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Stability and bifurcation analysis of two species amensalism model with Michaelis–Menten type harvesting and a cover for the first species

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Abstract

In this paper, a two species amensalism model with Michaelis–Menten type harvesting and a cover for the first species that takes the form

$$\frac{dx(t)}{dt} = a_1 x(t) - b_1 x^2(t) - c_1(1-k)x(t)y(t) - \frac{qE(1-k)x(t)}{m_1 E + m_2(1-k)x(t)},$$

$$\frac{dy(t)}{dt} = a_2 y(t) - b_2 y^2(t)$$

is investigated, where a_i, b_i, $i = 1, 2$, and c_1 are all positive constants, k is a cover provided for the species x, and $0 < k < 1$. The stability and bifurcation analysis for the system are taken into account. The existence and stability of all possible equilibria of the system are investigated. With the help of Sotomayor's theorem, we can prove that there exist two saddle-node bifurcations and two transcritical bifurcations under suitable conditions.

MSC: 34C25; 92D25; 34D20; 34D40

Keywords: Amensalism model; Nonlinear harvesting; Cover; Bifurcation analysis

1 Introduction

Amensalism and commensalism are common relations between the species. Here, amensalism is an interaction where a species inflicts harm on the other species without any costs or benefits received by the other, and commensalism is a relationship which is only favorable to the one side and has no influence on the other side.

During the last decade, many scholars investigated the dynamic behaviors of a commensalism model [1–13], while only recently did scholars pay attention to the dynamic behaviors of an amensalism model [1, 14–21].

Sun [14] for the first time proposed the two species amensalism model as follows:

$$\frac{dx}{dt} = r_1 x \left(\frac{k_1 - x - \alpha y}{k_1} \right),$$
$$\frac{dy}{dt} = r_2 y \left(\frac{k_2 - y}{k_2} \right).$$

The stability of all equilibria of the system is investigated. Zhu and Chen [15] studied the qualitative property of the following amensalism model which is a general form of the previous system:

$$\frac{dx}{dt} = x(r_1 - a_{11}x - a_{12}y),$$
$$\frac{dy}{dt} = y(r_2 - a_{22}y).$$

After that, some scholars focused on an amensalism model. Han et al. [16] investigated the two species non-autonomous amensalism model. Wu [17] investigated the dynamic behaviors of the two species amensalism symbiosis model with non-monotonic function response as follows:

$$\frac{dx}{dt} = x\big(a_1(t) - b_1(t)x - c_1(t)y\big),$$
$$\frac{dy}{dt} = y\big(a_2(t) - b_2(t)y\big).$$

With more works on the amensalism mode published, some scholars paid attention to the influence of refuge on the amensalism model. Xie et al. [18] considered a two species amensalism model with a partial cover for the first species to protect it from the second species, the model is as follows:

$$\frac{dx}{dt} = a_1 x - b_1 x^2 - c_1(1-k)xy,$$
$$\frac{dy}{dt} = a_2 y - b_2 y^2, \tag{1.1}$$

where $a_i, b_i, i = 1, 2$, and c_1 are all positive constants, k is a cover provided for the species x, and $0 < k < 1$. They showed that system (1.1) has four possible equilibria $E_0(0,0)$, $E_1(\frac{a_1}{b_1}, 0)$, $E_2(0, \frac{a_2}{b_2})$, and $E_3(x^*, y^*)$, where $x^* = \frac{a_1 b_2 - a_2 c_1(1-k)}{b_1 b_2}$, $y^* = \frac{a_2}{b_2}$. They showed that $E_0(0,0)$ and $E_1(\frac{a_1}{b_1}, 0)$ are unstable, if $0 < k < 1 - \frac{a_1 b_2}{a_2 c_1}$, then $E_3(x^*, y^*)$ is globally stable, if $1 - \frac{a_1 b_2}{a_2 c_1} < k < 1$, then $E_3(x^*, y^*)$ is globally stable. More precisely, the conditions which ensure the local stability of $E_2(0, \frac{a_2}{b_2})$ are enough to ensure its global stability, and once the positive equilibrium exists, it is globally stable. After that, Wu et al. [19] proposed a two species amensalism model with Holling II functional response and a cover for the first species and investigated the local and global stability property of possible equilibria of the system.

On the other hand, to obtain the resource for the development of a human being, considering the harvesting of species is necessary. During the last decade, many scholars investigated the influence of the harvesting to predator–prey or competition system [22–24].

Recently, some scholars started to focus on the influence of the harvesting on the amensalism or commensalism model [20, 25]. Chen [20] proposed a non-selective harvesting Lotka–Volterra amensalism model incorporating partial closure for the populations, investigated the existence, local stability, and the global stability property of the equilibrium solutions of the system and discussed the influence of the parameter m. Liu et al. [25] proposed a non-autonomous non-selective harvesting Lotka–Volterra commensalism model incorporating partial closure for the populations and investigated the extinction, partial survival, persistent property, and the global stability property of the solutions of the system.

There are three types of harvesting: (1) constant harvesting [26, 27]; (2) linear harvesting [28–32]; and (3) nonlinear harvesting [22–24, 33]. As we know, nonlinear harvesting is more realistic from biological and economical points of view [22]. Clark [23] first proposed a harvesting term $h = \frac{qEx}{cE+lx}$, which is named Michaelis–Menten type functional form of catch rate. Hu and Cao [24] studied the stability and bifurcation of the system with the Michaelis–Menten type harvesting in predator.

It brings to our attention that in [18] the dynamic behaviors of the amensalism model with a partial cover are simple and interesting; at the same time, the influence of a human actor to the nature is rising. No one has ever considered the amensalism model with a cover and nonlinear harvesting. Hence, studying the dynamic behavior of the amensalism model with cover and nonlinear harvesting is meaningful. Stimulated by the works of [18, 20], and [24], we propose the following two species amensalism model with Michaelis–Menten type harvesting and a cover for the first species:

$$
\begin{aligned}
\frac{dx}{dt} &= a_1 x - b_1 x^2 - c_1 (1-k) xy - \frac{qE(1-k)x}{m_1 E + m_2 (1-k)x}, \\
\frac{dy}{dt} &= a_2 y - b_2 y^2,
\end{aligned}
\tag{1.2}
$$

where a_i, b_i, m_i, $i = 1, 2$, q, E, and c_1 are all positive constants, where a_i represents the intrinsic growth rate of the ith species, b_i describes the intraspecific competition of the ith species, k is a cover provided for the species x, and $0 < k < 1$, E is the combined fishing effort used to harvest.

We take the following transformations as [26] to simplify system (1.2):

$$
\bar{t} = lt, \qquad \bar{x} = nx, \qquad \bar{y} = my,
$$

dropping the bars, system (1.2) can be written as

$$
\begin{aligned}
\frac{dx}{dt} &= x(1-x) - xy - \frac{\alpha x}{\gamma + x}, \\
\frac{dy}{dt} &= y\left(\delta - \frac{y}{\beta}\right),
\end{aligned}
\tag{1.3}
$$

where

$$
\delta = \frac{a_2}{a_1}, \qquad \alpha = \frac{b_1 qE}{a_1^2 m_2}, \qquad \beta = \frac{c_1 (1-k)}{b_2}, \qquad \gamma = \frac{m_1 b_1 E}{m_2 a_1 (1-k)}.
$$

The aim of this paper is to investigate the local stability property of the possible equilibria and bifurcation of system (1.3). We arrange the paper as follows: in the next section, we investigate the existence and local stability property of the equilibria of system (1.3). In Sect. 3, we discuss the saddle-node bifurcation and transcritical bifurcation of the system. Numeric simulations are presented in Sect. 4 to show the feasibility of the main results. We end this paper with a brief discussion.

2 The existence and stability of equilibria

2.1 The existence of equilibria

The equilibria of system (1.3) are determined by the system

$$x(1 - x) - xy - \frac{\alpha x}{\gamma + x} = 0,$$
$$y\left(\delta - \frac{y}{\beta}\right) = 0. \tag{2.1}$$

The system always admits the boundary equilibria $E_0(0,0)$, $E_1(0,\delta\beta)$, while for other possible boundary equilibria and positive equilibria, we need to consider the following situations:

(i) If $x \neq 0$, $y = 0$, we may have another boundary equilibrium $E_2(x_2, 0)$, where x_2 is the root of the following equation:

$$x^2 + (\gamma - 1)x + \alpha - \gamma = 0. \tag{2.2}$$

Let the discriminant of Eq. (2.2) be denoted by $\triangle_1$ and express $\triangle_1$ in terms of α, i.e.,

$$\begin{aligned} \triangle_1(\alpha) &= -4\alpha + (\gamma - 1)^2 + 4\gamma \\ &= -4\alpha + (\gamma + 1)^2, \end{aligned} \tag{2.3}$$

let α_1 be the root of $\triangle_1(\alpha)$. After calculating, we have

$$\alpha_1 = \frac{(\gamma + 1)^2}{4}.$$

(ii) If $x \neq 0$, $y \neq 0$, we may have a positive equilibrium $E_3(x_3, \delta\beta)$, where x_3 is the root of the following equation:

$$x^2 + (\delta\beta + \gamma - 1)x + \gamma\delta\beta + \alpha - \gamma = 0. \tag{2.4}$$

Let $\triangle_2$ denote the discriminant of Eq. (2.4) and express it as follows:

$$\begin{aligned} \triangle_2(\alpha) &= -4\alpha + (\delta\beta + \gamma - 1)^2 + 4\gamma - 4\gamma\delta\beta \\ &= -4\alpha + (\delta\beta - \gamma - 1)^2, \end{aligned} \tag{2.5}$$

let α_2 be the root of $\triangle_2(\alpha)$. After calculating, we have

$$\alpha_2 = \frac{(\delta\beta - \gamma - 1)^2}{4}.$$

Concerned with the number of equilibria of system (1.3), we have the following theorem.

Theorem 2.1 *For all positive parameters, there are two boundary equilibria* $E_0(0,0)$, $E_1(0,\delta\beta)$. *In system* (1.3), *for other possible boundary equilibria and positive equilibria, we have*:

  (i) *For other possible boundary equilibria*:
    (a) *If* $\alpha > \alpha_1$, *then system* (1.3) *has no other boundary equilibrium*;
    (b) *If* $\alpha = \alpha_1$ *and* $\gamma < 1$, *then there exists another boundary equilibrium* $E_{21}(x_{21}, 0)$, *where* $x_{21} = \frac{1-\gamma}{2}$;
    (c) *If* $\gamma < \alpha < \alpha_1$ *and* $\gamma < 1$, *then there exist two boundary equilibria* $E_{22}(x_{22}, 0)$, $E_{23}(x_{23}, 0)$, *where* $x_{22,23} = \frac{1-\gamma\mp\sqrt{\triangle_1}}{2}$;
    (d) *If* $\alpha = \gamma$ *and* $\gamma < 1$, *then* E_{22} *coincides with* E_0 *and there exists another boundary equilibrium* $E_{23}(x_{23}, 0)$, *where* $x_{23} = 1 - \gamma$;
    (e) *If* $0 < \alpha < \gamma$, *then there exists another boundary equilibrium* $E_{23}(x_{23}, 0)$, *where* x_{23} *is the same as in case* (c);
  (ii) *For the possible positive equilibria*:
    (a) *If* $\alpha > \alpha_2$, *then system* (1.3) *has no positive equilibria*;
    (b) *If* $\alpha = \alpha_2$ *and* $\gamma < 1 - \delta\beta$, *then there exists a unique positive equilibrium* $E_{31}(x_{31}, \delta\beta)$, *where* $x_{31} = \frac{1-\delta\beta-\gamma}{2}$;
    (c) *If* $\gamma - \gamma\delta\beta < \alpha < \alpha_2$ *and* $\gamma < 1 - \delta\beta$, *then there exist two distinct positive equilibria* $E_{32}(x_{32}, \delta\beta)$, $E_{33}(x_{33}, \delta\beta)$, *where* $x_{32,33} = \frac{1-\delta\beta-\gamma\mp\sqrt{\triangle_2}}{2}$;
    (d) *If* $\alpha = \gamma - \gamma\delta\beta$ *and* $\gamma < 1 - \delta\beta$, *then* E_{32} *coincides with* E_1 *and there exists a unique positive equilibrium* $E_{33}(x_{33}, \delta\beta)$, *where* $x_{33} = 1 - \delta\beta - \gamma$;
    (e) *If* $0 < \alpha < \gamma - \gamma\delta\beta$, *then there exists a unique positive equilibrium* $E_{33}(x_{33}, \delta\beta)$, *where* x_{33} *is the same as in case* (c).

Proof Let $f(x) = x^2 + (\gamma + 1)x + \alpha - \gamma$, we can get that if $\alpha > \alpha_1$, then $\triangle_1 < 0$, which means that the function has no zero point.

If $\alpha = \alpha_1$, then $\triangle_1 = 0$, which means that the function has a unique positive zero point $x_{21} = \frac{1-\gamma}{2}$, where $\gamma < 1$.

If $0 < \alpha < \alpha_1$, then $\triangle_1 > 0$, which means that the function has two distinct zero points $x_{22,23} = \frac{1-\gamma\mp\sqrt{\triangle_1}}{2}$, and we have to discuss the symbol of them.

If $f(0) > 0$ and the symmetry is positive, we have $x_{22,23} > 0$, as the conditions mention above, it follows that $\gamma < \alpha < \alpha_1$ and $\gamma > 1$. If $f(0) > 0$ and the symmetry is negative, we have $x_{22,23} < 0$. If $f(0) = 0$ and the symmetry is positive, that is, $\alpha = \gamma$ and $\gamma > 1$, we have $x_{22} = 0$, $x_{23} = 1 - \gamma$, that is to say, E_{22} coincides with E_0 and there exists another boundary equilibrium E_{23} where $\alpha = \gamma$ and $\gamma > 1$. If $f(0) = 0$ and the symmetry is negative, we have $x_{22} < 0$, $x_{23} = 0$. If $f(0) < 0$, no matter what the symbol of symmetry is, we have $x_{22} < 0$, $x_{23} > 0$. In other words, there exists another boundary equilibrium E_{23} where $0 < \alpha < \gamma$.

To discuss the possible positive equilibria, we use similar methods and get the results as in Theorem 2.1(ii).

This ends the proof of Theorem 2.1. □

2.2 Stability of the equilibria E_0, E_1

Then, we analyze the stability of the equilibrium of system (1.3).

Theorem 2.2 *For all positive parameters, there are two boundary equilibria* $E_0(0,0)$, $E_1(0,\delta\beta)$, *then*:

(1) $E_0(0,0)$ *is always unstable*;

(2) *For* $E_1(0,\delta\beta)$, *we have*:

    (i) *If* $1 - \delta\beta - \alpha/\gamma > 0$, *then* $E_1(0,\delta\beta)$ *is a saddle*;

    (ii) *If* $1 - \delta\beta - \alpha/\gamma < 0$, *then* $E_1(0,\delta\beta)$ *is a stable node*;

    (iii) *If* $1 - \delta\beta - \alpha/\gamma = 0$, *then*: *If* $\alpha = \gamma^2$, *then* $E_1(0,\delta\beta)$ *is a stable node*; *If* $\alpha \neq \gamma^2$, *then* $E_1(0,\delta\beta)$ *is a saddle node*.

Proof The Jacobian matrix of system (1.3) is calculated as

$$J(x,y) = \begin{pmatrix} 1 - 2x - y - \frac{\alpha\gamma}{(\gamma+x)^2} & -x \\ 0 & \delta - \frac{2y}{\beta} \end{pmatrix}. \tag{2.6}$$

(i) Then the Jacobian matrix of system (1.3) about the equilibrium $E_0(0,0)$ is given by

$$J(E_0) = \begin{pmatrix} 1 - \frac{\alpha}{\gamma} & 0 \\ 0 & \delta \end{pmatrix}. \tag{2.7}$$

The characteristic equation of Jacobian matrix (2.7) is

$$\left(\lambda - 1 + \frac{\alpha}{\gamma}\right)(\lambda - \delta) = 0,$$

from which it follows that two eigenvalues of $J(E_0)$ are $\lambda_1 = 1 - \frac{\alpha}{\gamma}$, $\lambda_2 = \delta > 0$. It is obvious that $E_0(0,0)$ is always unstable.

(ii) The Jacobian matrix of system (1.3) about the equilibrium $E_1(0,\delta\beta)$ is given by

$$J(E_1) = \begin{pmatrix} 1 - \delta\beta - \frac{\alpha}{\gamma} & 0 \\ 0 & -\delta \end{pmatrix}. \tag{2.8}$$

The characteristic equation of Jacobian matrix (2.8) is

$$\left(\lambda - 1 + \delta\beta + \frac{\alpha}{\gamma}\right)(\lambda + \delta) = 0,$$

from which it follows that two eigenvalues of $J(E_1)$ are $\lambda_1 = 1 - \delta\beta - \frac{\alpha}{\gamma}$, $\lambda_2 = -\delta < 0$. One could easily see that if $1 - \delta\beta - \frac{\alpha}{\gamma} > 0$, then $E_1(0,\delta\beta)$ is a saddle; if $1 - \delta\beta - \alpha/\gamma < 0$, then $E_1(0,\delta\beta)$ is a stable node; if $1 - \delta\beta - \frac{\alpha}{\gamma} = 0$, i.e., $\lambda_1 = 0, \lambda_2 = -\delta$, we cannot draw the conclusion easily.

In this case, Theorem 7.1 in Chap. 2 in [34] is used to determine the stability of the equilibrium E_1. We transform the equilibrium E_1 to the origin by translation $(X, Y) = (x, y - \delta\beta)$ at first, and then expand in power series up to the forth order around the origin, which makes the system to be of the following form:

$$\begin{aligned} \frac{dX}{dt} &= -\frac{\gamma^2 - \alpha}{\gamma^2}X^2 - XY - \frac{\alpha}{\gamma^3}X^3 + \frac{\alpha}{\gamma^4}X^4 + P_1(X, Y), \\ \frac{dY}{dt} &= -\delta Y - \frac{1}{\beta}Y^2, \end{aligned} \tag{2.9}$$

where $P_1(X, Y)$ is a power series in (X, Y) with terms $X^i Y^j$ satisfying $i + j \geq 5$.

Let $x = X$, $y = Y$, $\tau = -\delta t$, where τ is a new time variable, then we have

$$
\begin{aligned}
\frac{dx}{d\tau} &= \frac{\gamma^2 - \alpha}{\delta\gamma^2}x^2 + \frac{1}{\delta}xy + \frac{\alpha}{\delta\gamma^3}x^3 - \frac{\alpha}{\delta\gamma^4}X^4 + P_2(x,y), \\
\frac{dy}{d\tau} &= y + \frac{1}{\delta\beta}y^2,
\end{aligned}
\tag{2.10}
$$

where $P_2(x,y)$ is a power series in (x,y) with terms $x^i y^j$ satisfying $i + j \geq 5$.

From $y + \frac{1}{\delta\beta}y^2 = 0$, we get the implicit function $y = 0$, then

$$
\frac{dx}{d\tau} = \frac{\gamma^2 - \alpha}{\delta\gamma^2}x^2 + \frac{\alpha}{\delta\gamma^3}x^3 - \frac{\alpha}{\delta\gamma^4}X^4 + \cdots.
$$

According to Theorem 7.1 in Chap. 2 in [34], if the coefficient of x^2 is $\gamma^2 - \alpha \neq 0$, i.e., $m = 2$, then the equilibrium E_1 is a saddle node. If $\gamma^2 - \alpha = 0$, then we have $m = 3$, $a_m = \frac{\alpha}{\delta\gamma^3} > 0$, by Theorem 7.1 in Chap. 2 in [34], E_1 is an unstable node. Note that we have used the transform $\tau = -\delta t$, which means the orbits with time going in the opposite direction. That is, E_1 is a stable node.

This ends the proof of Theorem 2.2. □

2.3 Stability of the equilibria E_{21}, E_{22}, E_{23}

Theorem 2.3 *Assume that E_{21}, E_{22}, E_{23} exist, then they are all unstable.*

Proof Note that $E_{21}(x_{21}, 0)$ satisfies the equation

$$
1 - x_{21} - \frac{\alpha}{\gamma + x_{21}} = 0.
$$

The Jacobian matrix about the equilibrium E_{21} is given by

$$
\begin{pmatrix}
-x_{21} + \frac{\alpha x_{21}}{(\gamma + x_{21})^2} & -x_{21} \\
0 & \delta
\end{pmatrix}.
\tag{2.11}
$$

The eigenvalues of the above matrix are $\lambda_1 = -x_{21} + \frac{\alpha x_{21}}{(\gamma + x_{21})^2}$, $\lambda_2 = \delta > 0$. Hence, E_{21} is unstable. The stability of E_{22}, E_{23} can be proved by the same method. Obviously, E_{22}, E_{23} are unstable too.

This ends the proof of Theorem 2.3. □

2.4 Stability of the equilibrium E_{31}

Theorem 2.4 *Assume that $\alpha = \alpha_2$ and $\gamma < 1 - \delta\beta$ hold, system (1.3) has a unique positive equilibrium $E_{31}(x_{31}, \delta\beta)$, where $x_{31} = \frac{1 - \delta\beta - \gamma}{2}$, then E_{31} is a saddle node.*

Proof Note that $E_{31}(x_{31}, \delta\beta)$ satisfies the equation

$$
1 - x_{31} - \delta\beta - \frac{\alpha}{\gamma + x_{31}} = 0,
$$

where $\alpha = \alpha_2 = \frac{(\delta\beta - \gamma - 1)^2}{4}$.

Then the Jacobian matrix about the equilibrium E_{31} is given by

$$\begin{pmatrix} 0 & -x_{31} \\ 0 & -\delta \end{pmatrix}. \tag{2.12}$$

The eigenvalues of the above matrix are $\lambda_1 = 0$, $\lambda_2 = -\delta < 0$.

Then Theorem 7.1 in Chap. 2 in [34] is used to determine the stability of the equilibrium E_{31}. Now we transform the equilibrium E_{31} to the origin by translation $(X, Y) = (x - x_{31}, y - \delta\beta)$ and then expand in power series up to the forth order around the origin, which makes the system to be of the following form:

$$\begin{aligned}
\frac{dX}{dt} &= a_{01}Y + a_{20}X^2 - XY + a_{30}X^3 + a_{40}X^4 + Q_1(X, Y), \\
\frac{dY}{dt} &= -\delta Y - \frac{1}{\beta}Y^2,
\end{aligned} \tag{2.13}$$

where

$$a_{01} = -x_{31}, \qquad a_{20} = -\frac{2x_{31}}{1 + \gamma - \delta\beta}, \qquad a_{30} = -\frac{\gamma}{x_{31}^2}, \qquad a_{40} = -\frac{\gamma}{x_{31}^3}$$

and $Q_1(X, Y)$ is a power series in (X, Y) with terms $X^i Y^j$ satisfying $i + j \geq 5$.

In order to make the Jacobian matrix into a standard form, we use the transformation

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 1 & \frac{x_{31}}{2\delta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

then system (2.13) becomes

$$\begin{aligned}
\frac{dx}{dt} &= b_{20}x^2 + b_{11}xy + b_{02}y^2 + b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3 + Q_2(x, y), \\
\frac{dy}{dt} &= -\delta y - \frac{1}{\beta}y^2,
\end{aligned} \tag{2.14}$$

where

$$b_{20} = -\frac{x_{31}}{1 + \gamma - \delta\beta}, \qquad b_{11} = -\frac{4x_{31}^2 + \delta + \delta\gamma - \beta\gamma^2}{\delta(1 + \gamma - \delta\beta)},$$

$$b_{02} = -\frac{x_{31}(-\beta^2\delta^2 + \beta\delta^2 + \gamma\delta\beta + 2\beta x_{31}^2 + \delta\beta - \delta\gamma - \delta)}{\delta^2\beta(1 + \gamma - \delta\beta)},$$

$$b_{30} = -\frac{\gamma}{x_{31}^2}, \qquad b_{21} = -\frac{3\gamma}{x_{31}\delta}, \qquad b_{12} = -\frac{3\gamma}{\delta^2}, \qquad b_{03} = -\frac{3\gamma x_{31}}{\delta^3},$$

and $Q_2(x, y)$ is a power series in (x, y) with terms $x^i y^j$ satisfying $i + j \geq 4$.

Let $\tau = -\delta t$, where τ is a new time variable, then we have

$$\begin{aligned}
\frac{dx}{d\tau} &= c_{20}x^2 + c_{11}xy + c_{02}y^2 + c_{30}x^3 + c_{21}x^2y + c_{12}xy^2 + c_{03}y^3 + Q_3(x, y), \\
\frac{dy}{d\tau} &= y + \frac{1}{\delta\beta}y^2,
\end{aligned} \tag{2.15}$$

where $c_{ij} = -\frac{1}{\delta}b_{ij}$ and $P_2(x, y)$ is a power series in (x, y) with terms $x^i y^j$ satisfying $i + j \geq 4$.

From $y + \frac{1}{\delta\beta}y^2 = 0$, we get the implicit function $y = 0$, then

$$\frac{dx}{d\tau} = \frac{x_{31}}{\delta(1 + \gamma - \delta\beta)}x^2 + \frac{4\gamma}{\delta(1 + \gamma - \delta\beta)}x^3 + \cdots.$$

Because of $x_{31} > 0$, $1 + \gamma - \delta\beta > 2\gamma > 0$, the coefficient of x^2 is $\frac{x_{31}}{\delta(1+\gamma-\delta\beta)} > 0$ holding for all positive parameters, i.e., $m = 2, a_m > 0$. Hence, by Theorem 7.1 in Chap. 2 in [34], E_{31} is a saddle node.

This ends the proof of Theorem 2.4. □

2.5 Stability of the equilibria E_{32}, E_{33}

Theorem 2.5

(1) *Assume that*

$$\gamma - \gamma\delta\beta < \alpha < \alpha_2, \qquad \gamma < 1 - \delta\beta$$

holds, then E_{32} is unstable, E_{33} is locally asymptotically stable;

(2) *Assume that*

$$\alpha = \gamma - \gamma\delta\beta, \qquad \gamma < 1 - \gamma\delta\beta$$

holds, then E_{33} is locally asymptotically stable;

(3) *Assume that*

$$0 < \alpha < \gamma - \gamma\delta\beta$$

holds, then E_{33} is locally asymptotically stable if $\gamma < 1 - \delta\beta$, E_{33} is unstable if $\gamma > 1 - \delta\beta$.

Proof Note that $E_{32,33}(x_{32,33}, \delta\beta)$ satisfies the equation

$$1 - x_{32,33} - \delta\beta - \frac{\alpha}{\gamma + x_{32,33}} = 0.$$

The Jacobian matrix about the equilibria $E_{32,33}$ is given by

$$\begin{pmatrix} -x_{32,33} + \frac{\alpha x_{32,33}}{(\gamma + x_{32,33})^2} & -x_{32,33} \\ 0 & -\delta \end{pmatrix}. \tag{2.16}$$

The eigenvalues of the above matrix are $\lambda_1 = x_{32,33}(\frac{\alpha}{(\gamma + x_{32,33})^2} - 1)$, $\lambda_2 = -\delta < 0$. Hence, the stability of $E_{32,33}$ is determined by the symbol of λ_1.

(1) If $\gamma - \gamma\delta\beta < \alpha < \alpha_2$ and $\gamma < 1 - \delta\beta$, after simple calculation, we have:
   (a) For the equilibrium $E_{32}(\frac{1-\gamma-\sqrt{\Delta_1}}{2}, \delta\beta)$, since $\frac{\alpha}{(\gamma + x_{32,33})^2} > 1$, it follows that $\lambda_1 > 0$, that is, E_{32} is unstable;
   (b) For the equilibrium $E_{33}(\frac{1-\gamma+\sqrt{\Delta_1}}{2}, \delta\beta)$, since $\frac{\alpha}{(\gamma + x_{32,33})^2} < 1$, it follows that $\lambda_1 < 0$, that is, E_{33} is locally asymptotically stable.

Table 1 The existence and stability of equilibria of system (1.3), where $\gamma < 1 - \delta\beta$

Qualifications	Equilibria & Stability									
	E_0	E_1	E_{21}	E_{22}	E_{23}	E_{23*}	E_{31}	E_{32}	E_{33}	E_{33*}
$0 < \alpha < \gamma - \gamma\delta\beta$	saddle	unstable	–	–	unstable	–	–	–	stable	–
$\alpha = \gamma - \gamma\delta\beta$	saddle	unstable	–	–	unstable	–	–	–	–	stable
$\gamma - \gamma\delta\beta < \alpha < \gamma$	saddle	stable	–	–	unstable	–	–	unstable	stable	–
$\alpha = \gamma$	saddle	stable	–	–	–	unstable	–	unstable	stable	–
$\gamma < \alpha < \alpha_2$	saddle	stable	–	unstable	unstable	–	–	unstable	stable	–
$\alpha = \alpha_2$	saddle	stable	–	unstable	unstable	–	unstable	–	–	–
$\alpha_2 < \alpha < \alpha_1$	saddle	stable	–	unstable	unstable	–	–	–	–	–
$\alpha = \alpha_1$	saddle	stable	unstable	–	–	–	–	–	–	–
$\alpha_1 < \alpha$	saddle	stable	–	–	–	–	–	–	–	–

(2) If $\alpha = \gamma - \gamma\delta\beta$ and $\gamma < 1 - \delta\beta$, system (1.3) only has a unique equilibrium $E_{33}(1 - \delta\beta - \gamma, \delta\beta)$, after simple calculation, we have $\lambda_1 = \frac{(\delta\beta + \gamma - 1)^2}{\delta\beta - 1}$. And as $\gamma < 1 - \delta\beta$, we have $\delta\beta - 1 < 0$, $(\delta\beta + \gamma - 1)^2 > 0$, that is to say, $\lambda_1 < 0$. Hence E_{33} is locally asymptotically stable.

(3) If $0 < \alpha < \gamma - \gamma\delta\beta$, as discussed in case (1), for the equilibrium $E_{33}(\frac{1 - \gamma + \sqrt{\triangle_1}}{2}, \delta\beta)$, there are $\frac{\alpha}{(\gamma + x_{32,33})^2} < 1$, then:
  (a) If $\gamma < 1 - \delta\beta$, we have $\lambda_1 < 0$, that is, E_{33} is locally asymptotically stable;
  (b) If $\gamma > 1 - \delta\beta$, we have $\lambda_1 > 0$, that is, E_{33} is unstable.

This ends the proof of Theorem 2.5. □

The existence and stability of equilibria of system (1.3) are summarized in Table 1, where $\gamma < 1 - \delta\beta$ and $\gamma < \alpha_2$.

3 Bifurcation analysis
This section tries to discuss variable bifurcations of system (1.3) and obtains the conditions for the saddle-node bifurcation and the transcritical bifurcation.

3.1 Saddle-node bifurcation
The conditions which can ensure the existence of boundary equilibria E_{22}, E_{23} are given in Sect. 2. After observing the change of E_{22}, E_{23}, we could find that these boundary equilibria are distinct if $\gamma < \alpha < \alpha_1$, and E_{22} coincides with E_{23} if $\alpha = \alpha_1$; finally they will all disappear if $\alpha > \alpha_1$. Hence, the appearance or annihilation of equilibria may be due to the occurrence of a saddle-node bifurcation in the boundary equilibrium E_{21} which exists when

$$\alpha = \alpha_{SN1} = \frac{(\gamma + 1)^2}{4}.$$

We will prove the existence of a saddle-node bifurcation as follows.

Theorem 3.1 *System* (1.3) *undergoes a saddle-node bifurcation when the system parameters satisfy the restriction $\alpha = \alpha_{SN1}$ along with the condition $\gamma < 1$ which is given in Theorem* 2.1. *Here, $\alpha \equiv \alpha_{SN1} = \frac{(\gamma + 1)^2}{4}$ and α is seen as the bifurcation parameter.*

Proof In this section, we use Sotomayor's theorem [35] to prove the occurrence of a saddle-node bifurcation with the transversality condition $\alpha = \alpha_{SN1}$. The Jacobian matrix

about the equilibrium E_{21} is given by

$$J(E_{21}) = \begin{pmatrix} 0 & -x_{21} \\ 0 & \delta \end{pmatrix}. \tag{3.1}$$

Obviously, $J(E_{21})$ has a zero eigenvalue, named λ_1.

Let V and W be two eigenvectors respectively corresponding to the eigenvalue λ_1 for the matrices $J(E_{21})$ and $J(E_{21})^T$. After simple calculation, we have

$$V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \qquad W = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1-\gamma}{2\delta} \end{pmatrix}.$$

Moreover,

$$F_\alpha(E_{21}; \alpha_{SN1}) = \begin{pmatrix} -\frac{x}{\gamma+x} \\ 0 \end{pmatrix}_{x=x_{21}} = \begin{pmatrix} -\frac{1-\gamma}{\gamma+1} \\ 0 \end{pmatrix},$$

$$D^2 F(E_{21}; \alpha_{SN1})(V, V) = \begin{pmatrix} \frac{\partial^2 F_1}{\partial x^2} V_1^2 + 2\frac{\partial^2 F_1}{\partial x \partial y} V_1 V_2 + \frac{\partial^2 F_1}{\partial y^2} V_2^2 \\ \frac{\partial^2 F_2}{\partial x^2} V_1^2 + 2\frac{\partial^2 F_2}{\partial x \partial y} V_1 V_2 + \frac{\partial^2 F_2}{\partial y^2} V_2^2 \end{pmatrix}_{(E_{21}; \alpha_{SN1})}$$

$$= \begin{pmatrix} -2 + \frac{4\gamma}{\gamma+1} \\ 0 \end{pmatrix}.$$

One could easily see that V and W satisfy

$$W^T F_\alpha(E_{21}; \alpha_{SN1}) = -\frac{1-\gamma}{\gamma+1} \neq 0,$$

$$W^T \big[D^2 F(E_{21}; \alpha_{SN1})(V, V) \big] = -2 + \frac{4\gamma}{\gamma+1} \neq 0,$$

which means that when $\alpha = \alpha_{SN1}$, the saddle-node bifurcation occurs at E_{21}.

The proof of Theorem 3.1 is finished. □

Similarly the conditions which can ensure the existence of positive equilibria E_{32}, E_{33} are given in Sect. 2, and we could find that these positive equilibria are distinct if $\gamma - \gamma\delta\beta < \alpha < \alpha_2$, and E_{32} coincides with E_{33} if $\alpha = \alpha_2$; finally they will all disappear if $\alpha > \alpha_2$. Hence, the appearance or annihilation of equilibria may be due to the occurrence of a saddle-node bifurcation at the positive equilibrium E_{31} which exists when

$$\alpha = \alpha_{SN2} = \frac{(\delta\beta - \gamma - 1)^2}{4}.$$

We will prove the occurrence of a saddle-node bifurcation at the positive equilibrium E_{31} as follows.

Theorem 3.2 *System* (1.3) *undergoes a saddle-node bifurcation when the system parameters satisfy the restriction $\alpha = \alpha_{SN1}$ along with the condition $\gamma < 1$ which is given in Theorem* 2.1. *Here, $\alpha \equiv \alpha_{SN1} = \frac{(\gamma+1)^2}{4}$ and α is seen as the bifurcation parameter.*

Proof We also use Sotomayor's theorem [35] to prove the occurrence of a saddle-node bi-furcation with the transversality condition $\alpha = \alpha_{SN2}$. The Jacobian matrix about the equilibrium E_{31} is given by

$$J(E_{31}) = \begin{pmatrix} 0 & -x_{31} \\ 0 & \delta \end{pmatrix}. \tag{3.2}$$

Obviously, $J(E_{31})$ has a zero eigenvalue, named λ_1.

Let V and W be two eigenvectors respectively corresponding to the eigenvalue λ_1 for the matrices $J(E_{31})$ and $J(E_{31})^T$. After simple calculation, we have

$$V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \qquad W = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{1-\delta\beta-\gamma}{2\delta} \end{pmatrix}.$$

Moreover,

$$F_\alpha(E_{31}; \alpha_{SN2}) = \begin{pmatrix} -\frac{x}{\gamma+x} \\ 0 \end{pmatrix}_{x=x_{31}} = \begin{pmatrix} -\frac{1-\delta\beta-\gamma}{\gamma+1-\delta\beta} \\ 0 \end{pmatrix},$$

$$D^2 F(E_{31}; \alpha_{SN2})(V, V) = \begin{pmatrix} \frac{\partial^2 F_1}{\partial x^2} V_1^2 + 2\frac{\partial^2 F_1}{\partial x \partial y} V_1 V_2 + \frac{\partial^2 F_1}{\partial Y^2} V_2^2 \\ \frac{\partial^2 F_2}{\partial x^2} V_1^2 + 2\frac{\partial^2 F_2}{\partial x \partial y} V_1 V_2 + \frac{\partial^2 F_2}{\partial Y^2} V_2^2 \end{pmatrix}_{(E_{31}; \alpha_{SN2})}$$

$$= \begin{pmatrix} -2 + \frac{4\gamma}{\gamma+1-\delta\beta} \\ 0 \end{pmatrix}.$$

One could easily see that V and W satisfy

$$W^T F_\alpha(E_{31}; \alpha_{SN2}) = -\frac{1-\delta\beta-\gamma}{\gamma+1-\delta\beta} \neq 0,$$

$$W^T[D^2 F(E_{31}; \alpha_{SN2})(V, V)] = -2 + \frac{4\gamma}{\gamma+1-\delta\beta} \neq 0,$$

which means that when $\alpha = \alpha_{SN2}$, the saddle-node bifurcation occurs at E_{31}.

The proof of Theorem 3.2 is finished. □

For $\beta = 0.5$, $\gamma = 0.2$, $\delta = 0.4$, we get $\alpha_{SN1} = \alpha_1 = 0.36$. For $0.2 = \gamma < \alpha < \alpha_{SN1}$, system (1.3) has two distinct boundary equilibria E_{22}, E_{23} (see Fig. 1(e), (f), (g)) which coincide with each other for $\alpha = \alpha_{SN1}$ (see Fig. 1(h)) and no boundary equilibrium in the x-axis for $\alpha > \alpha_{SN1}$ (see Fig. 1(i)).

Meanwhile, we get $\alpha_{SN2} = \alpha_2 = 0.25$. For $0.16 = \gamma - \gamma\delta\beta < \alpha < \alpha_{SN2}$, system (1.3) has two distinct interior equilibria E_{22}, E_{23} (see Fig. 1(c), (d), (e)) which coincide with each other for $\alpha = \alpha_{SN1}$ (see Fig. 1(f)) and no interior equilibrium for $\alpha > \alpha_{SN1}$ (see Fig. 1(g), (h), (i)).

3.2 Transcritical bifurcation

In Sect. 2, it was easy for us to notice an interesting phenomenon: when $\alpha = \gamma$, E_{22} will coincide with E_0 if $\gamma < 1$ and E_{23} will coincide with E_0 if $\gamma > 1$. Hence, the appearance of the phenomenon may be due to the occurrence of a transcritical bifurcation in the boundary equilibrium E_0. Then we have the following.

Figure 1 (**a**) Dynamic behaviors of system (1.3), the parameters $(\alpha, \beta, \gamma, \delta) = (0.1, 0.5, 0.2, 0.4)$. (**b**) Dynamic behaviors of system (1.3), the parameters $(\alpha, \beta, \gamma, \delta) = (0.16, 0.5, 0.2, 0.4)$. (**c**) Dynamic behaviors of system (1.3), the parameters $(\alpha, \beta, \gamma, \delta) = (0.18, 0.5, 0.2, 0.4)$. (**d**) Dynamic behaviors of system (1.3), the parameters $(\alpha, \beta, \gamma, \delta) = (0.2, 0.5, 0.2, 0.4)$. (**e**) Dynamic behaviors of system (1.3), the parameters $(\alpha, \beta, \gamma, \delta) = (0.22, 0.5, 0.2, 0.4)$. (**f**) Dynamic behaviors of system (1.3), the parameters $(\alpha, \beta, \gamma, \delta) = (0.25, 0.5, 0.2, 0.4)$. (**g**) Dynamic behaviors of system (1.3), the parameters $(\alpha, \beta, \gamma, \delta) = (0.3, 0.5, 0.2, 0.4)$. (**h**) Dynamic behaviors of system (1.3), the parameters $(\alpha, \beta, \gamma, \delta) = (0.36, 0.5, 0.2, 0.4)$. (**i**) Dynamic behaviors of system (1.3), the parameters $(\alpha, \beta, \gamma, \delta) = (0.5, 0.5, 0.2, 0.4)$

Theorem 3.3 *System* (1.3) *undergoes a transcritical bifurcation when the system parameters satisfy the restriction* $\alpha = \alpha_{TC1} = \gamma$. *Here,* α *is seen as the bifurcation parameter.*

Proof In this section, we also use Sotomayor's theorem [35] to verify the occurrence of a transcritical bifurcation with the transversality condition $\alpha = \alpha_{TC1}$. The Jacobian matrix

Figure 1 Continued
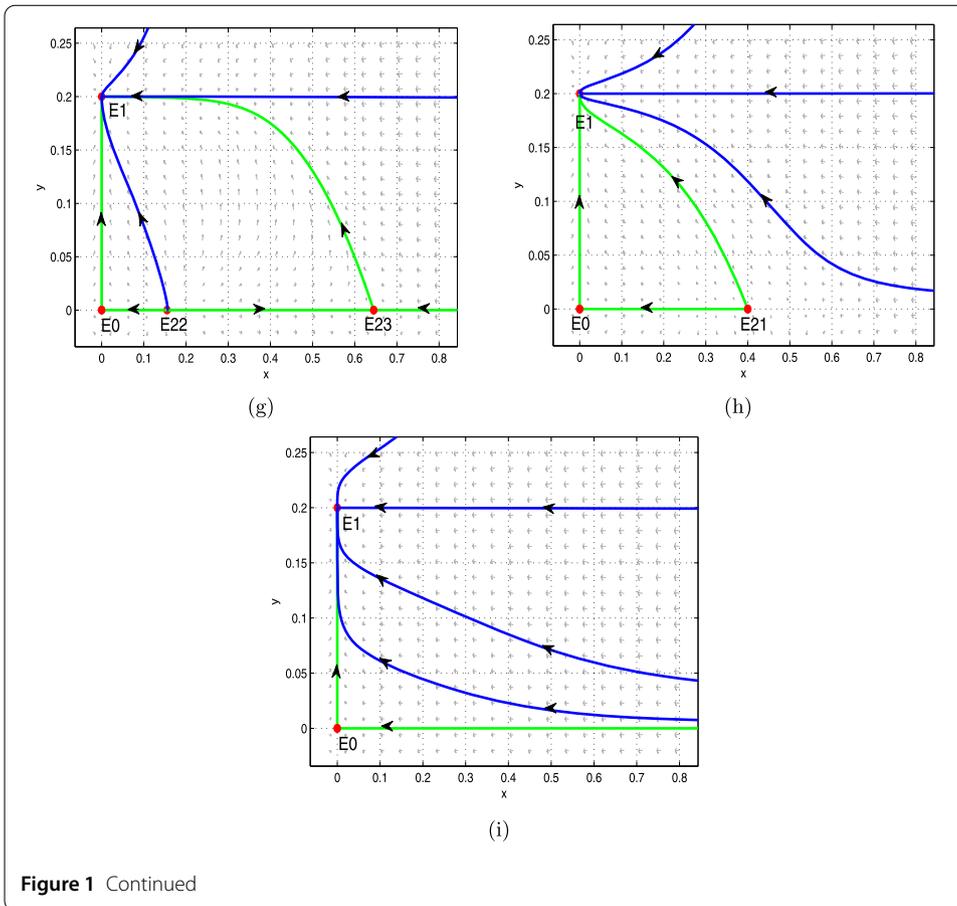
about the equilibrium E_0 is given by

$$J(E_0) = \begin{pmatrix} 0 & 0 \\ 0 & -\delta \end{pmatrix}. \tag{3.3}$$

Obviously, $J(E_0)$ has a zero eigenvalue, named λ_1.

Let V and W be two eigenvectors respectively corresponding to the eigenvalue λ_1 for the matrices $J(E_0)$ and $J(E_0)^T$. After simple calculation, we have

$$V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \qquad W = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Moreover,

$$F_\alpha(E_0; \alpha_{TC1}) = \begin{pmatrix} -\frac{x}{\gamma+x} \\ 0 \end{pmatrix}_{x=0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$DF_\alpha(E_0; \alpha_{TC1})V = \begin{pmatrix} -\frac{\gamma}{(\gamma+x)^2} & 0 \\ 0 & 0 \end{pmatrix}_{(E_0; \alpha_{TC1})} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\gamma} \\ 0 \end{pmatrix},$$

$$D^2 F(E_0; \alpha_{TC1})(V, V) = \begin{pmatrix} \frac{\partial^2 F_1}{\partial x^2} V_1^2 + 2\frac{\partial^2 F_1}{\partial x \partial y} V_1 V_2 + \frac{\partial^2 F_1}{\partial Y^2} V_2^2 \\ \frac{\partial^2 F_2}{\partial x^2} V_1^2 + 2\frac{\partial^2 F_2}{\partial x \partial y} V_1 V_2 + \frac{\partial^2 F_2}{\partial Y^2} V_2^2 \end{pmatrix}_{(E_0; \alpha_{TC1})}$$

$$= \begin{pmatrix} -2 + \frac{2}{\gamma} \\ 0 \end{pmatrix}.$$

Therefore V and W satisfy

$$W^T F_\alpha(E_0; \alpha_{TC1}) = 0,$$

$$W^T [DF_\alpha(E_0; \alpha_{TC1}) V] = -\frac{1}{\gamma} \neq 0,$$

$$W[D^2 F(E_0; \alpha_{TC1})(V, V)] = -2 + \frac{2}{\gamma} \neq 0,$$

which means that when $\alpha = \alpha_{TC1}$, the transcritical bifurcation occurs at E_0.

The proof of Theorem 3.3 is finished. □

This interesting phenomenon also takes place in the boundary equilibrium E_1: when $\alpha = \gamma - \gamma\delta\beta$, then E_{32} will coincide with E_1 if $\gamma < 1 - \delta\beta$ and E_{33} will coincide with E_1 if $\gamma > 1 - \delta\beta$. Hence, the appearance of the phenomenon may be due to the occurrence of a transcritical bifurcation in the boundary equilibrium E_1. Then we have the following.

Theorem 3.4 *System* (1.3) *undergoes a transcritical bifurcation when the system parameters satisfy the restriction* $\alpha = \alpha_{TC2} = \gamma - \gamma\delta\beta$. *Here,* α *is seen as the bifurcation parameter.*

Proof We also use Sotomayor's theorem [35] to prove the occurrence of a transcritical bifurcation with the transversality condition $\alpha = \alpha_{TC2}$. The Jacobian matrix about the equilibrium E_1 is given by

$$J(E_1) = \begin{pmatrix} 0 & 0 \\ 0 & -\delta \end{pmatrix}. \tag{3.4}$$

Obviously, $J(E_1)$ has a zero eigenvalue, named λ_1.

Let V and W be two eigenvectors respectively corresponding to the eigenvalue λ_1 for the matrices $J(E_1)$ and $J(E_1)^T$. After simple calculation, we have

$$V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \qquad W = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Moreover,

$$F_\alpha(E_1; \alpha_{TC2}) = \begin{pmatrix} -\frac{x}{\gamma + x} \\ 0 \end{pmatrix}_{(E_1; \alpha_{TC2})} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$DF_\alpha(E_1; \alpha_{TC2})V = \begin{pmatrix} -\frac{\gamma}{(\gamma + x)^2} & 0 \\ 0 & 0 \end{pmatrix}_{(E_1; \alpha_{TC2})} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\gamma} \\ 0 \end{pmatrix},$$

$$D^2F(E_1;\alpha_{TC2})(V,V) = \begin{pmatrix} \frac{\partial^2 F_1}{\partial x^2}V_1^2 + 2\frac{\partial^2 F_1}{\partial x \partial y}V_1 V_2 + \frac{\partial^2 F_1}{\partial Y^2}V_2^2 \\ \frac{\partial^2 F_2}{\partial x^2}V_1^2 + 2\frac{\partial^2 F_2}{\partial x \partial y}V_1 V_2 + \frac{\partial^2 F_2}{\partial Y^2}V_2^2 \end{pmatrix}_{(E_1;\alpha_{TC2})}$$

$$= \begin{pmatrix} -2 + \frac{2(1-\delta\beta)}{\gamma} \\ 0 \end{pmatrix}.$$

Therefore V and W satisfy

$$W^T F_\alpha(E_1;\alpha_{TC2}) = 0,$$

$$W^T\big[DF_\alpha(E_1;\alpha_{TC2})V\big] = -\frac{1}{\gamma} \neq 0,$$

$$W\big[D^2F(E_1;\alpha_{TC2})(V,V)\big] = -2 + \frac{2(1-\delta\beta)}{\gamma} \neq 0,$$

which means that when $\alpha = \alpha_{TC2}$, the transcritical bifurcation also occurs at E_1.

The proof of Theorem 3.4 is finished. □

4 Numeric simulations

Example 4.1 Consider the following system:

$$\frac{dx}{dt} = x(1-x) - xy - \frac{\alpha x}{0.2+x},$$
$$\frac{dy}{dt} = y\left(0.4 - \frac{y}{0.5}\right). \tag{4.1}$$

In this system, corresponding to system (1.3), we take $\beta = 0.5$, $\gamma = 0.2$, $\delta = 0.4$. Then we have $0.2 = \gamma < 1 - \delta\beta = 0.8$, $\gamma - \gamma\delta\beta = 0.16$, $\alpha_1 = \frac{(\gamma+1)^2}{4} = 0.36$, and $\alpha_2 = \frac{(\delta\beta-\gamma-1)^2}{4} = 0.25$. From Theorem 2.1, system (4.1) has two boundary equilibria $E_0(0,0)$, $E_1(0,0.2)$ for all positive parameters, and $E_0(0,0)$ is unstable from Theorem 2.2(1).

(1) Take $\alpha = 0.1$, then $0 < \alpha < \gamma - \gamma\delta\beta$, from Theorem 2.1(i)(e) and (ii)(e), system (4.1) has another boundary equilibrium $E_{23}(0.9099,0)$ and a unique positive equilibrium $E_{33}(0.6873,0.2)$. Furthermore, from Theorem 2.2(2)(i), Theorem 2.3, and Theorem 2.5(3), we have E_1 is a saddle, E_{23} is unstable, and E_{33} is locally asymptotically stable, see Fig. 1(a);

(2) Take $\alpha = 0.16$, then $\alpha = \gamma - \gamma\delta\beta < \gamma$, $\alpha \neq \gamma^2$, from Theorem 2.1(i)(e) and (ii)(d), system (4.1) has another boundary equilibrium $E_{23}(0.8472,0)$ and a unique positive equilibrium $E_{33*}(0.6,0.2)$. Furthermore, from Theorem 2.2(2)(iii), Theorem 2.3, and Theorem 2.5(2), we have E_1 is a saddle node, E_{23} is unstable, and E_{33*} is locally asymptotically stable, see Fig. 1(b);

(3) Take $\alpha = 0.18$, then $\gamma - \gamma\delta\beta < \alpha < \gamma$, from Theorem 2.1(i)(e) and (ii)(c), system (4.1) has another boundary equilibrium $E_{23}(0.8243,0)$ and two distinct positive equilibria $E_{32}(0.0354,0.2)$, $E_{33}(0.5646,0.2)$. Furthermore, from Theorem 2.2(2)(ii), Theorem 2.3, and Theorem 2.5(1), we have E_1 is a stable node, E_{23} is unstable, E_{32} is unstable, and E_{33} is locally asymptotically stable, see Fig. 1(c);

(4) Take $\alpha = 0.2$, then $\alpha = \gamma$, from Theorem 2.1(i)(d) and (ii)(c), system (4.1) has another boundary equilibrium $E_{23*}(0.8,0)$ and two distinct positive equilibria $E_{32}(0.0764,0.2)$, $E_{33}(0.5236,0.2)$. Furthermore, from Theorem 2.2(2)(ii),

Theorem 2.3, and Theorem 2.5(1), we have E_1 is a stable node, E_{23} is unstable, E_{32} is unstable, and E_{33} is locally asymptotically stable, see Fig. 1(d);

(5) Take $\alpha = 0.22$, then $\gamma < \alpha < \alpha_2$, from Theorem 2.1(i)(c) and (ii)(c), system (4.1) has other two boundary equilibria $E_{22}(0.0258, 0)$ $E_{23}(0.7741, 0)$ and two distinct positive equilibria $E_{32}(0.1268, 0.2)$, $E_{33}(0.4732, 0.2)$. Furthermore, from Theorem 2.2(2)(ii), Theorem 2.3, and Theorem 2.5(1), we have E_1 is a stable node, E_{22}, E_{23} is unstable, E_{32} is unstable, and E_{33} is locally asymptotically stable, see Fig. 1(e);

(6) Take $\alpha = 0.25$, then $\alpha = \alpha_2$, from Theorem 2.1(i)(c) and (ii)(b), system (4.1) has other two boundary equilibria $E_{22}(0.0683, 0)$ $E_{23}(0.7317, 0)$ and a unique positive equilibrium $E_{31}(0.3, 0.2)$. Furthermore, from Theorem 2.2(2)(ii), Theorem 2.3, and Theorem 2.4, we have E_1 is a stable node, E_{22}, E_{23} is unstable and E_{31} is a saddle node, see Fig. 1(f);

(7) Take $\alpha = 0.3$, then $\alpha_2 < \alpha < \alpha_1$, from Theorem 2.1(i)(c) and (ii)(a), system (4.1) has other two boundary equilibria $E_{22}(0.1551, 0)$ $E_{23}(0.6450, 0)$ and none positive equilibria. Furthermore, from Theorem 2.2(2)(ii), Theorem 2.3, we have E_1 is a stable node and E_{22}, E_{23} is unstable, see Fig. 1(g);

(8) Take $\alpha = 0.36$, then $\alpha = \alpha_1$, from Theorem 2.1(i)(b) and (ii)(a), system (4.1) has another boundary equilibrium $E_{21}(0.4, 0)$ and none positive equilibria. Furthermore, from Theorem 2.2(2)(ii), Theorem 2.3, we have E_1 is a stable node and E_{21} is unstable, see Fig. 1(h);

(9) Take $\alpha = 0.5$, then $\alpha_1 < \alpha$, from Theorem 2.1(i)(a) and (ii)(a), system (4.1) only has two boundary equilibria $E_0(0, 0)$, $E_1(0, 0.2)$. Furthermore, from Theorem 2.2 (2) (ii), we have E_1 is a stable node, see Fig. 1(i);

5 Conclusion

An amensalism model with Michaelis–Menten type harvesting and a cover for the first species is proposed and studied in this paper. Already Xie et al. [18] investigated the local and global stability property of the possible equilibria of the above model, where the harvesting term $h = 0$. By contrast, we can find some interesting phenomena about the dynamic behavior of system (1.3):

(a) At most, there are six equilibria for the system including two distinct interior points. While in [18] there just four equilibria and a unique interior point;

(b) With the increase in the number of equilibria, dynamic behavior of the system becomes more complex, such as the appearance of bifurcation;

(c) In [18], once the positive equilibrium exists, it is globally stable. While in system (1.3), if $\alpha = \alpha_2$, then there is a unique positive equilibrium E_{31}; from Theorem 2.4, we have E_{31} is a saddle node (see Fig. 1(h)).

That is, by introducing the harvesting, the dynamic behaviors of the system become complicated.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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