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Dynamic behaviors of a stage structure amensalism system with a cover for the first species

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Abstract
In this paper, we propose and study a two-species stage structured amensalism model with a cover for the first species. By developing a new analysis technique or, more precisely, by combining the differential inequality theory and the Lyapunov function method, we obtain sufficient conditions ensuring the global attractivity of positive and boundary equilibria, respectively. Our study shows that the final density of the first species is an increasing function of the partial cover, and if the stage structured species is globally asymptotically stable, then there exists a threshold such that if the cover is greater than this threshold, the species can still exist in the long run, whereas if the cover is too small, then the first species is driven to extinction.

MSC: 34C25; 92D25; 34D20; 34D40

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1 Introduction

The aim of this paper is to investigate the dynamic behavior of the following stage structure amensalism system with a cover for the first species:

$$\begin{aligned} \frac{dx_1}{dt} &= \alpha x_2 - \beta x_1 - \delta_1 x_1 - d_1(1-k)x_1 y, \\ \frac{dx_2}{dt} &= \beta x_1 - \delta_2 x_2 - \gamma x_2^2 - d_2(1-k)x_2 y, \\ \frac{dy}{dt} &= y(b_2 - a_2 y), \end{aligned} \tag{1.1}$$

where $\alpha, \beta, \delta_1, \delta_2, d_1, d_2, k, b_2, a_2$, and γ are positive constants, $x_1(t)$ and $x_2(t)$ are the densities of the immature and mature first species at time t, y is the density of the second species at time t, and $k \in (0,1)$ is a cover provided for the first species. The following assumptions are made in model (1.1):

1. The first species has two-stage structure, immature and mature. Its dynamic behavior is described by the equation system

$$\frac{dx_1}{dt} = \alpha x_2 - \beta x_1 - \delta_1 x_1,$$

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$$\frac{dx_2}{dt} = \beta x_1 - \delta_2 x_2 - \gamma x_2^2.$$

We refer to Khajanchi and Banerjee [1] for more background of this equation system.

2. There is a partial cover (represented by k) for the first species to protect it from the second species.
3. Both relationships between the immature species and the second species and between the mature species and the second species are bilinear: ($d_1(1-k)x_1y$ and $d_2(1-k)x_2y$).
4. The second species satisfies the logistic model.

During the last decades, many scholars investigated the dynamic behavior of the commensalism or amensalism model [2–19]. Such topics as the local stability of the equilibrium [2–4, 7, 8, 10–16, 18, 19], the existence of the positive periodic solution [5, 17] the existence and stability of the almost periodic solution [6], extinction of the species [8, 11, 14], and the influence of the cover [14, 16, 18] have been studied, and many excellent results are obtained. Recently, Xiong, Wang, and Zhang [13] proposed the following amensalism model:

$$\begin{aligned} \frac{dN_1}{dt} &= r_1N_1\left(1 - \frac{N_1}{P_1} - u\frac{N_2}{P_1}\right), \\ \frac{dN_2}{dt} &= r_2N_2\left(1 - \frac{N_2}{P_2}\right). \end{aligned} \tag{1.2}$$

The authors investigated the local stability property of the equilibria of the system.

Zhu and Chen [10] studied the qualitative property of the following two-species amensalism model:

$$\begin{aligned} \frac{dx}{dt} &= x(r_1 - a_{11}x - a_{12}y), \\ \frac{dy}{dt} &= y(r_2 - a_{22}y). \end{aligned} \tag{1.3}$$

They showed by a vector field that system (1.3) may have a positive equilibrium, and it is globally stable, or that the system has no positive equilibrium, and one of the boundary equilibria is globally stable.

Stimulated by the work [13], Chen [15] proposed the following nonselective harvesting Lotka–Volterra amensalism model incorporating partial closure for the populations:

$$\begin{aligned} \frac{dN_1}{dt} &= r_1N_1\left(1 - \frac{N_1}{P_1} - u\frac{N_2}{P_1}\right) - q_1EmN_1, \\ \frac{dN_2}{dt} &= r_2N_2\left(1 - \frac{N_2}{P_2}\right) - q_2EmN_2. \end{aligned} \tag{1.4}$$

Chen showed that after introducing the harvesting term, the dynamic behavior of system (1.4) becomes complicated, the system maybe extinction, partial survival, or two species can coexist in a stable state.

Some scholars argued that the functional response between the species is the essential factor to influence the dynamic behavior of the system. For example, Wu [19] proposed a two-species amensalism model with nonmonotonic functional response, which takes the form

$$\begin{aligned}\frac{dx_1}{dt} &= x_1\left(a_1 - b_1x_1 - \frac{c_1x_2}{d_1 + x_2^2}\right),\\ \frac{dx_2}{dt} &= x_2(a_2 - b_2x_2).\end{aligned} \tag{1.5}$$

They showed that the dynamic behavior of system (1.5) is similar to that of system (1.3).

It is well known that refuge plays an important role on the dynamic behavior of the predator–prey system (see [20–23]). Stimulated by the notion of the refuge, Xie, Chen, and He [18] studied the following two species amensalism model with a partial cover for the first species to protect it from the second species:

$$\begin{aligned}\frac{dx}{dt} &= a_1x(t) - b_1x^2(t) - a_{12}(1-k)x(t)y(t),\\ \frac{dy}{dt} &= a_2y(t) - b_2y^2(t).\end{aligned} \tag{1.6}$$

Their study indicates that the conditions that ensure the local stability of the boundary equilibrium are sufficient to ensure its global stability, and once a positive equilibrium exists, it is globally stable. Their results were then generalized by Wu, Zhao, and Lin [14] to a two-species amensalism model with Holling II functional response and a cover for the first species.

On the other hand, many scholars investigated the dynamic behavior of the stage structured species; see [1, 24–36] and the references therein. Many scholars [24–34] argued that a suitable stage-structured model should incorporate the time delay, which reflects the period of immature species to grow up to mature species. For example, Lin, Xie, and Chen [32] proposed the following stage-structured predator–prey model (stage structure for both predator and prey, respectively) with modified Leslie–Gower and Holling-type II schemes:

$$\begin{aligned}x_1'(t) &= r_1x_2(t) - d_{11}x_1(t) - r_1e^{-d_{11}\tau_1}x_2(t-\tau_1),\\ x_2'(t) &= r_1e^{-d_{11}\tau_1}x_2(t-\tau_1) - d_{12}x_2(t) - bx_2^2(t) - \frac{a_1y_2(t)x_2(t)}{x_2(t)+k_1},\\ y_1'(t) &= r_2y_2(t) - d_{22}y_1(t) - r_2e^{-d_{22}\tau_2}y_2(t-\tau_2),\\ y_2'(t) &= r_2e^{-d_{22}\tau_2}y_2(t-\tau_2) - d_{21}y_2(t) - \frac{a_2y_2^2(t)}{x_2(t)+k_2}.\end{aligned} \tag{1.7}$$

They showed that for a stage-structured predator–prey community, both stage structure and death rate of the mature species are the important factors that lead to the permanence or extinction of the system.

Many scholars [1, 35, 36] also proposed and studied the stage-structured ecosystem without time delay, that is, they assumed that there are proportional numbers of immature species that become mature species at time t. Recently, Khajanchi and Banerjee [1] proposed the following stage structure predator–prey model with ratio dependent functional response:

$$\begin{aligned} \frac{dx_1}{dt} &= \alpha x_2(t) - \beta x_1(t) - \delta_1 x_1(t), \\ \frac{dx_2}{dt} &= \beta x_1(t) - \delta_2 x_2(t) - \gamma x_2^2(t) - \frac{\eta(1-\theta)x_2(t)y(t)}{g(1-\theta)x_2(t) + hy(t)}, \\ \frac{dy}{dt} &= \frac{u\eta(1-\theta)x_2(t)y(t)}{g(1-\theta)x_2(t) + hy(t)} - \delta_3 y(t). \end{aligned} \tag{1.8}$$

The authors investigated the stability property of the positive equilibrium and boundary equilibrium.

Stimulated by the works of Chen [15] and Khajanchi and Banerjee [1], we [36] proposed the following single-species stage structure system incorporating partial closure for the populations and nonselective harvesting:

$$\begin{aligned} \frac{dx_1}{dt} &= \alpha x_2 - \beta x_1 - \delta_1 x_1 - q_1 E m x_1, \\ \frac{dx_2}{dt} &= \beta x_1 - \delta_2 x_2 - \gamma x_2^2 - q_2 E m x_2. \end{aligned} \tag{1.9}$$

We showed that the birth rate of the immature species and the fraction of the stocks for the harvesting play a crucial role on the dynamic behavior of the system.

It brings to our attention that, to this day, still no scholars propose and study the dynamic behaviors of the amensalism model with stage structure. This motivated us to propose system (1.1). We mention here that at first sight, system (1.1) is very simple, However, the third equation is independent of x_1 and x_2, and hence it is impossible to investigate the stability property of the system by constructing a suitable Lyapunov function. Also, since this is a three-dimensional system, we cannot investigate the stability property of the system by using the Dulac criterion.

The paper is arranged as follows. We investigate the existence and locally stability property of the equilibria of system (1.1) in Sect. 2. In Sect. 3, by applying the differential inequality theory and constructing some suitable Lyapunov function we are able to investigate the global attractivity of the positive and boundary equilibria. We then discuss the influence of partial cover to the final density of the first species in Sect. 4, and in Sect. 5, we present an example together with its numerical simulations to show the feasibility of the main results. We end this paper by a brief discussion.

2 Local stability of the equilibria

Before we study the local stability property of the equilibrium points of system (1.1), we introduce the stability of equilibrium of the following single-species stage-structured system:

$$\begin{aligned} \frac{dx_1}{dt} &= \alpha x_2 - \beta x_1 - \delta_1 x_1, \\ \frac{dx_2}{dt} &= \beta x_1 - \delta_2 x_2 - \gamma x_2^2, \end{aligned} \tag{2.1}$$

where $\alpha, \beta, \delta_1, \delta_2$, and γ are positive constants. The following lemma is Theorems 4.1 and 4.2 of [36].

Lemma 2.1 *If*

$$\alpha\beta < \delta_2(\beta + \delta_1), \tag{2.2}$$

then the boundary equilibrium $O(0,0)$ *of system* (2.1) *is globally stable*. *If*

$$\alpha\beta > \delta_2(\beta + \delta_1), \tag{2.3}$$

then the positive equilibrium $B(x_1^*, x_2^*)$ *of system* (2.1) *is globally stable*, *where*

$$x_1^* = \frac{\alpha x_2^*}{\beta + \delta_1}, \qquad x_2^* = \frac{\alpha\beta - \delta_2(\beta + \delta_1)}{\gamma(\beta + \delta_1)}.$$

Now we are in position to investigate the local stability property of system (1.1).
The equilibria of system (1.1) are determined by the system

$$\begin{aligned} &\alpha x_2 - \beta x_1 - \delta_1 x_1 - d_1(1-k)x_1 y = 0, \\ &\beta x_1 - \delta_2 x_2 - \gamma x_2^2 - d_2(1-k)x_2 y = 0, \\ &y(b_2 - a_2 y) = 0. \end{aligned} \tag{2.4}$$

The system always admits two boundary equilibria, $A_1(0,0,0)$ and $A_2(0,0,\frac{b_2}{a_2})$. If

$$\alpha\beta > \delta_2(\beta + \delta_1), \tag{2.5}$$

then the system admits boundary equilibrium $A_3(x_1^*, x_2^*, 0)$, where

$$x_1^* = \frac{\alpha x_2^*}{\beta + \delta_1}, \qquad x_2^* = \frac{\alpha\beta - \delta_2(\beta + \delta_1)}{\gamma(\beta + \delta_1)}. \tag{2.6}$$

If

$$\alpha\beta - \left(\delta_2 + \frac{d_2(1-k)b_2}{a_2}\right)\left(\beta + \delta_1 + \frac{d_1(1-k)b_2}{a_2}\right) > 0, \tag{2.7}$$

then system (1.1) admits a unique positive equilibrium $A_4(x_1^{**}, x_2^{**}, y^{**})$, where

$$\begin{aligned} x_1^{**} &= \frac{\alpha x_2^{**}}{\beta + \delta_1 + \frac{d_1(1-k)b_2}{a_2}}, \\ x_2^{**} &= \frac{\alpha\beta - (\delta_2 + \frac{d_2(1-k)b_2}{a_2})(\beta + \delta_1 + \frac{d_1(1-k)b_2}{a_2})}{(\beta + \delta_1 + \frac{d_1(1-k)b_2}{a_2})\gamma}, \\ y^{**} &= \frac{b_2}{a_2}. \end{aligned} \tag{2.8}$$

Obviously, x_1^{**}, x_2^{**}, and y^{**} satisfy the equations

$$\begin{aligned} &\alpha x_2^{**} - \beta x_1^{**} - \delta_1 x_1^{**} - d_1(1-k)x_1^{**} y^{**} = 0, \\ &\beta x_1^{**} - \delta_2 x_2^{**} - \gamma \left(x_2^{**}\right)^2 - d_2(1-k)x_2^{**} y^{**} = 0, \\ &b_2 - a_2 y^{**} = 0. \end{aligned} \tag{2.9}$$

We will now investigate the local stability of the above equilibria.

The variational matrix of system (1.1) is

$$J(x_1,x_2,y)=\begin{pmatrix}\Delta_1 & \alpha & -d_1(1-k)x_1\\ \beta & \Delta_2 & -d_2(1-k)x_2\\ 0 & 0 & -2a_2y+b_2\end{pmatrix}, \tag{2.10}$$

where

$$\Delta_1=-\beta-\delta_1-d_1(1-k)y,$$
$$\Delta_2=-\delta_2-2\gamma x_2-d_2(1-k)y.$$

Theorem 2.1 $A_1(0,0,0)$ *is unstable.*

Proof From (2.10) we can see that the Jacobian matrix of the system about the equilibrium point $A_1(0,0,0)$ is given by

$$\begin{pmatrix}-\beta-\delta_1 & \alpha & 0\\ \beta & -\delta_2 & 0\\ 0 & 0 & b_2\end{pmatrix}. \tag{2.11}$$

We can easily see that it has one positive characteristic root $\lambda_1=b_2$, and, consequently, $A_1(0,0,0)$ is unstable. This ends the proof of Theorem 2.1. □

Theorem 2.2 *If*

$$\left(\delta_2+\frac{d_2(1-k)b_2}{a_2}\right)\left(\beta+\delta_1+\frac{d_1(1-k)b_2}{a_2}\right)-\alpha\beta>0, \tag{2.12}$$

then $A_2(0,0,\frac{b_2}{a_2})$ *is locally asymptotically stable. If*

$$\left(\delta_2+\frac{d_2(1-k)b_2}{a_2}\right)\left(\beta+\delta_1+\frac{d_1(1-k)b_2}{a_2}\right)-\alpha\beta<0, \tag{2.13}$$

then $A_2(0,0,\frac{b_2}{a_2})$ *is unstable.*

Proof From (2.10) we can see that the Jacobian matrix of the system about the equilibrium point $A_2(0,0,\frac{b_2}{a_2})$ is given by

$$\begin{pmatrix}-\beta-\delta_1-\frac{d_1(1-k)b_2}{a_2} & \alpha & 0\\ \beta & -\delta_2-\frac{d_2(1-k)b_2}{a_2} & 0\\ 0 & 0 & -b_2\end{pmatrix}. \tag{2.14}$$

The characteristic equation of this matrix is

$$(\lambda+b_2)\left[\lambda^2+K_1\lambda+K_2\right]=0, \tag{2.15}$$

where

$$K_1 = \delta_1 + \delta_2 + \beta + \frac{d_1(1-k)b_2}{a_2} + \frac{d_2(1-k)b_2}{a_2},$$

$$K_2 = \left(\beta + \delta_1 + \frac{d_1(1-k)b_2}{a_2}\right)\left(\delta_2 + \frac{d_2(1-k)b_2}{a_2}\right) - \alpha\beta.$$

Hence it has one negative characteristic root $\lambda_1 = -b_2 < 0$; the other two characteristic roots are determined by the equation

$$\lambda^2 + K_1\lambda + K_2 = 0. \tag{2.16}$$

Note that the two characteristic roots of equation (2.16) satisfy

$$\lambda_2 + \lambda_3 = -K_1, \qquad \lambda_2\lambda_3 = K_2. \tag{2.17}$$

Under assumption (2.13), $\lambda_2\lambda_3 < 0$, hence at least one characteristic root is positive, and, consequently, $A_2(0, 0, \frac{b_2}{a_2})$ is unstable. Under assumption (2.12), $\lambda_2 < 0$ and $\lambda_3 < 0$. Thus three characteristic roots of matrix (2.14) are all negative, and hence $A_1(0, 0, \frac{b_2}{a_2})$ is locally asymptotically stable. This ends the proof of Theorem 2.2. □

Theorem 2.3 *$A_3(x_1^*, x_2^*, 0)$ is unstable.*

Proof From (2.10) we can see that the Jacobian matrix of the system about the equilibrium point $A_3(x_1^*, x_2^*, 0)$ is given by

$$\begin{pmatrix} -\beta - \delta_1 & \alpha & -d_1(1-k)x_1^* \\ \beta & -\delta_2 - 2\gamma x_2^* & -d_2(1-k)x_2^* \\ 0 & 0 & b_2 \end{pmatrix}. \tag{2.18}$$

From (2.18) we can easily see that the matrix has one positive characteristic root $\lambda_1 = b_2$, and, consequently, $A_3(x_1^*, x_2^*, 0)$ is unstable. This ends the proof of Theorem 2.3. □

Theorem 2.4 *If*

$$\left(\delta_2 + \frac{d_2(1-k)b_2}{a_2}\right)\left(\beta + \delta_1 + \frac{d_1(1-k)b_2}{a_2}\right) - \alpha\beta < 0, \tag{2.19}$$

then $A_4(x_1^{**}, x_2^{**}, y^{**})$ is locally asymptotically stable.

Proof From (2.10) we can see that the Jacobian matrix of the system about the equilibrium point $A_4(x_1^{**}, x_2^{**}, y^{**})$ is given by

$$\begin{pmatrix} \Gamma_1 & \alpha & -d_1(1-k)x_1^{**} \\ \beta & \Gamma_2 & -d_2(1-k)x_2^{**} \\ 0 & 0 & -2a_2y^{**} + b_2 \end{pmatrix}, \tag{2.20}$$

where

$$\Gamma_1 = -\beta - \delta_1 - d_1(1-k)y^{**},$$

$$\Gamma_2 = -\delta_2 - 2\gamma x_2^{**} - d_2(1-k)y^{**}.$$

Noting that

$$-2a_2y^{**} + b_2 = -2a_2\frac{b_2}{a_2} + b_2 = -b_2,$$

from the second equation of (2.9) we have

$$\begin{aligned}
&-\delta_2 - 2\gamma x_2^{**} - d_2(1-k)y^{**}\\
&\quad = -\frac{\beta x_1^{**}}{x_2^{**}} - \gamma x_2^{**}\\
&\quad = -\frac{\alpha\beta}{\beta + \delta_1 + \frac{d_1(1-k)b_2}{a_2}} - \gamma x_2^{**}.
\end{aligned}$$

The characteristic equation of above matrix is

$$(\lambda + b_2)\left[\lambda^2 + B_1\lambda + B_2\right] = 0, \tag{2.21}$$

where

$$\begin{aligned}
B_1 &= \beta + \delta_1 + d_1(1-k)y^{**} + \frac{\alpha}{\beta + \delta_1 + \frac{d_1(1-k)b_2}{a_2}} + \gamma x_2^{**},\\
B_2 &= \left(\beta + \delta_1 + d_1(1-k)y^{**}\right)\left(\frac{\beta\alpha}{\beta + \delta_1 + \frac{d_1(1-k)b_2}{a_2}} + \gamma x_2^{**}\right) - \alpha\beta.
\end{aligned}$$

Hence it has one negative characteristic root $\lambda_1 = -b_2 < 0$, and the other two characteristic roots are determined by the equation

$$\lambda^2 + B_1\lambda + B_2 = 0. \tag{2.22}$$

Note that by the expressions of x_2^{**} and y^{**} and by condition (2.19) the two characteristic roots of equation (2.22) satisfy

$$\begin{aligned}
&\lambda_2 + \lambda_3 = -B_1 < 0,\\
&\lambda_2\lambda_3 = \left(\beta + \delta_1 + d_1(1-k)y^{**}\right)L_1 - \alpha\beta\\
&\qquad = \left(\beta + \delta_1 + d_1(1-k)\frac{b_2}{a_2}\right)L_1 - \alpha\beta\\
&\qquad = \alpha\beta - \left(\delta_2 + \frac{d_2(1-k)b_2}{a_2}\right)\left(\beta + \delta_1 + \frac{d_1(1-k)b_2}{a_2}\right)\\
&\qquad > 0,
\end{aligned} \tag{2.23}$$

where

$$L_1 = \frac{\beta\alpha}{\beta + \delta_1 + \frac{d_1(1-k)b_2}{a_2}} + \gamma x_2^{**}$$

$$= \frac{2\alpha\beta - (\delta_2 + \frac{d_2(1-k)b_2}{a_2})(\beta + \delta_1 + \frac{d_1(1-k)b_2}{a_2})}{\beta + \delta_1 + \frac{d_1(1-k)b_2}{a_2}}.$$

Hence $\lambda_2 < 0$ and $\lambda_3 < 0$, and therefore all of three characteristic roots are negative. Consequently, $A_4(x_1^{**}, x_2^{**}, y^{**})$ is locally asymptotically stable. This ends the proof of Theorem 2.4. □

3 Global stability

As was shown in the previous section, under some suitable conditions, A_2 and A_4 can be locally asymptotically stable. In this section, we obtain some sufficient conditions that for the global asymptotical stability of the equilibria A_2 and A_4.

Theorem 3.1 *If*

$$\left(\delta_2 + \frac{d_2(1-k)b_2}{a_2}\right)\left(\beta + \delta_1 + \frac{d_1(1-k)b_2}{a_2}\right) - \alpha\beta > 0, \tag{3.1}$$

then $A_2(0, 0, \frac{b_2}{a_2})$ *is globally attractive, that is,*

$$\lim_{t\to+\infty} x_i(t) = 0, \quad i = 1, 2, \qquad \lim_{t\to+\infty} y(t) = \frac{b_2}{a_2}.$$

Proof For a small enough positive constant ε, it follows from (3.1) that

$$\left(\delta_2 + d_2(1-k)\left(\frac{b_2}{a_2} + \varepsilon\right)\right)\left(\beta + \delta_1 + d_1(1-k)\left(\frac{b_2}{a_2} + \varepsilon\right)\right) - \alpha\beta > 0. \tag{3.2}$$

Now let us consider the equation

$$\frac{dy}{dt} = y(b_2 - a_2 y). \tag{3.3}$$

From Theorem 2.1 of [37] we know that the unique positive equilibrium $y^* = \frac{b_2}{a_2}$ is globally stable, that is,

$$\lim_{t\to+\infty} y(t) = \frac{b_2}{a_2}. \tag{3.4}$$

Hence, for above $\varepsilon > 0$, there exists $T > 0$ such that

$$y(t) < \frac{b_2}{a_2} + \varepsilon \quad \text{for all } t \geq T. \tag{3.5}$$

Inequality (3.5), together with system (1.1), shows that, for $t > T$,

$$\begin{aligned}
\frac{dx_1}{dt} &\leq \alpha x_2 - \beta x_1 - \delta_1 x_1 - d_1(1-k)x_1\left(\frac{b_2}{a_2} + \varepsilon\right), \\
\frac{dx_2}{dt} &\leq \beta x_1 - \delta_2 x_2 - \gamma x_2^2 - d_2(1-k)x_2\left(\frac{b_2}{a_2} + \varepsilon\right).
\end{aligned} \tag{3.6}$$

Now let us consider the system

$$\begin{aligned} \frac{du_1}{dt} &= \alpha u_2 - \beta u_1 - \delta_1 u_1 - d_1(1-k)u_1\left(\frac{b_2}{a_2} + \varepsilon\right), \\ \frac{du_2}{dt} &= \beta u_1 - \delta_2 u_2 - \gamma u_2^2 - d_2(1-k)u_2\left(\frac{b_2}{a_2} + \varepsilon\right). \end{aligned} \tag{3.7}$$

It admits a boundary equilibrium $O(0,0)$. Now we show that, under assumption (3.1), $O(0,0)$ is globally asymptotically stable. We prove this by constructing some suitable Lyapunov function. Let us define the Lyapunov function

$$V_1(u_1,u_2) = \frac{\beta}{\beta + \delta_1 + d_1(1-k)(\frac{b_2}{a_2} + \varepsilon)} u_1 + u_2. \tag{3.8}$$

We can easily see that the function V_1 is zero at the boundary equilibrium $O(0,0)$ and is positive for all other positive values of u_1 and u_2. The time derivative of V_1 along the trajectories of (3.7) is

$$\begin{aligned} D^+V_1(t) = {} & \frac{\beta}{\beta + \delta_1 + d_1(1-k)(\frac{b_2}{a_2} + \varepsilon)} \\ & \times \left(\alpha u_2 - \beta u_1 - \delta_1 u_1 - d_1(1-k)u_1\left(\frac{b_2}{a_2} + \varepsilon\right)\right) \\ & + \beta u_1 - \delta_2 u_2 - \gamma u_2^2 - d_2(1-k)u_2\left(\frac{b_2}{a_2} + \varepsilon\right) \\ = {} & \Upsilon x_2 - \gamma x_2^2, \end{aligned} \tag{3.9}$$

where

$$\Upsilon = \frac{\alpha\beta}{\beta + \delta_1 + d_1(1-k)(\frac{b_2}{a_2} + \varepsilon)} - \delta_2 - d_2(1-k)\left(\frac{b_2}{a_2} + \varepsilon\right).$$

From (3.2) we have $\Upsilon < 0$. It then follows from (3.1) and (3.9) that $D^+V_1(t) < 0$ strictly for all $u_1, u_2 > 0$ except the boundary equilibrium $O(0,0)$, where $D^+V_1(t) = 0$. Thus, $V_1(u_1,u_2)$ satisfies Lyapunov's asymptotic stability theorem, and the boundary equilibrium $O(0,0)$ of system (3.7) is globally asymptotically stable, that is, if $(u_1(t), u_2(t))$ is any positive solution of system (3.7), then

$$\lim_{t\to+\infty} u_1(t) = 0, \qquad \lim_{t\to+\infty} u_2(t) = 0. \tag{3.10}$$

Let $(x_1(t), x_2(t), y(t))$ be any positive solution of system (1.1) with initial condition $(x_1(0), x_2(0), y(0)) = (x_{10}, x_{20}, y_0)$, and let $(u_1(t), u_2(t))$ be the positive solution of system (3.7) with initial condition $(u_1(0), u_2(0)) = (x_{10}, x_{20})$. It then follows from the differential inequality theory that

$$x_i(t) \le u_i(t) \quad \text{for all } t \ge 0. \tag{3.11}$$

The positivity of the solution of system (1.1), together with (3.10) and (3.11), leads to

$$0 \le \liminf_{t\to+\infty} x_i(t) \le \limsup_{t\to+\infty} x_i(t) \le \lim_{t\to+\infty} u_i(t) = 0, \quad i = 1, 2.$$

Hence

$$\lim_{t\to+\infty} x_i(t) = 0, \quad i = 1, 2. \tag{3.12}$$

Relation (3.4), together with (3.12), ends the proof of Theorem 3.1. □

Remark 3.1 Under the assumption $\alpha\beta < \delta_2(\beta + \delta_1)$, it follows from Lemma 2.1 that the first species will be driven to extinction. In this case, for all $0 < k < 1$, inequality (3.1) holds, and it follows from Theorem 3.1 that $A_2(0, 0, \frac{b_2}{a_2})$ is globally attractive, which means that the first species is still driven to extinction.

Remark 3.2 Under the assumption $\alpha\beta > \delta_2(\beta + \delta_1)$, it follows from Lemma 2.1 that the first species is globally asymptotically stable; however, if

$$k < 1 - \left(\frac{\alpha\beta}{\beta + \delta_1} - \delta_2\right)\frac{a_2}{b_2 d_2}, \tag{3.13}$$

then inequality (3.1) holds, and it follows from Theorem 3.1 that $A_2(0, 0, \frac{b_2}{a_2})$ is globally attractive, which means that the first species will be driven to extinction, that is, if the cover for the first species is not large enough, then with the influence of the second species, the first species will be driven to extinction.

Remark 3.3 At first sight, system (1.1) is not complicate, and we may conjecture that it is easy to investigate the stability of the equilibrium by constructing a suitable Lyapunov function as that of An and Lei [36]; however, this is impossible, since the term $-d_1(1-k)x_1 y$ in the first equation of system (1.1) cannot be dealt with directly. Here, by combining the differential inequality theory and the Lyapunov function we give a strict proof of Theorem 3.1. Such a method possibly could be applied to other situations.

Theorem 3.2 *If*

$$\alpha\beta - \left(\delta_2 + \frac{d_2(1-k)b_2}{a_2}\right)\left(\beta + \delta_1 + \frac{d_1(1-k)b_2}{a_2}\right) > 0, \tag{3.14}$$

then $A_4(x_1^{**}, x_2^{**}, y^{**})$ *is globally attractive.*

Proof By (3.14) there exists an small enough positive constant $\varepsilon_1 > 0$ (without loss of generality, we may assume that $\varepsilon_1 < \frac{b_2}{2a_2}$) such that

$$\alpha\beta - \left(\delta_2 + d_2(1-k)\left(\frac{b_2}{a_2} + \varepsilon_1\right)\right)\left(\beta + \delta_1 + d_1(1-k)\left(\frac{b_2}{a_2} + \varepsilon_1\right)\right) > 0 \tag{3.15}$$

and

$$\alpha\beta - \left(\delta_2 + d_2(1-k)\left(\frac{b_2}{a_2} - \varepsilon_1\right)\right)\left(\beta + \delta_1 + d_1(1-k)\left(\frac{b_2}{a_2} - \varepsilon_1\right)\right) > 0. \tag{3.16}$$

Now let us consider the equation

$$\frac{dy}{dt} = y(b_2 - a_2 y). \tag{3.17}$$

From Theorem 2.1 of [37] we know that the unique positive equilibrium $y^{**} = \frac{b_2}{a_2}$ is globally stable, that is,

$$\lim_{t \to +\infty} y(t) = \frac{b_2}{a_2}. \tag{3.18}$$

Hence, for above $\varepsilon_1 > 0$, there exists $T_1 > 0$ such that

$$\frac{b_2}{a_2} - \varepsilon_1 < y(t) < \frac{b_2}{a_2} + \varepsilon_1 \quad \text{for all } t \geq T_1. \tag{3.19}$$

The right-hand side of (3.19), together with system (1.1), shows that, for $t > T_1$,

$$\begin{aligned} \frac{dx_1}{dt} &\leq \alpha x_2 - \beta x_1 - \delta_1 x_1 - d_1(1-k)x_1\left(\frac{b_2}{a_2} + \varepsilon_1\right), \\ \frac{dx_2}{dt} &\leq \beta x_1 - \delta_2 x_2 - \gamma x_2^2 - d_2(1-k)x_2\left(\frac{b_2}{a_2} + \varepsilon_1\right). \end{aligned} \tag{3.20}$$

Now let us consider the system

$$\begin{aligned} \frac{du_1}{dt} &= \alpha u_2 - \beta u_1 - \delta_1 u_1 - d_1(1-k)u_1\left(\frac{b_2}{a_2} + \varepsilon_1\right), \\ \frac{du_2}{dt} &= \beta u_1 - \delta_2 u_2 - \gamma u_2^2 - d_2(1-k)u_2\left(\frac{b_2}{a_2} + \varepsilon_1\right). \end{aligned} \tag{3.21}$$

Since inequality (3.15) holds, system (3.21) admits a unique positive equilibrium $M(u_1^{**}, u_2^{**})$, where

$$\begin{aligned} u_1^{**} &= \frac{\alpha u_2^{**}}{\beta + \delta_1 + d_1(1-k)(\frac{b_2}{a_2} + \varepsilon_1)}, \\ u_2^{**} &= \frac{\alpha\beta - (\delta_2 + d_2(1-k)(\frac{b_2}{a_2} + \varepsilon_1))(\beta + \delta_1 + d_1(1-k)(\frac{b_2}{a_2} + \varepsilon_1))}{(\beta + \delta_1 + d_1(1-k)(\frac{b_2}{a_2} + \varepsilon_1))\gamma}. \end{aligned} \tag{3.22}$$

Obviously, u_1^{**} and u_2^{**} satisfy the equations

$$\begin{aligned} &\alpha u_2^{**} - \beta u_1^{**} - \delta_1 u_1^{**} - d_1(1-k)u_1^{**}\left(\frac{b_2}{a_2} + \varepsilon_1\right) = 0, \\ &\beta u_1^{**} - \delta_2 u_2^{**} - \gamma\left(u_2^{**}\right)^2 - d_2(1-k)u_2^{**}\left(\frac{b_2}{a_2} + \varepsilon_1\right) = 0. \end{aligned} \tag{3.23}$$

Now we show that $M(u_1^{**}, u_2^{**})$ is globally asymptotically stable. We will prove this assertion by constructing some suitable Lyapunov function. Let us define the Lyapunov function

$$V_2(u_1, u_2) = k_1\left(u_1 - u_1^{**} - u_1^{**}\ln\frac{u_1}{u_1^{**}}\right) + k_2\left(u_2 - u_2^{**} - u_2^{**}\ln\frac{u_2}{u_2^{**}}\right),$$

where k_1, k_2 are some positive constants to be determined later.

We can easily see that the function V_2 is zero at the equilibrium $M(u_1^{**}, u_2^{**})$ and is positive for all other positive values of u_1 and u_2. The time derivative of V_2 along the trajectories of (3.21) is

$$\begin{aligned} &D^+V_2(t) \\ &\quad = k_1\frac{u_1-u_1^{**}}{u_1}\dot{u}_1 + k_2\frac{u_2-u_2^{**}}{u_2}\dot{u}_2 \\ &\quad = k_1\frac{u_1-u_1^{**}}{u_1}\left(\alpha u_2 - \left(\beta+\delta_1+d_1(1-k)\left(\frac{b_2}{a_2}+\varepsilon_1\right)\right)u_1\right) \\ &\qquad + k_2\frac{u_2-u_2^{**}}{u_2}\left(\beta u_1 - \delta_2 u_2 - \gamma u_2^2 - d_2(1-k)\left(\frac{b_2}{a_2}+\varepsilon_1\right)u_2\right). \end{aligned} \tag{3.24}$$

Note that from the relationship of u_1^{**} and u_2^{**} (see (3.22) for more detail) we have

$$\begin{aligned} &\alpha u_2 - \left(\beta+\delta_1+d_1(1-k)\left(\frac{b_2}{a_2}+\varepsilon_1\right)\right)u_1 \\ &\quad = \frac{\alpha}{u_1^{**}}\left(u_2u_1^{**} - u_1u_2^{**}\right) + \alpha u_1\frac{u_2^{**}}{u_1^{**}} \\ &\qquad - \left(\beta+\delta_1+d_1(1-k)\left(\frac{b_2}{a_2}+\varepsilon_1\right)\right)u_1 \\ &\quad = \frac{\alpha}{u_1^{**}}\left(u_2u_1^{**} - u_2u_1 + u_2u_1 - u_1u_2^{**}\right) \\ &\quad = \frac{\alpha}{u_1^{**}}\left(-u_2\left(u_1-u_1^{**}\right) + u_1\left(u_2-u_2^{**}\right)\right). \end{aligned} \tag{3.25}$$

Also, from the expression of u_2^{**}, we have

$$\begin{aligned} &\beta u_1 - \delta_2 u_2 - \gamma u_2^2 - d_2(1-k)\left(\frac{b_2}{a_2}+\varepsilon_1\right)u_2 \\ &\quad = \frac{\beta}{u_2^{**}}\left(u_1u_2^{**} - u_2u_1^{**}\right) + \beta u_2\frac{u_1^{**}}{u_2^{**}} \\ &\qquad - \left(\delta_2 + d_2(1-k)\left(\frac{b_2}{a_2}+\varepsilon_1\right)\right)u_2 - \gamma u_2^2 \\ &\quad = \frac{\beta}{u_2^{**}}\left(u_1u_2^{**} - u_1u_2 + u_1u_2 - u_2u_1^{**}\right) - \gamma u_2^2 \\ &\qquad + \left(\frac{\alpha\beta}{\beta+\delta_1+d_1(1-k)(\frac{b_2}{a_2}+\varepsilon_1)} - \delta_2 - d_2(1-k)\left(\frac{b_2}{a_2}+\varepsilon_1\right)\right)u_2 \\ &\quad = \frac{\beta}{u_2^{**}}\left(u_1\left(u_2^{**}-u_2\right) + u_2\left(u_1-u_1^{**}\right)\right) + \gamma u_2^{**}u_2 - \gamma u_2^2. \end{aligned} \tag{3.26}$$

Applying (3.25) and (3.26) to (3.24) leads to

$$\begin{aligned} D^+V_2(t) = {} & k_1\frac{u_1-u_1^{**}}{u_1}\frac{\alpha}{u_1^{**}}\left(-u_2\left(u_1-u_1^{**}\right) + u_1\left(u_2-u_2^{**}\right)\right) \\ & + k_2\frac{u_2-u_2^{**}}{u_2}\frac{\beta}{u_2^{**}}\left(-u_1\left(u_2-u_2^{**}\right) + u_2\left(u_1-u_1^{**}\right)\right) \end{aligned}$$

$$\begin{aligned}
&- k_2\gamma\left(u_2-u_2^{**}\right)^2\\
&= -\frac{k_1\alpha u_2}{u_1u_1^{**}}\left(u_1-u_1^{**}\right)^2+\left(\frac{k_1\alpha}{u_1^{**}}+\frac{k_2\beta}{u_2^{**}}\right)\left(u_1-u_1^{**}\right)\left(u_2-u_2^{**}\right)\\
&- \frac{k_2\beta u_1}{u_2u_2^{**}}\left(u_2-u_2^{**}\right)^2-k_2\gamma\left(u_2-u_2^{**}\right)^2.
\end{aligned}$$

Now let us choose $k_2 = 1$ and $k_1 = \frac{\beta u_1^{**}}{u_2^{**}\alpha}$. Then

$$\begin{aligned}
D^+V_2(t) = {}& -\frac{\beta}{u_2^{**}}\left[\sqrt{\frac{u_2}{u_1}}\left(u_1-u_1^{**}\right)-\sqrt{\frac{u_1}{u_2}}\left(u_2-u_2^{**}\right)\right]^2\\
&- \gamma\left(u_2-u_2^{**}\right)^2.
\end{aligned} \tag{3.27}$$

Hence $D^+V_2(t) < 0$ strictly for all $u_1, u_2 > 0$ except the positive equilibrium $M(u_1^{**}, u_2^{**})$, where $D^+V_2(t) = 0$. Thus, $V_2(t)$ satisfies Lyapunov's asymptotic stability theorem, and the positive equilibrium $M(u_1^{**}, u_2^{**})$ of system (3.21) is globally asymptotically stable, that is, if $(u_1(t), u_2(t))$ is any positive solution of system (3.7), then

$$\lim_{t\to+\infty} u_1(t) = u_1^{**}, \qquad \lim_{t\to+\infty} u_2(t) = u_2^{**}. \tag{3.28}$$

Let $(x_1(t), x_2(t), y(t))$ be any positive solution of system (1.1) with initial condition $(x_1(0), x_2(0), y(0) = (x_{10}, x_{20}, y_0)$, and let $(u_1(t), u_2(t))$ be the positive solution of system (3.21) with initial condition $(u_1(0), u_2(0)) = (x_{10}, x_{20})$. It then follows from the differential inequality theory that

$$x_i(t) \le u_i(t) \quad \text{for all } t \ge 0. \tag{3.29}$$

The positivity of the solution of system (1.1), together with (3.28) and (3.29), leads to

$$\limsup_{t\to+\infty} x_i(t) \le \lim_{t\to+\infty} u_i(t) = u_i^{**}, \quad i = 1, 2. \tag{3.30}$$

On the other hand, the left-hand side of (3.19), together with system (1.1), shows that, for $t > T_1$,

$$\begin{aligned}
&\frac{dx_1}{dt} \ge \alpha x_2 - \beta x_1 - \delta_1 x_1 - d_1(1-k)x_1\left(\frac{b_2}{a_2}-\varepsilon_1\right),\\
&\frac{dx_2}{dt} \ge \beta x_1 - \delta_2 x_2 - \gamma x_2^2 - d_2(1-k)x_2\left(\frac{b_2}{a_2}-\varepsilon_1\right).
\end{aligned} \tag{3.31}$$

Now let us consider the system

$$\begin{aligned}
&\frac{dv_1}{dt} = \alpha v_2 - \beta v_1 - \delta_1 v_1 - d_1(1-k)v_1\left(\frac{b_2}{a_2}-\varepsilon_1\right),\\
&\frac{dv_2}{dt} = \beta v_1 - \delta_2 v_2 - \gamma u_2^2 - d_2(1-k)v_2\left(\frac{b_2}{a_2}-\varepsilon_1\right).
\end{aligned} \tag{3.32}$$

Since inequality (3.16) holds, system (3.32) admits a unique positive equilibrium $N(v_1^{**}, v_2^{**})$, where

$$\begin{aligned} v_1^{**} &= \frac{\alpha v_2^{**}}{\beta + \delta_1 + d_1(1-k)(\frac{b_2}{a_2} - \varepsilon_1)}, \\ v_2^{**} &= \frac{\alpha\beta - (\delta_2 + d_2(1-k)(\frac{b_2}{a_2} - \varepsilon_1))(\beta + \delta_1 + d_1(1-k)(\frac{b_2}{a_2} - \varepsilon_1))}{(\beta + \delta_1 + d_1(1-k)(\frac{b_2}{a_2} - \varepsilon_1))\gamma}. \end{aligned} \tag{3.33}$$

Obviously, v_1^{**} and v_2^{**} satisfy the equations

$$\begin{aligned} &\alpha v_2^{**} - \beta v_1^{**} - \delta_1 v_1^{**} - d_1(1-k)v_1^{**}\left(\frac{b_2}{a_2} - \varepsilon_1\right) = 0, \\ &\beta v_1^{**} - \delta_2 v_2^{**} - \gamma\left(v_2^{**}\right)^2 - d_2(1-k)v_2^{**}\left(\frac{b_2}{a_2} - \varepsilon_1\right) = 0. \end{aligned} \tag{3.34}$$

Now we show that $N(v_1^{**}, v_2^{**})$ is globally asymptotically stable. We will prove this assertion by constructing some suitable Lyapunov function. Let us define the Lyapunov function

$$V_3(v_1, v_2) = l_1\left(v_1 - v_1^{**} - v_1^{**}\ln\frac{v_1}{v_1^{**}}\right) + l_2\left(v_2 - v_2^{**} - v_2^{**}\ln\frac{v_2}{v_2^{**}}\right),$$

where l_1, l_2 are some positive constants to be determined later.

We easily see that the function V_3 is zero at the equilibrium $N(v_1^{**}, v_2^{**})$ and is positive for all other positive values of v_1 and v_2. The time derivative of V_3 along the trajectories of (3.32) is

$$\begin{aligned} &D^+V_3(t) \\ &\quad = l_1\frac{v_1 - v_1^{**}}{v_1}\left(\alpha v_2 - \left(\beta + \delta_1 + d_1(1-k)\left(\frac{b_2}{a_2} - \varepsilon_1\right)\right)v_1\right) \\ &\qquad + l_2\frac{v_2 - v_2^{**}}{v_2}\left(\beta v_1 - \delta_2 v_2 - \gamma v_2^2 - d_2(1-k)\left(\frac{b_2}{a_2} - \varepsilon_1\right)v_2\right). \end{aligned} \tag{3.35}$$

Note that from the relationship of v_1^{**} and v_2^{**} (see (3.34) for more detail) we have

$$\begin{aligned} &\alpha v_2 - \left(\beta + \delta_1 + d_1(1-k)\left(\frac{b_2}{a_2} + \varepsilon_1\right)\right)v_1 \\ &\quad = \frac{\alpha}{v_1^{**}}\left(-v_2\left(v_1 - v_1^{**}\right) + v_1\left(v_2 - v_2^{**}\right)\right). \end{aligned} \tag{3.36}$$

Also, from the expression of v_2^{**} we have

$$\begin{aligned} &\beta v_1 - \delta_2 v_2 - \gamma v_2^2 - d_2(1-k)\left(\frac{b_2}{a_2} - \varepsilon_1\right)v_2 \\ &\quad = \frac{\beta}{v_2^{**}}\left(v_1\left(v_2^{**} - v_2\right) + v_2\left(v_1 - v_1^{**}\right)\right) + \gamma v_2^{**}v_2 - \gamma v_2^2. \end{aligned} \tag{3.37}$$

Applying (3.36) and (3.37) to (3.35) leads to

$$D^+V_3(t) = -\frac{l_1\alpha v_2}{v_1 v_1^{**}}\left(v_1 - v_1^{**}\right)^2 + \left(\frac{l_1\alpha}{v_1^{**}} + \frac{l_2\beta}{v_2^{**}}\right)\left(v_1 - v_1^{**}\right)\left(v_2 - v_2^{**}\right) - \frac{l_2\beta v_1}{v_2 v_2^{**}}\left(v_2 - v_2^{**}\right)^2 - l_2\gamma\left(v_2 - v_2^{**}\right)^2.$$

Now let us choose $l_2 = 1$ and $l_1 = \frac{\beta v_1^{**}}{v_2^{**}\alpha}$. Then

$$D^+V_3(t) = -\frac{\beta}{v_2^{**}}\left[\sqrt{\frac{v_2}{v_1}}\left(v_1 - v_1^{**}\right) - \sqrt{\frac{v_1}{v_2}}\left(v_2 - v_2^{**}\right)\right]^2 - \gamma\left(v_2 - v_2^{**}\right)^2. \tag{3.38}$$

Hence $D^+V_3(t) < 0$ strictly for all $v_1, v_2 > 0$ except the positive equilibrium $N(v_1^{**}, v_2^{**})$, where $D^+V_3(t) = 0$. Thus, $V_3(t)$ satisfies Lyapunov's asymptotic stability theorem, and the positive equilibrium $N(v_1^{**}, v_2^{**})$ of system (3.32) is globally asymptotically stable, that is, if $(v_1(t), v_2(t))$ is any positive solution of system (3.32), then

$$\lim_{t\to+\infty} v_1(t) = v_1^{**}, \qquad \lim_{t\to+\infty} v_2(t) = v_2^{**}. \tag{3.39}$$

Let $(x_1(t), x_2(t), y(t))$ be any positive solution of system (1.1) with initial condition $(x_1(0), x_2(0), y(0) = (x_{10}, x_{20}, y_0)$, and let $(v_1(t), v_2(t))$ be the positive solution of system (3.32) with initial condition $(v_1(0), v_2(0)) = (x_{10}, x_{20})$. It then follows from the differential inequality theory that

$$x_i(t) \geq v_i(t) \quad \text{for all } t \geq 0. \tag{3.40}$$

The positivity of the solution of system (1.1), together with (3.39) and (3.40), leads to

$$\liminf_{t\to+\infty} x_i(t) \geq \lim_{t\to+\infty} v_i(t) = v_i^{**}, \quad i = 1, 2. \tag{3.41}$$

Relation (3.30), together with (3.41), leads to

$$v_i^{**} \leq \liminf_{t\to+\infty} x_i(t) \leq \limsup_{t\to+\infty} x_i(t) \leq u_i^{**}. \tag{3.42}$$

Noting that ε_1 is any small enough positive constant, from (2.8), (3.22), and (3.33) we have

$$u_i^{**} \to x_i^{**}, \qquad v_i^{**} \to x_i^{**} \quad \text{as } \varepsilon_1 \to 0. \tag{3.43}$$

Letting $\varepsilon_1 \to 0$ in (3.41), by (3.43) it immediately follows that

$$\lim_{t\to+\infty} x_i(t) = x_i^{**}, \quad i = 1, 2. \tag{3.44}$$

Relation (3.44), together with (3.17), ends the proof of Theorem 3.2. □

Remark 3.4 Condition (3.14) is necessary to ensure the existence of the positive equilibrium. Theorem 3.2 shows that if the positive equilibrium exists, then it is globally asymptotically stable, and hence it is impossible for the system to have the bifurcation phenomenon.

Remark 3.5 If $\alpha\beta > \delta_2(\beta + \delta_1)$, then for large enough k (k is close to 1) inequality (3.14) can hold, and from Lemma 2.1 we know that in this case, system (2.1) admits a unique positive equilibrium. In other words, if system (2.1) admits a unique positive equilibrium, then for the amensalism model, if the influence of the second species to the first species is limited, then the system still admits a unique globally asymptotically stable positive equilibrium.

4 The influence of the partial cover

From (2.8) we easily see that the final density of the immature and mature species are relevant to the partial cover, and hence one interesting issue is to find out a relationship between the final density of the species and the partial cover.

Note that from the second equality of (2.8) we have

$$x_2^{**} = \frac{1}{\gamma}\left(\frac{\alpha\beta}{\beta + \delta_1 + \frac{d_1(1-k)b_2}{a_2}} - \left(\delta_2 + \frac{d_2(1-k)b_2}{a_2}\right)\right). \tag{4.1}$$

Hence

$$\frac{dx_2^{**}(k)}{dk} = \frac{1}{\gamma}\left(\frac{\alpha\beta d_1 b_2}{(\beta + \delta_1 + \frac{d_1(1-k)b_2}{a_2})^2 a_2} + \frac{d_2 b_2}{a_2}\right) > 0. \tag{4.2}$$

Also, from the first equality of (2.8) and (4.2) we have

$$\begin{aligned} \frac{dx_1^{**}(k)}{dk} &= \frac{\alpha d_1 b_2 x_2^{**}}{(\beta + \delta_1 + \frac{d_1(1-k)b_2}{a_2})^2 a_2} \\ &\quad + \frac{\alpha}{\beta + \delta_1 + \frac{d_1(1-k)b_2}{a_2}} \frac{dx_2^{**}(k)}{dk} \\ &> 0, \end{aligned} \tag{4.3}$$

that is, increasing the partial cover will increase both the immature and mature first species density.

5 Example

Now let us consider the following example.

Example 5.1 Consider the two-species stage-structured amensalism model with a cover for the first species:

$$\begin{aligned} \frac{dx_1}{dt} &= 4x_2 - x_1 - x_1 - (1-k)x_1 y, \\ \frac{dx_2}{dt} &= x_1 - x_2 - x_2^2 - (1-k)x_2 y, \\ \frac{dy}{dt} &= y(1-y). \end{aligned} \tag{5.1}$$

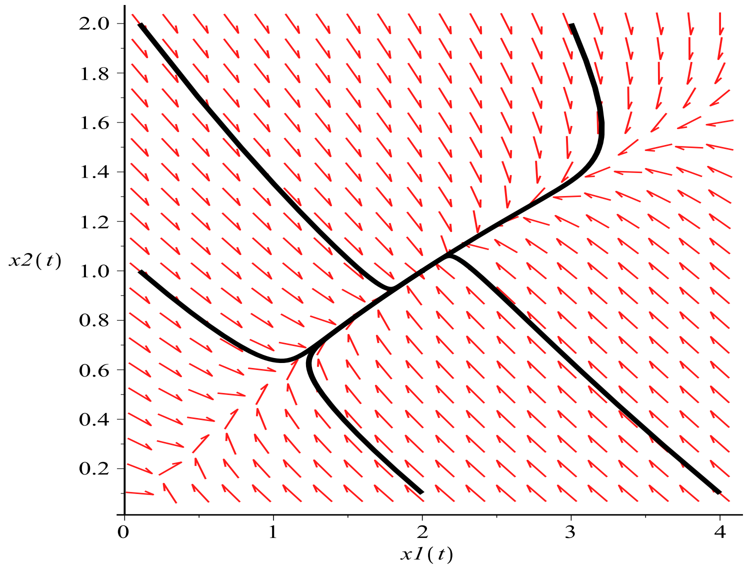

Figure 1 Dynamics behavior of system (5.2), the initial conditions $(x_1(0), x_2(0)) = (4, 0.1)$, $(3, 2)$, $(0.1, 2)$, $(0.1, 1)$, and $(2, 0.1)$, respectively

Here we choose $\alpha = 4, \beta = \delta_1 = \delta_2 = \gamma = a_2 = b_2 = d_1 = d_2 = 1$. Hence

$$\alpha\beta = 4 > 2 = \delta_2(\beta + \delta_1).$$

Then, for the subsystem

$$\begin{aligned} \frac{dx_1}{dt} &= 4x_2 - x_1 - x_1, \\ \frac{dx_2}{dt} &= x_1 - x_2 - x_2^2, \end{aligned} \tag{5.2}$$

it follows from Lemma 2.1 that the unique positive equilibrium $A(2,1)$ of system (5.2) is globally asymptotically stable, that is, without the influence of the second species, the first species will be globally asymptotically stable (Fig. 1 supports this assertion).

System (5.1) has two equilibria $O(0,0,1)$ and $M(x_1^*(k), x_2^*(k), 1)$, where

$$x_1^*(k) = -\frac{4(k^2 - 5k + 2)}{(k-3)^2}, \qquad x_2^*(k) = \frac{k^2 - 5k + 2}{k - 3}. \tag{5.3}$$

By simple computation we know that, for $0 < k < 0.43845$, the boundary equilibrium $O(0,0,1)$ is globally attractive, and for $0.43845 < k < 1$, the unique positive equilibrium $M(x_1^*(k), x_2^*(k), 1)$ is globally attractive.

(1) Now let us choose $k = 0.9$ in system (5.1). Then $M(1.53288, 0.80476, 1)$ is globally attractive. Figures 2–4 support this assertion.

(2) Now let us choose $k = 0.2$ in system (5.1). Then $M(0,0,1)$ is globally attractive. Figures 5–7 support this assertion.

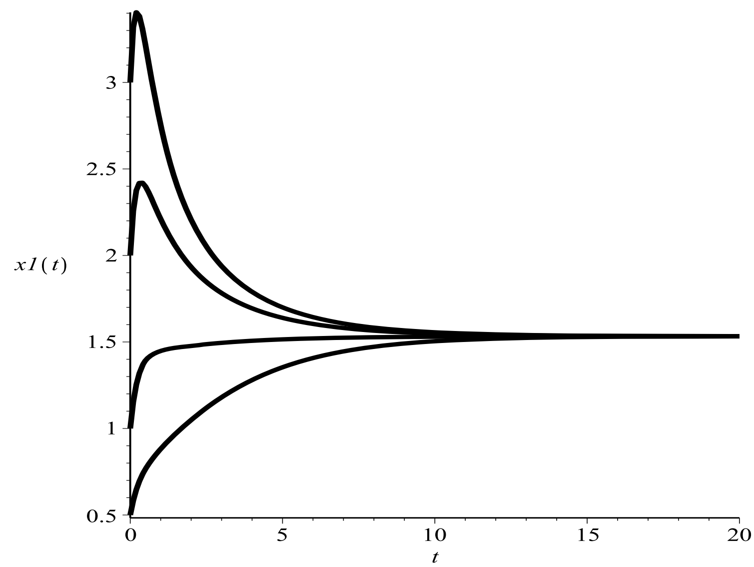

Figure 2 Dynamics behavior of the first component $x_1(t)$ of system (5.1); here we take $k = 0.9$ and the initial conditions $(x_1(0), x_2(0), y(0)) = (0.5, 0.5, 0.5), (2, 2, 2), (1, 1, 1)$, and $(3, 3, 3)$, respectively

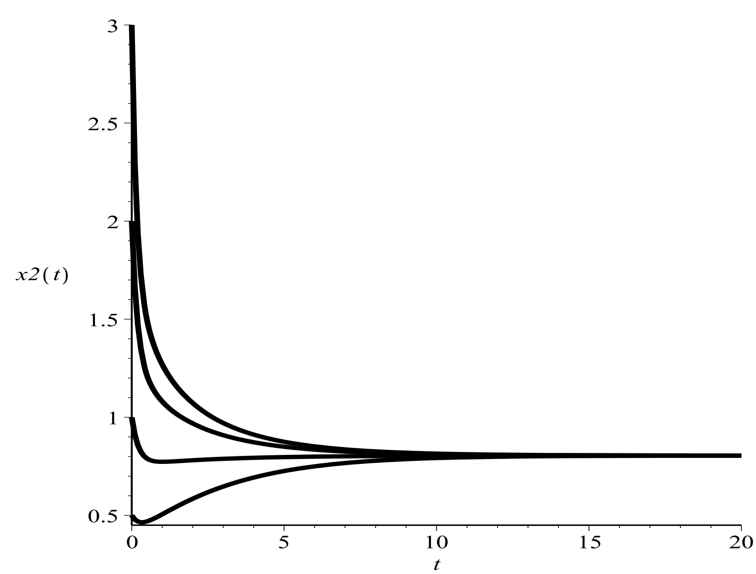

Figure 3 Dynamics behavior of the second component $x_2(t)$ of system (5.1); here we take $k = 0.9$ and the initial conditions $(x_1(0), x_2(0), y(0)) = (0.5, 0.5, 0.5), (2, 2, 2), (1, 1, 1)$, and $(3, 3, 3)$, respectively

6 Conclusion

During the lase decade, many scholars [13–19] studied the dynamic behavior of the amensalism model; however, only recently, scholars [14, 16, 18] studied the influence of the partial cover to the traditional two-species amensalism model. In this paper, for first time, we propose a two-species stage-structured amensalism model with a cover for the first species.

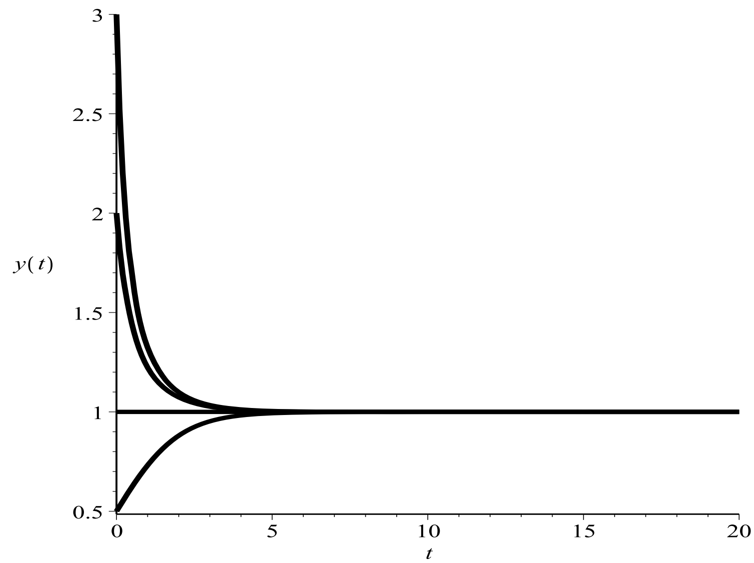

Figure 4 Dynamics behavior of the third component $y(t)$ of system (5.1); here we take $k = 0.9$ and the initial conditions $(x_1(0), x_2(0), y(0)) = (0.5, 0.5, 0.5), (2, 2, 2), (1, 1, 1)$, and $(3, 3, 3)$, respectively

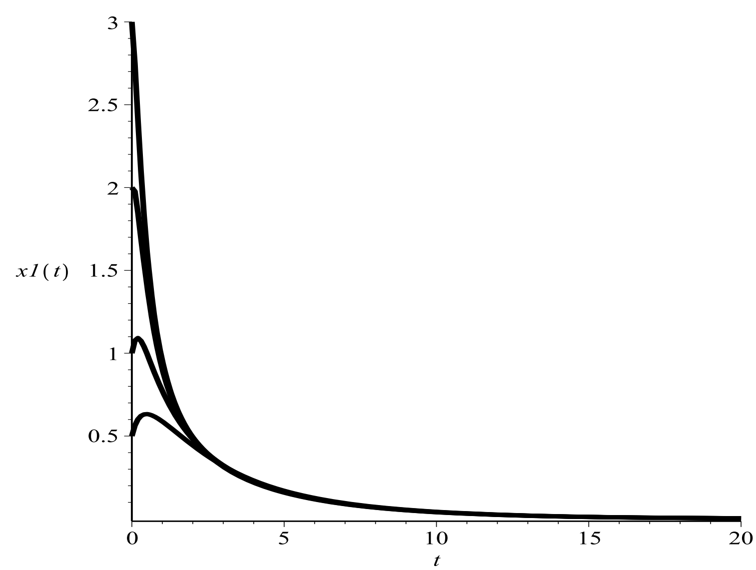

Figure 5 Dynamics behavior of the first component $x_1(t)$ of system (5.1); here we take $k = 0.2$ and the initial conditions $(x_1(0), x_2(0), y(0)) = (0.5, 0.5, 0.5), (2, 2, 2), (1, 1, 1)$, and $(3, 3, 3)$, respectively

Though at first sight, the system seems very simple, note that the third equation of system (1.1) is independent of x_1 and x_2, and thus the Lyapunov method cannot be applied directly to investigate the stability property of system (1.1). By combining the differential inequality theory and the Lyapunov function method we are able to investigate the global stability property of the boundary and positive equilibrium. Theorem 3.2 shows that if the positive equilibrium exists, then it is globally attractive, and the final density of the first species is an increasing function of the partial cover.

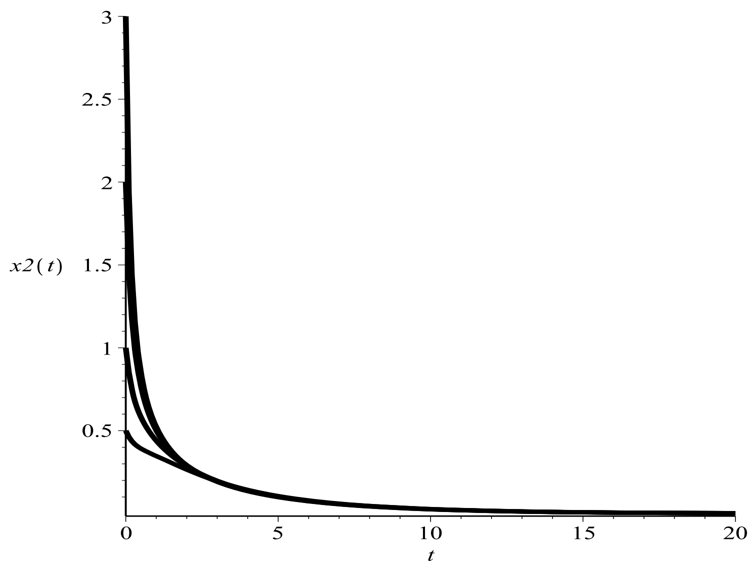

Figure 6 Dynamics behavior of the second component $x_2(t)$ of system (5.1); here we take $k = 0.2$ and the initial conditions $(x_1(0), x_2(0), y(0)) = (0.5, 0.5, 0.5), (2, 2, 2), (1, 1, 1)$, and $(3, 3, 3)$, respectively

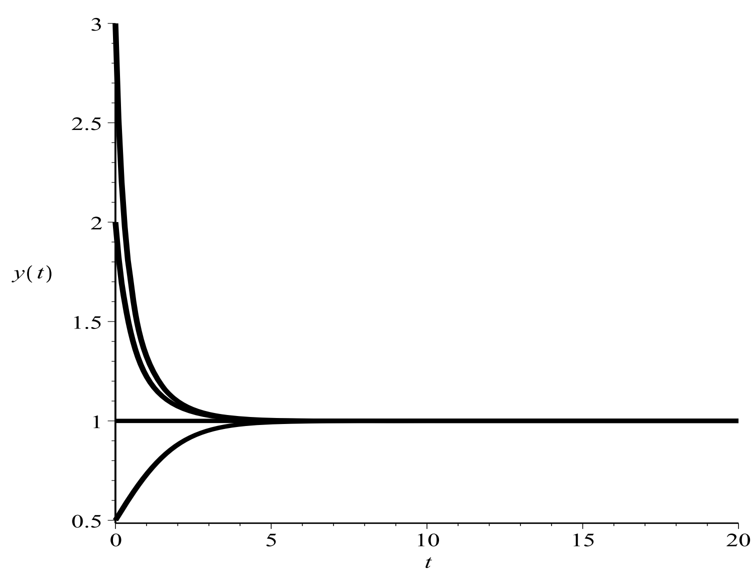

Figure 7 Dynamics behavior of the third component $y(t)$ of system (5.1); here we take $k = 0.2$ and the initial conditions $(x_1(0), x_2(0), y(0)) = (0.5, 0.5, 0.5), (2, 2, 2), (1, 1, 1)$, and $(3, 3, 3)$, respectively

We mention here that the method used in this paper can be applied to investigate the stability property of the other ecosystem. We leave this for the future study.

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Competing interests
The author declares that there is no conflict of interests.

Authors' contributions
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