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Sums of finite products of Chebyshev polynomials of the third and fourth kinds

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Abstract

In this paper, we study sums of finite products of Chebyshev polynomials of the third and fourth kinds and obtain Fourier series expansions of functions associated with them. Then from these Fourier series expansions we will be able to express those sums of finite products as linear combinations of Bernoulli polynomials and to have some identities from those expressions.

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1 Introduction and preliminaries

The Chebyshev polynomials $T_n(x)$, $U_n(x)$, $V_n(x)$, and $W_n(x)$ of the first, second, third, and fourth kinds are respectively defined by the recurrence relations as follows (see [2, 6, 7, 12, 14]):

$$T_{n+2}(x) = 2xT_{n+1}(x) - T_n(x) \quad (n \geq 0), \quad T_0(x) = 1, \quad T_1(x) = x, \quad (1.1)$$

$$U_{n+2}(x) = 2xU_{n+1}(x) - U_n(x) \quad (n \geq 0), \quad U_0(x) = 1, \quad U_1(x) = 2x, \quad (1.2)$$

$$V_{n+2}(x) = 2xV_{n+1}(x) - V_n(x) \quad (n \geq 0), \quad V_0(x) = 1, \quad V_1(x) = 2x - 1, \quad (1.3)$$

$$W_{n+2}(x) = 2xW_{n+1}(x) - W_n(x) \quad (n \geq 0), \quad W_0(x) = 1, \quad W_1(x) = 2x + 1. \quad (1.4)$$

It can be easily seen from (1.1), (1.2), (1.3), and (1.4) that the generating functions for $T_n(x)$, $U_n(x)$, $V_n(x)$, and $W_n(x)$ are respectively given by (see [6, 7, 12])

$$\frac{1 - xt}{1 - 2xt + t^2} = \sum_{n=0}^{\infty} T_n(x)t^n, \quad (1.5)$$

$$\frac{1}{1 - 2xt + t^2} = \sum_{n=0}^{\infty} U_n(x)t^n, \quad (1.6)$$

$$F(t, x) = \frac{1 - t}{1 - 2xt + t^2} = \sum_{n=0}^{\infty} V_n(x)t^n, \quad (1.7)$$

$$G(t, x) = \frac{1 + t}{1 - 2xt + t^2} = \sum_{n=0}^{\infty} W_n(x)t^n. \quad (1.8)$$

The Bernoulli polynomials $B_m(x)$ are given by the generating function

$$\frac{t}{e^t - 1} e^{xt} = \sum_{m=0}^{\infty} B_m(x) \frac{t^m}{m!}. \tag{1.9}$$

For any real number x , we let

$$\langle x \rangle = x - [x] \in [0, 1) \tag{1.10}$$

denote the fractional part of x , where $[x]$ indicates the greatest integer $\leq x$.

We also recall here that

(a) for $m \geq 2$,

$$B_m(\langle x \rangle) = -m! \sum_{n=-\infty, n \neq 0}^{\infty} \frac{e^{2\pi i n x}}{(2\pi i n)^m}; \tag{1.11}$$

(b) for $m = 1$,

$$- \sum_{n=-\infty, n \neq 0}^{\infty} \frac{e^{2\pi i n x}}{2\pi i n} = \begin{cases} B_1(\langle x \rangle), & \text{for } x \in \mathbb{R} - \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases} \tag{1.12}$$

For any integers m, r with $m, r \geq 1$, we set

$$\alpha_{m,r}(x) = \sum_{l=0}^m \sum_{c_1+c_2+\dots+c_{r+1}=l} \binom{r-1+m-l}{r-1} V_{c_1}(x) \cdots V_{c_{r+1}}(x), \tag{1.13}$$

where the inner sum runs over all nonnegative integers c_1, c_2, \dots, c_{r+1} with $c_1 + c_2 + \dots + c_{r+1} = l$.

Then we will consider the function $\alpha_{m,r}(\langle x \rangle)$ and derive its Fourier series expansions. As an immediate corollary to these Fourier series expansions, we will be able to express $\alpha_{m,r}(x)$ as a linear combination of Bernoulli polynomials $B_m(x)$. We state our result here as Theorem 1.1.

Theorem 1.1 *For any integers m, r with $m, r \geq 1$, we let*

$$\Delta_{m,r} = \frac{1}{r!} \sum_{k=1}^m (-1)^{k+1} \binom{m+2r+k}{2r+2k} (k+r)_r 2^k.$$

Then we have the identity

$$\begin{aligned} & \sum_{l=0}^m \sum_{c_1+c_2+\dots+c_{r+1}=l} \binom{r-1+m-l}{r-1} V_{c_1}(x) \cdots V_{c_{r+1}}(x) \\ &= \frac{1}{2^r} \sum_{j=0}^m 2^j \binom{r+j-1}{r-1} \Delta_{m-j+1, r+j-1} B_j(x). \end{aligned} \tag{1.14}$$

Here $(x)_r = x(x-1) \cdots (x-r+1)$, for $r \geq 1$, and $(x)_0 = 1$.

Also, for any integers m, r with $m, r \geq 1$, we put

$$\beta_{m,r}(x) = \sum_{l=0}^m \sum_{c_1+c_2+\dots+c_{r+1}=l} (-1)^{m-l} \binom{r-1+m-l}{r-1} W_{c_1}(x) \cdots W_{c_{r+1}}(x), \tag{1.15}$$

where the inner sum is over all nonnegative integers c_1, c_2, \dots, c_{r+1} with $c_1 + c_2 + \dots + c_{r+1} = l$.

Then we will consider the function $\beta_{m,r}(\langle x \rangle)$ and derive its Fourier series expansions. Again, as a corollary to these, we can express $\beta_{m,r}(x)$ in terms of Bernoulli polynomials. Indeed, our result here is as follows.

Theorem 1.2 *For any integers m, r with $m, r \geq 1$, we let*

$$\Omega_{m,r} = \frac{2m + 2r + 1}{r!} \sum_{k=1}^m (-1)^{k+1} \frac{2^k}{2k + 2r + 1} \binom{m + 2r + k}{2k + 2r} (k + r)_r.$$

Then we have the identity

$$\begin{aligned} & \sum_{l=0}^m \sum_{c_1+c_2+\dots+c_{r+1}=l} (-1)^{m-l} \binom{r-1+m-l}{r-1} W_{c_1}(x) \cdots W_{c_{r+1}}(x) \\ &= \frac{1}{2r} \sum_{j=0}^m 2^j \binom{r+j-1}{r-1} \Omega_{m-j+1,r+j-1} B_j(x). \end{aligned} \tag{1.16}$$

Here we cannot go without saying that neither $V_n(x)$ nor $W_n(x)$ is an Appell polynomial, whereas all our related results have been only about Appell polynomials (see [1, 8–11]).

We mentioned in the above that, using the Fourier series expansions of $\alpha_{m,r}(\langle x \rangle)$ and $\beta_{m,r}(\langle x \rangle)$, we can express $\alpha_{m,r}(x)$ and $\beta_{m,r}(x)$ as linear combinations of Bernoulli polynomials as stated in Theorem 1.1 and Theorem 1.2.

In addition, we will express $\alpha_{m,r}(x)$ and $\beta_{m,r}(x)$ as linear combinations of Euler polynomials by using a simple formula (see (4.1), (4.3)).

Finally, as an application of our results, we will derive some interesting identities from Theorem 1.1 and Theorem 1.2 together with some well-known identities connecting the four kinds of Chebyshev polynomials (see (5.1)–(5.3)).

It was mentioned in [11] that studying these kinds of sums of finite products of special polynomials can be well justified by the following example. Let us put

$$\gamma_m(x) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)} B_k(x) B_{m-k}(x) \quad (m \geq 2). \tag{1.17}$$

Then just as in [14] and [16] we can express $\gamma_m(x)$ in terms of Bernoulli polynomials from the Fourier series expansions of $\gamma_m(\langle x \rangle)$. Further, some simple modification of this gives the famous Faber–Pandharipande–Zagier identity (see [4]) and a slightly different variant of Miki’s identity (see [3, 5, 13, 15]).

Our methods here are simple, whereas others are quite involved. Indeed, Dunne and Schubert in [3] use the asymptotic expansions of some special polynomials coming from quantum field theory computations, the work of Gessel in [5] is based on two different

expressions for the Stirling numbers of the second kind, Miki in [13] utilizes a formula for the Fermat quotient $\frac{a^p - a}{p}$ modulo p^2 , and Shiratani and Yokoyama in [15] employ p -adic analysis.

The reader may refer to the papers [1, 8–11] for some related recent results.

2 Fourier series expansions for functions associated with the Chebyshev polynomials of the third kind

The following lemma, which expresses the sums of products in (1.13) neatly, will play a crucial role in this section. The corresponding ones for Chebyshev polynomials of the second kind and of Fibonacci polynomials are respectively stated in [16] and [17].

Lemma 2.1 *Let n, r be integers with $n \geq 0, r \geq 1$. Then we have the identity*

$$\sum_{l=0}^n \sum_{c_1+c_2+\dots+c_{r+1}=l} \binom{r-1+n-l}{r-1} V_{c_1}(x) \cdots V_{c_{r+1}}(x) = \frac{1}{2^r r!} V_{n+r}^{(r)}(x),$$

where the inner sum runs over all nonnegative integers c_1, c_2, \dots, c_{r+1} with $c_1 + c_2 + \dots + c_{r+1} = l$.

Proof By differentiating (1.7) r times, we have

$$\frac{\partial^r F(t, x)}{\partial x^r} = \frac{(2t)^r r! (1-t)}{(1-2xt+t^2)^{r+1}}, \tag{2.1}$$

$$\frac{\partial^r F(t, x)}{\partial x^r} = \sum_{n=r}^{\infty} V_n^{(r)}(x) t^n = \sum_{n=0}^{\infty} V_{n+r}^{(r)}(x) t^{n+r}. \tag{2.2}$$

Equating (2.1) and (2.2) gives

$$\frac{2^r r! (1-t)}{(1-2xt+t^2)^{r+1}} = \sum_{n=0}^{\infty} V_{n+r}^{(r)}(x) t^n. \tag{2.3}$$

On the other hand, using (1.7) and (2.3) we note that

$$\begin{aligned} & \sum_{n=0}^{\infty} \left(\sum_{c_1+c_2+\dots+c_{r+1}=n} V_{c_1}(x) \cdots V_{c_{r+1}}(x) \right) t^n \\ &= \left(\sum_{n=0}^{\infty} V_n(x) t^n \right)^{r+1} \\ &= \frac{(1-t)^{r+1}}{(1-2xt+t^2)^{r+1}} \\ &= \frac{1}{2^r r!} (1-t)^r \sum_{n=0}^{\infty} V_{n+r}^{(r)}(x) t^n. \end{aligned} \tag{2.4}$$

From (2.4), we obtain

$$\begin{aligned}
 & \frac{1}{2^r r!} \sum_{n=0}^{\infty} V_{n+r}^{(r)}(x) t^n \\
 &= (1-t)^{-r} \sum_{m=0}^{\infty} \left(\sum_{c_1+c_2+\dots+c_{r+1}=m} V_{c_1}(x) \cdots V_{c_{r+1}}(x) \right) t^m \\
 &= \sum_{m=0}^{\infty} \binom{r+m-1}{r-1} t^m \sum_{l=0}^{\infty} \left(\sum_{c_1+c_2+\dots+c_{r+1}=l} V_{c_1}(x) \cdots V_{c_{r+1}}(x) \right) t^l \\
 &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^n \sum_{c_1+c_2+\dots+c_{r+1}=l} \binom{r+n-l-1}{r-1} V_{c_1}(x) \cdots V_{c_{r+1}}(x) \right) t^n, \tag{2.5}
 \end{aligned}$$

from which our result follows. □

It is well known that the Chebyshev polynomials of the third kind $V_n(x)$ are given by (see [6])

$$\begin{aligned}
 V_n(x) &= {}_2F_1\left(-n, n+1; \frac{1}{2}; \frac{1-x}{2}\right) \\
 &= \sum_{k=0}^n \binom{n+k}{2k} 2^k (x-1)^k, \tag{2.6}
 \end{aligned}$$

where ${}_2F_1(a, b; c; z)$ is the hypergeometric function.

The r th derivative of (2.6) is given by

$$V_n^{(r)}(x) = \sum_{k=r}^n \binom{n+k}{2k} 2^k (k)_r (x-1)^{k-r} \quad (0 \leq r \leq n). \tag{2.7}$$

Combining (2.1) and (2.7), we obtain the following lemma.

Lemma 2.2 *For integers n, r , with $n \geq 0, r \geq 1$, we have the following identity:*

$$\begin{aligned}
 & \sum_{l=0}^n \sum_{c_1+c_2+\dots+c_{r+1}=l} \binom{r-1+n-l}{r-1} V_{c_1}(x) \cdots V_{c_{r+1}}(x) \\
 &= \frac{1}{r!} \sum_{k=0}^n \binom{n+2r+k}{2r+2k} 2^k (k+r)_r (x-1)^k. \tag{2.8}
 \end{aligned}$$

For integers m, r with $m, r \geq 1$, as in (1.13), we let

$$\alpha_{m,r}(x) = \sum_{l=0}^m \sum_{c_1+c_2+\dots+c_{r+1}=l} \binom{r-1+m-l}{r-1} V_{c_1}(x) \cdots V_{c_{r+1}}(x). \tag{2.9}$$

Then we will consider the function

$$\alpha_{m,r}(\langle x \rangle) = \sum_{l=0}^m \sum_{c_1+c_2+\dots+c_{r+1}=l} \binom{r-1+m-l}{r-1} V_{c_1}(\langle x \rangle) \cdots V_{c_{r+1}}(\langle x \rangle), \tag{2.10}$$

defined on \mathbb{R} , which is periodic with period 1.

The Fourier series of $\alpha_{m,r}(\cdot)$ is

$$\sum_{n=-\infty}^{\infty} A_n^{(m,r)} e^{2\pi i n x}, \tag{2.11}$$

where

$$A_n^{(m,r)} = \int_0^1 \alpha_{m,r}(\cdot) e^{-2\pi i n x} dx = \int_0^1 \alpha_{m,r}(x) e^{-2\pi i n x} dx. \tag{2.12}$$

For $m, r \geq 1$, we let

$$\begin{aligned} \Delta_{m,r} &= \alpha_{m,r}(1) - \alpha_{m,r}(0) \\ &= \sum_{l=0}^m \sum_{c_1+c_2+\dots+c_{r+1}=l} \binom{r-1+m-l}{r-1} \\ &\quad \times (V_{c_1}(1) \cdots V_{c_{r+1}}(1) - V_{c_1}(0) \cdots V_{c_{r+1}}(0)). \end{aligned} \tag{2.13}$$

Then, from (2.8) and (2.13), we see that

$$\Delta_{m,r} = \frac{1}{r!} \sum_{k=1}^m (-1)^{k+1} \binom{m+2r+k}{2r+2k} 2^k (k+r)_r, \tag{2.14}$$

where we observe that

$$\alpha_{m,r}(1) = \binom{m+2r}{2r}. \tag{2.15}$$

Now, from (2.1), we note the following:

$$\begin{aligned} \frac{d}{dx} \alpha_{m,r}(x) &= \frac{d}{dx} \left(\frac{1}{2^r r!} V_{m+r}^{(r)}(x) \right) \\ &= \frac{1}{2^r r!} V_{m+r}^{(r+1)}(x) \\ &= 2(r+1) \alpha_{m-1,r+1}(x). \end{aligned} \tag{2.16}$$

Thus we have shown that

$$\frac{d}{dx} \alpha_{m,r}(x) = 2(r+1) \alpha_{m-1,r+1}(x). \tag{2.17}$$

Replacing m by $m+1$, r by $r-1$, from (2.17) we obtain

$$\frac{d}{dx} \left(\frac{\alpha_{m+1,r-1}(x)}{2r} \right) = \alpha_{m,r}(x), \tag{2.18}$$

$$\int_0^1 \alpha_{m,r}(x) dx = \frac{1}{2r} \Delta_{m+1,r-1}, \tag{2.19}$$

$$\alpha_{m,r}(0) = \alpha_{m,r}(1) \iff \Delta_{m,r} = 0. \tag{2.20}$$

We are now going to determine the Fourier coefficients $A_n^{(m,r)}$.

Case 1: $n \neq 0$.

$$\begin{aligned}
 A_n^{(m,r)} &= \int_0^1 \alpha_{m,r}(x)e^{-2\pi inx} dx \\
 &= -\frac{1}{2\pi in} [\alpha_{m,r}(x)e^{-2\pi inx}]_0^1 + \frac{1}{2\pi in} \int_0^1 \left(\frac{d}{dx}\alpha_{m,r}(x)\right)e^{-2\pi inx} dx \\
 &= -\frac{1}{2\pi in} (\alpha_{m,r}(1) - \alpha_{m,r}(0)) + \frac{2(r+1)}{2\pi in} \int_0^1 \alpha_{m-1,r+1}(x)e^{-2\pi inx} dx \\
 &= \frac{2(r+1)}{2\pi in} A_n^{(m-1,r+1)} - \frac{1}{2\pi in} \Delta_{m,r} \\
 &= \frac{2(r+1)}{2\pi in} \left(\frac{2(r+2)}{2\pi in} A_n^{(m-2,r+2)} - \frac{1}{2\pi in} \Delta_{m-1,r+1}\right) - \frac{1}{2\pi in} \Delta_{m,r} \\
 &= \frac{2^2(r+2)_2}{(2\pi in)^2} A_n^{(m-2,r+2)} - \sum_{j=1}^2 \frac{2^{j-1}(r+j-1)_{j-1}}{(2\pi in)^j} \Delta_{m-j+1,r+j-1} \\
 &= \dots \\
 &= \frac{2^m(r+m)_m}{(2\pi in)^m} A_n^{(0,r+m)} - \sum_{j=1}^m \frac{2^{j-1}(r+j-1)_{j-1}}{(2\pi in)^j} \Delta_{m-j+1,r+j-1} \\
 &= -\sum_{j=1}^m \frac{2^{j-1}(r+j-1)_{j-1}}{(2\pi in)^j} \Delta_{m-j+1,r+j-1} \\
 &= -\frac{1}{2r} \sum_{j=1}^m \frac{2^j(r+j-1)_j}{(2\pi in)^j} \Delta_{m-j+1,r+j-1}. \tag{2.21}
 \end{aligned}$$

Case 2: $n = 0$.

$$A_0^{(m,r)} = \int_0^1 \alpha_{m,r}(x) dx = \frac{1}{2r} \Delta_{m+1,r-1}. \tag{2.22}$$

From (1.11), (1.12), (2.21), and (2.22), we get the Fourier series of $\alpha_{m,r}(\langle x \rangle)$ as follows:

$$\begin{aligned}
 &\frac{1}{2r} \Delta_{m+1,r-1} - \sum_{n=-\infty, n \neq 0}^{\infty} \left(\frac{1}{2r} \sum_{j=1}^m \frac{2^j(r+j-1)_j}{(2\pi in)^j} \Delta_{m-j+1,r+j-1}\right) e^{2\pi inx} \\
 &= \frac{1}{2r} \Delta_{m+1,r-1} + \frac{1}{2r} \sum_{j=1}^m 2^j \binom{r+j-1}{r-1} \Delta_{m-j+1,r+j-1} \\
 &\quad \times \left(-j! \sum_{n=-\infty, n \neq 0}^{\infty} \frac{e^{2\pi inx}}{(2\pi in)^j}\right) \\
 &= \frac{1}{2r} \Delta_{m+1,r-1} + \frac{1}{2r} \sum_{j=2}^m 2^j \binom{r+j-1}{r-1} \Delta_{m-j+1,r+j-1} B_j(\langle x \rangle) \\
 &\quad + \Delta_{m,r} \times \begin{cases} B_1(\langle x \rangle), & \text{for } x \in \mathbb{R} - \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z} \end{cases}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2r} \sum_{j=0, j \neq 1}^m 2^j \binom{r+j-1}{r-1} \Delta_{m-j+1, r+j-1} B_j(\langle x \rangle) \\
 &+ \Delta_{m,r} \times \begin{cases} B_1(\langle x \rangle), & \text{for } x \in \mathbb{R} - \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases} \tag{2.23}
 \end{aligned}$$

$\alpha_{m,r}(\langle x \rangle)$ ($m, r \geq 1$) is piecewise C^∞ . Moreover, $\alpha_{m,r}(\langle x \rangle)$ is continuous for those positive integers m, r with $\Delta_{m,r} = 0$, and discontinuous with jump discontinuities at integers for those positive integers m, r with $\Delta_{m,r} \neq 0$. Thus, for $\Delta_{m,r} = 0$, the Fourier series of $\alpha_{m,r}(\langle x \rangle)$ converges uniformly to $\alpha_{m,r}(\langle x \rangle)$. Whereas, for $\Delta_{m,r} \neq 0$, the Fourier series of $\alpha_{m,r}(\langle x \rangle)$ converges pointwise to $\alpha_{m,r}(\langle x \rangle)$, for $x \in \mathbb{R} - \mathbb{Z}$, and converges to

$$\begin{aligned}
 \frac{1}{2}(\alpha_{m,r}(0) + \alpha_{m,r}(1)) &= \alpha_{m,r}(1) - \frac{1}{2} \Delta_{m,r} \\
 &= \binom{m+2r}{2r} - \frac{1}{2} \Delta_{m,r}, \quad \text{for } x \in \mathbb{Z} \text{ (see (2.15)).} \tag{2.24}
 \end{aligned}$$

From these observations, together with (2.23) and (2.24), we obtain the next two theorems.

Theorem 2.3 *For any integers m, r with $m, r \geq 1$, we let*

$$\Delta_{m,r} = \frac{1}{r!} \sum_{k=1}^m (-1)^{k+1} \binom{m+2r+k}{2r+2k} 2^k (k+r)_r.$$

Assume that $\Delta_{m,r} = 0$ for some positive integers m, r . Then we have the following.

(a) $\sum_{l=0}^m \sum_{c_1+c_2+\dots+c_{r+1}=l} \binom{r-1+m-l}{r-1} V_{c_1}(\langle x \rangle) \cdots V_{c_{r+1}}(\langle x \rangle)$ has the Fourier series expansion

$$\begin{aligned}
 &\sum_{l=0}^m \sum_{c_1+c_2+\dots+c_{r+1}=l} \binom{r-1+m-l}{r-1} V_{c_1}(\langle x \rangle) \cdots V_{c_{r+1}}(\langle x \rangle) \\
 &= \frac{1}{2r} \Delta_{m+1, r-1} - \sum_{n=-\infty, n \neq 0}^{\infty} \left(\frac{1}{2r} \sum_{j=1}^m \frac{2^j (r+j-1)_j}{(2\pi i n)^j} \Delta_{m-j+1, r+j-1} \right) e^{2\pi i n x}, \tag{2.25}
 \end{aligned}$$

for all $x \in \mathbb{R}$, where the convergence is uniform.

(b)

$$\begin{aligned}
 &\sum_{l=0}^m \sum_{c_1+c_2+\dots+c_{r+1}=l} \binom{r-1+m-l}{r-1} V_{c_1}(\langle x \rangle) \cdots V_{c_{r+1}}(\langle x \rangle) \\
 &= \frac{1}{2r} \sum_{j=0, j \neq 1}^m 2^j \binom{r+j-1}{r-1} \Delta_{m-j+1, r+j-1} B_j(\langle x \rangle) \tag{2.26}
 \end{aligned}$$

for all x in \mathbb{R} .

Theorem 2.4 *For any integers m, r with $m, r \geq 1$, we let*

$$\Delta_{m,r} = \frac{1}{r!} \sum_{k=1}^m (-1)^{k+1} \binom{m+2r+k}{2r+2k} 2^k (k+r)_r.$$

Assume that $\Delta_{m,r} \neq 0$ for some positive integers m, r . Then we have the following.

(a)

$$\begin{aligned} & \frac{1}{2r} \Delta_{m+1,r-1} - \sum_{n=-\infty, n \neq 0}^{\infty} \left(\frac{1}{2r} \sum_{j=1}^m \frac{2^j (r+j-1)_j}{(2\pi in)^j} \Delta_{m-j+1,r+j-1} \right) e^{2\pi inx} \\ &= \begin{cases} \sum_{l=0}^m \sum_{c_1+c_2+\dots+c_{r+1}=l} \binom{r-1+m-l}{r-1} V_{c_1}(\langle x \rangle) \cdots V_{c_{r+1}}(\langle x \rangle), & \text{for } x \notin \mathbb{Z}, \\ \binom{m+2r}{2r} - \frac{1}{2} \Delta_{m,r}, & \text{for } x \in \mathbb{Z}. \end{cases} \end{aligned} \tag{2.27}$$

(b)

$$\begin{aligned} & \frac{1}{2r} \sum_{j=0}^m 2^j \binom{r+j-1}{r-1} \Delta_{m-j+1,r+j-1} B_j(\langle x \rangle) \\ &= \sum_{l=0}^m \sum_{c_1+c_2+\dots+c_{r+1}=l} \binom{r-1+m-l}{r-1} V_{c_1}(\langle x \rangle) \cdots V_{c_{r+1}}(\langle x \rangle), \\ & \quad \text{for } x \in \mathbb{R} - \mathbb{Z}; \\ & \frac{1}{2r} \sum_{j=0, j \neq 1}^m 2^j \binom{r+j-1}{r-1} \Delta_{m-j+1,r+j-1} B_j(\langle x \rangle) \\ &= \binom{m+2r}{2r} - \frac{1}{2} \Delta_{m,r}, \quad \text{for } x \in \mathbb{Z}. \end{aligned} \tag{2.28}$$

Finally, we observe that, from Theorems 2.3 and 2.4, we immediately obtain the result in Theorem 1.1 expressing $\alpha_{m,r}(x)$ in terms of Bernoulli polynomials.

3 Fourier series expansions for functions associated with the Chebyshev polynomials of the fourth kind

Here we will omit the details for the results in this section, as all of them can be obtained analogously to the previous section. We start with the following important lemma.

Lemma 3.1 *Let n, r be integers with $n \geq 0, r \geq 1$. Then we have the following identity:*

$$\sum_{l=0}^n \sum_{c_1+c_2+\dots+c_{r+1}=l} (-1)^{n-l} \binom{r-1+n-l}{r-1} W_{c_1}(x) \cdots W_{c_{r+1}}(x) = \frac{1}{2^r r!} W_{n+r}^{(r)}(x),$$

where the inner sum runs over all nonnegative integers c_1, c_2, \dots, c_{r+1} with $c_1 + c_2 + \dots + c_{r+1} = l$.

As is well known, the Chebyshev polynomials of the fourth kind $W_n(x)$ are explicitly given by (see [6])

$$\begin{aligned} W_n(x) &= (2n+1) {}_2F_1 \left(-n, n+1; \frac{3}{2}; \frac{1-x}{2} \right) \\ &= (2n+1) \sum_{k=0}^n \frac{2^k}{2k+1} \binom{n+k}{2k} (x-1)^k, \end{aligned} \tag{3.1}$$

where ${}_2F_1(a, b; c; z)$ is the hypergeometric function.

The r th derivative of (3.1) is given by

$$W_n^{(r)}(x) = (2n + 1) \sum_{k=r}^n \frac{2^k}{2k + 1} \binom{n + k}{2k} (k)_r (x - 1)^{k-r} \quad (0 \leq r \leq n). \tag{3.2}$$

Combining (3.1) and (3.2) yields the following lemma.

Lemma 3.2 *For integers n, r with $n, r \geq 1$, we have the following identity:*

$$\begin{aligned} & \sum_{l=0}^n \sum_{c_1+c_2+\dots+c_{r+1}=l} (-1)^{n-l} \binom{r-1+n-l}{r-1} W_{c_1}(x) \cdots W_{c_{r+1}}(x) \\ &= \frac{1}{r!} (2n + 2r + 1) \sum_{k=0}^n \frac{2^k}{2k + 2r + 1} \binom{n + 2r + k}{2k + 2r} (k + r)_r (x - 1)^k. \end{aligned} \tag{3.3}$$

As in (1.15), for $m, r \geq 1$, we let

$$\beta_{m,r}(x) = \sum_{l=0}^m \sum_{c_1+c_2+\dots+c_{r+1}=l} (-1)^{m-l} \binom{r-1+m-l}{r-1} W_{c_1}(x) \cdots W_{c_{r+1}}(x). \tag{3.4}$$

Then the Fourier series of $\beta_{m,r}(\langle x \rangle)$ is

$$\sum_{n=-\infty}^{\infty} B_n^{(m,r)} e^{2\pi i n x}, \tag{3.5}$$

where

$$B_n^{(m,r)} = \int_0^1 \beta_{m,r}(\langle x \rangle) e^{-2\pi i n x} dx = \int_0^1 \beta_{m,r}(x) e^{-2\pi i n x} dx. \tag{3.6}$$

For $m, r \geq 1$, we put

$$\begin{aligned} \Omega_{m,r} &= \beta_{m,r}(1) - \beta_{m,r}(0) \\ &= \sum_{l=0}^m \sum_{c_1+c_2+\dots+c_{r+1}=l} (-1)^{m-l} \binom{r-1+m-l}{r-1} \\ &\quad \times (W_{c_1}(1) \cdots W_{c_{r+1}}(1) - W_{c_1}(0) \cdots W_{c_{r+1}}(0)). \end{aligned} \tag{3.7}$$

Then, from (3.3) and (3.7), we have

$$\Omega_{m,r} = \frac{2m + 2r + 1}{r!} \sum_{k=1}^m (-1)^{k+1} \frac{2^k}{2k + 2r + 1} \binom{m + 2r + k}{2k + 2r} (k + r)_r, \tag{3.8}$$

where we note that

$$\beta_{m,r}(1) = \frac{2m + 2r + 1}{2r + 1} \binom{m + 2r}{2r}. \tag{3.9}$$

From (3.1), we can deduce the following:

$$\frac{d}{dx} \beta_{m,r}(x) = 2(r+1)\beta_{m-1,r+1}(x), \tag{3.10}$$

$$\frac{d}{dx} \left(\frac{\beta_{m+1,r-1}(x)}{2r} \right) = \beta_{m,r}(x), \tag{3.11}$$

$$\int_0^1 \beta_{m,r}(x) dx = \frac{1}{2r} \Omega_{m+1,r-1}, \tag{3.12}$$

$$\beta_{m,r}(0) = \beta_{m,r}(1) \iff \Omega_{m,r} = 0. \tag{3.13}$$

The Fourier series expansion of $\beta_{m,r}(\langle x \rangle)$ is as follows:

$$\begin{aligned} & \frac{1}{2r} \Omega_{m+1,r-1} - \sum_{n=-\infty, n \neq 0}^{\infty} \left(\frac{1}{2r} \sum_{j=1}^m \frac{2^j(r+j-1)_j}{(2\pi in)^j} \Omega_{m-j+1,r+j-1} \right) e^{2\pi inx} \\ &= \frac{1}{2r} \sum_{j=0, j \neq 1}^m 2^j \binom{r+j-1}{r-1} \Omega_{m-j+1,r+j-1} B_j(\langle x \rangle) \\ &+ \Omega_{m,r} \times \begin{cases} B_1(\langle x \rangle), & \text{for } x \in \mathbb{R} - \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases} \end{aligned} \tag{3.14}$$

We also observe that

$$\begin{aligned} \frac{1}{2}(\beta_{m,r}(0) + \beta_{m,r}(1)) &= \beta_{m,r}(1) - \frac{1}{2} \Omega_{m,r} \\ &= \frac{2m+2r+1}{2r+1} \binom{m+2r}{2r} - \frac{1}{2} \Omega_{m,r}. \end{aligned} \tag{3.15}$$

Now, the next two theorems follow from (3.14) and (3.15).

Theorem 3.3 *For any integers m, r with $m, r \geq 1$, we let*

$$\Omega_{m,r} = \frac{2m+2r+1}{r!} \sum_{k=1}^m (-1)^{k+1} \frac{2^k}{2k+2r+1} \binom{m+2r+k}{2k+2r} (k+r)_r.$$

Assume that $\Omega_{m,r} = 0$ for some positive integers m, r . Then we have the following.

(a) $\sum_{l=0}^m \sum_{c_1+c_2+\dots+c_{r+1}=l} (-1)^{m-l} \binom{r-1+m-l}{r-1} W_{c_1}(\langle x \rangle) \dots W_{c_{r+1}}(\langle x \rangle)$ has the Fourier series expansion

$$\begin{aligned} & \sum_{l=0}^m \sum_{c_1+c_2+\dots+c_{r+1}=l} (-1)^{m-l} \binom{r-1+m-l}{r-1} W_{c_1}(\langle x \rangle) \dots W_{c_{r+1}}(\langle x \rangle) \\ &= \frac{1}{2r} \Omega_{m+1,r-1} - \sum_{n=-\infty, n \neq 0}^{\infty} \left(\frac{1}{2r} \sum_{j=1}^m \frac{2^j(r+j-1)_j}{(2\pi in)^j} \Omega_{m-j+1,r+j-1} \right) e^{2\pi inx}, \end{aligned} \tag{3.16}$$

for all $x \in \mathbb{R}$, where the convergence is uniform.

(b)

$$\begin{aligned} & \sum_{l=0}^m \sum_{c_1+c_2+\dots+c_{r+1}=l} (-1)^{m-l} \binom{r-1+m-l}{r-1} W_{c_1}(\langle x \rangle) \cdots W_{c_{r+1}}(\langle x \rangle) \\ &= \frac{1}{2r} \sum_{j=0, j \neq 1}^m 2^j \binom{r+j-1}{r-1} \Omega_{m-j+1, r+j-1} B_j(\langle x \rangle) \end{aligned} \tag{3.17}$$

for all x in \mathbb{R} .

Theorem 3.4 For any integers m, r with $m, r \geq 1$, we let

$$\Omega_{m,r} = \frac{2m+2r+1}{r!} \sum_{k=1}^m (-1)^{k+1} \frac{2^k}{2k+2r+1} \binom{m+2r+k}{2k+2r} (k+r)_r.$$

Assume that $\Omega_{m,r} \neq 0$ for some positive integers m, r . Then we have the following:

(a)

$$\begin{aligned} & \frac{1}{2r} \Omega_{m+1, r-1} - \sum_{n=-\infty, n \neq 0}^{\infty} \left(\frac{1}{2r} \sum_{j=1}^m \frac{2^j (r+j-1)_j}{(2\pi i n)^j} \Omega_{m-j+1, r+j-1} \right) e^{2\pi i n x} \\ &= \begin{cases} \sum_{l=0}^m \sum_{c_1+c_2+\dots+c_{r+1}=l} (-1)^{m-l} \binom{r-1+m-l}{r-1} W_{c_1}(\langle x \rangle) \cdots W_{c_{r+1}}(\langle x \rangle), \\ \text{for } x \in \mathbb{R} - \mathbb{Z}, \\ \frac{2m+2r+1}{2r+1} \binom{m+2r}{2r} - \frac{1}{2} \Omega_{m,r}, \text{ for } x \in \mathbb{Z}. \end{cases} \end{aligned} \tag{3.18}$$

(b)

$$\begin{aligned} & \frac{1}{2r} \sum_{j=0}^m 2^j \binom{r+j-1}{r-1} \Omega_{m-j+1, r+j-1} B_j(\langle x \rangle) \\ &= \sum_{l=0}^m \sum_{c_1+c_2+\dots+c_{r+1}=l} (-1)^{m-l} \binom{r-1+m-l}{r-1} W_{c_1}(\langle x \rangle) \cdots W_{c_{r+1}}(\langle x \rangle), \\ & \text{for } x \in \mathbb{R} - \mathbb{Z}; \\ & \frac{1}{2r} \sum_{j=0, j \neq 1}^m 2^j \binom{r+j-1}{r-1} \Omega_{m-j+1, r+j-1} B_j(\langle x \rangle) \\ &= \frac{2m+2r+1}{2r+1} \binom{m+2r}{2r} - \frac{1}{2} \Omega_{m,r}, \text{ for } x \in \mathbb{Z}. \end{aligned} \tag{3.19}$$

Now, as an immediate corollary to Theorems 3.3 and 3.4, we get the stated result in Theorem 1.2 expressing $\beta_{m,r}(x)$ in terms of Bernoulli polynomials.

4 Expressions in terms of Euler polynomials

Let $p(x) \in \mathbb{C}[x]$ be a polynomial of degree m . Then it is known that

$$p(x) = \sum_{k=0}^m b_k E_k(x), \tag{4.1}$$

where $E_k(x)$ are the Euler polynomials given by

$$\frac{2}{e^t + 1} e^{xt} = \sum_{k=0}^{\infty} E_k(x) \frac{t^k}{k!}, \tag{4.2}$$

and

$$b_k = \frac{1}{2k!} (p^{(k)}(1) + p^{(k)}(0)), \quad k = 0, 1, \dots, m. \tag{4.3}$$

Applying (4.1) and (4.3) to $p(x) = \alpha_{m,r}(x)$, from (2.17) we have

$$\alpha_{m,r}^{(k)}(x) = 2^k (r+k)_k \alpha_{m-k,r+k}(x). \tag{4.4}$$

Hence, from (4.4) we see that

$$\begin{aligned} b_k &= \frac{1}{2k!} (\alpha_{m,r}^{(k)}(1) + \alpha_{m,r}^{(k)}(0)) \\ &= 2^{k-1} \binom{r+k}{r} (\alpha_{m-k,r+k}(1) + \alpha_{m-k,r+k}(0)) \\ &= 2^{k-1} \binom{r+k}{r} (2\alpha_{m-k,r+k}(1) - \Delta_{m-k,r+k}). \end{aligned} \tag{4.5}$$

Now, we note from (2.15) that

$$\alpha_{m-k,r+k}(1) = \binom{m+2r+k}{2r+2k}. \tag{4.6}$$

Combining (4.5) and (4.6), we finally obtain

$$b_k = 2^{k-1} \binom{r+k}{r} \left(2 \binom{m+2r+k}{2r+2k} - \Delta_{m-k,r+k} \right). \tag{4.7}$$

Thus we have the following theorem from (4.1) and (4.7).

Theorem 4.1 *For any integers m, r with $m, r \geq 1$, we have the following:*

$$\begin{aligned} &\sum_{l=0}^m \sum_{c_1+c_2+\dots+c_{r+1}=l} \binom{r-1+m-l}{r-1} V_{c_1}(x) \cdots V_{c_{r+1}}(x) \\ &= \sum_{k=0}^m 2^{k-1} \binom{r+k}{r} \left(2 \binom{m+2r+k}{2r+2k} - \Delta_{m-k,r+k} \right) E_k(x), \end{aligned} \tag{4.8}$$

where $\Delta_{m,r}$ is given by (2.14).

The next theorem follows analogously to the previous discussion and hence the details will be left to the reader.

Theorem 4.2 For any integers m, r with $m, r \geq 1$, we have the following:

$$\begin{aligned} & \sum_{l=0}^m \sum_{c_1+c_2+\dots+c_{r+1}=l} (-1)^{m-l} \binom{r-1+m-l}{r-1} W_{c_1}(x) \cdots W_{c_{r+1}}(x) \\ &= \sum_{k=0}^m 2^{k-1} \binom{r+k}{r} \left(\frac{2(2m+2r+1)}{2r+2k+1} \binom{m+2r+k}{2r+2k} - \Omega_{m-k,r+k} \right) E_k(x), \end{aligned} \tag{4.9}$$

where $\Omega_{m,r}$ is given by (3.8).

5 Applications

Let $T_n(x), U_n(x)$ ($n \geq 0$) be the Chebyshev polynomials of the first kind and of the second kind respectively given by (1.1) or (1.5), and (1.2) or (1.6). The statement in the following lemma is from equations (1.2) and (1.3) of [12].

Lemma 5.1 Let $u = [\frac{1}{2}(1+x)]^{\frac{1}{2}}$. Then we have the following identities:

$$W_n(x) = (-1)^n V_n(-x), \tag{5.1}$$

$$V_n(x) = u^{-1} T_{2n+1}(u), \tag{5.2}$$

$$W_n(x) = U_{2n}(u). \tag{5.3}$$

From (1.14) and (5.2) and with u as in Lemma 5.1, we have

$$\begin{aligned} & \sum_{l=0}^m \sum_{c_1+c_2+\dots+c_{r+1}=l} \binom{r-1+m-l}{r-1} T_{2c_1+1}(u) \cdots T_{2c_{r+1}+1}(u) \\ &= \frac{1}{2r} u^{r+1} \sum_{j=0}^m 2^j \binom{r+j-1}{r-1} \Delta_{m-j+1,r+j-1} B_j(x). \end{aligned} \tag{5.4}$$

In particular, for $x = 0$, (5.4) yields

$$\begin{aligned} & \sum_{l=0}^m \sum_{c_1+c_2+\dots+c_{r+1}=l} \binom{r-1+m-l}{r-1} T_{2c_1+1}\left(\frac{1}{\sqrt{2}}\right) \cdots T_{2c_{r+1}+1}\left(\frac{1}{\sqrt{2}}\right) \\ &= \frac{1}{2^{\frac{r+3}{2}} r} \sum_{j=0}^m 2^j \binom{r+j-1}{r-1} \Delta_{m-j+1,r+j-1} B_j. \end{aligned} \tag{5.5}$$

On the other hand, from (1.16) and (5.3) and with u as in Lemma 5.1, we get

$$\begin{aligned} & \sum_{l=0}^m \sum_{c_1+c_2+\dots+c_{r+1}=l} (-1)^{m-1} \binom{r-1+m-l}{r-1} U_{2c_1}(u) \cdots U_{2c_{r+1}}(u) \\ &= \frac{1}{2r} \sum_{j=1}^m 2^j \binom{r+j-1}{r-1} \Omega_{m-j+1,r+j-1} B_j(x). \end{aligned} \tag{5.6}$$

Now, letting $x = 0$ in (5.6) gives

$$\begin{aligned} &\sum_{l=0}^m \sum_{c_1+c_2+\dots+c_{r+1}=l} (-1)^{m-l} \binom{r-1+m-l}{r-1} U_{2c_1}\left(\frac{1}{\sqrt{2}}\right) \cdots U_{2c_{r+1}}\left(\frac{1}{\sqrt{2}}\right) \\ &= \frac{1}{2^r} \sum_{j=1}^m 2^j \binom{r+j-1}{r-1} \Omega_{m-j+1,r+j-1} B_j. \end{aligned} \tag{5.7}$$

Finally, from (1.14), (1.16), and (5.1), we obtain

$$\begin{aligned} &\sum_{l=0}^m \sum_{c_1+c_2+\dots+c_{r+1}=l} (-1)^l \binom{r-1+m-l}{r-1} W_{c_1}(x) \cdots W_{c_{r+1}}(x) \\ &= \frac{1}{2^r} \sum_{j=0}^m 2^j \binom{r+j-1}{r-1} \Delta_{m-j+1,r+j-1} B_j(-x) \\ &= \frac{1}{2^r} (-1)^m \sum_{j=0}^m 2^j \binom{r+j-1}{r-1} \Omega_{m-j+1,r+j-1} B_j(x). \end{aligned} \tag{5.8}$$

In particular, when $x = 0$, (5.8) gives

$$\begin{aligned} &\sum_{j=0}^m 2^j \binom{r+j-1}{r-1} \Delta_{m-j+1,r+j-1} B_j \\ &= (-1)^m \sum_{j=0}^m 2^j \binom{r+j-1}{r-1} \Omega_{m-j+1,r+j-1} B_j. \end{aligned} \tag{5.9}$$

In addition, from (5.8) and using the well-known fact that $B_j(-x) = (-1)^j(B_j(x) + jx^{j-1})$, we have

$$\begin{aligned} &\sum_{j=1}^m (-2)^j j \binom{r+j-1}{r-1} \Delta_{m-j+1,r+j-1} x^{j-1} \\ &= (-1)^m \sum_{j=0}^{m-1} 2^j \binom{r+j-1}{r-1} (\Omega_{m-j+1,r+j-1} + (-1)^{m-j+1} \Delta_{m-j+1,r+j-1}) B_j(x), \end{aligned} \tag{5.10}$$

where we observed that

$$\Omega_{1,r+m-1} - \Delta_{1,r+m-1} = 0. \tag{5.11}$$

6 Results and discussion

Let $\alpha_{m,r}(x)$, $\beta_{m,r}(x)$ denote the following sums of finite products given by

$$\begin{aligned} \alpha_{m,r}(x) &= \sum_{l=0}^m \sum_{c_1+c_2+\dots+c_{r+1}=l} \binom{r-1+m-l}{r-1} V_{c_1}(x) \cdots V_{c_{r+1}}(x), \\ \beta_{m,r}(x) &= \sum_{l=0}^m \sum_{c_1+c_2+\dots+c_{r+1}=l} (-1)^{m-l} \binom{r-1+m-l}{r-1} W_{c_1}(x) \cdots W_{c_{r+1}}(x), \end{aligned} \tag{6.1}$$

where $V_n(x)$, $W_n(x)$ ($n \geq 0$) are respectively the Chebyshev polynomials of the third kind and of the fourth kind. Then we considered the functions $\alpha_{m,r}(\langle x \rangle)$, $\beta_{m,r}(\langle x \rangle)$ which are obtained by extending by periodicity 1 from $\alpha_{m,r}(x)$, $\beta_{m,r}(x)$ on $[0, 1)$. We obtained the Fourier series expansions of $\alpha_{m,r}(\langle x \rangle)$, $\beta_{m,r}(\langle x \rangle)$, and expressed $\alpha_{m,r}(x)$, $\beta_{m,r}(x)$ as linear combinations of the usual Bernoulli polynomials. Moreover, we expressed $\alpha_{m,r}(x)$, $\beta_{m,r}(x)$ in terms of the usual Euler polynomials and derived several identities involving the four kinds of Chebyshev polynomials. We expect that we can get similar results for some other orthogonal polynomials.

7 Conclusion

In this paper, we studied sums of finite products of Chebyshev polynomials of the third kind and of the fourth kind. In recent years, we have obtained similar results for many other special polynomials. However, all of them have been the Appell polynomials, whereas Chebyshev polynomials are the classical orthogonal polynomials. Studying these kinds of sums of finite products of special polynomials can be well justified by the following example. Let us put

$$\gamma_m(x) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)} B_k(x) B_{m-k}(x) \quad (m \geq 2). \quad (7.1)$$

Then we can express $\gamma_m(x)$ in terms of Bernoulli polynomials from the Fourier series expansions of $\gamma_m(\langle x \rangle)$, just as we did these for $\alpha_{m,r}(x)$, $\beta_{m,r}(x)$. Further, some simple modification of this gives us the famous Faber–Pandharipande–Zagier identity and a slightly different variant of Miki's identity. We note here that all the other known derivations of Faber–Pandharipande–Zagier identity and Miki's identity are quite involved. But our approach to (7.1) via Fourier series is elementary but powerful enough to get many results. We expect that we can get similar results for some other orthogonal polynomials.

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Authors' contributions

All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

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