - b_1e^{-d_{11}\tau_1}x_1(t - \tau_1), \\
 \dot{x}_2(t) &= b_2e^{-d_{22}\tau_2}x_2(t - \tau_2) - d_{21}x_2(t) - \frac{a_{22}x_2^2(t)}{c_2 + f_2x_1(t)} - a_{21}x_2^2(t), \\
 \dot{y}_2(t) &= b_2x_2(t) - d_{22}y_2(t) - b_2e^{-d_{22}\tau_2}x_2(t - \tau_2).
 \end{aligned}
 \tag{1.10}$$

They showed that with introduction of the stage structure, the May type cooperative system may admit partial survival property, that is, despite the cooperation between the species, the species may still be driven to extinction due to the stage structure. They finally drew the conclusion: the cooperation between the species has no influence on the persistence property of the system.

Another way to construct a stage-structured ecosystem is to assume that there is a proportional number of immature species that become mature species [40, 41, 43, 44]. Recently, Khajanchi and Banerjee [41] proposed the following stage-structured predator-prey model with ratio-dependent functional response:

$$\begin{aligned}
 \frac{dx_1}{dt} &= \alpha x_2(t) - \beta x_1(t) - \delta_1 x_1(t), \\
 \frac{dx_2}{dt} &= \beta x_1(t) - \delta_2 x_2(t) - \gamma x_2^2(t) - \frac{\eta(1 - \theta)x_2(t)y(t)}{g(1 - \theta)x_2(t) + hy(t)}, \\
 \frac{dy}{dt} &= \frac{u\eta(1 - \theta)x_2(t)y(t)}{g(1 - \theta)x_2(t) + hy(t)} - \delta_3 y_1(t)
 \end{aligned}
 \tag{1.11}$$

The authors investigated the stability property of the positive equilibrium and boundary equilibrium. In system (1.11), without predator species, the system will reduce to the following single species stage-structured system:

$$\begin{aligned}
 \frac{dx_1}{dt} &= \alpha x_2(t) - \beta x_1(t) - \delta_1 x_1(t), \\
 \frac{dx_2}{dt} &= \beta x_1(t) - \delta_2 x_2(t) - \gamma x_2^2(t).
 \end{aligned}
 \tag{1.12}$$

Though system (1.12) seems very simple, in [43], we showed that if

$$\alpha\beta < \delta_2(\beta + \delta_1)
 \tag{1.13}$$

holds, the equilibrium  $O(0,0)$  is globally asymptotically stable, which means extinction of the species. Therefore, the dynamic behaviors of system (1.12) are very different to the dynamic behaviors of system (1.8). It is in this sense that we need to do more work on a non-delay stage-structured ecosystem.

Recently, in [43], we proposed the following single species stage-structured system incorporating partial closure for the populations and non-selective harvesting:

$$\begin{aligned} \frac{dx_1}{dt} &= \alpha x_2 - \beta x_1 - \delta_1 x_1 - q_1 E m x_1, \\ \frac{dx_2}{dt} &= \beta x_1 - \delta_2 x_2 - \gamma x_2^2 - q_2 E m x_2, \end{aligned} \tag{1.14}$$

where  $\alpha, \beta, \delta_1, \delta_2, q_1, q_2, E$ , and  $\gamma$  are all positive constants,  $x_1(t)$  and  $x_2(t)$  are the densities of the immature and mature species at time  $t$ . Our study showed that the birth rate of the immature species and the fraction of the stocks for the harvesting play a crucial role in the dynamic behaviors of the system.

One could easily see that in system (1.1), without the cooperation of the second species, the first species is described by (1.12). Hence, one interesting issue is proposed:

*Assume that without the cooperation of the second species, the first species in system (1.1) will be driven to extinction, i.e., assume inequality (1.13) holds. Is it possible for the species 1 to avoid the extinction due to the cooperation of the second species, or will the first species be driven to extinction despite the cooperation of the second species, just like the dynamic behaviors of system (1.10)?*

We will try to find the answer in the rest of the paper.

The paper is arranged as follows. We investigate the existence and local stability property of the equilibria of system (1.1) in Sect. 2. In Sect. 3, by constructing some suitable Lyapunov function, we are able to investigate the global stability property of the equilibria. Section 4 presents some numerical simulations to show the feasibility of the main results. We end this paper with a brief discussion.

## 2 Local stability of the equilibria

Before we study the local stability property of the equilibrium points of system (1.1), we would like to introduce the stability result of equilibrium of system (1.12). The following lemma is Theorems 4.1 and 4.2 of [43].

**Lemma 2.1** *Assume that*

$$\alpha\beta < \delta_2(\beta + \delta_1) \tag{2.1}$$

*holds, then the boundary equilibrium  $O(0, 0)$  of system (1.12) is globally stable. Assume that*

$$\alpha\beta > \delta_2(\beta + \delta_1) \tag{2.2}$$

*holds, then the positive equilibrium  $B(x_1^*, x_2^*)$  of system (1.12) is globally stable, where*

$$x_1^* = \frac{\alpha x_2^*}{\beta + \delta_1}, \quad x_2^* = \frac{\alpha\beta - \delta_2(\beta + \delta_1)}{\gamma(\beta + \delta_1)}.$$

**Lemma 2.2**  $R_3^+$  *is the invariant set of system (1.1).*

*Proof* Note that from system (1.1), for all  $x_1, x_2, y > 0$ , one has

$$\dot{x}_1|_{x_1=0} = \alpha x_2 > 0,$$

$$\begin{aligned} \dot{x}_2|_{x_2=0} &= \beta x_1 > 0, \\ y(t) &= y(0) \exp \int_0^t (b_2 - a_2 y(s)) ds > 0. \end{aligned}$$

It immediately follows that  $R_3^+$  is the invariant set of system (1.1). □

Now we are in a position to investigate the local stability property of system (1.1). The equilibria of system (1.1) are determined by the following system:

$$\begin{aligned} \alpha x_2 - \beta x_1 - \delta_1 x_1 &= 0, \\ \beta x_1 - \delta_2 x_2 - \gamma x_2^2 + dx_2 y &= 0, \\ y(b_2 - a_2 y) &= 0. \end{aligned} \tag{2.3}$$

The system always admits two boundary equilibria:  $A_1(0, 0, 0)$ ,  $A_2(0, 0, \frac{b_2}{a_2})$ . Also, if

$$\alpha\beta > \delta_2(\beta + \delta_1), \tag{2.4}$$

then the system admits another boundary equilibrium  $A_3(x_1^*, x_2^*, 0)$ , where

$$x_1^* = \frac{\alpha x_2^*}{\beta + \delta_1}, \quad x_2^* = \frac{\alpha\beta - \delta_2(\beta + \delta_1)}{\gamma(\beta + \delta_1)}. \tag{2.5}$$

Assume that

$$\alpha\beta - \left(\delta_2 - \frac{db_2}{a_2}\right)(\beta + \delta_1) > 0, \tag{2.6}$$

then system (1.1) admits a unique positive equilibrium  $A_4(x_1^{**}, x_2^{**}, y^{**})$ , where

$$\begin{aligned} x_1^{**} &= \frac{\alpha x_2^{**}}{\beta + \delta_1}, \\ x_2^{**} &= \frac{\alpha\beta - (\delta_2 - \frac{db_2}{a_2})(\beta + \delta_1)}{(\beta + \delta_1)\gamma}, \\ y^{**} &= \frac{b_2}{a_2}. \end{aligned} \tag{2.7}$$

Obviously,  $x_1^{**}$ ,  $x_2^{**}$ , and  $y^{**}$  satisfy the equations

$$\begin{aligned} \alpha x_2^{**} - \beta x_1^{**} - \delta_1 x_1^{**} &= 0, \\ \beta x_1^{**} - \delta_2 x_2^{**} - \gamma (x_2^{**})^2 + dx_2^{**} y^{**} &= 0, \\ b_2 - a_2 y^{**} &= 0. \end{aligned} \tag{2.8}$$

We shall now investigate the local stability property of the above equilibria. The variational matrix of system (1.1) is

$$J(x_1, x_2, y) = \begin{pmatrix} -\beta - \delta_1 & \alpha & 0 \\ \beta & -\delta_2 - 2\gamma x_2 + dy & dx_2 \\ 0 & 0 & -2a_2 y + b_2 \end{pmatrix}. \tag{2.9}$$

**Theorem 2.1**  $A_1(0, 0, 0)$  is unstable.

*Proof* From (2.9) we could see that the Jacobian matrix of the system about the equilibrium point  $A_1(0, 0, 0)$  is given by

$$\begin{pmatrix} -\beta - \delta_1 & \alpha & 0 \\ \beta & -\delta_2 & 0 \\ 0 & 0 & b_2 \end{pmatrix}. \tag{2.10}$$

The characteristic equation of the above matrix is

$$(\lambda - b_2)(\lambda^2 + (\delta_1 + \delta_2 + \beta)\lambda + \beta\delta_2 + \delta_1\delta_2 - \alpha\beta) = 0. \tag{2.11}$$

Hence, it has one positive characteristic root  $\lambda_1 = b_2$ ; consequently,  $A_1(0, 0, 0)$  is unstable. This ends the proof of Theorem 2.1.  $\square$

*Remark 2.1* Theorem 2.1 shows that it is impossible for the system to be driven to extinction, that is, the two species in system (1.1) could not be extinct at the same time.

**Theorem 2.2** Assume that

$$(\beta + \delta_1)\left(\delta_2 - \frac{db_2}{a_2}\right) - \alpha\beta > 0, \tag{2.12}$$

then  $A_2(0, 0, \frac{b_2}{a_2})$  is locally asymptotically stable. Assume that

$$(\beta + \delta_1)\left(\delta_2 - \frac{db_2}{a_2}\right) - \alpha\beta < 0, \tag{2.13}$$

then  $A_2(0, 0, \frac{b_2}{a_2})$  is unstable.

*Proof* From (2.9) we could see that the Jacobian matrix of the system about the equilibrium point  $A_2(0, 0, \frac{b_2}{a_2})$  is given by

$$\begin{pmatrix} -\beta - \delta_1 & \alpha & 0 \\ \beta & \frac{db_2}{a_2} - \delta_2 & 0 \\ 0 & 0 & -b_2 \end{pmatrix}. \tag{2.14}$$

The characteristic equation of the above matrix is

$$(\lambda + b_2)\left[\lambda^2 + \left(\delta_1 + \delta_2 + \beta - \frac{db_2}{a_2}\right)\lambda + (\beta + \delta_1)\left(\delta_2 - \frac{db_2}{a_2}\right) - \alpha\beta\right] = 0. \tag{2.15}$$

Hence, it has one negative characteristic root  $\lambda_1 = -b_2 < 0$ , the other two characteristic roots are determined by the equation

$$\lambda^2 + \left(\delta_1 + \delta_2 + \beta - \frac{db_2}{a_2}\right)\lambda + (\beta + \delta_1)\left(\delta_2 - \frac{db_2}{a_2}\right) - \alpha\beta = 0. \tag{2.16}$$

Note that the two characteristic roots of Eq. (2.16) satisfy

$$\begin{aligned} \lambda_2 + \lambda_3 &= -\left(\delta_1 + \delta_2 + \beta - \frac{db_2}{a_2}\right), \\ \lambda_2\lambda_3 &= (\beta + \delta_1)\left(\delta_2 - \frac{db_2}{a_2}\right) - \alpha\beta. \end{aligned} \tag{2.17}$$

Under assumption (2.13),  $\lambda_2\lambda_3 < 0$ , hence at least one characteristic root is positive; consequently,  $A_2(0, 0, \frac{b_2}{a_2})$  is unstable. Under assumption (2.12), it implies that  $\delta_2 > \frac{db_2}{a_2}$ , and so, from (2.17), one has  $\lambda_2 + \lambda_3 < 0$ ,  $\lambda_2\lambda_3 > 0$ . Hence,  $\lambda_2 < 0, \lambda_3 < 0$ . That is, under assumption (2.12), three characteristic roots of matrix (2.14) are all negative; consequently,  $A_1(0, 0, \frac{b_2}{a_2})$  is locally asymptotically stable. This ends the proof of Theorem 2.2.  $\square$

*Remark 2.2* Assume that (2.1) holds, then for the system without cooperation, it follows from Lemma 2.1 that the first species will be driven to extinction. If the cooperative coefficient  $d$  is small enough, then inequality (2.12) holds. It follows from the first part of Theorem 2.2 that in this case the first species will still be driven to extinction despite the cooperation of the second species.

**Theorem 2.3**  $A_3(x_1^*, x_2^*, 0)$  is unstable.

*Proof* From (2.9) we could see that the Jacobian matrix of the system about the equilibrium point  $A_3(x_1^*, x_2^*, 0)$  is given by

$$\begin{pmatrix} -\beta - \delta_1 & \alpha & 0 \\ \beta & -\delta_2 - 2\gamma x_2^* & dx_2^* \\ 0 & 0 & b_2 \end{pmatrix}. \tag{2.18}$$

The characteristic equation of the above matrix is

$$(\lambda - b_2)(\lambda^2 + (\delta_1 + 2\gamma x_2^* + \beta)\lambda + (\beta + \delta_1)(\delta_2 + 2\gamma x_2^*) - \alpha\beta) = 0. \tag{2.19}$$

Hence, it has one positive characteristic root  $\lambda_1 = b_2$ ; consequently,  $A_3(x_1^*, x_2^*, 0)$  is unstable. This ends the proof of Theorem 2.3.  $\square$

*Remark 2.3* Theorem 2.3 shows that it is impossible for the second species to be driven to extinction while the first species is asymptotically stable.

**Theorem 2.4** Assume that (2.6) holds, then  $A_4(x_1^{**}, x_2^{**}, y^{**})$  is locally asymptotically stable.

*Proof* From (2.9) we could see that the Jacobian matrix of the system about the equilibrium point  $A_4(x_1^{**}, x_2^{**}, y^{**})$  is given by

$$\begin{pmatrix} -\beta - \delta_1 & \alpha & 0 \\ \beta & -\delta_2 - 2\gamma x_2^{**} + dy^{**} & dx_2^{**} \\ 0 & 0 & -2a_2y^{**} + b_2 \end{pmatrix}. \tag{2.20}$$



Noting that

$$-2a_2y^{**} + b_2 = -2a_2 \frac{b_2}{a_2} + b_2 = -b_2,$$

also, from the second equation of (2.6), we have

$$\begin{aligned} -\delta_2 - 2\gamma x_2^{**} + dy^{**} &= -\frac{\beta x_1^{**}}{x_2^{**}} - \gamma x_2^{**} \\ &= -\frac{\alpha\beta}{\beta + \delta_1} - \gamma x_2^{**}. \end{aligned}$$

The characteristic equation of the above matrix is

$$(\lambda + b_2)[\lambda^2 + B_1\lambda + B_2] = 0, \tag{2.21}$$

where

$$B_1 = \beta + \delta_1 + \frac{\beta\alpha}{\beta + \delta_1} + \gamma x_2^{**},$$

$$B_2 = (\beta + \delta_1)\left(\frac{\beta\alpha}{\beta + \delta_1} + \gamma x_2^{**}\right) - \alpha\beta.$$

Hence, it has one negative characteristic root  $\lambda_1 = -b_2 < 0$ , the other two characteristic roots are determined by the equation

$$\lambda^2 + B_1\lambda + B_2 = 0. \tag{2.22}$$

Note that from the expression of  $x_2^{**}$  and condition (2.6), the two characteristic roots of Eq. (2.22) satisfy

$$\begin{aligned} \lambda_2 + \lambda_3 &= -B_1 < 0, \\ \lambda_2\lambda_3 &= (\beta + \delta_1)\left(\frac{\beta\alpha}{\beta + \delta_1} + \gamma x_2^{**}\right) - \alpha\beta \\ &= (\beta + \delta_1)\left(\frac{\beta\alpha}{\beta + \delta_1} + \frac{\alpha\beta - \delta_2(\beta + \delta_1)}{\beta + \delta_1}\right) - \alpha\beta \\ &= \alpha\beta - \delta_2(\beta + \delta_1) > 0. \end{aligned} \tag{2.23}$$

Hence,  $\lambda_2 < 0, \lambda_3 < 0$ , therefore, all of the three characteristic roots are negative. Consequently,  $A_4(x_1^{**}, x_2^{**}, y^{**})$  is locally asymptotically stable. This ends the proof of Theorem 2.4. □

*Remark 2.4* Condition (2.6) is necessary to ensure the existence of the positive equilibrium. Theorem 2.4 shows that if the positive equilibrium exists, it is locally asymptotically stable.

### 3 Global stability

We showed in Sect. 2 that  $A_1(0, 0, 0)$  and  $A_3(x_1^*, x_2^*, 0)$  are unstable, while under assumption (2.12),  $A_2(0, 0, \frac{b_2}{a_2})$  is locally asymptotically stable; and if the positive equilibrium exists, it is locally asymptotically stable. One interesting issue is to investigate the global stability property of the equilibria. In this section we will try to obtain some sufficient conditions which could ensure the global asymptotic stability of the equilibria  $A_2$  and  $A_4$  of system (1.1).

**Theorem 3.1** *Assume that*

$$(\beta + \delta_1) \left( \delta_2 - \frac{db_2}{a_2} \right) - \alpha\beta > 0, \tag{3.1}$$

*then  $A_2(0, 0, \frac{b_2}{a_2})$  is globally asymptotically stable.*

*Proof* We will prove Theorem 3.1 by constructing some suitable Lyapunov function. Let us define a Lyapunov function

$$V_1(x_1, x_2, y) = \frac{\beta}{\beta + \delta_1} x_1 + x_2 + \frac{d^2}{4\gamma a_2} \left( y - y_1 - y_1 \ln \frac{y}{y_1} \right), \tag{3.2}$$

where

$$y_1 = \frac{b_2}{a_2}. \tag{3.3}$$

One could easily see that the function  $V_1$  is zero at the boundary equilibrium  $A_2(0, 0, \frac{b_2}{a_2})$  and is positive for all other positive values of  $x_1$  and  $x_2$ . The time derivative of  $V_1$  along the trajectories of (1.1) is

$$\begin{aligned} D^+ V_1(t) &= \frac{\beta}{\beta + \delta_1} (\alpha x_2 - \beta x_1 - \delta x_1) \\ &\quad + \beta x_1 - \delta_2 x_2 - \gamma x_2^2 + dx_2 \\ &\quad + \frac{d^2}{4\gamma a_2} (y - y_1)(b_2 - a_2 y) \\ &= \left( \frac{\alpha\beta}{\beta + \delta_1} - \delta_2 \right) x_2 - \gamma x_2^2 + dx_2 \\ &\quad + \frac{d^2}{4\gamma a_2} (y - y_1)(b_2 - a_2 y) \\ &= \left( \frac{\alpha\beta}{\beta + \delta_1} - \delta_2 + \frac{db_2}{a_2} \right) x_2 - \gamma x_2^2 + dx_2 (y - y_1) \\ &\quad + \frac{d^2}{4\gamma a_2} (y - y_1)(a_2 y_1 - a_2 y) \\ &= \frac{1}{\beta + \delta_1} \left( \alpha\beta - \left( \delta_2 - \frac{db_2}{a_2} \right) (\beta + \delta_1) \right) x_2 - \gamma x_2^2 \\ &\quad + dx_2 (y - y_1) - \frac{d^2}{4\gamma} (y - y_1)^2 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{\beta + \delta_1} \left( \alpha\beta - \left( \delta_2 - \frac{db_2}{a_2} \right) (\beta + \delta_1) \right) x_2 \\
 &\quad - \gamma \left[ x_2 - \frac{d}{2\gamma} (y - y_1) \right]^2.
 \end{aligned} \tag{3.4}$$

It then follows from (3.1) and (3.4) that  $D^+ V_1(t) < 0$  strictly for all  $x_1, x_2, y > 0$  except the boundary equilibrium  $A_2(0, 0, \frac{b_2}{a_2})$ , where  $D^+ V_1(t) = 0$ . Thus,  $V_1(x_1, x_2, y)$  satisfies Lyapunov’s asymptotic stability theorem, and the boundary equilibrium  $A_2(0, 0, \frac{b_2}{a_2})$  of system (1.1) is globally asymptotically stable.

This completes the proof of Theorem 3.1. □

*Remark 3.1* Under the assumption  $\alpha\beta < \delta_2(\beta + \delta_1)$ , it follows from Lemma 2.1 that the first species will be driven to extinction. Obviously, if  $\frac{db_2}{a_2}$  is small enough, then inequality (3.1) holds. Moreover, it follows from Theorem 3.1 that  $A_2(0, 0, \frac{b_2}{a_2})$  is globally asymptotically stable, which means that the first species is still driven to extinction. That is, if the cooperation is limited, then, despite the cooperation between the two species, the species is still driven to extinction.

**Theorem 3.2** *Assume that*

$$\alpha\beta - \left( \delta_2 - \frac{db_2}{a_2} \right) (\beta + \delta_1) > 0 \tag{3.5}$$

*holds, then  $A_4(x_1^{**}, x_2^{**}, y^{**})$  is globally asymptotically stable.*

*Proof* We will prove Theorem 3.2 by constructing some suitable Lyapunov function. Let us define a Lyapunov function

$$\begin{aligned}
 V_2(x_1, x_2, y) &= k_1 \left( x_1 - x_1^{**} - x_1^{**} \ln \frac{x_1}{x_1^{**}} \right) \\
 &\quad + k_2 \left( x_2 - x_2^{**} - x_2^{**} \ln \frac{x_2}{x_2^{**}} \right) \\
 &\quad + k_3 \left( y - y^{**} - y^{**} \ln \frac{y}{y^{**}} \right),
 \end{aligned} \tag{3.6}$$

where  $k_1, k_2, k_3$  are some positive constants to be determined later.

One could easily see that the function  $V_2$  is zero at the equilibrium  $A_4(x_1^{**}, x_2^{**}, y^{**})$  and is positive for all other positive values of  $x_1, x_2$ , and  $y$ . The time derivative of  $V_2$  along the trajectories of (1.1) is

$$\begin{aligned}
 D^+ V_2(t) &= k_1 \frac{x_1 - x_1^{**}}{x_1} \dot{x}_1 + k_2 \frac{x_2 - x_2^{**}}{x_2} \dot{x}_2 + k_3 \frac{y - y^{**}}{y} \dot{y} \\
 &= k_1 \frac{x_1 - x_1^{**}}{x_1} (\alpha x_2 - (\beta + \delta_1) x_1) \\
 &\quad + k_2 \frac{x_2 - x_2^{**}}{x_2} (\beta x_1 - \delta_2 x_2 - \gamma x_2^2 + dyx_2) \\
 &\quad + k_3 (y - y^{**})(b_2 - a_2 y).
 \end{aligned} \tag{3.7}$$

Note that from the relationship of  $x_1^{**}$ ,  $x_2^{**}$ , and  $y^{**}$  (see (2.8)) we have

$$\alpha x_2 - (\beta + \delta_1)x_1 = \frac{\alpha}{x_1^{**}}(-x_2(x_1 - x_1^{**}) + x_1(x_2 - x_2^{**})). \tag{3.8}$$

Also, from (2.7) and (2.8), we have

$$\begin{aligned} & \beta x_1 - \delta_2 x_2 - \gamma x_2^2 - dux_2 \\ &= \frac{\beta}{x_2^{**}}(x_1 x_2^{**} - x_2 x_1^{**}) + \beta x_2 \frac{x_1^{**}}{x_2^{**}} - \delta_2 x_2 - \gamma x_2^2 + dyx_2 \\ &= \frac{\beta}{x_2^{**}}(x_1 x_2^{**} - x_1 x_2 + x_1 x_2 - x_2 x_1^{**}) \\ &\quad + \left(\frac{\alpha\beta}{\beta + \delta_1} - \delta_2 + \frac{db_2}{a_2}\right)x_2 - \gamma x_2^2 + dx_2(y - y^{**}) \\ &= \frac{\beta}{x_2^{**}}(x_1(x_2^{**} - x_2) + x_2(x_1 - x_1^{**})) \\ &\quad + \left(\frac{\alpha\beta}{\beta + \delta_1} - \delta_2 + \frac{db_2}{a_2}\right)x_2 - \gamma x_2^2 + dx_2(y - y^{**}) \\ &= \frac{\beta}{x_2^{**}}(x_1(x_2^{**} - x_2) + x_2(x_1 - x_1^{**})) \\ &\quad - \gamma x_2^2 + \gamma x_2 x_2^{**} + dx_2(y - y^{**}) \\ &= \frac{\beta}{x_2^{**}}(x_1(x_2^{**} - x_2) + x_2(x_1 - x_1^{**})) \\ &\quad - \gamma x_2(x_2 - x_2^{**}) + dx_2(y - y^{**}), \end{aligned} \tag{3.9}$$

from the third equation of (2.8), we have

$$b_2 - a_2 y = a_2 y^{**} - a_2 y = -a_2(y - y^{**}). \tag{3.10}$$

Applying (3.8)–(3.10) to (3.7) leads to

$$\begin{aligned} D^+ V_2(t) &= k_1 \frac{x_1 - x_1^{**}}{x_1} \frac{\alpha}{x_1^{**}}(-x_2(x_1 - x_1^{**}) + x_1(x_2 - x_2^{**})) \\ &\quad + k_2 \frac{x_2 - x_2^{**}}{x_2} \frac{\beta}{x_2^{**}}(x_1(x_2^{**} - x_2) + x_2(x_1 - x_1^{**})) \\ &\quad - k_2 \gamma x_2 \frac{x_2 - x_2^{**}}{x_2}(x_2 - x_2^{**}) + k_2 dx_2 \frac{x_2 - x_2^{**}}{x_2}(y - y^{**}) \\ &\quad - k_3 a_2 (y - y^{**})^2 \\ &= -\frac{k_1 \alpha x_2}{x_1 x_1^{**}}(x_1 - x_1^{**})^2 + \left(\frac{k_1 \alpha}{x_1^{**}} + \frac{k_2 \beta}{x_2^{**}}\right)(x_1 - x_1^{**})(x_2 - x_2^{**}) \\ &\quad - \frac{k_2 \beta x_1}{x_2 x_2^{**}}(x_2 - x_2^{**})^2 - k_2 \gamma (x_2 - x_2^{**})^2 \\ &\quad + k_2 d(x_2 - x_2^{**})(y - y^{**}) - k_3 a_2 (y - y^{**})^2. \end{aligned}$$

Now let us choose  $k_2 = 1, k_1 = \frac{\beta x_{1*}}{x_{2*}\alpha}, k_3 = \frac{d^2}{4\gamma a_2}$ , then

$$\begin{aligned}
 D^+ V_2(t) &= -\frac{\beta x_2}{x_1 x_{2*}}(x_1 - x_{1*})^2 + \frac{2\beta}{x_{2*}}(x_1 - x_{1*})(x_2 - x_{2*}) \\
 &\quad - \frac{\beta x_1}{x_2 x_{2*}}(x_2 - x_{2*})^2 - \gamma(x_2 - x_{2*})^2 \\
 &\quad + d(x_2 - x_{2*}^{**})(y - y^{**}) - \frac{d^2}{4\gamma a_2} a_2 (y - y^{**})^2 \\
 &= -\frac{\beta}{x_{2*}} \left[ \sqrt{\frac{x_2}{x_1}}(x_1 - x_{1*}) - \sqrt{\frac{x_1}{x_2}}(x_2 - x_{2*}) \right]^2 \\
 &\quad - \gamma \left[ (x_2 - x_{2*}) - \frac{d}{2\gamma}(y - y^{**}) \right]^2. \tag{3.11}
 \end{aligned}$$

Hence,  $D^+ V_2(t) < 0$  strictly for all  $x_1, x_2, y > 0$  except the positive equilibrium  $A_4(x_1^{**}, x_2^{**}, y^{**})$ , where  $D^+ V_2(t) = 0$ . Thus,  $V_2(x_1, x_2, y)$  satisfies Lyapunov’s asymptotic stability theorem, and the positive equilibrium  $A_4(x_1^{**}, x_2^{**}, y^{**})$  of system (1.1) is globally asymptotically stable.

This completes the proof of Theorem 3.2. □

*Remark 3.2* Condition (3.5) is necessary to ensure the existence of positive equilibrium. Theorem 3.2 shows that if the positive equilibrium exists, it is globally asymptotically stable. Hence, it is impossible for the system to have a bifurcation phenomenon.

*Remark 3.3* Assume that  $\alpha\beta > \delta_2(\beta + \delta_1)$  holds, then inequality (3.5) always holds. From Lemma 2.1, we know that in this case system (1.12) admits a unique positive equilibrium. That is, if system (1.12) admits the unique positive equilibrium, then for the commensalism model, the system still admits the unique positive equilibrium which is globally asymptotically stable.

*Remark 3.4* Assume that  $\alpha\beta < \delta_2(\beta + \delta_1)$  holds, then if  $\frac{db_2}{a_2}$  is large enough, inequality (3.5) still holds. From Lemma 2.1, we know that in this case the boundary equilibrium  $O(0, 0)$  of system (1.12) is globally asymptotically stable, which means extinction of the species. Then, for the commensalism model, if the cooperative effect is large enough, then the system admits the unique positive equilibrium which is globally asymptotically stable, which means the species is permanent. Therefore, for the endangered species, the intensity of cooperation between the species plays the essential role in the persistence property of the species.

*Remark 3.5* Theorems 3.1 and 3.2 depict a very intuitive biological phenomenon. From Zhang et al. [45], we can regard  $\frac{\alpha}{\delta_2}$  as a relative birth rate of the first mature species,  $\frac{\beta}{\beta + \delta_1}$  as a relative transformation rate of the first immature species. Then conditions (3.1) and (3.5) are equivalent to

$$\frac{\alpha}{\delta_2 - \frac{db_2}{a_2}} \frac{\beta}{\beta + \delta_1} < 1 \tag{3.12}$$

and

$$\frac{\alpha}{\delta_2 - \frac{db_2}{a_2}} \frac{\beta}{\beta + \delta_1} > 1, \tag{3.13}$$

respectively. Hence, with the help of the second species, the relative birth rate of the first mature species is increasing, this finally increases the chance of the survival of the first species.

#### 4 Numeric simulations

Now let us consider the following example.

*Example 4.1* Let us consider the two species stage-structured commensalism model:

$$\begin{aligned} \frac{dx_1}{dt} &= x_2 - x_1 - x_1, \\ \frac{dx_2}{dt} &= x_1 - x_2 - x_2^2 + dx_2y, \\ \frac{dy}{dt} &= y(1 - y). \end{aligned} \tag{4.1}$$

Here we choose  $\alpha = \beta = \delta_1 = \delta_2 = \gamma = a_2 = b_2 = 1$ . Hence

$$\alpha\beta = 1 < 2 = \delta_2(\beta + \delta_1).$$

It follows from Lemma 2.1 that the boundary equilibrium  $O(0, 0)$  of the following system is globally asymptotically stable.

$$\begin{aligned} \frac{dx_1}{dt} &= x_2 - x_1 - x_1, \\ \frac{dx_2}{dt} &= x_1 - x_2 - x_2^2. \end{aligned} \tag{4.2}$$

That is, without the cooperation of the second species, the first species will be driven to extinction (Fig. 1 supports this assertion).

- (1) Now let us choose  $d = 0.25$  in system (4.1), then

$$(\beta + \delta_1) \left( \delta_2 - \frac{db_2}{a_2} \right) - \alpha\beta = 0.5 > 0, \tag{4.3}$$

and it follows from Theorem 3.1 that  $A_2(0, 0, 1)$  is globally asymptotically stable.

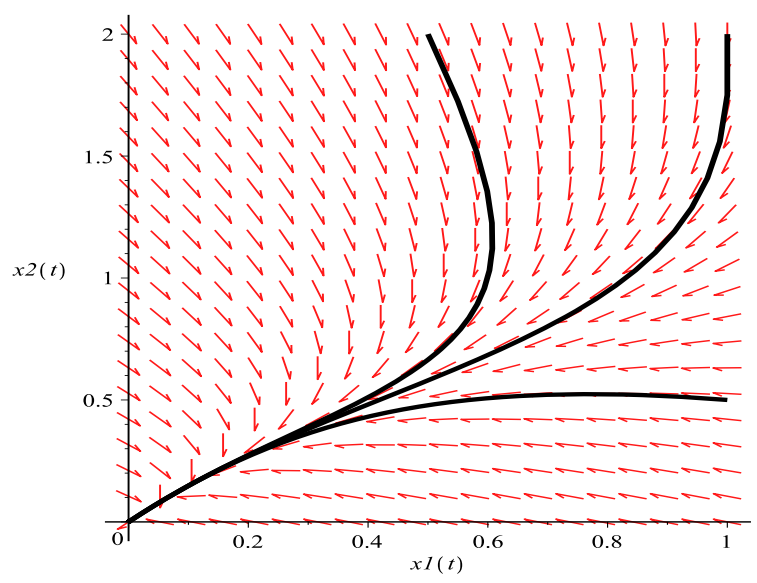
Figure 2–Fig. 4 support this assertion.

- (2) Now let us choose  $d = 2$  in system (4.1), then

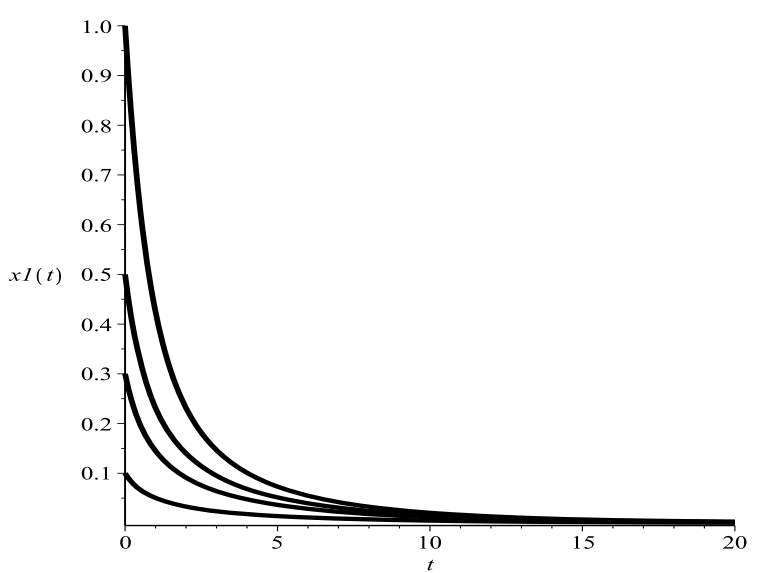
$$-\alpha\beta - (\beta + \delta_1) \left( \delta_2 - \frac{db_2}{a_2} \right) = 3 > 0, \tag{4.4}$$

and it follows from Theorem 3.2 that  $A_4(\frac{3}{4}, \frac{3}{2}, 1)$  is globally asymptotically stable.

Figure 5–Fig. 7 support this assertion.



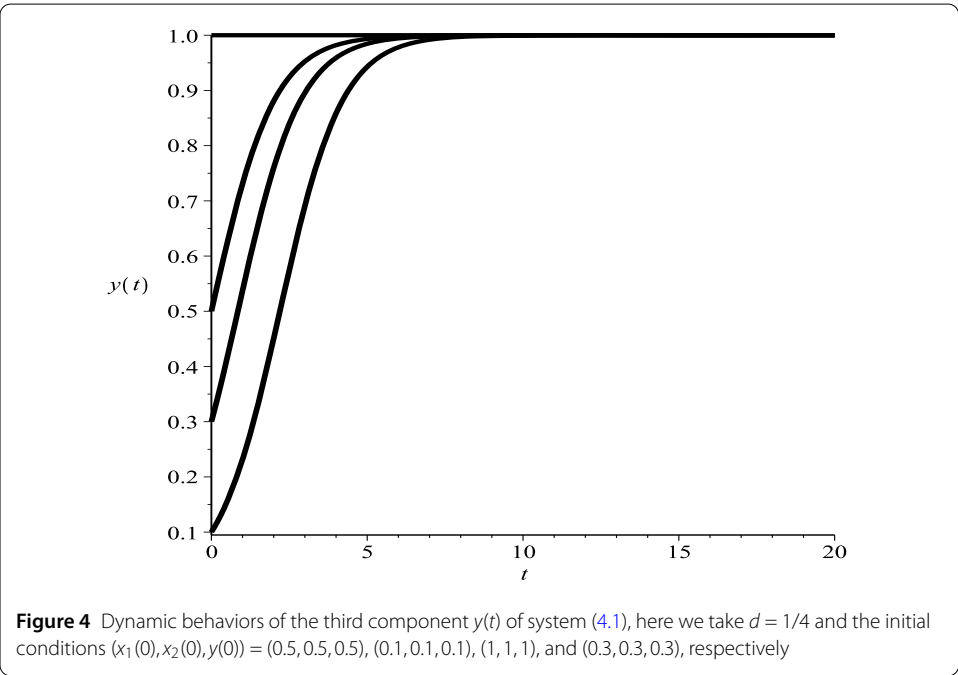
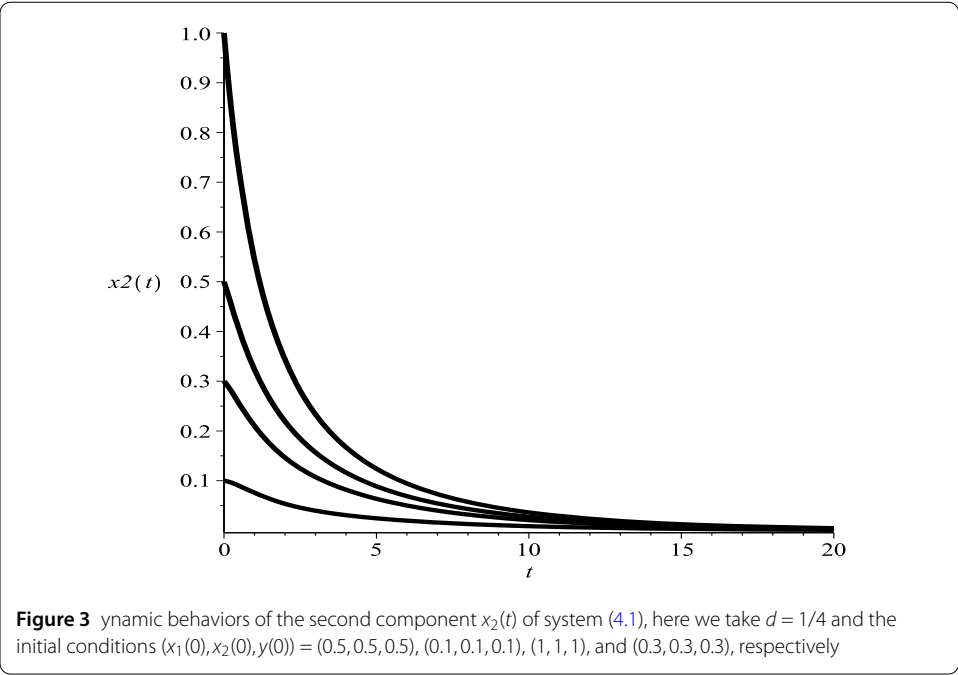
**Figure 1** Dynamic behaviors of system (4.2), the initial conditions  $(x_1(0), x_2(0)) = (0.5, 2), (1, 2),$  and  $(1, 0.5),$  respectively



**Figure 2** Dynamic behaviors of the first component  $x_1(t)$  of system (4.1), here we take  $d = 1/4$  and the initial conditions  $(x_1(0), x_2(0), y(0)) = (0.5, 0.5, 0.5), (0.1, 0.1, 0.1), (1, 1, 1),$  and  $(0.3, 0.3, 0.3),$  respectively

**5 Conclusion**

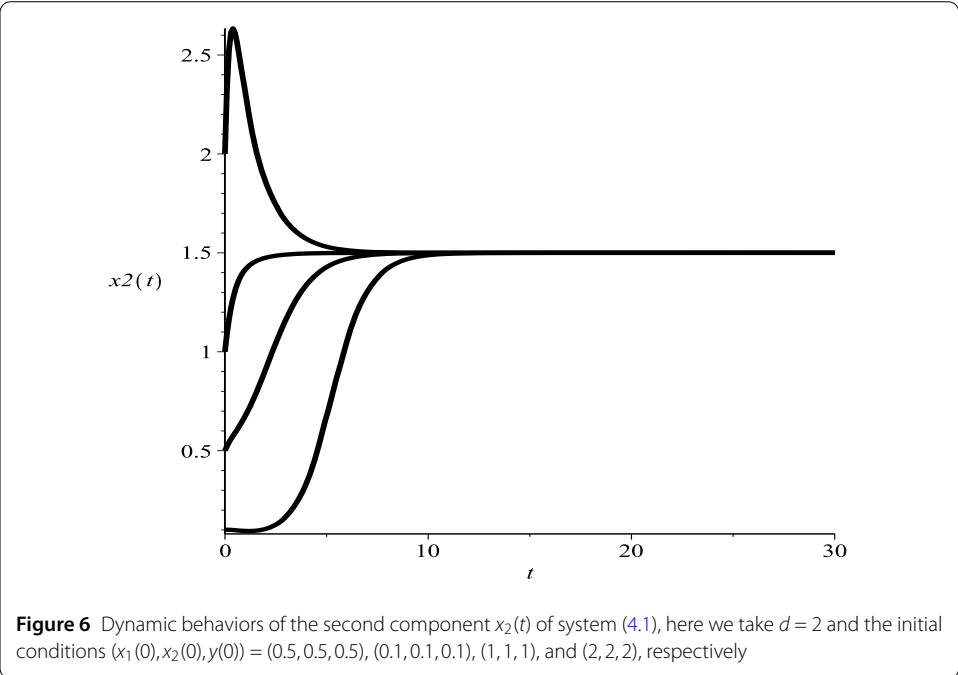
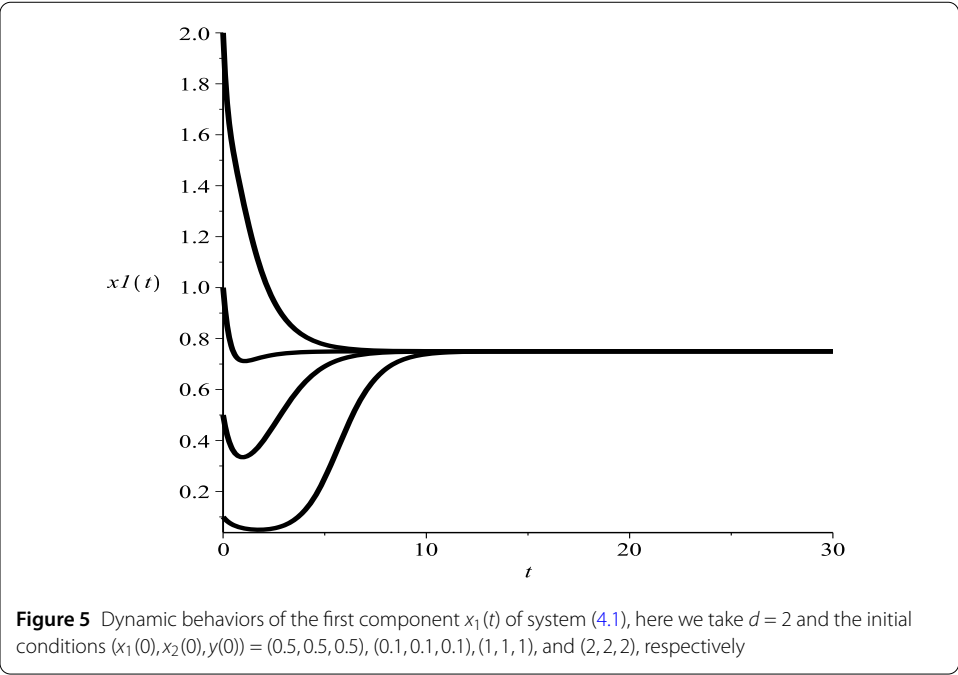
Recently, many scholars investigated the dynamic behaviors of the mutualism and commensalism model [1–21]. The traditional Lotka–Volterra commensalism model was studied by many scholars, and in [15], by constructing some suitable Lyapunov function, Han and Chen showed that system (1.2) admits the unique positive equilibrium. However, to this day, still no scholar has proposed and studied the dynamic behaviors of the stage-



structured commensalism model. This motivated us to propose system (1.1), which is the most simple commensalism system with stage structure.

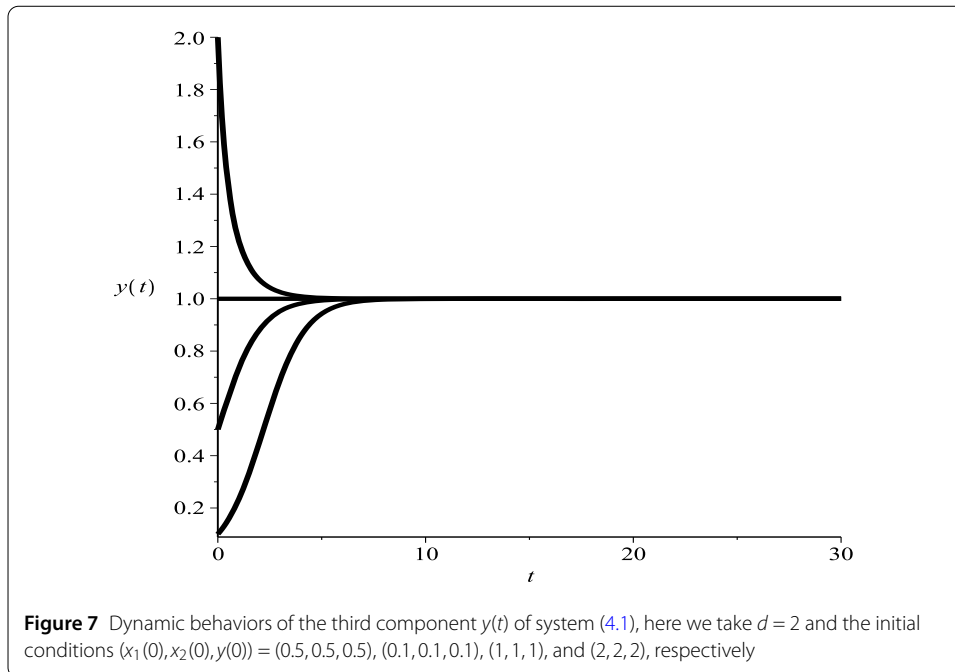
In system (1.1), without the cooperation of the second species, depending on the relationship of the coefficients, the species may be driven to extinction or become persistent in the long run. Such dynamic behaviors are different to those of the stage-structured system (1.8), which was introduced by Aeillo and Freedman. We argue that such kind of





property (the species could be driven to extinction) is one of the new characters due to the introduction of the stage structure.

For the extinction case, we show that depending on the intensity of cooperation, the species may still be driven to extinction or become persistent. Therefore, the cooperation between the two species is one of the essential factors that lead to the persistence of species. Such a property is quite different to that of the stage-structured cooperative



system (1.10), which was proposed by Chen et al. [34]. They showed that the cooperation between species has no influence on the persistence property of the system.

To sum up, to increase the intensity of cooperation between the species is one of the essential methods to avoid the extinction of the endangered species.

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**Competing interests**

The authors declare that there is no conflict of interests.

**Authors' contributions**

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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