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Lyapunov-type inequalities for certain higher-order difference equations with mixed non-linearities

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Abstract

In this paper, we establish some new Lyapunov-type inequalities for some higher-order difference equations with boundary conditions. The obtained inequalities generalize the existing results in the literature.

Keywords: Lyapunov-type inequality; Difference equation; Higher-order; Anti-periodic boundary conditions

1 Introduction

During the past decades, continuous and discrete integral inequalities have attracted the attention of many researchers (see [1–59] and the references therein). Particularly, there have been plenty of references focused on the Lyapunov-type inequality and many of its generalizations due to its broad applications in the study of various properties of solutions of differential and difference equations such as oscillation theory, disconjugacy, and eigenvalue problems (see [1, 2, 5–7, 9, 13, 15, 21, 24, 27–29, 37, 39, 45, 48, 57, 59] and the references therein).

Compared with a large number of references devoted to continuous Lyapunov-type inequalities, there is not much done for discrete Lyapunov-type inequalities (see [6, 13, 21, 29, 39, 59] and the references therein). For example, Zhang and Tang [29] considered the following even order difference equation:

$$\triangle^{2k}u(n)+(-1)^{k-1}q(n)u(n+1)=0, \tag{1}$$

where $\triangle$ is the usual forward difference operator defined by $\triangle u(n)=u(n+1)-u(n)$, $k\in\mathbb{N}$, $n\in\mathbb{Z}$ and $q(n)$ is a real-valued function defined on $\mathbb{Z}$. Under the following boundary conditions

$$\triangle^{2i}u(a)=\triangle^{2i}u(b)=0,\quad i=0,1,\dots,k-1;\qquad u(n)\not\equiv 0,\quad n\in\mathbb{Z}[a,b], \tag{2}$$

where $a,b\in\mathbb{N}$, $\mathbb{Z}[a,b]=\{a,a+1,\dots,b-1,b\}$, they obtained the following result:

Assume that $k \in \mathbb{N}$ and $q(n)$ is a real-valued function on $\mathbb{Z}$. If (1) has a solution $u(n)$ satisfying the boundary conditions (2), then

$$\sum_{n=a}^{b-1}\bigl[\bigl|q(n)\bigr|(n-a+1)(b-n-1)\bigr] \geq \frac{2^{3(k-1)}}{(b-a)^{2k-3}}. \tag{3}$$

Recently, Liu and Tang [21] studied the following m-order difference equation:

$$\bigl|\triangle^m u(n)\bigr|^{p-2}\triangle^m u(n) + r(n)\bigl|u(n)\bigr|^{p-2}u(n) = 0, \tag{4}$$

where $m \in \mathbb{N}$, $n \in \mathbb{Z}$ and $r(n)$ is a real-valued function defined on $\mathbb{Z}$, $p > 1$ is a constant, and $u(n)$ satisfies the following anti-periodic boundary conditions:

$$\triangle^i u(a) + \triangle^i u(b) = 0, \quad i = 0, 1, \ldots, m-1; \qquad u(n) \not\equiv 0, \quad n \in \mathbb{Z}[a,b], \tag{5}$$

and they obtained the following result:

If (4) has a nonzero solution $u(n)$ satisfying the anti-periodic boundary conditions (5), then

$$\sum_{n=a}^{b-1}\bigl|r(n)\bigr|^q \geq \frac{2^{mp}}{(b-a)^{mp-1}}, \tag{6}$$

where q is a conjugate exponent of p.

In the present paper, we shall establish a new discrete Lyapunov-type inequality for the following m-order difference equation with mixed nonlinearities:

$$\bigl|\triangle^m u(n)\bigr|^{p-2}\triangle^m u(n) + \sum_{i=0}^{m-1} r_i(n)\bigl|\triangle^i u(n)\bigr|^{p-2}\triangle^i u(n) = 0, \tag{7}$$

with the anti-periodic boundary conditions (5), where $m \in \mathbb{N}$, $n \in \mathbb{Z}$, $p > 1$ is a constant and $r_i(n)$ $(i = 0, 1, \ldots, m-1)$ are real-valued functions defined on $\mathbb{Z}$. Further, we will also prove a new Lyapunov-type inequality for the $2m$-order difference equation

$$\bigl|\triangle^{2m} u(n)\bigr|^{p-2}\triangle^{2m} u(n) + (-1)^{m-1} r(n)\bigl|u(n+1)\bigr|^{q-2}u(n+1) = 0, \tag{8}$$

with the following boundary conditions:

$$\triangle^{2i} u(a) = \triangle^{2i} u(b) = 0, \quad i = 0, 1, \ldots, m-1; \qquad u(n) \not\equiv 0, \quad n \in \mathbb{Z}[a,b], \tag{9}$$

where $m \in \mathbb{N}$, $p \geq q > 2$ are constants, $n \in \mathbb{Z}$ and $r(n)$ is a real-valued function defined on $\mathbb{Z}$. Our works extend the results in [21] and [29].

2 Main results

Lemma 2.1 ([1]) *If A is positive and B, z are nonnegative, then*

$$Az^{2\tau} - Bz^{\sigma} + \Gamma_{\sigma\tau}A^{-\sigma/(2-\sigma)}B^{2\tau/(2-\sigma)} \geq 0 \tag{10}$$

for any $\sigma \in (0, 2\tau)$, *where*

$$\Gamma_{\sigma\tau} = (2\tau - \sigma)\sigma^{\sigma/(2\tau-\sigma)}\tau^{-2\tau/(2\tau-\sigma)}2^{-2\tau/(2\tau-\sigma)} > 0$$

with equality holding if and only if $B = z = 0$.

Lemma 2.2 ([29]) *Assume that* $u(n)$ *is a real-valued function on* $\mathbb{Z}[a,b]$, $u(a) = u(b) = 0$. *Then*

$$|u(n)| \le \frac{(n-a)(b-n)}{b-a}\sum_{s=a}^{b-1}|\triangle^2 u(s)|, \quad \forall n \in \mathbb{Z}(a, b-1), \tag{11}$$

$$\sum_{n=a}^{b-1}|u(n)| \le \frac{1}{2}\sum_{n=a}^{b-1}\bigl[(n-a+1)(b-n-1)|\triangle^2 u(n)|\bigr] \le \frac{(b-a)^2}{8}\sum_{n=a}^{b-1}|\triangle^2 u(n)|. \tag{12}$$

We now state the main theorem of this paper.

Theorem 2.1 *If* $u(n)$ *is a nonzero solution of Eq.* (7) *satisfying the anti-periodic boundary conditions* (5), *then*

$$\sum_{i=0}^{m-1}\frac{(b-a)^{(m-i-1/p)(p-1)}}{2^{(m-i)(p-1)}}\left(\sum_{n=a}^{b-1}|r_i(n)|^q\right)^{1/q} \ge 1, \tag{13}$$

where q is the Hölder conjugate exponent of p, i.e., $1/p + 1/q = 1$.

Proof Since the nonzero solution $u(n)$ of Eq. (7) satisfies the anti-periodic boundary conditions (5), then $u(a) + u(b) = 0$. For $n \in \mathbb{Z}[a,b]$, we have

$$\begin{aligned} u(n) = u(n) - \frac{1}{2}\bigl[u(a) + u(b)\bigr] &= \frac{1}{2}\sum_{k=a}^{n-1}\bigl[u(k+1) - u(k)\bigr] - \frac{1}{2}\sum_{k=n}^{b-1}\bigl[u(k+1) - u(k)\bigr] \\ &= \frac{1}{2}\sum_{k=a}^{n-1}\triangle u(k) - \frac{1}{2}\sum_{k=n}^{b-1}\triangle u(k). \end{aligned} \tag{14}$$

Then

$$|u(n)| \le \frac{1}{2}\sum_{k=a}^{b-1}|\triangle u(k)|. \tag{15}$$

Applying discrete Hölder's inequality

$$\sum_{k=a}^{b-1}|f(k)g(k)| \le \left(\sum_{k=a}^{b-1}|f(k)|^\alpha\right)^{1/\alpha}\left(\sum_{k=a}^{b-1}|g(k)|^\beta\right)^{1/\beta} \tag{16}$$

to (15) with $f(k) = 1$, $g(k) = |\triangle u(k)|$, $\alpha = q$, and $\beta = p$, we obtain that

$$|u(n)| \le \frac{1}{2}\sum_{k=a}^{b-1}|\triangle u(k)| \le \frac{1}{2}(b-a)^{1/q}\left(\sum_{k=a}^{b-1}|\triangle u(k)|^p\right)^{1/p}. \tag{17}$$

Similarly, we get

$$\begin{aligned} \left|\triangle^i u(n)\right| &\le \frac{1}{2}\sum_{k=a}^{b-1}\left|\triangle^{i+1}u(k)\right| \\ &\le \frac{1}{2}(b-a)^{1/q}\left(\sum_{k=a}^{b-1}\left|\triangle^{i+1}u(k)\right|^p\right)^{1/p}, \quad i=1,2,\ldots,m-1. \end{aligned} \tag{18}$$

Then

$$\left|\triangle^i u(n)\right|^p \le \left(\frac{1}{2}\right)^p (b-a)^{p/q}\sum_{k=a}^{b-1}\left|\triangle^{i+1}u(k)\right|^p, \quad i=1,2,\ldots,m-1. \tag{19}$$

Summing (19) from a to $b-1$, we have

$$\sum_{n=a}^{b-1}\left|\triangle^i u(n)\right|^p \le (b-a)\left(\frac{1}{2}\right)^p (b-a)^{p/q}\sum_{k=a}^{b-1}\left|\triangle^{i+1}u(k)\right|^p, \quad i=1,2,\ldots,m-1, \tag{20}$$

i.e.,

$$\left(\sum_{n=a}^{b-1}\left|\triangle^i u(n)\right|^p\right)^{1/p} \le \frac{b-a}{2}\left(\sum_{k=a}^{b-1}\left|\triangle^{i+1}u(k)\right|^p\right)^{1/p}, \quad i=1,2,\ldots,m-1. \tag{21}$$

From (21), we obtain

$$\begin{aligned} &\left(\sum_{n=a}^{b-1}\left|\triangle^i u(n)\right|^p\right)^{1/p} \\ &\quad\le \frac{b-a}{2}\left(\sum_{k=a}^{b-1}\left|\triangle^{i+1}u(k)\right|^p\right)^{1/p} \\ &\quad\le \left(\frac{b-a}{2}\right)^2\left(\sum_{k=a}^{b-1}\left|\triangle^{i+2}u(k)\right|^p\right)^{1/p} \\ &\quad\le \cdots \\ &\quad\le \left(\frac{b-a}{2}\right)^{m-i}\left(\sum_{k=a}^{b-1}\left|\triangle^{m}u(k)\right|^p\right)^{1/p}, \quad i=1,2,\ldots,m-1. \end{aligned} \tag{22}$$

Then, from (17) and (22) for $i=1$, we obtain

$$\left|u(n)\right| \le \frac{1}{2}(b-a)^{1/q}\left(\frac{b-a}{2}\right)^{m-1}\left(\sum_{k=a}^{b-1}\left|\triangle^{m}u(k)\right|^p\right)^{1/p}, \tag{23}$$

and by (18) and (22), we get

$$\left|\triangle^i u(n)\right| \le \frac{1}{2}(b-a)^{1/q}\left(\frac{b-a}{2}\right)^{m-i-1}\left(\sum_{k=a}^{b-1}\left|\triangle^{m}u(k)\right|^p\right)^{1/p}, \quad i=1,2,\ldots,m-1. \tag{24}$$

Multiplying (7) by $\triangle^m u(n)$, we have

$$\left|\triangle^m u(n)\right|^p + \sum_{i=0}^{m-1} r_i(n)\left|\triangle^i u(n)\right|^{p-2}\triangle^i u(n)\triangle^m u(n) = 0. \tag{25}$$

Then we get

$$\begin{aligned}\left|\triangle^m u(n)\right|^p &= -\sum_{i=0}^{m-1} r_i(n)\left|\triangle^i u(n)\right|^{p-2}\triangle^i u(n)\triangle^m u(n)\\ &\le \sum_{i=0}^{m-1}\left|r_i(n)\right|\left|\triangle^i u(n)\right|^{p-2}\left|\triangle^i u(n)\right|\left|\triangle^m u(n)\right|\\ &= \sum_{i=0}^{m-1}\left|r_i(n)\right|\left|\triangle^i u(n)\right|^{p-1}\left|\triangle^m u(n)\right|.\end{aligned} \tag{26}$$

Summing (26) from a to $b-1$, we have

$$\begin{aligned}\sum_{n=a}^{b-1}\left|\triangle^m u(n)\right|^p &\le \sum_{i=0}^{m-1}\sum_{n=a}^{b-1}\left|r_i(n)\right|\left|\triangle^i u(n)\right|^{p-1}\left|\triangle^m u(n)\right|\\ &= \sum_{n=a}^{b-1}\left|r_0(n)\right|\left|u(n)\right|^{p-1}\left|\triangle^m u(n)\right|\\ &\quad + \sum_{i=1}^{m-1}\sum_{n=a}^{b-1}\left|r_i(n)\right|\left|\triangle^i u(n)\right|^{p-1}\left|\triangle^m u(n)\right|.\end{aligned} \tag{27}$$

For the first summation on the right-hand side of (27), from (23) and Hölder's inequality (16), we obtain that

$$\begin{aligned}&\sum_{n=a}^{b-1}\left|r_0(n)\right|\left|u(n)\right|^{p-1}\left|\triangle^m u(n)\right|\\ &\quad\le \left(\frac{1}{2}\right)^{p-1}(b-a)^{(p-1)/q}\left(\frac{b-a}{2}\right)^{(m-1)(p-1)}\\ &\qquad\cdot\left(\sum_{n=a}^{b-1}\left|\triangle^m u(n)\right|^p\right)^{(p-1)/p}\sum_{n=a}^{b-1}\left|r_0(n)\right|\left|\triangle^m u(n)\right|\\ &\quad= \frac{(b-a)^{(m-1/p)(p-1)}}{2^{m(p-1)}}\left(\sum_{n=a}^{b-1}\left|\triangle^m u(n)\right|^p\right)^{(p-1)/p}\sum_{n=a}^{b-1}\left|r_0(n)\right|\left|\triangle^m u(n)\right|\\ &\quad\le \frac{(b-a)^{(m-1/p)(p-1)}}{2^{m(p-1)}}\left(\sum_{n=a}^{b-1}\left|\triangle^m u(n)\right|^p\right)^{(p-1)/p}\\ &\qquad\cdot\left(\sum_{n=a}^{b-1}\left|r_0(n)\right|^q\right)^{1/q}\left(\sum_{n=a}^{b-1}\left|\triangle^m u(n)\right|^p\right)^{1/p}\\ &\quad= \frac{(b-a)^{(m-1/p)(p-1)}}{2^{m(p-1)}}\left(\sum_{n=a}^{b-1}\left|\triangle^m u(n)\right|^p\right)\left(\sum_{n=a}^{b-1}\left|r_0(n)\right|^q\right)^{1/q}.\end{aligned} \tag{28}$$

On the other hand, for the second summation on the right-hand side of (27), from (24) and Hölder's inequality (16), we have that

$$\begin{aligned}
&\sum_{n=a}^{b-1}\left|r_i(n)\right|\left|\triangle^i u(n)\right|^{p-1}\left|\triangle^m u(n)\right| \\
&\quad\le \left(\frac{1}{2}\right)^{p-1}(b-a)^{(p-1)/q}\left(\frac{b-a}{2}\right)^{(m-i-1)(p-1)} \\
&\qquad\cdot\left(\sum_{n=a}^{b-1}\left|\triangle^m u(n)\right|^p\right)^{(p-1)/p}\sum_{n=a}^{b-1}\left|r_i(n)\right|\left|\triangle^m u(n)\right| \\
&\quad= \frac{(b-a)^{(m-i-1/p)(p-1)}}{2^{(m-i)(p-1)}}\left(\sum_{n=a}^{b-1}\left|\triangle^m u(n)\right|^p\right)^{(p-1)/p}\sum_{n=a}^{b-1}\left|r_i(n)\right|\left|\triangle^m u(n)\right| \\
&\quad\le \frac{(b-a)^{(m-i-1/p)(p-1)}}{2^{(m-i)(p-1)}}\left(\sum_{n=a}^{b-1}\left|\triangle^m u(n)\right|^p\right)^{(p-1)/p} \\
&\qquad\cdot\left(\sum_{n=a}^{b-1}\left|r_i(n)\right|^q\right)^{1/q}\left(\sum_{n=a}^{b-1}\left|\triangle^m u(n)\right|^p\right)^{1/p} \\
&\quad= \frac{(b-a)^{(m-i-1/p)(p-1)}}{2^{(m-i)(p-1)}}\left(\sum_{n=a}^{b-1}\left|\triangle^m u(n)\right|^p\right)\left(\sum_{n=a}^{b-1}\left|r_i(n)\right|^q\right)^{1/q}, \quad i=1,2,\ldots,m-1, \qquad (29)
\end{aligned}$$

and then

$$\begin{aligned}
&\sum_{i=1}^{m-1}\sum_{n=a}^{b-1}\left|r_i(n)\right|\left|\triangle^i u(n)\right|^{p-1}\left|\triangle^m u(n)\right| \\
&\quad\le \sum_{i=1}^{m-1}\frac{(b-a)^{(m-i-1/p)(p-1)}}{2^{(m-i)(p-1)}}\left(\sum_{n=a}^{b-1}\left|\triangle^m u(n)\right|^p\right)\left(\sum_{n=a}^{b-1}\left|r_i(n)\right|^q\right)^{1/q} \\
&\quad= \left(\sum_{n=a}^{b-1}\left|\triangle^m u(n)\right|^p\right)\sum_{i=1}^{m-1}\frac{(b-a)^{(m-i-1/p)(p-1)}}{2^{(m-i)(p-1)}}\left(\sum_{n=a}^{b-1}\left|r_i(n)\right|^q\right)^{1/q}. \qquad (30)
\end{aligned}$$

By (27), (28), and (30), we get

$$\begin{aligned}
&\sum_{n=a}^{b-1}\left|\triangle^m u(n)\right|^p \\
&\quad\le \frac{(b-a)^{(m-1/p)(p-1)}}{2^{m(p-1)}}\left(\sum_{n=a}^{b-1}\left|\triangle^m u(n)\right|^p\right)\left(\sum_{n=a}^{b-1}\left|r_0(n)\right|^q\right)^{1/q} \\
&\qquad+\left(\sum_{n=a}^{b-1}\left|\triangle^m u(n)\right|^p\right)\sum_{i=1}^{m-1}\frac{(b-a)^{(m-i-1/p)(p-1)}}{2^{(m-i)(p-1)}}\left(\sum_{n=a}^{b-1}\left|r_i(n)\right|^q\right)^{1/q}. \qquad (31)
\end{aligned}$$

Now, we claim that $\sum_{n=a}^{b-1}|\triangle u(n)|^p > 0$. In fact, if the above inequality is not true, we have $\sum_{n=a}^{b-1}|\triangle u(n)|^p = 0$, then $\triangle u(n) = 0$ for $n \in \mathbb{Z}[a,b-1]$. By the anti-periodic conditions (5), we obtain $u(n) = 0$ for $n \in \mathbb{Z}[a,b]$, which contradicts $u(n) \not\equiv 0$, $n \in \mathbb{Z}[a,b]$. From (22), we get $\sum_{n=a}^{b-1}|\triangle^m u(n)|^p > 0$. Thus, dividing both sides of (31) by $\sum_{n=a}^{b-1}|\triangle^m u(n)|^p$, we obtain

$$\begin{aligned}1 &\leq \frac{(b-a)^{(m-1/p)(p-1)}}{2^{m(p-1)}}\left(\sum_{n=a}^{b-1}\left|r_0(n)\right|^q\right)^{1/q}\\&\quad+\sum_{i=1}^{m-1}\frac{(b-a)^{(m-i-1/p)(p-1)}}{2^{(m-i)(p-1)}}\left(\sum_{n=a}^{b-1}\left|r_i(n)\right|^q\right)^{1/q}\\&=\sum_{i=0}^{m-1}\frac{(b-a)^{(m-i-1/p)(p-1)}}{2^{(m-i)(p-1)}}\left(\sum_{n=a}^{b-1}\left|r_i(n)\right|^q\right)^{1/q}.\end{aligned}$$

This completes the proof of Theorem 2.1. □

Remark If $r_i(n) \equiv 0$, $i = 1, 2, \ldots, m-1$, then Theorem 2.1 coincides with Theorem 1 in [21].

Let $p = 2$, $m = 2k$, $k \in \mathbb{N}$ in Theorem 2.1, we have the following corollary.

Corollary 2.1 *If $u(n)$ is a nonzero solution of*

$$\triangle^{2k}u(n)+\sum_{i=0}^{2k-1}r_i(n)\triangle^i u(n)=0 \tag{32}$$

and satisfies the anti-periodic boundary conditions

$$\triangle^i u(a)+\triangle^i u(b)=0,\quad i=0,1,\ldots,2k-1;\qquad u(n)\not\equiv 0,\quad n\in\mathbb{Z}[a,b], \tag{33}$$

then

$$\sum_{i=0}^{2k-1}\frac{(b-a)^{2k-i-1/2}}{2^{2k-i}}\left(\sum_{n=a}^{b-1}\left|r_i(n)\right|^2\right)^{1/2}\geq 1.$$

Let $p = 2$, $m = 2k-1$, $k \in \mathbb{N}$ in Theorem 2.1, we have the following corollary.

Corollary 2.2 *If $u(n)$ is a nonzero solution of*

$$\triangle^{2k-1}u(n)+\sum_{i=0}^{2k-2}r_i(n)\triangle^i u(n)=0 \tag{34}$$

and satisfies the anti-periodic boundary conditions

$$\triangle^i u(a)+\triangle^i u(b)=0,\quad i=0,1,\ldots,2k-2;\qquad u(n)\not\equiv 0,\quad n\in\mathbb{Z}[a,b], \tag{35}$$

then

$$\sum_{i=0}^{2k-2}\frac{(b-a)^{2k-i-3/2}}{2^{2k-1-i}}\left(\sum_{n=a}^{b-1}\left|r_i(n)\right|^2\right)^{1/2}\geq 1.$$

Let $m = 2$ in Theorem 2.1, we have the following corollary.

Corollary 2.3 *If $u(n)$ is a nonzero solution of*

$$\left|\triangle^2 u(n)\right|^{p-2}\triangle^2 u(n)+\sum_{i=0}^{1} r_i(n)\left|\triangle^i u(n)\right|^{p-2}\triangle^i u(n)=0 \tag{36}$$

and satisfies the anti-periodic boundary conditions

$$\triangle^i u(a)+\triangle^i u(b)=0,\quad i=0,1;\qquad u(n)\not\equiv 0,\quad n\in\mathbb{Z}[a,b], \tag{37}$$

then

$$\sum_{i=0}^{1}\frac{(b-a)^{(2-i-1/p)(p-1)}}{2^{(2-i)(p-1)}}\left(\sum_{n=a}^{b-1}\left|r_i(n)\right|^q\right)^{1/q}\geq 1.$$

Next, we establish a Lyapunov-type inequality for Eq. (8).

Theorem 2.2 *If $u(n)$ is a nonzero solution of Eq.* (8) *satisfying the anti-periodic boundary conditions* (9), *then*

$$\left(\sum_{n=a}^{b-1}\left|r(n)\right|\right)^{1/(p-1)}>\frac{2^{3m-2}}{(b-a)^{2m-1/(p-1)}}\frac{1}{\sqrt{\Gamma_{\frac{q-1}{p-1}1}}}, \tag{38}$$

where

$$\Gamma_{\frac{q-1}{p-1}1}=\left(\frac{2p-q-1}{p-1}\right)\left(\frac{q-1}{p-1}\right)^{(q-1)/(2p-q-1)}2^{2(1-p)/(2p-q-1)}. \tag{39}$$

Proof Choose $c\in\mathbb{Z}[a,b]$ such that $|u(c)|=\max_{n\in\mathbb{Z}[a,b]}|u(n)|$. Since (9), it follows from Lemma 2.2 that

$$\left|u(c)\right|\leq\frac{(c-a)(b-c)}{b-a}\sum_{n=a}^{b-1}\left|\triangle^2 u(n)\right|\leq\frac{b-a}{4}\sum_{n=a}^{b-1}\left|\triangle^2 u(n)\right| \tag{40}$$

and

$$\sum_{n=a}^{b-1}\left|\triangle^{2i}u(n)\right|\leq\frac{(b-a)^2}{8}\sum_{n=a}^{b-1}\left|\triangle^{2i+2}u(n)\right|,\quad i=1,2,\ldots,m-1. \tag{41}$$

From (40) and (41), we obtain

$$\begin{aligned}\left|u(c)\right|&\leq\frac{b-a}{4}\sum_{n=a}^{b-1}\left|\triangle^2 u(n)\right|\\&\leq\frac{b-a}{4}\frac{(b-a)^2}{8}\sum_{n=a}^{b-1}\left|\triangle^4 u(n)\right|\\&\leq\frac{b-a}{4}\left(\frac{(b-a)^2}{8}\right)^2\sum_{n=a}^{b-1}\left|\triangle^6 u(n)\right|\end{aligned}$$

$$\leq \cdots$$
$$\leq \frac{b-a}{4}\left(\frac{(b-a)^2}{8}\right)^{m-1}\sum_{n=a}^{b-1}\left|\triangle^{2m}u(n)\right|. \tag{42}$$

Applying discrete Hölder's inequality (16) to the summation on the right-hand side of (42) with $f(n) = 1$, $g(n) = |\triangle^{2m}u(n)|$, $\alpha = \frac{p-1}{p-2}$, and $\beta = p-1$, we obtain that

$$\begin{aligned}\left|u(c)\right| &\leq \frac{b-a}{4}\left(\frac{(b-a)^2}{8}\right)^{m-1}(b-a)^{(p-2)/(p-1)}\left(\sum_{k=a}^{b-1}\left|\triangle^{2m}u(n)\right|^{p-1}\right)^{1/(p-1)} \\ &= \frac{(b-a)^{2m-1/(p-1)}}{2^{3m-1}}\left(\sum_{k=a}^{b-1}\left|\triangle^{2m}u(n)\right|^{p-1}\right)^{1/(p-1)}.\end{aligned} \tag{43}$$

On the other hand, from (8), we have

$$\left|\triangle^{2m}u(n)\right|^{p-2}\triangle^{2m}u(n) = (-1)^m r(n)\left|u(n+1)\right|^{q-2}u(n+1), \tag{44}$$

then

$$\left|\triangle^{2m}u(n)\right|^{p-1} = \left|r(n)\right|\left|u(n+1)\right|^{q-1}. \tag{45}$$

Summing (45) from a to $b-1$, we have

$$\sum_{n=a}^{b-1}\left|\triangle^{2m}u(n)\right|^{p-1} = \sum_{n=a}^{b-1}\left|r(n)\right|\left|u(n+1)\right|^{q-1}, \tag{46}$$

then

$$\left(\sum_{n=a}^{b-1}\left|\triangle^{2m}u(n)\right|^{p-1}\right)^{1/(p-1)} = \left(\sum_{n=a}^{b-1}\left|r(n)\right|\left|u(n+1)\right|^{q-1}\right)^{1/(p-1)}. \tag{47}$$

From (43) and (47), we have

$$\begin{aligned}\left|u(c)\right| &\leq \frac{(b-a)^{2m-1/(p-1)}}{2^{3m-1}}\left(\sum_{n=a}^{b-1}\left|r(n)\right|\left|u(n+1)\right|^{q-1}\right)^{1/(p-1)} \\ &\leq \frac{(b-a)^{2m-1/(p-1)}}{2^{3m-1}}\left(\sum_{n=a}^{b-1}\left|r(n)\right|\right)^{1/(p-1)}\left|u(c)\right|^{(q-1)/(p-1)} \\ &\leq \mathcal{K}\left|u(c)\right|^{(q-1)/(p-1)},\end{aligned} \tag{48}$$

where

$$\mathcal{K} = \frac{(b-a)^{2m-1/(p-1)}}{2^{3m-1}}\left(\sum_{n=a}^{b-1}\left|r(n)\right|\right)^{1/(p-1)}. \tag{49}$$

Using inequality (10) in Lemma 2.1 with $A = B = 1$, $z = |u(c)|$, $\tau = 1$, $\sigma = \frac{q-1}{p-1}$, we have

$$\left|u(c)\right|^2 - \left|u(c)\right|^{(q-1)/(p-1)} + \Gamma_{\frac{q-1}{p-1}1} > 0. \tag{50}$$

From (48) and (50), we get

$$\left|u(c)\right|^2 - \frac{1}{\mathcal{K}}\left|u(c)\right| + \Gamma_{\frac{q-1}{p-1}1} > 0. \tag{51}$$

This is possible only if

$$\frac{1}{\mathcal{K}^2} - 4\Gamma_{\frac{q-1}{p-1}1} < 0, \tag{52}$$

i.e.,

$$\mathcal{K} > \frac{1}{2\sqrt{\Gamma_{\frac{q-1}{p-1}1}}}. \tag{53}$$

From (49) and (53), we obtain

$$\frac{(b-a)^{2m-1/(p-1)}}{2^{3m-1}}\left(\sum_{n=a}^{b-1}\left|r(n)\right|\right)^{1/(p-1)} > \frac{1}{2\sqrt{\Gamma_{\frac{q-1}{p-1}1}}}. \tag{54}$$

Thus, (38) holds. This completes the proof of Theorem 2.2. □

For $p > q = 2$, using a method similar to Theorem 2.2, we have the following theorem.

Theorem 2.3 *If $u(n)$ is a nonzero solution of*

$$\left|\triangle^{2m}u(n)\right|^{p-2}\triangle^{2m}u(n) + (-1)^{m-1}r(n)u(n+1) = 0, \tag{55}$$

satisfying the anti-periodic boundary conditions (9), *then*

$$\left(\sum_{n=a}^{b-1}\left|r(n)\right|\right)^{1/(p-1)} > \frac{2^{3m-2}}{(b-a)^{2m-1/(p-1)}}\frac{1}{\sqrt{\Gamma_{\frac{1}{p-1}1}}}, \tag{56}$$

where

$$\Gamma_{\frac{1}{p-1}1} = \left(\frac{2p-3}{p-1}\right)\left(\frac{1}{p-1}\right)^{1/(2p-3)}2^{2(1-p)/(2p-3)}. \tag{57}$$

Remark For $p = q = 2$, using a method similar to Theorem 2.2, we have that the result coincides with Corollary 2.3 in [29].

Let $m = 1$ in Theorem 2.2, we have the following corollary.

Corollary 2.4 *If $u(n)$ is a nonzero solution of*

$$\left|\triangle^2 u(n)\right|^{p-2}\triangle^2 u(n) + r(n)\left|u(n+1)\right|^{q-2}u(n+1) = 0 \tag{58}$$

and satisfies the anti-periodic boundary conditions

$$u(a) = u(b) = 0, \qquad u(n) \not\equiv 0, \quad n \in \mathbb{Z}[a,b], \tag{59}$$

then

$$\left(\sum_{n=a}^{b-1}\left|r(n)\right|\right)^{1/(p-1)} > \frac{2}{(b-a)^{2-1/(p-1)}}\frac{1}{\sqrt{\Gamma_{\frac{q-1}{p-1}1}}},$$

where $\Gamma_{\frac{q-1}{p-1}1}$ is defined as in (39).

Let $m = 1$ in Theorem 2.3, we have the following corollary.

Corollary 2.5 *If $u(n)$ is a nonzero solution of*

$$\left|\triangle^2 u(n)\right|^{p-2}\triangle^2 u(n) + r(n)u(n+1) = 0 \tag{60}$$

and satisfies the anti-periodic boundary conditions

$$u(a) = u(b) = 0, \qquad u(n) \not\equiv 0, \quad n \in \mathbb{Z}[a,b], \tag{61}$$

then

$$\left(\sum_{n=a}^{b-1}\left|r(n)\right|\right)^{1/(p-1)} > \frac{2}{(b-a)^{2-1/(p-1)}}\frac{1}{\sqrt{\Gamma_{\frac{1}{p-1}1}}},$$

where $\Gamma_{\frac{1}{p-1}1}$ is defined as in (57).

3 Applications

In this section, we investigate the nonexistence and uniqueness for solutions of certain BVPs. First, we consider the nonexistence for solutions of the BVP consisting of (7) and the boundary conditions (5).

Theorem 3.1 *Assume*

$$\sum_{i=0}^{m-1}\frac{(b-a)^{(m-i-1/p)(p-1)}}{2^{(m-i)(p-1)}}\left(\sum_{n=a}^{b-1}\left|r_i(n)\right|^q\right)^{1/q} < 1, \tag{62}$$

where q is the Hölder conjugate exponent of p, i.e., $1/p + 1/q = 1$. Then BVP (7), (5) has no nontrivial solution.

Proof Assume the contrary. Then BVP (7), (5) has a nontrivial solution $u(n)$. By Theorem 2.1, inequality (13) holds. This contradicts assumption (62). □

Next, we consider the uniqueness for solutions of nonhomogeneous BVP consisting of the equation

$$\triangle^{2k}u(n)+\sum_{i=0}^{2k-1}r_i(n)\triangle^i u(n)=f(n),\quad n\in\mathbb{Z}[A,B], \tag{63}$$

and the boundary conditions

$$\triangle^i u(a)+\triangle^i u(b)=M_i,\quad i=0,1,\dots,2k-1;\qquad n\in\mathbb{Z}[a,b], \tag{64}$$

where $k\in\mathbb{N}$, $n\in\mathbb{Z}$, and f, $r_i(n)$ $(i=0,1,\dots,2k-1)$ are real-valued functions defined on $\mathbb{Z}$, $A,B,a,b\in\mathbb{N}$, $A<a<b<B$, and $M_i\in\mathbb{R}$, $i=0,1,\dots,2k-1$.

Theorem 3.2 *Assume*

$$\sum_{i=0}^{2k-1}\frac{(B-A)^{(2k-i-1/2)}}{2^{(2k-i)}}\left(\sum_{n=A}^{B-1}\left|r_i(n)\right|^2\right)^{1/2}<1. \tag{65}$$

Then BVP (63), (64) *has at most one solution on* (A,B) *for any* $a,b\in(A,B)$, $M_i\in\mathbb{R}$, $i=0,1,\dots,2k-1$.

Proof Let $u_1(n)$ and $u_2(n)$ be two solutions of BVP (63), (64) in (A,B). Define $u(n)=u_1(n)-u_2(n)$. Then $u(n)$ is a solution of BVP (32), (33). Then, by Theorem 3.1 with $p=2$ and $m=2k$, we have $u(n)\equiv 0$, i.e., $u_1(n)\equiv u_2(n)$. This shows that BVP (63), (64) has at most one solution on (A,B). □

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Competing interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Authors' contributions

HDL organized and wrote this paper. Further, he examined all the steps of the proofs in this research. The author read and approved the final manuscript.

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