Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

# New concepts of fractional Hahn's $q,\omega$-derivative of Riemann–Liouville type and Caputo type and applications

Yizhu Wang[1], Yiding Liu[1] and Chengmin Hou[1*]

[*]Correspondence:
cmhou@foxmail.com
[1]Department of Mathematics, Yanbian University, Yanji, P.R. China

**Abstract**

In this paper, we give the definitions of $q,\omega$-exponential function and $q,\omega$-gamma function. New concepts of fractional Hahn's $q,\omega$-derivative of Riemann–Liouville type and Caputo type are introduced, meanwhile we discuss some properties. As applications of the new concepts, we give the existence result of positive solutions for boundary value problem of fractional $q,\omega$-derivatives equations. We also definite certain $q,\omega$-Mittag-Leffler function by solving the initial value problem of fractional $q,\omega$-equations.

**Keywords:** Quantum calculus; $q,\omega$-derivative; Boundary value problem; Initial value problem

## 1 Introduction

The quantum calculus began with Jackson in the early twentieth century. The $q$-calculus appeared as a connection between mathematics and physics. It has a lot of applications in different mathematical areas such as number theory, combinatorics, and other sciences quantum theory. The book by Kac and Cheung [1] covers many of the fundamental aspects of quantum calculus. In recent years, the topic of $q$-calculus has attracted the attention of several researchers and a variety of new results can be found in papers [2–6].

The fractional $q$-calculus has developed into a popular mathematical modeling tool for many real world phenomena. Early developments for fractional $q$-calculus can be found in the work of Al-Salam [7] and Agarwal [8]. Moreover, the author in [9] generalizes the notions of fractional $q$-integral and $q$-derivative by introducing variable lower limit of integration.

Quantum difference operators have also been receiving an increasing interest due to their applications. In [10], Hahn introduced the quantum difference operator $D_{q,\omega}$, where $q \in ]0,1[$ and $\omega > 0$ are fixed. The Hahn operator unifies the two most well-known quantum difference operators: the Jackson $q$-difference derivative $D_q$, where $q \in ]0,1[$; and the forward difference $\triangle_\omega$, where $\omega > 0$. Hahn's difference operator has been applied successfully in the construction of families of orthogonal polynomials as well as in approximation problems. The right inverse of Hahn's difference operator was introduced by Aldwoah [11] who defined the right inverse of $D_{q,\omega}$ in terms of both the Jackson $q$-integral containing the right inverse of $D_q$ and the Nörlund sum involving the right inverse of $\triangle_\omega$. Malinowska and Torres [12] introduced the Hahn quantum variational calculus, while Malinowska and Martins [13] studied the generalized transversality conditions for the Hahn quantum variational calculus. In [14], some concepts of fractional quantum calculus were introduced in terms of a $q$-shifting operator ${}_a\Phi_q(m) = qm + (1-q)a$. As a research motivation, we redefine the fractional Hahn $q,\omega$-calculus with a $q$-shifting operator on the interval $[a,b]$.

In this paper, we recall some basic concepts on fractional $q$-calculus, which will be used in the later section. After preliminaries, in the third section the definitions of $q,\omega$-exponential function and $q,\omega$-gamma function are presented. In the fourth section we discuss some properties of $q,\omega$-calculus. After giving the basic properties, the fractional Hahn $q,\omega$-derivative of Riemann–Liouville type is introduced in the fifth section. Then we change the order of operators, we give the fractional $q,\omega$-derivative of Caputo type in the sixth section. And then, we consider boundary value problems of fractional $q,\omega$-derivative equations. We give the corresponding Green's function of the boundary value problem and its properties. By using Krasnoselskii's fixed point theorem, an existence result of positive solutions to the boundary value problem is enunciated. Finally, we define a certain $q,\omega$-Mittag-Leffler function by solving the initial value problem of fractional $q,\omega$-equations.

Let $q \in ]0,1[$ and $\omega \in ]0,+\infty[$ be given. Define $\omega_0 = \frac{\omega}{1-q}$. Throughout the paper, we assume $I$ to be an interval of $\mathbb{R}$ containing $\omega_0$.

**Definition 1.1** (Hahn's difference operator [10]) Let $f : I \to \mathbb{R}$. Hahn's difference operator is defined by

$$D_{q,\omega}f(t) := \begin{cases} \frac{f(qt+\omega)-f(t)}{(qt+\omega)-t}, & \text{if } t \neq \omega_0, \\ f'(t), & \text{if } t = \omega_0, \end{cases}$$

provided that f is differentiable at $\omega_0$.

**Definition 1.2** ([12]) Let I be a closed interval of $\mathbb{R}$ such that $\omega_0$, $a, b \in I$. For $f : I \to \mathbb{R}$, we define the $q,\omega$-integral of f from a to b by

$$\int_a^b f(t)\,d_{q,\omega}t := \int_{\omega_0}^b f(t)\,d_{q,\omega}t - \int_{\omega_0}^a f(t)\,d_{q,\omega}t, \tag{1}$$

where

$$\int_{\omega_0}^x f(t)\,d_{q,\omega}t := \bigl(x(1-q)-\omega\bigr)\sum_{k=0}^{\infty} q^k f\bigl(xq^k + [k]_{q,\omega}\bigr), \quad x \in I, \tag{2}$$

with $[k]_{q,\omega} := \frac{\omega(1-q^k)}{1-q}$ for $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, provided that the series converges at $x = a$ and $x = b$. In this case, f is called $q,\omega$-integrable on $[a,b]$. We say that f is $q,\omega$-integrable over I iff it is $q,\omega$-integrable over $[a,b]$ for all $a, b \in I$.

**Lemma 1.3** ([12]) *Assume $f : I \to \mathbb{R}$ to be continuous at $\omega_0$. Define*

$$F(x) := \int_{\omega_0}^x f(t)\,d_{q,\omega}t.$$

*Then $F$ is continuous at $\omega_0$. Furthermore, $D_{q,w}F(x)$ exists for every $x \in I$ and $D_{q,w}F(x) = f(x)$. Conversely,*

$$\int_a^b D_{q,w}F(x)\, d_{q,\omega}t = f(b) - f(a) \tag{3}$$

*for all $a, b \in I$.*

**Lemma 1.4** ([12]) *If $f, g : I \to \mathbb{R}$ are continuous at $\omega_0$, then*

$$\int_a^b f(t) D_{q,w}g(t)\, d_{q,\omega}t = f(t)g(t)|_{t=a}^{t=b} - \int_a^b D_{q,w}\bigl(f(t)\bigr)g(qt+\omega)\, d_{q,\omega}t \tag{4}$$

*for all $a, b \in I$. This is the integration by parts formula of the Jackson–Nörlund integral.*

## 2 Preliminary

To make this paper self-contained, below we recall some known facts on fractional $q$-calculus. The presentation here can be found in, for example, [14].

For $q \in ]0, 1[$, define

$$[m]_q = \frac{1-q^m}{1-q}, \quad m \in \mathbb{R}.$$

The $q$-analog of the power function $(n-m)^k$ with $k \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$ is

$$(n-m)^{(0)} = 1, \qquad (n-m)^{(k)} = \prod_{i=0}^{k-1}\bigl(n - mq^i\bigr), \quad k \in \mathbb{N}, n, m \in \mathbb{R}.$$

More generally, if $\gamma \in \mathbb{R}$, then

$$(n-m)^{(\gamma)} = \prod_{i=0}^{\infty} \frac{n - mq^i}{n - mq^{\gamma+i}}, \quad n \neq 0.$$

Note if $m = 0$, then $n^{(\gamma)} = n^\gamma$. We also use $0^{(\gamma)} = 0$ for $\gamma > 0$. The $q$-gamma function is defined by

$$\Gamma_q(t) = \frac{(1-q)^{(t-1)}}{(1-q)^{t-1}}, \quad t \in \mathbb{R}\backslash\{0, -1, -2, \ldots\}. \tag{5}$$

Obviously, $\Gamma_q(t+1) = [t]_q \Gamma_q(t)$.

**Lemma 2.1** ([9]) *For $\mu, \alpha, \beta \in \mathbb{R}^+$, the following identity is valid*:

$$\sum_{n=0}^{\infty} \frac{(1-\mu q^{1-n})^{(\alpha-1)}(1-q^{1+n})^{(\beta-1)}}{(1-q)^{(\alpha-1)}(1-q)^{(\beta-1)}} q^{\alpha n} = \frac{(1-\mu q)^{(\alpha+\beta-1)}}{(1-q)^{(\alpha+\beta-1)}}. \tag{6}$$

## 3 The $q,\omega$-exponential function and the $q,\omega$-gamma function

Let us define a $q$-shifting operator as

$$ {}_a\Phi_q(m) = qm + (1-q)a. $$

For any positive integer $k$, we have

$$ {}_a\Phi_q^k(m) = {}_a\Phi_q^{k-1}\big({}_a\Phi_q(m)\big) \quad \text{and} \quad {}_a\Phi_q^0(m) = m. $$

We also define the new power of $q$-shifting operator as

$$ (n-m)_a^{(0)} = 1, (n-m)_a^{(k)} = \prod_{i=0}^{k-1}\big(n - {}_a\Phi_q^i(m)\big), \quad k \in \mathbb{N}\cup\{\infty\}. $$

More generally, if $\gamma \in \mathbb{R}$, then

$$ (n-m)_a^{(\gamma)} = \prod_{i=0}^{\infty} \frac{n - {}_a\Phi_q^i(m)}{n - {}_a\Phi_q^{i+\gamma}(m)}, \tag{7} $$

with ${}_a\Phi_q^\gamma(m) = q^\gamma m + (1-q^\gamma)a$, $\gamma \in \mathbb{R}$.

For any $\gamma, n \in \mathbb{R}$, we have

$$ \big(n - {}_c\Phi_q(s)\big)_c^{(\gamma)} = (n-c)^\gamma \prod_{i=0}^{\infty} \frac{1 - q^{1+i}(\frac{s-c}{n-c})}{1 - q^{1+i+\gamma}(\frac{s-c}{n-c})}. \tag{8} $$

Using Definition 1.1 and (7), we may obtain the very useful examples of the $q,\omega$-derivatives of the following functions:

$$ \begin{aligned} {}_xD_{q,\omega}(x-a)_{\omega_0}^{(\alpha)} &= [\alpha]_q (x-a)_{\omega_0}^{(\alpha-1)}, \\ {}_tD_{q,\omega}(x-t)_{\omega_0}^{(\alpha)} &= -[\alpha]_q \big(x - {}_{\omega_0}\Phi_q(t)\big)_{\omega_0}^{(\alpha-1)}. \end{aligned} $$

**Definition 3.1** The $q,\omega$-exponential function is defined as

$$ e_{q,\omega}(t) = \prod_{n=0}^{\infty} \frac{1 - {}_{\omega_0}\Phi_q^n(t)}{1-\omega_0}, \quad e_{q,\omega}(\omega_0) = 1. $$

Note that $e_{q,\omega}(1) = 0$ and

$$ D_{q,\omega} e_{q,\omega}(t) = -\frac{1}{(1-\omega_0)(1-q)} e_{q,\omega}\big({}_{\omega_0}\Phi_q(t)\big). $$

We are now in a position to give the integral representation of the $q,\omega$-gamma function. Let $\alpha \in \mathbb{R}\backslash\{0,-1,-2,\dots\}$. Definite the $q,\omega$-gamma function by

$$ \Gamma_{q,\omega}(\alpha) = \int_{\omega_0}^{1} \left(\frac{1}{(1-\omega_0)(1-q)}\right)^{\alpha} \big(t - {}_{\omega_0}\Phi_q(\omega_0)\big)_{\omega_0}^{(\alpha-1)} e_{q,\omega}\big({}_{\omega_0}\Phi_q(t)\big)\, d_{q,\omega}t. $$

**Lemma 3.2** *For* $\alpha \in \mathbb{R}$,

$$\Gamma_{q,\omega}(\alpha+1) = [\alpha]_q \Gamma_{q,\omega}(\alpha), \quad \Gamma_{q,\omega}(1) = 1.$$

*For any positive integer* $k$,

$$\Gamma_{q,\omega}(k+1) = [k]! = \frac{(1-q^k)}{(1-q)} \frac{(1-q^{k-1})}{(1-q)} \cdots \frac{(1-q^2)}{(1-q)} \frac{(1-q)}{(1-q)}.$$

*Proof* By Lemma 1.4 and Definition 3.1, we can get

$$\begin{aligned}
&\Gamma_{q,\omega}(\alpha+1) \\
&= \int_{\omega_0}^{1} \left(\frac{1}{(1-\omega_0)(1-q)}\right)^{\alpha+1} \left(t - {}_{\omega_0}\Phi_q(\omega_0)\right)_{\omega_0}^{(\alpha)} e_{q,\omega}\left({}_{\omega_0}\Phi_q(t)\right) d_{q,\omega}t \\
&= \int_{\omega_0}^{1} \left(\frac{1}{(1-\omega_0)(1-q)}\right)^{\alpha+1} \left(t - {}_{\omega_0}\Phi_q(\omega_0)\right)_{\omega_0}^{(\alpha)} \left[-D_{q,\omega}(1-\omega_0)(1-q)e_{q,\omega}(t)\right] d_{q,\omega}t \\
&= -\left(\frac{1}{(1-\omega_0)(1-q)}\right)^{\alpha} \left[\left(t - {}_{\omega_0}\Phi_q(\omega_0)\right)_{\omega_0}^{(\alpha)} e_{q,\omega}(t)\right]\Big|_{\omega_0}^{1} \\
&\quad + \int_{\omega_0}^{1} \left(\frac{1}{(1-\omega_0)(1-q)}\right)^{\alpha} e_{q,\omega}\left({}_{\omega_0}\Phi_q(t)\right) D_{q,\omega}\left(t - {}_{\omega_0}\Phi_q(\omega_0)\right)_{\omega_0}^{(\alpha)} d_{q,\omega}t \\
&= [\alpha]_q \int_{\omega_0}^{1} \left(\frac{1}{(1-\omega_0)(1-q)}\right)^{\alpha} \left(t - {}_{\omega_0}\Phi_q(\omega_0)\right)_{\omega_0}^{(\alpha-1)} e_{q,\omega}\left({}_{\omega_0}\Phi_q(t)\right) d_{q,\omega}t \\
&= [\alpha]_q \Gamma_{q,\omega}(\alpha),
\end{aligned}$$

and

$$\begin{aligned}
\Gamma_{q,\omega}(1) &= \int_{\omega_0}^{1} \left(\frac{1}{(1-\omega_0)(1-q)}\right) e_{q,\omega}\left({}_{\omega_0}\Phi_q(t)\right) d_{q,\omega}t \\
&= \int_{\omega_0}^{1} \left(\frac{1}{(1-\omega_0)(1-q)}\right) \left[-D_{q,\omega}(1-\omega_0)(1-q)e_{q,\omega}(t)\right] d_{q,\omega}t \\
&= -e_{q,\omega}(t)\big|_{\omega_0}^{1} = -e_{q,\omega}(1) + e_{q,\omega}(\omega_0) = 1.
\end{aligned}$$

□

*Remark* 3.3 In [15], authors introduced the $q$-gamma function as

$$\Gamma_q(\alpha) = \frac{1}{1-q} \int_0^1 \left(\frac{t}{1-q}\right)^{\alpha-1} e_q(qt)\, d_q t,$$

where $e_q(t) = \prod_{n=0}^{\infty}(1-q^n t)$.

To see that $\Gamma_{q,\omega}(\alpha) = \Gamma_q(\alpha)$, in fact

$$\begin{aligned}
&\Gamma_{q,\omega}(\alpha) \\
&= \left(\frac{1}{(1-\omega_0)(1-q)}\right)^{\alpha} \int_{\omega_0}^{1} \left(t - {}_{\omega_0}\Phi_q(\omega_0)\right)_{\omega_0}^{(\alpha-1)} e_{q,\omega}\left({}_{\omega_0}\Phi_q(t)\right) d_{q,\omega}t
\end{aligned}$$

$$= \left(\frac{1}{(1-\omega_0)(1-q)}\right)^{\alpha}\big((1-q)-\omega\big)\sum_{k=0}^{\infty} q^k\big({}_{\omega_0}\Phi_q^k(1)-{}_{\omega_0}\Phi_q(\omega_0)\big)_{\omega_0}^{(\alpha-1)} e_{q,\omega}\big({}_{\omega_0}\Phi_q^{k+1}(1)\big)$$

$$= \left(\frac{1}{(1-\omega_0)(1-q)}\right)^{\alpha-1}\sum_{k=0}^{\infty} q^k\big(q^k\big)^{\alpha-1}(1-\omega_0)^{\alpha-1}\prod_{n=0}^{\infty}\big(1-q^{n+k+1}\big)$$

$$= \sum_{k=0}^{\infty} q^k\left(\frac{q^k}{1-q}\right)^{\alpha-1}\prod_{n=0}^{\infty}\big(1-q^{n+k+1}\big)$$

$$= \frac{1}{1-q}(1-q)\sum_{k=0}^{\infty} q^k\left(\frac{q^k}{1-q}\right)^{\alpha-1} e_q\big(q^{k+1}\big)$$

$$= \frac{1}{1-q}\int_0^1\left(\frac{t}{1-q}\right)^{\alpha-1} e_q(qt)\,d_qt = \Gamma_q(\alpha).$$

## 4 Concepts of fractional Hahn's quantum integration

In the section, we recall some basic concepts of fractional Hahn's quantum integration.

**Definition 4.1** ([14]) Let $\nu \geq 0$ and f be a function defined on $[c,b]$. The fractional $q$-integral of Riemann–Liouville type is given by $({}_cI_q^0 f)(t) = h(t)$ and

$$\big({}_cI_q^{\nu} f\big)(t) = \frac{1}{\Gamma_q(\nu)}\int_c^t \big(t - {}_c\Phi_q(s)\big)_c^{(\nu-1)} h(s)\,{}_cd_qs, \quad \nu > 0, t \in [c,b].$$

**Lemma 4.2** ([14]) *Let $\alpha,\beta \in \mathbb{R}^+$, and $f$ be a continuous function on $[c,b]$, $c \geq 0$. The Riemann–Liouville fractional $q$-integral has the following semi-group property*:

$$ {}_cI_q^{\beta}\,{}_cI_q^{\alpha} f(t) = {}_cI_q^{\alpha}\,{}_cI_q^{\beta} f(t) = {}_cI_q^{\alpha+\beta} f(t). \tag{9}$$

*Remark $c = \omega_0$, we have*

$$\big({}_{\omega_0}I_q^{\nu} f\big)(t) = \frac{1}{\Gamma_q(\nu)}\int_{\omega_0}^t \big(t - {}_{\omega_0}\Phi_q(s)\big)_{\omega_0}^{(\nu-1)} f(s)_{\omega_0}d_qs, \quad \nu > 0, t \in [c,b], \tag{10}$$

*and*

$$ {}_{\omega_0}I_q^{\beta}\,{}_{\omega_0}I_q^{\alpha} f(t) = {}_{\omega_0}I_q^{\alpha}\,{}_{\omega_0}I_q^{\beta} f(t) = {}_{\omega_0}I_q^{\alpha+\beta} f(t). \tag{11}$$

**Definition 4.3** ([16]) Let $\nu \geq 0$ and $f$ be a function defined on $[a,b]$. Hahn's fractional integration of Riemann–Liouville type is given by $({}_aI_q^0 f)(t) = f(t)$ and

$$\big({}_aI_{q,\omega}^{\nu} f\big)(t) = \frac{1}{\Gamma_q(\nu)}\int_a^t \big(t - {}_{\omega_0}\Phi_q(s)\big)_{\omega_0}^{(\nu-1)} f(s)\,d_{q,\omega}s, \quad \nu > 0, t \in [a,b]. \tag{12}$$

**Lemma 4.4** *For $\alpha,\beta \in \mathbb{R}^+$, the following is valid*:

$$\int_{\omega_0}^a \big(x - {}_{\omega_0}\Phi_q(t)\big)_{\omega_0}^{(\beta-1)}\big({}_aI_{q,\omega}^{\alpha} f\big)(t)\,d_{q,\omega}t = 0 \quad (\omega_0 < a < x < b).$$

*Proof* Using (2), we have

$$\begin{aligned}
&\big({}_aI^{\alpha}_{q,\omega}f\big)\big(aq^n+[n]_{q,\omega}\big)\\
&=\frac{1}{\Gamma_q(\alpha)}\int_a^{aq^n+[n]_{q,\omega}}\big(aq^n+[n]_{q,\omega}-{}_{\omega_0}\Phi_q(t)\big)^{(\beta-1)}_{\omega_0}f(t)\,d_{q,\omega}t\\
&=\frac{1}{\Gamma_q(\alpha)}\big[(1-q)\big(aq^n+[n]_{q,\omega}\big)-\omega\big]\\
&\quad\times\sum_{k=0}^{\infty}q^k\big(aq^n+[n]_{q,\omega}-{}_{\omega_0}\Phi_q\big(aq^{n+k}+[n+k]_{q,\omega}\big)\big)^{(\beta-1)}_{\omega_0}f\big(aq^{n+k}+[n+k]_{q,\omega}\big)\\
&\quad-\frac{1}{\Gamma_q(\alpha)}\big[(1-q)a-\omega\big]\\
&\quad\times\sum_{k=0}^{\infty}q^k\big(aq^n+[n]_{q,\omega}-{}_{\omega_0}\Phi_q\big(aq^{k}+[k]_{q,\omega}\big)\big)^{(\beta-1)}_{\omega_0}f\big(aq^{k}+[k]_{q,\omega}\big)\\
&=-\frac{1}{\Gamma_q(\alpha)}\big[(1-q)a-\omega\big]\\
&\quad\times\sum_{k=0}^{n-1}q^k\big(aq^n+[n]_{q,\omega}-{}_{\omega_0}\Phi_q\big(aq^{k}+[k]_{q,\omega}\big)\big)^{(\beta-1)}_{\omega_0}f\big(aq^{k}+[k]_{q,\omega}\big)=0,
\end{aligned}$$

since

$$\begin{aligned}
&\big(aq^n+[n]_{q,\omega}-{}_{\omega_0}\Phi_q\big(aq^k+[k]_{q,\omega}\big)\big)^{(\beta-1)}_{\omega_0}\\
&=\frac{\prod_{i=0}^{\infty}(aq^n+[n]_{q,\omega}-{}_{\omega_0}\Phi^i_q({}_{\omega_0}\Phi_q(aq^k+[k]_{q,\omega})))}{\prod_{i=0}^{\infty}(aq^n+[n]_{q,\omega}-{}_{\omega_0}\Phi^{i+\nu}_q({}_{\omega_0}\Phi_q(aq^k+[k]_{q,\omega})))}=0
\end{aligned}$$

for $0\le k\le n-1$.

Then, according to the definition of $q,\omega$-integral, it follows

$$\begin{aligned}
&\int_{\omega_0}^{a}\big(x-{}_{\omega_0}\Phi_q(t)\big)^{(\beta-1)}_{\omega_0}\big({}_aI^{\alpha}_{q,\omega}f\big)(t)\,d_{q,\omega}t\\
&=\big[(1-q)a-\omega\big]\sum_{k=0}^{\infty}q^k\big(x-{}_{\omega_0}\Phi_q\big(aq^k+[k]_{q,\omega}\big)\big)^{(\beta-1)}_{\omega_0}\big({}_aI^{\alpha}_{q,\omega}f\big)\big(aq^k+[k]_{q,\omega}\big)\\
&=0.
\end{aligned}$$

□

**Theorem 4.5** *Let $\alpha,\beta\in\mathbb{R}^+$. Hahn's fractional integration has the following semi-group property*:

$$\big({}_aI^{\beta}_{q,\omega}\,{}_aI^{\alpha}_{q,\omega}f\big)(x)=\big({}_aI^{\alpha+\beta}_{q,\omega}f\big)(x)\quad(\omega_0<a<x<b).$$

*Proof* By previous Lemma 4.4, we have

$$\begin{aligned}
&\big({}_aI^{\beta}_{q,\omega}\,{}_aI^{\alpha}_{q,\omega}f\big)(x)\\
&=\frac{1}{\Gamma_q(\beta)}\int_{\omega_0}^{x}\big(x-{}_{\omega_0}\Phi_q(t)\big)^{(\beta-1)}_{\omega_0}\big({}_aI^{\alpha}_{q,\omega}f\big)(t)\,d_{q,\omega}t
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)} \int_{\omega_0}^{x} \big(x - {}_{\omega_0}\Phi_q(t)\big)_{\omega_0}^{(\beta-1)} \int_{\omega_0}^{t} \big(t - {}_{\omega_0}\Phi_q(u)\big)_{\omega_0}^{(\alpha-1)} f(u)\, d_{q,\omega}u\, d_{q,\omega}t \\
&\quad - \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)} \int_{\omega_0}^{x} \big(x - {}_{\omega_0}\Phi_q(t)\big)_{\omega_0}^{(\beta-1)} \int_{\omega_0}^{a} \big(t - {}_{\omega_0}\Phi_q(u)\big)_{\omega_0}^{(\alpha-1)} f(u)\, d_{q,\omega}u\, d_{q,\omega}t.
\end{aligned}
$$

Using Lemma 4.2, we conclude that

$$
\begin{aligned}
\big({}_aI_{q,\omega}^{\beta}\, {}_aI_{q,\omega}^{\alpha} f\big)(x) &= \big({}_{\omega_0}I_{q,\omega}^{\alpha+\beta} f\big)(x) - \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)} \\
&\quad \times \int_{\omega_0}^{x} \big(x - {}_{\omega_0}\Phi_q(t)\big)_{\omega_0}^{(\beta-1)} \int_{\omega_0}^{a} \big(t - {}_{\omega_0}\Phi_q(u)\big)_{\omega_0}^{(\alpha-1)} f(u)\, d_{q,\omega}u\, d_{q,\omega}t.
\end{aligned}
$$

Furthermore, we can write

$$
\begin{aligned}
&\big({}_aI_{q,\omega}^{\beta}\, {}_aI_{q,\omega}^{\alpha} f\big)(x) \\
&\quad = \big({}_aI_{q,\omega}^{\alpha+\beta} f\big)(x) + \frac{1}{\Gamma_q(\alpha+\beta)} \int_{\omega_0}^{a} \big(x - {}_{\omega_0}\Phi_q(t)\big)_{\omega_0}^{(\alpha+\beta-1)} f(t)\, d_{q,\omega}t \\
&\quad\quad - \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)} \int_{\omega_0}^{x} \big(x - {}_{\omega_0}\Phi_q(t)\big)_{\omega_0}^{(\beta-1)} \int_{\omega_0}^{a} \big(t - {}_{\omega_0}\Phi_q(u)\big)_{\omega_0}^{(\alpha-1)} f(u)\, d_{q,\omega}u\, d_{q,\omega}t.
\end{aligned}
$$

Wherefrom it follows

$$
\big({}_aI_{q,\omega}^{\beta}\, {}_aI_{q,\omega}^{\alpha} f\big)(x) = \big({}_aI_{q,\omega}^{\alpha+\beta} f\big)(x) + \big[(1-q)a - \omega\big] \sum_{k=0}^{\infty} c_j q^k f\big(aq^k + [k]_{q,\omega}\big),
$$

with

$$
\begin{aligned}
c_j &= \frac{(x - {}_{\omega_0}\Phi_q(aq^k + [k]_{q,\omega}))_{\omega_0}^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+\beta)} - \frac{(1-q)x - \omega}{\Gamma_q(\alpha)\Gamma_q(\beta)} \\
&\quad \times \sum_{n=0}^{\infty} q^n \big(x - {}_{\omega_0}\Phi_q\big(xq^n + [n]_{q,\omega}\big)\big)_{\omega_0}^{(\beta-1)} \big(xq^n + [n]_{q,\omega} - {}_{\omega_0}\Phi_q^{k+1}(a)\big)_{\omega_0}^{(\alpha-1)}.
\end{aligned}
$$

By using formulas (5) and (7), we get

$$
\begin{aligned}
c_j &= \big((x-\omega_0)(1-q)\big)^{\alpha+\beta-1} \\
&\quad \times \left[ \frac{(1 - \frac{a-\omega_0}{x-\omega_0} q^{k+1})^{(\alpha+\beta-1)}}{(1-q)^{(\alpha+\beta-1)}} - \sum_{n=0}^{\infty} \frac{(1-q^{n+1})^{(\beta-1)} (1 - \frac{a-\omega_0}{x-\omega_0} q^{k-n+1})^{(\alpha-1)}}{(1-q)^{(\beta-1)}(1-q)^{(\alpha-1)}} q^{n\alpha} \right].
\end{aligned}
$$

Putting $\mu = q^k \frac{a-\omega_0}{x-\omega_0}$ into Lemma 2.1, we see that $c_j = 0$ for all $j \in \mathbb{N}$, which completes the proof. □

**Lemma 4.6** *For $\alpha \in \mathbb{R}^+$, $\lambda \in (-1, \infty)$, the following is valid:*

$$
{}_aI_{q,\omega}^{\alpha}\big((x-a)_{\omega_0}^{(\lambda)}\big) = \frac{\Gamma_q(\lambda+1)}{\Gamma_q(\alpha+\lambda+1)} (x-a)_{\omega_0}^{(\alpha+\lambda)} \quad (\omega_0 < a < x < b). \tag{13}
$$

*Proof* For $\lambda \neq 0$, according to Definition 4.3, we have

$$\begin{aligned} {}_aI^{\alpha}_{q,\omega}\big((x-a)^{(\lambda)}_{\omega_0}\big) = \frac{1}{\Gamma_q(\alpha)}\bigg[\int_{\omega_0}^{x}\big(x - {}_{\omega_0}\Phi_q(t)\big)^{(\alpha-1)}_{\omega_0}(t-a)^{(\lambda)}_{\omega_0}\,d_{q,\omega}t \\ - \int_{\omega_0}^{a}\big(x - {}_{\omega_0}\Phi_q(t)\big)^{(\alpha-1)}_{\omega_0}(t-a)^{(\lambda)}_{\omega_0}\,d_{q,\omega}t\bigg]. \end{aligned}$$

Also, the following is valid:

$$\begin{aligned} &\int_{\omega_0}^{a}\big(x - {}_{\omega_0}\Phi_q(t)\big)^{(\alpha-1)}_{\omega_0}(t-a)^{(\lambda)}_{\omega_0}\,d_{q,\omega}t \\ &\quad = \big[(1-q)a-\omega\big]\sum_{k=0}^{\infty}q^k\big(x - {}_{\omega_0}\Phi_q\big(aq^k+[k]_{q,\omega}\big)\big)^{(\alpha-1)}_{\omega_0}\big(aq^k+[k]_{q,\omega}-a\big)^{(\lambda)}_{\omega_0} \\ &\quad = \big[(1-q)a-\omega\big]\sum_{k=0}^{\infty}q^k\big(x - {}_{\omega_0}\Phi_q^{k+1}(a)\big)^{(\alpha-1)}_{\omega_0}\big({}_{\omega_0}\Phi_q^{k}(a)-a\big)^{(\lambda)}_{\omega_0} = 0. \end{aligned}$$

Therefrom, using Lemma 2.1, we get

$$\begin{aligned} &\int_{\omega_0}^{x}\big(x - {}_{\omega_0}\Phi_q(t)\big)^{(\alpha-1)}_{\omega_0}(t-a)^{(\lambda)}_{\omega_0}\,d_{q,\omega}t \\ &\quad = \big[(1-q)x-\omega\big]\sum_{k=0}^{\infty}q^k\big(x - {}_{\omega_0}\Phi_q^{k+1}(x)\big)^{(\alpha-1)}_{\omega_0}\big({}_{\omega_0}\Phi_q^{k}(x)-a\big)^{(\lambda)}_{\omega_0} \\ &\quad = (1-q)(x-\omega_0)^{\alpha+\lambda}(1-q)^{(\alpha-1)}(1-q)^{(\lambda)}\sum_{k=0}^{\infty}\frac{(1-q^{k+1})^{(\alpha-1)}(1-q^{-k}\frac{a-\omega_0}{x-\omega_0})^{(\lambda)}}{(1-q)^{(\alpha-1)}(1-q)^{(\lambda)}}q^{(\lambda+1)k} \\ &\quad = (1-q)(x-\omega_0)^{\alpha+\lambda}(1-q)^{(\alpha-1)}(1-q)^{(\lambda)}\frac{(1-\frac{a-\omega_0}{x-\omega_0})^{(\alpha+\lambda)}}{(1-q)^{(\alpha+\lambda)}} \\ &\quad = (1-q)\frac{(1-q)^{(\alpha-1)}(1-q)^{(\lambda)}}{(1-q)^{(\alpha+\lambda)}}(x-a)^{(\alpha+\lambda)}_{\omega_0}. \end{aligned}$$

Using (5), we obtain the required formula (13).

Particularly, for $\lambda = 0$, since the $q,\omega$-derivative over the variable $t$ is

$$ {}_tD_{q,\omega}(x-t)^{(\alpha)}_{\omega_0} = -[\alpha]_q\big(x - {}_{\omega_0}\Phi_q(t)\big)^{(\alpha-1)}_{\omega_0} \tag{14}$$

and using Lemma 1.3, we have

$$\big({}_aI^{\alpha}_{q,\omega}1\big)(x) = \frac{1}{\Gamma_q(\alpha)}\int_a^x\big(x - {}_{\omega_0}\Phi_q(t)\big)^{(\alpha-1)}_{\omega_0}\,d_{q,\omega}t = \frac{1}{\Gamma_q(\alpha+1)}(x-a)^{(\alpha)}_{\omega_0}. \qquad \square$$

## 5 The fractional $q,\omega$-derivative of Riemann–Liouville type

We define the *fractional $q,\omega$-derivative* of Riemann–Liouville type by

$$\big({}_aD^{\alpha}_{q,\omega}f\big)(x) = \begin{cases} ({}_aI^{-\alpha}_{q,\omega}f)(x), & \alpha < 0, \\ f(x), & \alpha = 0, \\ (D^{\lceil\alpha\rceil}_{q,\omega}\,{}_aI^{\lceil\alpha\rceil-\alpha}_{q,\omega}f)(x), & \alpha > 0, \end{cases} \tag{15}$$

where $\lceil\alpha\rceil$ denotes the smallest integer greater or equal to $\alpha$.

**Theorem 5.1** *For* $\alpha \in \mathbb{R}^+$, $\lambda \in (-1,\infty)$, *the following is valid*:

$$ {}_aD^{\alpha}_{q,\omega}\big((x-a)^{(\lambda)}_{\omega_0}\big) = \frac{\Gamma_q(\lambda+1)}{\Gamma_q(\lambda-\alpha+1)}(x-a)^{(\lambda-\alpha)}_{\omega_0}, \quad (\omega_0 < a < x < b). $$

*Proof* For $\lambda \neq 0$, according to (15) and Lemma 4.6, we have

$$ \begin{aligned} {}_aD^{\alpha}_{q,\omega}\big((x-a)^{(\lambda)}_{\omega_0}\big) &= D^{\lceil\alpha\rceil}_{q,\omega}\, {}_aI^{\lceil\alpha\rceil-\alpha}_{q,\omega}(x-a)^{(\lambda)}_{\omega_0} \\ &= D^{\lceil\alpha\rceil}_{q,\omega}\left(\frac{\Gamma_q(\lambda+1)}{\Gamma_q(\lambda+\lceil\alpha\rceil-\alpha+1)}(x-a)^{(\lambda+\lceil\alpha\rceil-\alpha)}_{\omega_0}\right) \\ &= \frac{\Gamma_q(\lambda+1)}{\Gamma_q(\lambda+\lceil\alpha\rceil-\alpha+1)} D^{\lceil\alpha\rceil-1}_{q,\omega}\big[\lambda+\lceil\alpha\rceil-\alpha\big]_q (x-a)^{(\lambda+\lceil\alpha\rceil-\alpha-1)}_{\omega_0} \\ &= \frac{\Gamma_q(\lambda+1)}{\Gamma_q(\lambda+\lceil\alpha\rceil-\alpha+1)}\big[\lambda+\lceil\alpha\rceil-\alpha\big]_q\big[\lambda+\lceil\alpha\rceil-\alpha-1\big]_q\cdots \\ &\quad \times \big[\lambda+\lceil\alpha\rceil-\alpha-\big(\lceil\alpha\rceil-1\big)\big]_q (x-a)^{(\lambda-\alpha)}_{\omega_0} \\ &= \frac{\Gamma_q(\lambda+1)}{\Gamma_q(\lambda-\alpha+1)}(x-a)^{(\lambda-\alpha)}_{\omega_0}. \end{aligned} $$

□

**Lemma 5.2** *For* $\alpha \in \mathbb{R}\setminus\mathbb{N}_0$, *the following is valid*:

$$ \big(D_{q,\omega}\, {}_aD^{\alpha}_{q,\omega}f\big)(x) = \big({}_aD^{\alpha+1}_{q,\omega}f\big)(x), \quad (\omega_0 < a < x < b). \tag{16} $$

*Proof* We will consider three cases. For $\alpha \leq -1$, according to Theorem 4.5, we have

$$ \begin{aligned} \big(D_{q,\omega}\, {}_aD^{\alpha}_{q,\omega}f\big)(x) &= \big(D_{q,\omega}\, {}_aI^{-\alpha}_{q,\omega}f\big)(x) = \big(D_{q,\omega}\, {}_aI^{1-\alpha-1}_{q,\omega}f\big)(x) \\ &= \big(D_{q,\omega}I_{q,\omega}\, {}_aI^{-\alpha-1}_{q,\omega}f\big)(x) = \big({}_aI^{-(\alpha+1)}_{q,\omega}f\big)(x) \\ &= \big({}_aD^{\alpha+1}_{q,\omega}f\big)(x). \end{aligned} $$

In the case $-1 < \alpha < 0$, i.e., $0 < \alpha + 1 < 1$, we obtain

$$ \big(D_{q,\omega}\, {}_aD^{\alpha}_{q,\omega}f\big)(x) = \big(D_{q,\omega}\, {}_aI^{-\alpha}_{q,\omega}f\big)(x) = \big(D_{q,\omega}\, {}_aI^{1-\alpha-1}_{q,\omega}f\big)(x) = \big({}_aD^{\alpha+1}_{q,\omega}f\big)(x). $$

For $\alpha > 0$, we get

$$ \begin{aligned} \big(D_{q,\omega}\, {}_aD^{\alpha}_{q,\omega}f\big)(x) &= \big(D_{q,\omega}D^{\lceil\alpha\rceil}_{q,\omega}\, {}_aI^{\lceil\alpha\rceil-\alpha}_{q,\omega}f\big)(x) \\ &= \big(D^{\lceil\alpha\rceil+1}_{q,\omega}\, {}_aI^{\lceil\alpha\rceil-\alpha}_{q,\omega}f\big)(x) = \big({}_aD^{\alpha+1}_{q,\omega}f\big)(x). \end{aligned} $$

□

**Theorem 5.3** *For* $\alpha \in \mathbb{R}\setminus\mathbb{N}_0$, *the following is valid*:

$$ \big(D_{q,\omega}\, {}_aD^{\alpha}_{q,\omega}f\big)(x) - \big({}_aD^{\alpha}_{q,\omega}D_{q,\omega}f\big)(x) = \frac{f(a)}{\Gamma_q(-\alpha)}(x-a)^{(-\alpha-1)}_{\omega_0} \quad (\omega_0 < a < x < b). $$

*Proof* We will use Lemma 1.3, Theorem 4.5, and Lemma 4.6 to prove the statement. Let us consider two cases. If $\alpha < 0$, then

$$\begin{aligned}
\big(D_{q,\omega}\,{}_aD_{q,\omega}^{\alpha}f\big)(x) &= \big(D_{q,\omega}\,{}_aI_{q,\omega}^{-\alpha}f\big)(x) = D_{q,\omega}\,{}_aI_{q,\omega}^{-\alpha}\big(({}_aI_{q,\omega}D_{q,\omega}f)(x) + f(a)\big) \\
&= \big(D_{q,\omega}\,{}_aI_{q,\omega}^{-\alpha}\,{}_aI_{q,\omega}D_{q,\omega}f\big)(x) + f(a)\big(D_{q,\omega}\,{}_aI_{q,\omega}^{-\alpha}1\big)(x) \\
&= \big(D_{q,\omega}\,{}_aI_{q,\omega}^{-\alpha+1}D_{q,\omega}f\big)(x) + f(a)D_{q,\omega}\frac{(x-a)_{\omega_0}^{(-\alpha)}}{\Gamma_q(-\alpha+1)} \\
&= \big(D_{q,\omega}\,{}_aI_{q,\omega}\,{}_aI_{q,\omega}^{-\alpha}D_{q,\omega}f\big)(x) + f(a)\frac{[-\alpha]_q(x-a)_{\omega_0}^{(-\alpha-1)}}{\Gamma_q(-\alpha+1)} \\
&= \big({}_aD_{q,\omega}^{\alpha}D_{q,\omega}f\big)(x) + \frac{f(a)}{\Gamma_q(-\alpha)}(x-a)_{\omega_0}^{(-\alpha-1)}.
\end{aligned}$$

If $\alpha > 0$, there exists $l \in \mathbb{N}_0$ such that $\alpha \in (l, l+1)$. Then, applying a similar procedure, we get

$$\begin{aligned}
\big(D_{q,\omega}\,{}_aD_{q,\omega}^{\alpha}f\big)(x) &= \big(D_{q,\omega}D_{q,\omega}^{l+1}\,{}_aI_{q,\omega}^{l+1-\alpha}f\big)(x) \\
&= D_{q,\omega}^{l+2}\,{}_aI_{q,\omega}^{l+1-\alpha}\big(({}_aI_{q,\omega}D_{q,\omega}f)(x) + f(a)\big) \\
&= \big(D_{q,\omega}^{l+1}D_{q,\omega}\,{}_aI_{q,\omega}\,{}_aI_{q,\omega}^{l+1-\alpha}D_{q,\omega}f\big)(x) \\
&\quad + \frac{f(a)}{\Gamma_q(l+2-\alpha)}D_{q,\omega}^{l+1}\big((x-a)_{\omega_0}^{(l+1-\alpha)}\big) \\
&= \big({}_aD_{q,\omega}^{\alpha}D_{q,\omega}f\big)(x) + \frac{f(a)}{\Gamma_q(-\alpha)}(x-a)_{\omega_0}^{(-\alpha-1)}.
\end{aligned}$$

□

**Lemma 5.4** *Let $f(x)$ be a function defined on an interval $(\omega_0, b)$ and $\alpha \in \mathbb{R}^+$. Then the following is valid*:

$$\big({}_aD_{q,\omega}^{\alpha}\,{}_aI_{q,\omega}^{\alpha}f\big)(x) = f(x) \quad (\omega_0 < a < x < b).$$

*Proof* For $\alpha > 0$, we have

$$\begin{aligned}
\big({}_aD_{q,\omega}^{\alpha}\,{}_aI_{q,\omega}^{\alpha}f\big)(x) &= \big(D_{q,\omega}^{\lceil\alpha\rceil}\,{}_aI_{q,\omega}^{\lceil\alpha\rceil-\alpha}\,{}_aI_{q,\omega}^{\alpha}f\big)(x) = \big(D_{q,\omega}^{\lceil\alpha\rceil}\,{}_aI_{q,\omega}^{\lceil\alpha\rceil-\alpha+\alpha}f\big)(x) \\
&= \big({}_aD_{q,\omega}^{\lceil\alpha\rceil}\,{}_aI_{q,\omega}^{\lceil\alpha\rceil}f\big)(x) = f(x).
\end{aligned}$$

□

**Lemma 5.5** *Let $\alpha \in (0,1)$, then*

$$\big({}_aI_{q,\omega}^{\alpha}\,{}_aD_{q,\omega}^{\alpha}f\big)(x) = f(x) + K(a)(x-a)_{\omega_0}^{(\alpha-1)} \quad (\omega_0 < a < x < b),$$

*where $K(a)$ does not depend on $x$.*

*Proof* Let $A(x) = ({}_aI_{q,\omega}^{\alpha}{}_aD_{q,\omega}^{\alpha}f)(x) - f(x)$. Applying ${}_aD_{q,\omega}^{\alpha}$ to both sides of the above expression and using Lemma 5.4, we get

$$\begin{aligned}
\big({}_aD_{q,\omega}^{\alpha}A\big)(x) &= \big({}_aD_{q,\omega}^{\alpha}\,{}_aI_{q,\omega}^{\alpha}{}_aD_{q,\omega}^{\alpha}f\big)(x) - \big({}_aD_{q,\omega}^{\alpha}f\big)(x) \\
&= \big(\big({}_aD_{q,\omega}^{\alpha}\,{}_aI_{q,\omega}^{\alpha}\big){}_aD_{q,\omega}^{\alpha}f\big)(x) - \big({}_aD_{q,\omega}^{\alpha}f\big)(x) = 0.
\end{aligned}$$

On the other hand, according to Lemma 4.6, we obtain

$$ {}_aD^{\alpha}_{q,\omega}(x-a)^{(\alpha-1)}_{\omega_0} = D_{q,\omega}\,{}_aI^{1-\alpha}_{q,\omega}(x-a)^{(\alpha-1)}_{\omega_0} = \Gamma_q(\alpha)(D_{q,\omega}1)(x) = 0. $$

Hence, we conclude that $A(x)$ is a function of the form

$$ A(x) = K(a)(x-a)^{(\alpha-1)}_{\omega_0}. \qquad \square $$

**Theorem 5.6** *Let $\alpha \in (N-1, N]$. Then, for some constants $c_i \in \mathbb{R}, i = 1, 2, \ldots, N$, the following equality holds*:

$$ \bigl({}_aI^{\alpha}_{q,\omega}\,{}_aD^{\alpha}_{q,\omega}f\bigr)(x) = f(x) + c_1(x-a)^{(\alpha-1)}_{\omega_0} + c_2(x-a)^{(\alpha-2)}_{\omega_0} + \cdots + c_N(x-a)^{(\alpha-N)}_{\omega_0}. $$

*Proof* By Lemma 5.5, we have

$$ \begin{aligned} \bigl({}_aI^{\alpha}_{q,\omega}\,{}_aD^{\alpha}_{q,\omega}f\bigr)(x) &= \bigl({}_aI^{\alpha}_{q,\omega}D^{N}_{q,\omega}\,{}_aI^{N-\alpha}_{q,\omega}f\bigr)(x) \\ &= \bigl({}_aI^{\alpha-1}_{q,\omega}\,{}_aI_{q,\omega}D_{q,\omega}D^{N-1}_{q,\omega}\,{}_aI^{N-\alpha}_{q,\omega}f\bigr)(x) \\ &= \bigl({}_aI^{\alpha-1}_{q,\omega}D^{N-1}_{q,\omega}\,{}_aI^{N-\alpha}_{q,\omega}f\bigr)(x) - D^{N-1}_{q,\omega}\,{}_aI^{N-\alpha}_{q,\omega}f(a)\bigl({}_aI^{\alpha-1}_{q,\omega}1\bigr)(x) \\ &= \bigl({}_aI^{\alpha-1}_{q,\omega}D^{N-1}_{q,\omega}\,{}_aI^{N-\alpha}_{q,\omega}f\bigr)(x) - \frac{D^{N-1}_{q,\omega}\,{}_aI^{N-\alpha}_{q,\omega}f(a)}{\Gamma_q(\alpha)}(x-a)^{(\alpha-1)}_{\omega_0} \\ &= \bigl({}_aI^{\alpha-2}_{q,\omega}D^{N-2}_{q,\omega}\,{}_aI^{N-\alpha}_{q,\omega}f\bigr)(x) - \frac{D^{N-2}_{q,\omega}\,{}_aI^{N-\alpha}_{q,\omega}f(a)}{\Gamma_q(\alpha-1)}(x-a)^{(\alpha-2)}_{\omega_0} \\ &\quad - \frac{D^{N-1}_{q,\omega}\,{}_aI^{N-\alpha}_{q,\omega}f(a)}{\Gamma_q(\alpha)}(x-a)^{(\alpha-1)}_{\omega_0} \\ &= \cdots \\ &= \bigl({}_aI^{\alpha-N+1}_{q,\omega}\,{}_aD^{\alpha-N+1}_{q,\omega}f\bigr)(x) - \frac{D_{q,\omega}\,{}_aI^{N-\alpha}_{q,\omega}f(a)}{\Gamma_q(\alpha-N+2)}(x-a)^{(\alpha-N+1)}_{\omega_0} \\ &\quad - \cdots - \frac{D^{N-1}_{q,\omega}\,{}_aI^{N-\alpha}_{q,\omega}f(a)}{\Gamma_q(\alpha)}(x-a)^{(\alpha-1)}_{\omega_0} \\ &= f(x) + c_1(x-a)^{(\alpha-1)}_{\omega_0} + \cdots + c_N(x-a)^{(\alpha-N)}_{\omega_0}. \qquad \square \end{aligned} $$

**Lemma 5.7**

$$ D_{q,\omega}\int_{\omega_0}^{t} f(t,s)\,d_{q,\omega}s = \int_{\omega_0}^{t} D_{q,\omega}f(t,s)\,d_{q,\omega}s + f(qt+\omega, t). $$

*Proof* Using (2) and Definition 1.1, we have

$$ \begin{aligned} &D_{q,\omega}\int_{\omega_0}^{t} f(t,s)\,d_{q,\omega}s \\ &\quad = D_{q,\omega}\Biggl[\bigl(t(1-q)-\omega\bigr)\sum_{k=0}^{\infty} q^k f\bigl(t, q^k t + [k]_{q,\omega}\bigr)\Biggr] \end{aligned} $$

$$
\begin{aligned}
&= \frac{1}{qt+\omega-t}\Bigg[\big((qt+\omega)(1-q)-\omega\big)\sum_{k=0}^{\infty} q^k f\big(qt+\omega,(qt+\omega)q^k+[k]_{q,\omega}\big) \\
&\quad - \big(t(1-q)-\omega\big)\sum_{k=0}^{\infty} q^k f\big(t, tq^k+[k]_{q,\omega}\big)\Bigg] \\
&= -\sum_{k=0}^{\infty} q^{k+1} f\big(qt+\omega,(qt+\omega)q^k+[k]_{q,\omega}\big) + \sum_{k=0}^{\infty} q^k f\big(t,tq^k+[k]_{q,\omega}\big) \\
&= -\sum_{k=0}^{\infty} q^{k} f\big(qt+\omega,(qt+\omega)q^{k-1}+[k-1]_{q,\omega}\big) + \sum_{k=0}^{\infty} q^k f\big(t,tq^k+[k]_{q,\omega}\big) \\
&\quad + f(qt+\omega,t) \\
&= \big(t(1-q)-\omega\big)\sum_{k=0}^{\infty} q^k \frac{f(qt+\omega,tq^k+[k]_{q,\omega})-f(t,tq^k+[k]_{q,\omega})}{qt+\omega-t} \\
&\quad + f(qt+\omega,t) \\
&= \int_{\omega_0}^{t} D_{q,\omega} f(t,s)\, d_{q,\omega}s + f(qt+\omega,t). \qquad \square
\end{aligned}
$$

**Theorem 5.8** *If $f(t)$ is defined and finite, then for $\nu > 0$ with $N-1 < \nu \le N$,*

$$
\big(D_{q,\omega}^{\nu} f\big)(t) = \frac{1}{\Gamma_q(-\nu)} \int_a^t \big(t - {}_{\omega_0}\Phi_q(s)\big)_{\omega_0}^{(-\nu-1)} f(s)\, d_{q,\omega}s.
$$

*Proof* Using Definition 4.3 and Lemma 5.7, we have

$$
\begin{aligned}
\big(D_{q,\omega}^{\nu} f\big)(t) &= D_{q,\omega}^{N}\, {}_aI_{q,\omega}^{N-\nu} f(t) \\
&= D_{q,\omega}^{N}\left(\frac{1}{\Gamma_q(N-\nu)}\int_a^t \big(t - {}_{\omega_0}\Phi_q(s)\big)_{\omega_0}^{(N-\nu-1)} f(s)\, d_{q,\omega}s\right) \\
&= D_{q,\omega}^{N-1} D_{q,\omega}\left(\frac{1}{\Gamma_q(N-\nu)}\int_a^t \big(t - {}_{\omega_0}\Phi_q(s)\big)_{\omega_0}^{(N-\nu-1)} f(s)\, d_{q,\omega}s\right) \\
&= D_{q,\omega}^{N-1}\Bigg[\frac{1}{\Gamma_q(N-\nu)}\int_a^t D_{q,\omega}\big(t - {}_{\omega_0}\Phi_q(s)\big)_{\omega_0}^{(N-\nu-1)} f(s)\, d_{q,\omega}s \\
&\quad + \big(qt+\omega - {}_{\omega_0}\Phi_q(t)\big)_{\omega_0}^{(N-\nu-1)} f(t)\Bigg] \\
&= D_{q,\omega}^{N-1}\frac{[N-\nu-1]_q}{\Gamma_q(N-\nu)}\int_a^t \big(t - {}_{\omega_0}\Phi_q(s)\big)_{\omega_0}^{(N-\nu-2)} f(s)\, d_{q,\omega}s \\
&= D_{q,\omega}^{N-1}\frac{1}{\Gamma_q(N-\nu-1)}\int_a^t \big(t - {}_{\omega_0}\Phi_q(s)\big)_{\omega_0}^{(N-\nu-2)} f(s)\, d_{q,\omega}s \\
&= \cdots \\
&= \frac{1}{\Gamma_q(-\nu)}\int_a^t \big(t - {}_{\omega_0}\Phi_q(s)\big)_{\omega_0}^{(-\nu-1)} f(s)\, d_{q,\omega}s. \qquad \square
\end{aligned}
$$

## 6 The fractional $q,\omega$-derivative of Caputo type

If we change the order of operators, we can introduce another type of fractional $q,\omega$-derivative.

The *fractional* $q,\omega$*-derivative* of Caputo type is

$$\left({}_a^cD_{q,\omega}^{\alpha}f\right)(x)=\begin{cases}\left({}_aI_{q,\omega}^{-\alpha}f\right)(x), & \alpha<0,\\ f(x), & \alpha=0,\\ \left({}_aI_{q,\omega}^{\lceil\alpha\rceil-\alpha}D_{q,\omega}^{\lceil\alpha\rceil}f\right)(x), & \alpha>0,\end{cases}\tag{17}$$

where $\lceil\alpha\rceil$ denotes the smallest integer greater or equal to $\alpha$.

**Theorem 6.1** *For* $\alpha\in\mathbb{R}\setminus\mathbb{N}_0$ *and* $\omega_0<a<x<b$, *the following is valid*:

$$\left({}_a^cD_{q,\omega}^{\alpha+1}f\right)(x)-\left({}_a^cD_{q,\omega}^{\alpha}D_{q,\omega}f\right)(x)=\begin{cases}\frac{f(a)}{\Gamma_q(-\alpha)}(x-a)_{\omega_0}^{(-\alpha-1)}, & \alpha\le-1,\\ 0, & \alpha>-1.\end{cases}\tag{18}$$

*Proof* Clearly, (18) holds for $\alpha=-1$. Next, we will consider three cases.

Case 1. $\alpha<-1$, according to (17) and (3), we have

$$\begin{aligned}\left({}_a^cD_{q,\omega}^{\alpha+1}f\right)(x)&=\left({}_aI_{q,\omega}^{-\alpha-1}f\right)(x)={}_aI_{q,\omega}^{-\alpha-1}\left(\left({}_aI_{q,\omega}D_{q,\omega}f\right)(x)+f(a)\right)\\&=\left({}_aI_{q,\omega}^{-\alpha}D_{q,\omega}f\right)(x)+f(a)\left({}_aI_{q,\omega}^{-\alpha-1}1\right)(x)\\&=\left({}_a^cD_{q,\omega}^{\alpha}D_{q,\omega}f\right)(x)+\frac{f(a)}{\Gamma_q(-\alpha)}(x-a)_{\omega_0}^{(-\alpha-1)}.\end{aligned}$$

Case 2. $-1<\alpha\le0$, we obtain

$$\begin{aligned}\left({}_a^cD_{q,\omega}^{\alpha+1}f\right)(x)&=\left({}_aI_{q,\omega}^{1-(\alpha+1)}D_{q,\omega}f\right)(x)\\&=\left({}_aI_{q,\omega}^{-\alpha}D_{q,\omega}f\right)(x)=\left({}_a^cD_{q,\omega}^{\alpha}D_{q,\omega}f\right)(x).\end{aligned}$$

Case 3. $\alpha>0$, we assume $\alpha=n+\varepsilon$, $n\in\mathbb{N}_0$, $0<\varepsilon<1$, then $\alpha+1\in(n+1,n+2)$, so we obtain

$$\left({}_a^cD_{q,\omega}^{\alpha+1}f\right)(x)=\left({}_aI_{q,\omega}^{1-\varepsilon}D_{q,\omega}^{n+2}f\right)(x)=\left({}_aI_{q,\omega}^{1-\varepsilon}D_{q,\omega}^{n+1}D_{q,\omega}f\right)(x)=\left({}_a^cD_{q,\omega}^{\alpha}D_{q,\omega}f\right)(x).\qquad\square$$

**Theorem 6.2** *For* $\alpha\in\mathbb{R}\setminus\mathbb{N}_0$ *and* $\omega_0<a<x<b$, *the following is valid*:

$$\left(D_{q,\omega}\,{}_a^cD_{q,\omega}^{\alpha}f\right)(x)-\left({}_a^cD_{q,\omega}^{\alpha+1}f\right)(x)=\begin{cases}0, & \alpha\le-1,\\ \frac{D_{q,\omega}^{\lceil\alpha\rceil}f(a)}{\Gamma_q(\lceil\alpha\rceil-\alpha)}(x-a)_{\omega_0}^{(-\alpha-1)}, & \alpha>-1.\end{cases}\tag{19}$$

*Proof* We will consider two cases.

Case 1. $\alpha<0$, using Lemma 1.3, Lemma 4.6, and (17), we obtain

$$\begin{aligned}\left(D_{q,\omega}\,{}_a^cD_{q,\omega}^{\alpha}f\right)(x)&=\left(D_{q,\omega}\,{}_aI_{q,\omega}^{-\alpha}f\right)(x)=D_{q,\omega}\,{}_aI_{q,\omega}^{-\alpha}\left(\left({}_aI_{q,\omega}D_{q,\omega}f\right)(x)+f(a)\right)\\&=\left(D_{q,\omega}\,{}_aI_{q,\omega}\,{}_aI_{q,\omega}^{-\alpha}D_{q,\omega}f\right)(x)+f(a)\frac{[-\alpha]_q(x-a)_{\omega_0}^{(-\alpha-1)}}{\Gamma_q(-\alpha+1)}\\&=\left({}_a^cD_{q,\omega}^{\alpha+1}f\right)(x)+\frac{f(a)}{\Gamma_q(-\alpha)}(x-a)_{\omega_0}^{(-\alpha-1)}.\end{aligned}$$

By Theorem 5.1, the required equalities are valid both for $\alpha \leq -1$ and $-1 < \alpha < 0$.

Case 2. $\alpha > 0$, we assume $\alpha = n + \varepsilon$, $n \in \mathbb{N}_0$, $0 < \varepsilon < 1$, then $\alpha + 1 \in (n+1, n+2)$, by Lemma 1.3, Lemma 4.6, and (17), we obtain

$$\begin{aligned}\big(D_{q,\omega}\,{}_a^cD^{\alpha}_{q,\omega}f\big)(x) &= \big(D_{q,\omega}\,{}_aI^{1-\varepsilon}_{q,\omega}D^{n+1}_{q,\omega}f\big)(x)\\ &= \big(D_{q,\omega}\,{}_aI^{2-\varepsilon}_{q,\omega}D^{n+2}_{q,\omega}f\big)(x) + D^{n+1}_{q,\omega}f(a)\big(D_{q,\omega}\,{}_aI^{1-\varepsilon}_{q,\omega}1\big)(x)\\ &= \big({}_a^cD^{\alpha+1}_{q,\omega}f\big)(x) + \frac{D^{n+1}_{q,\omega}f(a)}{\Gamma_q(n+1-\alpha)}(x-a)^{(n-\alpha)}_{\omega_0}. \qquad \square\end{aligned}$$

**Theorem 6.3** *Let $\alpha \in (N-1, N]$. Then, for some constants $c_i \in \mathbb{R}, i = 1, 2, \ldots, N-1$, the following equality holds:*

$$\big({}_aI^{\alpha}_{q,\omega}\,{}_a^cD^{\alpha}_{q,\omega}f\big)(x) = f(x) + c_0 + c_1(x-a)^{(1)}_{\omega_0} + c_2(x-a)^{(2)}_{\omega_0} + \cdots + c_{N-1}(x-a)^{(N-1)}_{\omega_0}.$$

*Proof* By (17) and Lemma 1.3, Lemma 4.6, we have

$$\begin{aligned}\big({}_aI^{\alpha}_{q,\omega}\,{}_a^cD^{\alpha}_{q,\omega}f\big)(x) &= \big({}_aI^{\alpha}_{q,\omega}\,{}_aI^{N-\alpha}_{q,\omega}D^{N}_{q,\omega}f\big)(x) = \big({}_aI^{N}_{q,\omega}D^{N}_{q,\omega}f\big)(x)\\ &= {}_aI^{N-1}_{q,\omega}\big(\big(D^{N-1}_{q,\omega}f\big)(x) - \big(D^{N-1}_{q,\omega}f\big)(a)\big)\\ &= \big({}_aI^{N-1}_{q,\omega}D^{N-1}_{q,\omega}f\big)(x) - \frac{D^{N-1}_{q,\omega}f(a)}{\Gamma_q(N)}(x-a)^{(N-1)}_{\omega_0}\\ &= \big({}_aI^{N-2}_{q,\omega}D^{N-2}_{q,\omega}f\big)(x) - \frac{D^{N-2}_{q,\omega}f(a)}{\Gamma_q(N-1)}(x-a)^{(N-2)}_{\omega_0}\\ &\quad - \frac{D^{N-1}_{q,\omega}f(a)}{\Gamma_q(N)}(x-a)^{(N-1)}_{\omega_0}\\ &= \cdots\\ &= f(x) - \sum_{k=0}^{N-1}\frac{D^{k}_{q,\omega}f(a)}{\Gamma_q(k+1)}(x-a)^{(k)}_{\omega_0}. \qquad \square\end{aligned}$$

## 7 The application

In this section, we deal with the nonlocal $q,\omega$-integral boundary value problem of nonlinear fractional $q,\omega$-derivative equations:

$$\big({}_aD^{\alpha}_{q,\omega}u\big)(t) + f\big(qt+\omega, u(qt+\omega)\big) = 0, \tag{20}$$

$$u(a) = 0, \qquad u(b) = \mu\big({}_aI^{\beta}_{q,\omega}u\big)(\eta), \tag{21}$$

where $q \in (0,1)$, $1 < \alpha \leq 2$, $0 < \beta \leq 2$, $0 < \eta < 1$, and $\mu > 0$ is a parameter, ${}_aD^{\alpha}_{q,\omega}$ is the Riemann–Liouville $q,\omega$-derivative of Hahn quantum operator type of order $\alpha$, $f : [a,b] \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is continuous.

**Lemma 7.1** *Let $M = \Gamma_q(\alpha+\beta)(b-a)^{(\alpha-1)}_{\omega_0} - \mu\Gamma_q(\alpha)(\eta-a)^{(\alpha+\beta-1)}_{\omega_0} > 0$. Then, for given $y \in C[a,b]$, the unique solution of the boundary value problem*

$$\big({}_aD^{\alpha}_{q,\omega}u\big)(t) + y(qt+\omega) = 0, \quad 1 < \alpha \leq 2,$$

*subject to the boundary condition*

$$u(a) = 0, \qquad u(b) = \mu\left({}_aI^{\beta}_{q,\omega}u\right)(\eta), \quad 0 < \beta \le 2, 0 < \eta < 1,$$

*is given by*

$$u(t) = \int_a^b G\left(t, {}_{\omega_0}\Phi_q(s)\right)y(qs+\omega)\,d_{q,\omega}s, \quad t \in [a,b],$$

*where*

$$G\left(t, {}_{\omega_0}\Phi_q(s)\right) = g\left(t, {}_{\omega_0}\Phi_q(s)\right) + \frac{\mu(t-a)^{(\alpha-1)}_{\omega_0}}{M}H\left(\eta, {}_{\omega_0}\Phi_q(s)\right),$$

$$g\left(t, {}_{\omega_0}\Phi_q(s)\right) = \frac{1}{\Gamma_q(\alpha)}\begin{cases} \frac{(t-a)^{(\alpha-1)}_{\omega_0}}{(b-a)^{(\alpha-1)}_{\omega_0}}(b - {}_{\omega_0}\Phi_q(s))^{(\alpha-1)}_{\omega_0} - (t - {}_{\omega_0}\Phi_q(s))^{(\alpha-1)}_{\omega_0}, \\ \qquad a \le_{\omega_0} \Phi_q(s) \le t \le b, \\ \frac{(t-a)^{(\alpha-1)}_{\omega_0}}{(b-a)^{(\alpha-1)}_{\omega_0}}(b - {}_{\omega_0}\Phi_q(s))^{(\alpha-1)}_{\omega_0}, \quad a \le t \le_{\omega_0} \Phi_q(s) \le b, \end{cases}$$

$$H\left(\eta, {}_{\omega_0}\Phi_q(s)\right) = \begin{cases} \frac{(\eta-a)^{(\alpha+\beta-1)}_{\omega_0}}{(b-a)^{(\alpha-1)}_{\omega_0}}(b - {}_{\omega_0}\Phi_q(s))^{(\alpha-1)}_{\omega_0} - (\eta - {}_{\omega_0}\Phi_q(s))^{(\alpha+\beta-1)}_{\omega_0}, \\ \qquad a \le_{\omega_0} \Phi_q(s) \le \eta \le b, \\ \frac{(\eta-a)^{(\alpha+\beta-1)}_{\omega_0}}{(b-a)^{(\alpha-1)}_{\omega_0}}(b - {}_{\omega_0}\Phi_q(s))^{(\alpha-1)}_{\omega_0}, \quad a \le \eta \le_{\omega_0} \Phi_q(s) \le b. \end{cases}$$

*Proof* In view of Theorem 5.6,

$$u(t) = c_1(t-a)^{(\alpha-1)}_{\omega_0} + c_2(t-a)^{(\alpha-2)}_{\omega_0} - \frac{1}{\Gamma_q(\alpha)}\int_a^t \left(t - {}_{\omega_0}\Phi_q(s)\right)^{(\alpha-1)}_{\omega_0} y(qs+\omega)\,d_{q,\omega}s,$$

where $c_1, c_2$ does not depend on $t$, $t \in [a,b]$. Since $u(a) = 0$, we have $c_2 = 0$. From the boundary condition $u(b) = \mu({}_aI^{\beta}_{q,\omega}u)(\eta)$, we get

$$\begin{aligned} c_1 = \frac{\Gamma_q(\alpha+\beta)}{M}\Bigg[&\frac{1}{\Gamma_q(\alpha)}\int_a^b \left(b - {}_{\omega_0}\Phi_q(s)\right)^{(\alpha-1)}_{\omega_0} y(qs+\omega)\,d_{q,\omega}s \\ &- \frac{\mu}{\Gamma_q(\alpha+\beta)}\int_a^{\eta} \left(\eta - {}_{\omega_0}\Phi_q(s)\right)^{(\alpha+\beta-1)}_{\omega_0} y(qs+\omega)\,d_{q,\omega}s\Bigg]. \end{aligned}$$

Hence

$$\begin{aligned} u(t) = {}& \frac{\Gamma_q(\alpha+\beta)(t-a)^{(\alpha-1)}_{\omega_0}}{M}\Bigg[\frac{1}{\Gamma_q(\alpha)}\int_a^b \left(b - {}_{\omega_0}\Phi_q(s)\right)^{(\alpha-1)}_{\omega_0} y(qs+\omega)\,d_{q,\omega}s \\ & - \frac{\mu}{\Gamma_q(\alpha+\beta)}\int_a^{\eta} \left(\eta - {}_{\omega_0}\Phi_q(s)\right)^{(\alpha+\beta-1)}_{\omega_0} y(qs+\omega)\,d_{q,\omega}s\Bigg] \\ & - \frac{1}{\Gamma_q(\alpha)}\int_a^t \left(t - {}_{\omega_0}\Phi_q(s)\right)^{(\alpha-1)}_{\omega_0} y(qs+\omega)\,d_{q,\omega}s \\ = {}& \int_a^b g\left(t, {}_{\omega_0}\Phi_q(s)\right)y(qs+\omega)\,d_{q,\omega}s \end{aligned}$$

$$+\frac{\mu(t-a)_{\omega_0}^{(\alpha-1)}}{M}\int_a^b H\big(\eta,{}_{\omega_0}\Phi_q(s)\big)y(qs+\omega)\,d_{q,\omega}s$$
$$=\int_a^b G\big(t,{}_{\omega_0}\Phi_q(s)\big)y(qs+\omega)\,d_{q,\omega}s. \qquad \square$$

**Lemma 7.2** *The functions $g(t,{}_{\omega_0}\Phi_q(s))$ and $H(\eta,{}_{\omega_0}\Phi_q(s))$ satisfy the following properties*:

(a) $g(t,{}_{\omega_0}\Phi_q(s))\ge 0$, $g(t,{}_{\omega_0}\Phi_q(s))\le g({}_{\omega_0}\Phi_q(s),{}_{\omega_0}\Phi_q(s))$, $a\le t,{}_{\omega_0}\Phi_q(s)\le b$.

(b) $H(\eta,{}_{\omega_0}\Phi_q(s))\ge 0$, $a\le {}_{\omega_0}\Phi_q(s),\eta\le b$.

*Proof* Let

$$g_1\big(t,_{\omega_0}\Phi_q(s)\big)=\frac{1}{\Gamma_q(\alpha)}\left[\frac{(t-a)_{\omega_0}^{(\alpha-1)}}{(b-a)_{\omega_0}^{(\alpha-1)}}\big(b-{}_{\omega_0}\Phi_q(s)\big)_{\omega_0}^{(\alpha-1)}-\big(t-{}_{\omega_0}\Phi_q(s)\big)_{\omega_0}^{(\alpha-1)}\right],$$
$$g_2\big(t,_{\omega_0}\Phi_q(s)\big)=\frac{1}{\Gamma_q(\alpha)}\left[\frac{(t-a)_{\omega_0}^{(\alpha-1)}}{(b-a)_{\omega_0}^{(\alpha-1)}}\big(b-{}_{\omega_0}\Phi_q(s)\big)_{\omega_0}^{(\alpha-1)}\right].$$

We can easily get $g_2(t,{}_{\omega_0}\Phi_q(s))\ge 0$, according to property (7), we have

$$\begin{aligned}
&\frac{(t-a)_{\omega_0}^{(\alpha-1)}}{(b-a)_{\omega_0}^{(\alpha-1)}}\big(b-{}_{\omega_0}\Phi_q(s)\big)_{\omega_0}^{(\alpha-1)}-\big(t-{}_{\omega_0}\Phi_q(s)\big)_{\omega_0}^{(\alpha-1)}\\
&\quad=\prod_{i=0}^{\infty}\frac{(t-{}_{\omega_0}\Phi_q^{i}(a))}{(t-{}_{\omega_0}\Phi_q^{i+\alpha-1}(a))}\frac{(b-{}_{\omega_0}\Phi_q^{i+\alpha-1}(a))}{(b-{}_{\omega_0}\Phi_q^{i}(a))}\times\prod_{i=0}^{\infty}\frac{(b-{}_{\omega_0}\Phi_q^{i+1}(s))}{(b-{}_{\omega_0}\Phi_q^{i+\alpha}(s))}\\
&\qquad-\prod_{i=0}^{\infty}\frac{(t-{}_{\omega_0}\Phi_q^{i+1}(s))}{(t-{}_{\omega_0}\Phi_q^{i+\alpha}(s))}.
\end{aligned}$$

Since

$$\begin{aligned}
&\left(\frac{(b-{}_{\omega_0}\Phi_q^{i+1}(s))}{(b-{}_{\omega_0}\Phi_q^{i+\alpha}(s))}\frac{(t-{}_{\omega_0}\Phi_q^{i+\alpha}(s))}{(t-{}_{\omega_0}\Phi_q^{i+1}(s))}\right)_s'\\
&\quad=\frac{1}{(b-{}_{\omega_0}\Phi_q^{i+\alpha}(s))^2(t-{}_{\omega_0}\Phi_q^{i+1}(s))^2}\Big[\big(-q^{i+1}\big(t-{}_{\omega_0}\Phi_q^{i+\alpha}(s)\big)-q^{i+\alpha}\big(b-{}_{\omega_0}\Phi_q^{i+1}(s)\big)\big)\\
&\qquad\times\big(b-{}_{\omega_0}\Phi_q^{i+\alpha}(s)\big)\big(t-{}_{\omega_0}\Phi_q^{i+1}(s)\big)-\big(-q^{i+\alpha}\big(t-{}_{\omega_0}\Phi_q^{i+1}(s)\big)-q^{i+1}\big(b-{}_{\omega_0}\Phi_q^{i+\alpha}(s)\big)\big)\\
&\qquad\times\big(b-{}_{\omega_0}\Phi_q^{i+1}(s)\big)\big(t-{}_{\omega_0}\Phi_q^{i+\alpha}(s)\big)\Big]\\
&\quad=\frac{1}{(b-{}_{\omega_0}\Phi_q^{i+\alpha}(s))^2(t-{}_{\omega_0}\Phi_q^{i+1}(s))^2}\Big[-q^{i+1}t^2b+q^{i+\alpha}t^2b-q^{i+\alpha}b^2t+q^{i+1}b^2t\\
&\qquad+_{\omega_0}\Phi_q^{i+\alpha}(s)q^{i+1}t^2-{}_{\omega_0}\Phi_q^{i+1}(s)q^{i+\alpha}t^2+_{\omega_0}\Phi_q^{i+\alpha}(s)q^{i+1}b^2-{}_{\omega_0}\Phi_q^{i+1}(s)q^{i+\alpha}b^2\\
&\qquad+_{\omega_0}\Phi_q^{i+\alpha}(s)_{\omega_0}\Phi_q^{i+\alpha}(s)q^{i+1}(b-t)-{}_{\omega_0}\Phi_q^{i+1}(s)_{\omega_0}\Phi_q^{i+1}(s)q^{i+\alpha}(b-t)\Big]\\
&\quad=\frac{1}{(b-{}_{\omega_0}\Phi_q^{i+\alpha}(s))^2(t-{}_{\omega_0}\Phi_q^{i+1}(s))^2}\Big[\big(q^{i+1}-q^{i+\alpha}\big)\big(b^2t-t^2b\big)+\big(q^{i+1}-q^{i+\alpha}\big)\omega_0t^2\\
&\qquad+\big(q^{i+1}-q^{i+\alpha}\big)\omega_0b^2+\big(q^{i+1}-q^{i+\alpha}\big)\big(\omega_0^2b-\omega_0^2t\big)\Big]>0,
\end{aligned}$$

we may see that function $\frac{(b-{}_{\omega_0}\Phi_q^{i+1}(s))}{(b-{}_{\omega_0}\Phi_q^{i+\alpha}(s))}\frac{(t-{}_{\omega_0}\Phi_q^{i+\alpha}(s))}{(t-{}_{\omega_0}\Phi_q^{i+1}(s))}$ on $s$ is non-decreasing. Thus, we can get

$$\begin{aligned}&\frac{(t-{}_{\omega_0}\Phi_q^{i}(a))(b-{}_{\omega_0}\Phi_q^{i+\alpha-1}(a))}{(t-{}_{\omega_0}\Phi_q^{i+\alpha-1}(a))(b-{}_{\omega_0}\Phi_q^{i}(a))}\frac{(b-{}_{\omega_0}\Phi_q^{i}({}_{\omega_0}\Phi_q(s)))(t-{}_{\omega_0}\Phi_q^{i+\alpha-1}({}_{\omega_0}\Phi_q(s)))}{(b-{}_{\omega_0}\Phi_q^{i+\alpha-1}({}_{\omega_0}\Phi_q(s)))(t-{}_{\omega_0}\Phi_q^{i}({}_{\omega_0}\Phi_q(s)))}\\&\quad\geq\frac{(t-{}_{\omega_0}\Phi_q^{i}(a))(b-{}_{\omega_0}\Phi_q^{i+\alpha-1}(a))}{(t-{}_{\omega_0}\Phi_q^{i+\alpha-1}(a))(b-{}_{\omega_0}\Phi_q^{i}(a))}\frac{(b-{}_{\omega_0}\Phi_q^{i}(a))(t-{}_{\omega_0}\Phi_q^{i+\alpha-1}(a))}{(b-{}_{\omega_0}\Phi_q^{i+\alpha-1}(a))(t-{}_{\omega_0}\Phi_q^{i}(a))}=1,\end{aligned}$$

so $g_1(t,{}_{\omega_0}\Phi_q(s))\geq 0$, we can get $g(t,{}_{\omega_0}\Phi_q(s))\geq 0$. Clearly $g(t,{}_{\omega_0}\Phi_q(s))\leq g({}_{\omega_0}\Phi_q(s),{}_{\omega_0}\Phi_q(s))$ and $H(\eta,{}_{\omega_0}\Phi_q(s))\geq 0$ holds trivially. □

According to the property of being non-increasing of $(t-{}_{\omega_0}\Phi_q(s))_{\omega_0}^{(\alpha-1)}$ on $s$ and Lemma 7.2, we may easily obtain Lemma 7.3 as follows.

**Lemma 7.3** *The function $G(t,{}_{\omega_0}\Phi_q(s))$ satisfies the following properties:*

(a) *$G$ is a continuous function and $G(t,{}_{\omega_0}\Phi_q(s))\geq 0$, $(t,{}_{\omega_0}\Phi_q(s))\in[a,b]\times[a,b]$.*
(b) *There exists a positive function $\rho\in C(a,b)$, $(a,+\infty)$ such that $\max_{a\leq t\leq b}G(t,{}_{\omega_0}\Phi_q(s))\leq g({}_{\omega_0}\Phi_q(s),{}_{\omega_0}\Phi_q(s))+\frac{\mu}{M}H(\eta,{}_{\omega_0}\Phi_q(s))=:\rho({}_{\omega_0}\Phi_q(s))$, ${}_{\omega_0}\Phi_q(s),s\in(a,b)$.*

**Lemma 7.4** (Krasnoselskii [12]) *Let $\mathbb{E}$ be a Banach space, and let $P\subset\mathbb{E}$ be a cone. Assume $\Omega_1,\Omega_2$ are open subsets of $\mathbb{E}$ with $\theta\in\Omega_1\subset\overline{\Omega}_1\subset\Omega_2$, and let $T:P\cap(\overline{\Omega}_2\setminus\Omega_1)\longrightarrow P$ be a completely continuous operator such that*

$$\|Tu\|\geq\|u\|,\quad u\in P\cap\partial\Omega_1\quad\textit{and}\quad\|Tu\|\leq\|u\|,\quad u\in P\cap\partial\Omega_2.$$

*Then $T$ has at least one fixed point in $P\cap(\overline{\Omega}_2\setminus\Omega_1)$.*

Let $X=C[a,b]$ be a Banach space endowed with the norm $\|u\|_X=\max_{a\leq t\leq b}|u(t)|$. Define the cone $P\subset\{u\in X:u(t)\geq 0,a\leq t\leq b\}$.

Define the operator $T:P\longrightarrow X$ as follows:

$$(Tu)(t)=\int_a^b G\big(t,{}_{\omega_0}\Phi_q(s)\big)f\big(qt+\omega,u(qt+\omega)\big)\,d_{q,\omega}s.\tag{22}$$

It follows from the non-negativeness and continuity of $G$ and $f$ that the operator $T:P\longrightarrow X$ satisfies $T(P)\subset P$ and is completely continuous.

**Theorem 7.5** *Let $f(t,u)$ be a nonnegative continuous function on $[a,b]\times\mathbb{R}^+$. In addition, we assume that:*

($H_1$) *There exists a positive constant $r_1$ such that*

$$f(t,u)\geq\kappa r_1,\quad\textit{for }(t,u)\in[\tau_1,\tau_2]\times[a,r_1],$$

*where $a\leq\tau_1\leq\tau_2\leq b$ and*

$$\kappa\geq\left[\int_{\tau_1}^{\tau_2}\left(g\big({}_{\omega_0}\Phi_q(s),{}_{\omega_0}\Phi_q(s)\big)+\frac{\mu(s-a)_{\omega_0}^{(\alpha-1)}}{M}H\big(\eta,{}_{\omega_0}\Phi_q(s)\big)\right)d_{q,\omega}s\right]^{-1}.$$

$(H_2)$ *There exists a positive constant $r_2$ with $r_2 > r_1$ such that*

$$f(t,u) \leq Lr_2, \quad \textit{for } (t,u) \in [a,b] \times [a,r_2],$$

*where*

$$L = \left[\int_a^b \left(\frac{(b - {}_{\omega_0}\Phi_q(s))_{\omega_0}^{(\alpha-1)}}{\Gamma_q(\alpha)(b-a)_{\omega_0}^{(\alpha-1)}} + \frac{\mu}{M} H(\eta, {}_{\omega_0}\Phi_q(s))\right) d_{q,\omega}s\right]^{-1}.$$

*Then the boundary value problem* (20), (21) *has at least one positive solution $u_0$ satisfying* $a \leq r_1 \leq \|u_0\|_X \leq r_2$.

*Proof* By Lemma 7.2, we obtain $\max_{a\leq t\leq b} g(t, {}_{\omega_0}\Phi_q(s)) = g({}_{\omega_0}\Phi_q(s), {}_{\omega_0}\Phi_q(s))$. Let $\Omega_1 = \{u \in X : \|u\|_X < r_1\}$. For any $u \in P \cap \partial\Omega_1$, according to $(H_1)$, we have

$$\begin{aligned}
\|Tu\|_X &= \max_{a\leq t\leq b}\left[\int_a^b g(t, {}_{\omega_0}\Phi_q(s)) f(qs+\omega, u(qs+\omega))\, d_{q,\omega}s\right.\\
&\quad \left.+ \int_a^b \frac{\mu(t-a)_{\omega_0}^{(\alpha-1)}}{M} H(\eta, {}_{\omega_0}\Phi_q(s)) f(qs+\omega, u(qs+\omega))\, d_{q,\omega}s\right]\\
&\geq \int_a^b \left[g({}_{\omega_0}\Phi_q(s), {}_{\omega_0}\Phi_q(s)) + \frac{\mu(s-a)_{\omega_0}^{(\alpha-1)}}{M} H(\eta, {}_{\omega_0}\Phi_q(s))\right]\\
&\quad \times f(qs+\omega, u(qs+\omega))\, d_{q,\omega}s\\
&\geq \kappa r_1 \int_{\tau_1}^{\tau_2} \left[g({}_{\omega_0}\Phi_q(s), {}_{\omega_0}\Phi_q(s)) + \frac{\mu(s-a)_{\omega_0}^{(\alpha-1)}}{M} H(\eta, {}_{\omega_0}\Phi_q(s))\right] d_{q,\omega}s\\
&= r_1 = \|u\|_X.
\end{aligned}$$

Let $\Omega_2 = \{u \in X : \|u\|_X < r_2\}$. For any $u \in P \cap \partial\Omega_2$, by $(H_2)$, we have

$$\begin{aligned}
\|Tu\|_X &= \max_{a\leq t\leq b} \int_a^b G(t, {}_{\omega_0}\Phi_q(s)) f(qs+\omega, u(qs+\omega))\, d_{q,\omega}s\\
&\leq Lr_2 \int_a^b \rho({}_{\omega_0}\Phi_q(s))\, d_{q,\omega}s\\
&\leq \|u\|_X = r_2.
\end{aligned}$$

Now, an application of Lemma 7.4 concludes the proof. □

## 8 Mittag-Leffler function

*Example* 8.1 Let $0 < \alpha \leq 1$ and consider the fractional $q,\omega$-difference equation

$$({}^cD_{q,\omega}^\alpha y)(qt+\omega) = \lambda y(t) + f(t), \quad y(0) = a_0.$$

If we apply ${}_aI_{q,\omega}^\alpha$ on the equation, we see that

$$y(t) = a_0 + \lambda_a I_{q,\omega}^\alpha y({}_{\omega_0}\Phi_q^{-1}(t)) +_a I_{q,\omega}^\alpha g(t),$$

where $g(t) = f({}_{\omega_0}\Phi_q^{-1}(t))$.

To obtain an explicit clear solution, we apply the method of successive approximation. Set $y_0(t) = a_0$ and

$$y_m(t) = a_0 + \lambda_a I_{q,\omega}^{\alpha} y_{m-1}\big({}_{\omega_0}\Phi_q^{-1}(t)\big) +_a I_{q,\omega}^{\alpha} g(t), \quad m = 1, 2, \ldots.$$

For $m = 1$, we have

$$\begin{aligned} y_1(t) &= a_0 + \frac{\lambda}{\Gamma_q(\alpha)} \int_a^{\omega_0 \Phi_q^{-1}(t)} \big({}_{\omega_0}\Phi_q^{-1}(t) - {}_{\omega_0}\Phi_q(s)\big)_{\omega_0}^{(\alpha-1)} y_0(s)\, d_{q,\omega}s +_a I_{q,\omega}^{\alpha} g(t) \\ &= a_0\bigg[1 + \frac{\lambda}{\Gamma_q(\alpha+1)}\big({}_{\omega_0}\Phi_q^{-1}(t) - a\big)_{\omega_0}^{(\alpha)}\bigg] +_a I_{q,\omega}^{\alpha} g(t), \\ y_2(t) &= a_0\bigg[1 + \frac{\lambda}{\Gamma_q(\alpha+1)}\big({}_{\omega_0}\Phi_q^{-1}(t) - a\big)_{\omega_0}^{(\alpha)} + \frac{\lambda^2}{\Gamma_q(2\alpha+1)}\big({}_{\omega_0}\Phi_q^{-1}(t) - a\big)_{\omega_0}^{(2\alpha)}\bigg] \\ &\quad + \lambda_a I_{q,\omega}^{\alpha}\big({}_a I_{q,\omega}^{\alpha} g\big)\big({}_{\omega_0}\Phi_q^{-1}(t)\big) +_a I_{q,\omega}^{\alpha} g(t). \end{aligned}$$

If we proceed inductively and let $m \to \infty$, we obtain the solution

$$\begin{aligned} y(t) &= a_0\Bigg[1 + \sum_{k=1}^{\infty} \frac{\lambda^k}{\Gamma_q(k\alpha+1)}\big({}_{\omega_0}\Phi_q^{-1}(t) - a\big)_{\omega_0}^{(k\alpha)}\Bigg] \\ &\quad + \int_a^t \sum_{k=0}^{\infty} \frac{\lambda^k (t - {}_{\omega_0}\Phi_q(s))_{\omega_0}^{(k\alpha+\alpha-1)}}{\Gamma_q(k\alpha+\alpha)} f\big({}_{\omega_0}\Phi_q^{-1}(t)\big)\, d_{q,\omega}s. \end{aligned}$$

**Definition 8.2** For $z, z_0 \in \mathbb{C}$, the Mittag-Leffler function for fractional $q,\omega$-difference equation is defined by

$$E_{\alpha,\beta}(\lambda, z - z_0) = \sum_{k=0}^{\infty} \frac{\lambda^k (z - z_0)^{(k\alpha+\beta-1)}}{\Gamma_q(k\alpha+\beta)}.$$

When $\beta = 1$, we simply use $E_\alpha(\lambda, z - z_0) = E_{\alpha,1}(\lambda, z - z_0)$.

According to Definition 8.2 above, the solution of the fractional $q,\omega$-difference equation in Example 8.1 is expressed by

$$y(t) = a_0 E_\alpha\big(\lambda, t - {}_{\omega_0}\Phi_q(a)\big) + \int_a^t E_{\alpha,\alpha}\big(\lambda, t - {}_{\omega_0}\Phi_q(s)\big) f\big({}_{\omega_0}\Phi_q^{-1}(s)\big)\, d_{q,\omega}s.$$

## 9 Conclusion

The paper [16] focuses on essentials of fractional calculus on a special discrete time scale forming the background for the so-called $(q,h)$-calculus, which can be reduced to the quantum calculus (the case $h = 0$) or to the difference calculus (the case $q = h = 1$). From this point, in this article, we give a more general extension. The fractional $q,\omega$-calculus is defined in the set of real numbers. The main advantage of our results is that the corresponding properties can deal with the fractional $q,\omega$-calculus equation, the existence result of the solutions is solved, which is important in physical systems. We trust that the field here initiated will prove to be fruitful for further research.

**Acknowledgements**
The authors would like to thank the referee for invaluable comments and insightful suggestions.

**Funding**
This work is supported by the Natural Science Foundation of China (No. 11161049).

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors read and approved the final manuscript.

## Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 13 December 2017 Accepted: 14 August 2018 Published online: 28 August 2018

**References**
1. Kac, V., Cheung, P.: Quantum Calculus. Springer, New York (2002)
2. Bangerezako, G.: Variational $q$-calculus. J. Math. Anal. Appl. **289**(2), 650–665 (2004)
3. Bangerezako, G.: Variational calculus on $q$-nonuniform lattices. J. Math. Anal. Appl. **306**(1), 161–179 (2005)
4. Cresson, J., Frederico, G.S.F., Torres, D.F.M.: Constants of motion for non-differentiable quantum variational problems. Topol. Methods Nonlinear Anal. **33**(2), 217–231 (2009)
5. Annaby, M.H., Mansour, Z.S.: $q$-Fractional Calculus and Equations. Lecture Notes in Mathematics, vol. 2056. Springer, Berlin (2012). https://doi.org/10.1007/978-3-642-30898-7
6. El-Shahed, M., Hassan, H.A.: Positive solutions of $q$-difference equation. Proc. Am. Math. Soc. **138**(5), 1733–1738 (2010)
7. Al-Salam, W.A.: Some fractional $q$-integrals and $q$-derivatives. Proc. Edinb. Math. Soc. **15**, 135–140 (1966)
8. Agarwal, R.P.: Certain fractional $q$-integrals and $q$-derivatives. Math. Proc. Camb. Philos. Soc. **66**, 365–370 (1969)
9. Rajković, P.M., Marinković, S.D., Stanković, M.S.: Fractional integrals and derivatives in $q$-calculus. Appl. Anal. Discrete Math. **1**(1), 311–323 (2007). https://doi.org/10.2298/AADM0701311R
10. Hahn, W.: Über orthogonalpolynome, die $q$-differenzenlgleichungen genügen. Math. Nachr. **2**, 4–34 (1949)
11. Aldwoah, K.A.: Generalized time scales and associated difference equations. Ph.D. Thesis, Cairo University (2009)
12. Malinowska, A.B., Torres, D.F.M.: The Hahn quantum variational calculus. J. Optim. Theory Appl. **147**, 419–442 (2010)
13. Malinowska, A.B., Martins, N.: Generalized transversality conditions for the Hahn quantum variational calculus. Optimization **62**(3), 323–344 (2013)
14. Tariboon, J., Ntouyas, S.K., Agarwal, P.: New concepts of fractional quantum calculus and applications to impulsive fractional $q$-difference equations. Adv. Differ. Equ. **2015**, Article ID 18 (2015). https://doi.org/10.1186/s13662-014-0348-8
15. Atici, F.M., Eloe, P.W.: Fractional $q$-calculus on a time scale. J. Nonlinear Math. Phys. **14**, 341–352 (2007)
16. Čremák, J., Nechvátal, L.: On $(q,h)$-analogue of fractional calculus. J. Nonlinear Math. Phys. **17**(1), 51–68 (2010). https://doi.org/10.1142/S1402925110000593