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Monotonicity results for h-discrete fractional operators and application

Iyad Suwan[1], Shahd Owies[1] and Thabet Abdeljawad[2*]

*Correspondence:
tabdeljawad@psu.edu.sa
[2]Department of Mathematics and General Sciences, Prince Sultan University, Riyadh, Saudi Arabia
Full list of author information is available at the end of the article

Abstract

In this article, we formulate nabla fractional sums and differences of order $0 < \alpha \leq 1$ on the time scale $h\mathbb{Z}$, where $0 < h \leq 1$. Then, we prove that if the nabla h-Riemann–Liouville (RL) fractional difference operator $({}_a\nabla_h^\alpha y)(t) > 0$, then $y(t)$ is α-increasing. Conversely, if $y(t)$ is α-increasing and $y(a) > 0$, then $({}_a\nabla_h^\alpha y)(t) > 0$. The monotonicity results for the nabla h-Caputo fractional difference operator are also concluded by using the relation between h-nabla RL and Caputo fractional difference operators. It is observed that the reported monotonicity coefficient is not affected by the step h. We formulate a nabla h-fractional difference initial value problem as well. Finally, we furniture our results by proving a fractional difference version of the Mean Value Theorem (MVT) on $h\mathbb{Z}$.

Keywords: Nabla Riemann–Liouville h-fractional difference; Nabla h-Caputo fractional difference; h-Discrete fractional mean value theorem

1 Introduction

Due to their successful applications in many branches of science and engineering, techniques of fractional calculus have been under focus by many researchers in the past and in the present decades [1–10]. The theory of fractional sums with delta operator and the fractional differences with nabla operators were firstly introduced in [11]. Extensive development of the theory can be found in [12–31]. The monotonicity properties of delta and nabla fractional operators were studied in [32–36]. It is worth mentioning that Atıcı et al. in [34] studied the monotonicity properties of delta fractional differences and obtained a delta-fractional difference version of the mean-value theorem. In [37], interesting monotonicity results were provided using the dual identities related to delta and nabla fractional operators. In [38], the authors studied the relationship between the discrete sequential fractional operators and monotonicity. Recently, in [39–45], fractional operators with Mittag–Leffler and exponential "non-singular" kernels have been studied together with their discrete versions, and the monotonicity has been investigated. Motivated by all the above-mentioned works, we prove interesting monotonicity results with mean value theorem as an application in this paper for nabla fractional differences in the time scale $h\mathbb{Z}$, $0 < h \leq 1$ (called h-nabla fractional differences), which is considered as a generalized form of the monotonicity results when $h = 1$. Also, working on $h\mathbb{Z}$, $0 < h < 1$ guarantees more accurate approximations for the solutions of fractional dynamical systems.

The article is organized as follows. In Sect. 2, we present the main definitions and important information needed in this study, and in the third section we present the monotonicity analysis of the fractional difference operator. In Sect. 4 we formulate an initial value h-fractional difference problem in the sense of Riemann. Finally, we have the application on the mean value theorem and then the conclusions are presented.

2 Definitions and preliminary results

Definition 2.1 The backward difference operator on $h\mathbb{Z}$ is defined by

$$\nabla_h f(t) = \frac{f(t) - f(t-h)}{h},$$

and the forward difference operator on $h\mathbb{Z}$ is defined by

$$\Delta_h f(t) = \frac{f(t+h) - f(t)}{h}.$$

Definition 2.2 The backward jump operator on the time scale $h\mathbb{Z}$ is defined by $\rho_h(t) = t - h$ and the forward jump operator is defined by $\sigma_h(t) = t + h$.

For $a, b \in \mathbb{R}$ with $a < b$, $\frac{b-a}{h} \in \mathbb{N}$ and $0 < h \leq 1$, we use the notations $\mathbb{N}_{a,h} = \{a, a+h, a+2h, \ldots\}$ and ${}_{b,h}\mathbb{N} = \{b, b-h, b-2h, \ldots\}$.

Definition 2.3 Let $\alpha \in \mathbb{R}$ and $0 < h \leq 1$, the nabla h-factorial of t is defined by

$$t_h^{\overline{\alpha}} = h^\alpha \frac{\Gamma(\frac{t}{h} + \alpha)}{\Gamma(\frac{t}{h})}$$

such that $t \in \mathbf{R} - \{\ldots, -2h, -h, 0\}$, $0_h^{\overline{\alpha}} = 0$ and dividing by poles leads to zero.

Lemma 2.1 *For $\alpha > 0$ and $h > 0$, $t_h^{\overline{\alpha}}$ is increasing on $\mathbb{N}_{0,h}$.*

Proof

$$\begin{aligned}
\nabla_h t_h^{\overline{\alpha}} &= \frac{t_h^{\overline{\alpha}} - (t-h)_h^{\overline{\alpha}}}{h} = \frac{1}{h}\bigl(t_h^{\overline{\alpha}} - (t-h)_h^{\overline{\alpha}}\bigr) = \frac{1}{h}\left(h^\alpha \frac{\Gamma(\frac{t}{h} + \alpha)}{\Gamma(\frac{t}{h})} - h^\alpha \frac{\Gamma(\frac{(t-h)}{h} + \alpha)}{\Gamma(\frac{t-h}{h})}\right) \\
&= \frac{h^\alpha}{h}\left(\frac{\Gamma(\frac{t}{h} + \alpha)}{\Gamma(\frac{t}{h})} - \frac{\Gamma(\frac{t}{h} + \alpha - 1)}{\Gamma(\frac{t}{h} - 1)}\right) = h^{\alpha-1}\left(\frac{\Gamma(\frac{t}{h} + \alpha - 1)}{\Gamma(\frac{t}{h} - 1)}\right)\left(\frac{\frac{t}{h} + \alpha - 1}{\frac{t}{h} - 1} - 1\right) \\
&= h^{\alpha-1}\left(\frac{\Gamma(\frac{t}{h} + \alpha - 1)}{\Gamma(\frac{t}{h} - 1)}\right)\left(\frac{\frac{t}{h} + \alpha - 1 - \frac{t}{h} + 1}{\frac{t}{h} - 1}\right) = h^{\alpha-1}\left(\frac{\Gamma(\frac{t}{h} + \alpha - 1)}{\Gamma(\frac{t}{h} - 1)}\right)\left(\frac{\alpha}{\frac{t}{h} - 1}\right) \\
&= \alpha h^{\alpha-1}\frac{\Gamma(\frac{t}{h} + (\alpha - 1))}{\Gamma(\frac{t}{h})} = \alpha t_h^{\overline{\alpha-1}}.
\end{aligned}$$

Notice that since $\alpha, h > 0$, then $\nabla_h t_h^{\overline{\alpha}} = \frac{t_h^{\overline{\alpha}} - (t-h)_h^{\overline{\alpha}}}{h} = \alpha t_h^{\overline{\alpha-1}} \geq 0$, and hence the proof is completed. □

Definition 2.4 (Nabla h-fractional sums) For a function $f : \mathbb{N}_{a,h} = \{a, a+h, a+2h, \dots\} \to \mathbb{R}$, the nabla left h-fractional sum of order $\alpha > 0$ is defined by

$$\begin{aligned}\left({}_a\nabla_h^{-\alpha} f\right)(t) &= \frac{1}{\Gamma(\alpha)} \int_a^t \left(t - \rho_h(s)\right)_h^{\overline{\alpha-1}} f(s) \nabla_h s \\ &= \frac{1}{\Gamma(\alpha)} \sum_{k=a/h+1}^{t/h} \left(t - \rho_h(kh)\right)_h^{\overline{\alpha-1}} f(kh)h, \quad t \in \mathbb{N}_{a,h}.\end{aligned}$$

For a function $f : {}_{b,h}\mathbb{N} = \{b, b-h, b-2h, \dots\} \to \mathbb{R}$, the nabla right h-fractional sum of order $\alpha > 0$ is defined by

$$\begin{aligned}\left({}_h\nabla_b^{-\alpha} f\right)(t) &= \frac{1}{\Gamma(\alpha)} \int_t^b \left(s - \rho_h(t)\right)_h^{\overline{\alpha-1}} f(s) \Delta_h s \\ &= \frac{1}{\Gamma(\alpha)} \sum_{k=t/h}^{b/h-1} \left(kh - \rho_h(t)\right)_h^{\overline{\alpha-1}} f(kh)h, \quad t \in {}_{b,h}\mathbb{N}.\end{aligned}$$

Definition 2.5 (Nabla h-RL fractional differences)

The nabla left h-fractional difference of order $0 < \alpha \leq 1$ (starting from a) is defined by

$$\left({}_a\nabla_h^{\alpha} f\right)(t) = \left(\nabla_h\, {}_a\nabla_h^{-(1-\alpha)} f\right)(t), \quad \text{which is}$$

$$\left({}_a\nabla_h^{\alpha} f\right)(t) = \frac{1}{\Gamma(1-\alpha)} \nabla_h \sum_{k=a/h+1}^{t/h} \left(t - \rho_h(kh)\right)_h^{\overline{-\alpha}} f(kh)h, \quad t \in \mathbb{N}_{a+h,h}.$$

The nabla right h-fractional difference of order $0 < \alpha \leq 1$ (ending at b) is defined by

$$\left({}_h\nabla_b^{\alpha} f\right)(t) = \left(-\Delta_h\, {}_h\nabla_b^{-(1-\alpha)} f\right)(t), \quad \text{which is}$$

$$\left({}_h\nabla_b^{\alpha} f\right)(t) = \frac{-1}{\Gamma(1-\alpha)} \Delta_h \sum_{k=t/h}^{b/h-1} \left(kh - \rho_h(t)\right)_h^{\overline{-\alpha}} f(kh)h, \quad t \in {}_{b-h,h}\mathbb{N}.$$

Definition 2.6 (Nabla h-Caputo fractional differences) Assume that $0 < \alpha \leq 1$, $0 < h \leq 1$, $a, b \in \mathbb{R}$, and $a < b$, f is defined on $\mathbb{N}_{a,h} = \{a, a+h, a+2h, \dots\}$ and on ${}_{b,h}\mathbb{N} = \{b, b-h, b-2h, \dots\}$. Then:

the left h-Caputo fractional difference of order α starting at a is defined by

$$\left({}_a^C\nabla_h^{\alpha} f\right)(t) = \left({}_a\nabla_h^{-(1-\alpha)} \nabla_h f\right)(t), \quad t \in \mathbb{N}_{a+h,h};$$

the right h-Caputo fractional difference of order α ending at b is defined by

$$\left({}_h^C\nabla_b^{\alpha} f\right)(t) = \left({}_h\nabla_b^{-(1-\alpha)} (-\Delta_h f)\right)(t), \quad t \in {}_{b-h,h}\mathbb{N}.$$

Proposition 2.2 (The relation between nabla h-RL fractional difference and h-Caputo fractional difference)

(i) $\left({}_a^C\nabla_h^{\alpha} f\right)(t) = \left({}_a\nabla_h^{\alpha} f\right)(t) - \dfrac{1}{\Gamma(1-\alpha)} (t-a)_h^{\overline{-\alpha}} f(a);$

(ii) $\left({}_h^C\nabla_b^{\alpha} f\right)(t) = \left({}_h\nabla_b^{\alpha} f\right)(t) - \dfrac{1}{\Gamma(1-\alpha)} (b-t)_h^{\overline{-\alpha}} f(b).$

Proof

$$
\begin{aligned}
\text{(i)}\quad & \left({}_a^C\nabla_h^\alpha f\right)(t) \\
&= \left({}_a\nabla_h^{-(1-\alpha)}\nabla_h f\right)(t) \\
&= {}_a\nabla_h^{-(1-\alpha)}\left(\frac{f(t)-f(t-h)}{h}\right) \\
&= {}_a\nabla_h^{-(1-\alpha)}\left(\frac{f(t)}{h}\right) - {}_a\nabla_h^{-(1-\alpha)}\left(\frac{f(t-h)}{h}\right) \\
&= \frac{1}{\Gamma(1-\alpha)}\sum_{k=a/h+1}^{t/h}\left(t-\rho_h(kh)\right)_h^{\overline{-\alpha}}\frac{f(kh)}{h}h \\
&\quad - \frac{1}{\Gamma(1-\alpha)}\sum_{k=(a-h)/h+1}^{(t-h)/h}\left(t-h-\rho_h(kh)\right)_h^{\overline{-\alpha}}\frac{f(kh)}{h}h \\
&= \frac{1}{h\Gamma(1-\alpha)}\sum_{k=a/h+1}^{t/h}\left(t-\rho_h(kh)\right)_h^{\overline{-\alpha}}f(kh)h \\
&\quad - \frac{1}{h\Gamma(1-\alpha)}\sum_{k=a/h}^{t/h-1}\left((t-h)-(kh-h)\right)_h^{\overline{-\alpha}}f(kh)h \\
&= \frac{1}{h\Gamma(1-\alpha)}\sum_{k=a/h+1}^{t/h}\left(t-\rho_h(kh)\right)_h^{\overline{-\alpha}}f(kh)h \\
&\quad - \frac{1}{h\Gamma(1-\alpha)}\sum_{k=a/h+1}^{t/h-1}\left(t-h-\rho_h(kh)\right)_h^{\overline{-\alpha}}f(kh)h \\
&\quad - \frac{1}{h\Gamma(1-\alpha)}\left(t-h-\frac{a}{h}(h)+h\right)_h^{\overline{-\alpha}}f\left(\frac{a}{h}h\right)h \\
&= \frac{1}{\Gamma(1-\alpha)} \\
&\quad \times\left(\frac{\sum_{k=a/h+1}^{t/h}(t-\rho_h(kh))_h^{\overline{-\alpha}}f(kh)h - \sum_{k=a/h+1}^{t/h-1}((t-h)-\rho_h(kh))_h^{\overline{-\alpha}}f(kh)h}{h}\right) \\
&\quad - \frac{1}{\Gamma(1-\alpha)}(t-a)_h^{\overline{-\alpha}}f(a) \\
&= \frac{1}{\Gamma(1-\alpha)}\nabla_h\sum_{k=a/h+1}^{t/h}\left(t-\rho_h(kh)\right)_h^{\overline{-\alpha}}f(kh)h - \frac{1}{\Gamma(1-\alpha)}(t-a)_h^{\overline{-\alpha}}f(a) \\
&= \left({}_a\nabla_h^\alpha f\right)(t) - \frac{1}{\Gamma(1-\alpha)}(t-a)_h^{\overline{-\alpha}}f(a).
\end{aligned}
$$

(ii) The proof is similar to that in (i), and hence we omit it. □

The following lemma is a generalization of Lemma 3.3 in [20].

Lemma 2.3 *Let* $\alpha > 0$, $\mu > -1$, $h > 0$, *and* $t \in \mathbb{N}_{a,h}$. *Then*

$$
{}_a\nabla_h^{-\alpha}(t-a)_h^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+1+\alpha)}(t-a)_h^{\overline{\alpha+\mu}}. \tag{1}
$$

Proof Using Lemma 3.3 in [20], we have

$$
\begin{aligned}
&\nabla_{a/h}^{-\alpha}\left(\frac{t}{h}-\frac{a}{h}\right)^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}\left(\frac{t}{h}-\frac{a}{h}\right)^{\overline{\alpha+\mu}},\\
&\frac{1}{\Gamma(\alpha)}\sum_{a/h+1}^{t/h}\left(\frac{t}{h}-\rho(k)\right)^{\overline{\alpha-1}}\left(k-\frac{a}{h}\right)^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}\left(\frac{t-a}{h}\right)^{\overline{\alpha+\mu}},\\
&\frac{1}{\Gamma(\alpha)}\sum_{a/h+1}^{t/h}\left(\frac{t}{h}-k+1\right)^{\overline{\alpha-1}}\left(k-\frac{a}{h}\right)^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}\frac{\Gamma(\frac{t-a}{h}+\alpha+\mu)}{\Gamma(\frac{t-a}{h})},\\
&\frac{1}{\Gamma(\alpha)}\sum_{a/h+1}^{t/h}\left(\frac{t-kh+h}{h}\right)^{\overline{\alpha-1}}\left(\frac{kh-a}{h}\right)^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}\frac{(t-a)_h^{\overline{\alpha+\mu}}}{h^{\alpha+\mu}},\\
&\frac{1}{\Gamma(\alpha)}\sum_{a/h+1}^{t/h}\frac{\Gamma(\frac{t-kh+h}{h}+\alpha-1)}{\Gamma(\frac{t-kh+h}{h})}\frac{\Gamma(\frac{kh-a}{h}+\mu)}{\Gamma(\frac{kh-a}{h})} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}\frac{(t-a)_h^{\overline{\alpha+\mu}}}{h^{\alpha+\mu}},\\
&\frac{1}{\Gamma(\alpha)}\sum_{a/h+1}^{t/h}\frac{(t-kh+h)_h^{\overline{\alpha-1}}}{h^{\alpha-1}}\frac{(kh-a)_h^{\overline{\mu}}}{h^{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}\frac{(t-a)_h^{\overline{\alpha+\mu}}}{h^{\alpha}h^{\mu}},\\
&\frac{1}{\Gamma(\alpha)}\sum_{a/h+1}^{t/h}\bigl(t-\rho(kh)\bigr)_h^{\overline{\alpha-1}}(kh-a)_h^{\overline{\mu}}h = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}(t-a)_h^{\overline{\alpha+\mu}},\\
&{}_a\nabla_h^{-\alpha}(t-a)_h^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+1+\alpha)}(t-a)_h^{\overline{\alpha+\mu}}.
\end{aligned}
$$

□

3 The monotonicity results

The following two definitions are the $h\mathbb{Z}$ versions of monotonicity definitions given in [34].

Definition 3.1 Let $y : \mathbb{N}_{a,h} \to \mathbb{R}$ be a function satisfying $y(a) \geq 0$, and let $0 \leq h < 1$. Then $y(t)$ is called an α-increasing function on $\mathbb{N}_{a,h}$ if $y(t+h) \geq \alpha y(t)$ $\forall t \in \mathbb{N}_{a,h}$.

Note that if $y(t)$ is increasing on $\mathbb{N}_{a,h}$ ($y(t+h) \geq y(t)$ $\forall t \in \mathbb{N}_{a,h}$), then $y(t)$ is an α-increasing function on $\mathbb{N}_{a,h}$, and if $\alpha = 1$, then the increasing and α-increasing concepts coincide.

Definition 3.2 Let $y : \mathbb{N}_{a,h} \to \mathbb{R}$ be a function satisfying $y(a) \leq 0$, and $0 \leq h < 1$. Then $y(t)$ is called an α-decreasing function on $\mathbb{N}_{a,h}$ if $y(t+h) \leq \alpha y(t)$ $\forall t \in \mathbb{N}_{a,h}$.

Note that if $y(t)$ is decreasing on $\mathbb{N}_{a,h}$ ($y(t+h) \leq y(t)$ $\forall t \in \mathbb{N}_{a,h}$), then $y(t)$ is an α-decreasing function on $\mathbb{N}_{a,h}$, and if $\alpha = 1$, then the decreasing and α-decreasing concepts coincide.

Theorem 3.1 *Let $y : \mathbb{N}_{a-h,h} \to \mathbb{R}$, and suppose that $({}_{a-h}\nabla_h^{\alpha}y)(t) \geq 0$ for $0 < \alpha \leq 1$, and $0 < h \leq 1$, $t \in \mathbb{N}_{a-h,h}$. Then $y(t)$ is α-increasing.*

Proof First we recall that

$$\begin{aligned}\big({}_{a-h}\nabla_h^\alpha y\big)(t) &= \frac{1}{\Gamma(1-\alpha)}\nabla_h \sum_{k=(a-h)/h+1}^{t/h} \big(t-\rho_h(kh)\big)_h^{\overline{-\alpha}} y(kh)h \\ &= \frac{1}{\Gamma(1-\alpha)}\nabla_h \sum_{k=a/h}^{t/h} \big(t-\rho_h(kh)\big)_h^{\overline{-\alpha}} y(kh)h.\end{aligned}$$

Let

$$S(t) = \sum_{k=a/h}^{t/h} \big(t-\rho_h(kh)\big)_h^{\overline{-\alpha}} y(kh)h.$$

Then, from the assumption, we have $\nabla_h S(t) \geq 0$. That is,

$$\begin{aligned}\nabla_h S(t) &= \frac{S(t)-S(t-h)}{h} \\ &= \frac{\sum_{k=a/h}^{t/h}(t-\rho_h(kh))_h^{\overline{-\alpha}}y(kh)h - \sum_{k=a/h}^{t/h-1}(t-h-\rho_h(kh))_h^{\overline{-\alpha}}y(kh)h}{h} \\ &= \frac{\sum_{k=a/h}^{t/h-1}(t-\rho_h(kh))_h^{\overline{-\alpha}}y(kh)h - \sum_{k=a/h}^{t/h-1}(t-h-\rho_h(kh))_h^{\overline{-\alpha}}y(kh)h}{h} \\ &\quad + \frac{(t-\rho_h(t))_h^{\overline{-\alpha}}y(t)h}{h} \\ &= (t-t+h)_h^{\overline{-\alpha}}y(t) + \sum_{k=a/h}^{t/h-1}\left(\frac{(t-\rho_h(kh))_h^{\overline{-\alpha}} - (t-h-\rho_h(kh))_h^{\overline{-\alpha}}}{h}\right)y(kh)h \\ &= h_h^{\overline{-\alpha}}y(t) + \sum_{k=a/h}^{t/h-1} y(kh)h\nabla_h\big(t-\rho_h(kh)\big)_h^{\overline{-\alpha}} \\ &= h_h^{\overline{-\alpha}}y(t) + \sum_{k=a/h}^{t/h-1} y(kh)h\frac{(t-\rho_h(kh))_h^{\overline{-\alpha}} - (t-h-\rho_h(kh))_h^{\overline{-\alpha}}}{h} \\ &= h_h^{\overline{-\alpha}}y(t) + \sum_{k=a/h}^{t/h-1} y(kh)h\frac{(t-kh+h)_h^{\overline{-\alpha}} - (t-h-kh+h)_h^{\overline{-\alpha}}}{h} \\ &= h_h^{\overline{-\alpha}}y(t) + \sum_{k=a/h}^{t/h-1} y(kh)h\left(\frac{\Gamma(\frac{t-kh+h}{h}-\alpha)}{\Gamma(\frac{t-kh+h}{h})}h^{-\alpha} - \frac{\Gamma(\frac{t-kh}{h}-\alpha)}{\Gamma(\frac{t-kh}{h})}h^{-\alpha}\right)\frac{1}{h} \\ &= h_h^{\overline{-\alpha}}y(t) + \sum_{k=a/h}^{t/h-1} y(kh)h\left(\frac{\Gamma(\frac{t}{h}-k+1-\alpha)}{\Gamma(\frac{t}{h}-k+1)}h^{-\alpha} - \frac{\Gamma(\frac{t}{h}-k-\alpha)}{\Gamma(\frac{t}{h}-k)}h^{-\alpha}\right)\frac{1}{h} \\ &= h_h^{\overline{-\alpha}}y(t) + \sum_{k=a/h}^{t/h-1} y(kh)h\left(\frac{(\frac{t}{h}-k-\alpha)}{(\frac{t}{h}-k)} - 1\right)\frac{\Gamma(\frac{t}{h}-k-\alpha)}{\Gamma(\frac{t}{h}-k)}\frac{h^{-\alpha}}{h} \\ &= h_h^{\overline{-\alpha}}y(t) + \sum_{k=a/h}^{t/h-1} y(kh)h\left(\frac{\frac{t}{h}-k-\alpha-\frac{t}{h}+k}{\frac{t}{h}-k}\right)\frac{\Gamma(\frac{t}{h}-k-\alpha)}{\Gamma(\frac{t}{h}-k)}h^{-\alpha-1} \\ &= h_h^{\overline{-\alpha}}y(t) + \sum_{k=a/h}^{t/h-1} y(kh)h\left(\frac{-\alpha}{\frac{t}{h}-k}\right)\frac{\Gamma(\frac{t}{h}-k-\alpha)}{\Gamma(\frac{t}{h}-k)}h^{-\alpha-1}\end{aligned}$$

$$= h_h^{\overline{-\alpha}} y(t) + \sum_{k=a/h}^{t/h-1} y(kh)h(-\alpha)\frac{\Gamma(\frac{t}{h}-k+1+(-\alpha-1))}{\Gamma(\frac{t}{h}-k+1)}h^{-\alpha-1}$$

$$= h_h^{\overline{-\alpha}} y(t) + \sum_{k=a/h}^{t/h-1} y(kh)h(-\alpha)\bigl(t-\rho_h(kh)\bigr)_h^{\overline{-\alpha-1}}$$

$$= h_h^{\overline{-\alpha}} y(t) - \alpha \sum_{k=a/h}^{t/h-1} \bigl(t-\rho_h(kh)\bigr)_h^{\overline{-\alpha-1}} y(kh)h \geq 0.$$

Therefore

$$\nabla_h S(t) = h_h^{\overline{-\alpha}} y(t) - \alpha \sum_{k=a/h}^{t/h-1} \bigl(t-\rho_h(kh)\bigr)_h^{\overline{-\alpha-1}} y(kh)h \geq 0. \tag{2}$$

When $t = a$, we have

$$\nabla_h S(a) = h_h^{\overline{-\alpha}} y(a) = h^{-\alpha}\Gamma(1-\alpha)y(a) \geq 0, \quad \text{and hence } y(a) \geq 0.$$

When $t = a + h$, we get

$$\begin{aligned}
\nabla_h S(a+h) &= h_h^{\overline{-\alpha}} y(a+h) - \alpha\bigl(a+h-\rho_h(a)\bigr)_h^{\overline{-\alpha-1}} y(a)h \\
&= h_h^{\overline{-\alpha}} y(a+h) - \alpha(a+h-a+h)_h^{\overline{-\alpha-1}} y(a)h \\
&= h_h^{\overline{-\alpha}} y(a+h) - \alpha(2h)_h^{\overline{-\alpha-1}} y(a)h \\
&= h^{-\alpha}\frac{\Gamma(\frac{h}{h}-\alpha)}{\Gamma(\frac{h}{h})} y(a+h) - \alpha h^{-\alpha-1}\frac{\Gamma(\frac{2h}{h}-\alpha-1)}{\Gamma(\frac{2h}{h})} y(a)h \\
&= h^{-\alpha}\frac{\Gamma(1-\alpha)}{\Gamma(1)} y(a+h) - \alpha h^{-\alpha}\frac{\Gamma(2-\alpha-1)}{\Gamma(2)} y(a) \\
&= h^{-\alpha}\Gamma(1-\alpha)y(a+h) - \alpha h^{-\alpha}\Gamma(1-\alpha)y(a) \geq 0,
\end{aligned}$$

hence, $y(a+h) \geq \alpha y(a)$.

Now we follow inductively to show that

$$y(t+h) \geq \alpha y(t), \quad \forall t \in \mathbb{N}_{a,h}.$$

Assume $y(k+h) \geq \alpha y(k) \geq 0$, $\forall k < t$ such that $k, t \in \mathbb{N}_{a,h}$. We need to show that $y(t+h) \geq \alpha y(t)$. We know that

$$\nabla_h S(t) = h_h^{\overline{-\alpha}} y(t) - \alpha \sum_{k=a/h}^{t/h-1} \bigl(t-\rho_h(kh)\bigr)_h^{\overline{-\alpha-1}} y(kh)h \geq 0. \tag{3}$$

In (3) replace t by $t + h$. Then we have

$$h_h^{\overline{-\alpha}} y(t+h) - \alpha \sum_{k=a/h}^{t/h} \bigl(t+h-\rho_h(kh)\bigr)_h^{\overline{-\alpha-1}} y(kh)h \geq 0, \quad \text{which implies that}$$

$$\begin{aligned}&h_h^{\overline{-\alpha}}y(t+h)-\alpha\big(\big(t+h-\rho_h(a)\big)_h^{\overline{-\alpha-1}}y(a)h+\big(t+h-\rho_h(a+h)\big)_h^{\overline{-\alpha-1}}y(a+h)h\\&\quad+\cdots+\big(t+h-\rho_h(t)\big)_h^{\overline{-\alpha-1}}y(t)h\big)\geq 0,\end{aligned}$$

or

$$\begin{aligned}h_h^{\overline{-\alpha}}y(t+h)&\geq\alpha\big(\big(t+h-\rho_h(a)\big)_h^{\overline{-\alpha-1}}y(a)h+\big(t+h-\rho_h(a+h)\big)_h^{\overline{-\alpha-1}}y(a+h)h\\&\qquad+\cdots+\big(t+h-\rho_h(t)\big)_h^{\overline{-\alpha-1}}y(t)h\big)\\&=\alpha\big(t+h-\rho_h(a)\big)_h^{\overline{-\alpha-1}}y(a)h+\cdots+\alpha\big(t+h-\rho_h(t)\big)_h^{\overline{-\alpha-1}}y(t)h\\&\geq\alpha\big(t+h-\rho_h(t)\big)_h^{\overline{-\alpha-1}}y(t)h\\&=\alpha(2h)_h^{\overline{-\alpha-1}}y(t)h.\end{aligned}\tag{4}$$

Then from Definition 2.3 it follows that

$$h^{-\alpha}\Gamma(1-\alpha)y(t+h)\geq\alpha h^{-\alpha}\Gamma(1-\alpha)y(t).$$

Hence, $y(t+h)\geq\alpha y(t)$, and the proof is completed. □

Using Proposition 2.2 and Theorem 3.1, we can state the following h-Caputo fractional difference monotonicity result.

Corollary 3.2 *Let $y:\mathbb{N}_{a-h,h}\to\mathbb{R}$, and suppose that*

$$\big({}_{a-h}^{C}\nabla_h^{\alpha}y\big)(t)\geq\frac{-1}{\Gamma(1-\alpha)}(t-a+h)_h^{\overline{-\alpha}}y(a-h),\quad t\in\mathbb{N}_{a-h,h}$$

for $0<\alpha\leq 1$, and $0<h\leq 1$ then $y(t)$ is α-increasing.

Theorem 3.3 *Assume that the function $y:\mathbb{N}_{a-h,h}\to\mathbb{R}$ satisfies $y(a)\geq 0$ and assume $0<\alpha\leq 1$ and $0<h\leq 1$. If y is increasing on $\mathbb{N}_{a,h}$, then we have*

$$\big({}_{a-h}\nabla_h^{\alpha}y\big)(t)\geq 0,\quad\forall t\in\mathbb{N}_{a-h,h}.$$

Proof Since we have

$$\big({}_{a-h}\nabla_h^{\alpha}y\big)(t)=\frac{1}{\Gamma(1-\alpha)}\nabla_hS(t),\quad t\in\mathbb{N}_{a-h,h},$$

it is enough to show that

$$S(t)=\sum_{k=(a-h)/h+1}^{t/h}\big(t-\rho_h(kh)\big)_h^{\overline{-\alpha}}y(kh)h=\sum_{k=a/h}^{t/h}\big(t-\rho_h(kh)\big)_h^{\overline{-\alpha}}y(kh)h$$

is increasing on $\mathbb{N}_{a,h}$. In reference to the proof of Theorem 3.1, when $t=a$ we have

$$\nabla_hS(a)=h_h^{\overline{-\alpha}}y(a)=h^{-\alpha}\Gamma(1-\alpha)y(a),$$

and since $y(a) \geq 0$, $h^{-\alpha} > 0$, and $\Gamma(1-\alpha) > 0$, then

$$\nabla_h S(a) = h^{-\alpha}\Gamma(1-\alpha)y(a) \geq 0.$$

Assume that $\nabla_h S(i) \geq 0$, $\forall i < t$. We shall show that $\nabla_h S(t) \geq 0$.

From the assumption that $y(t)$ is increasing, it follows that $y(t) \geq y(t-h) \geq y(a) \geq 0$, $\forall t \in \mathbb{N}_{a,h}$.

From (2), we recall that

$$\nabla_h S(t) = h_h^{\overline{-\alpha}} y(t) - \alpha \sum_{k=a/h}^{t/h-1} \big(t-\rho_h(kh)\big)_h^{\overline{-\alpha-1}} y(kh)h \geq 0.$$

Then we have

$$\begin{aligned}
\nabla_h S(t) &= h_h^{\overline{-\alpha}} y(t) - \alpha\big(t-\rho_h(t-h)\big)_h^{\overline{-\alpha-1}} y(t-h)h - \alpha \sum_{k=a/h}^{t/h-2} \big(t-\rho_h(kh)\big)_h^{\overline{-\alpha-1}} y(kh)h \\
&= h_h^{\overline{-\alpha}} y(t) - \alpha(t-t+h+h)_h^{\overline{-\alpha-1}} y(t-h)h - \alpha \sum_{k=a/h}^{t/h-2} \big(t-\rho_h(kh)\big)_h^{\overline{-\alpha-1}} y(kh)h \\
&= h_h^{\overline{-\alpha}} y(t) - \alpha(2h)_h^{\overline{-\alpha-1}} y(t-h)h - \alpha \sum_{k=a/h}^{t/h-2} \big(t-\rho_h(kh)\big)_h^{\overline{-\alpha-1}} y(kh)h \\
&\quad - \alpha \sum_{k=a/h}^{t/h-2} \big(t-\rho_h(kh)\big)_h^{\overline{-\alpha-1}} y(t-h)h + \alpha \sum_{k=a/h}^{t/h-2} \big(t-\rho_h(kh)\big)_h^{\overline{-\alpha-1}} y(t-h)h \\
&= h_h^{\overline{-\alpha}} y(t) - \alpha(2h)_h^{\overline{-\alpha-1}} y(t-h)h + \alpha \sum_{k=a/h}^{t/h-2} \big(t-\rho_h(kh)\big)_h^{\overline{-\alpha-1}} \big(y(t-h)-y(kh)\big)h \\
&\quad - \alpha \sum_{k=a/h}^{t/h-2} \big(t-\rho_h(kh)\big)_h^{\overline{-\alpha-1}} y(t-h)h.
\end{aligned}$$

Note that since $y(t)$ is increasing, then $y(t-h) - y(kh) \geq 0$, $\forall k = \frac{a}{h}, \frac{a}{h}+1, \ldots, \frac{t}{h}-2$, from which it follows that

$$\begin{aligned}
\nabla_h S(t) &\geq h_h^{\overline{-\alpha}} y(t) - \alpha(2h)_h^{\overline{-\alpha-1}} y(t-h)h - \alpha \sum_{k=a/h}^{t/h-2} \big(t-\rho_h(kh)\big)_h^{\overline{-\alpha-1}} y(t-h)h \\
&= h_h^{\overline{-\alpha}} y(t) - \alpha \sum_{k=a/h}^{t/h-1} \big(t-\rho_h(kh)\big)_h^{\overline{-\alpha-1}} y(t-h)h \\
&= h_h^{\overline{-\alpha}} y(t) - h_h^{\overline{-\alpha}} y(t-h) + h_h^{\overline{-\alpha}} y(t-h) - \alpha y(t-h)h \sum_{k=a/h}^{t/h-1} \big(t-\rho_h(kh)\big)_h^{\overline{-\alpha-1}} \\
&= h_h^{\overline{-\alpha}}\big(y(t)-y(t-h)\big) + y(t-h)\left(h_h^{\overline{-\alpha}} - \alpha h \sum_{k=a/h}^{t/h-1} \big(t-\rho_h(kh)\big)_h^{\overline{-\alpha-1}}\right) \\
&\geq y(t-h)\left(h_h^{\overline{-\alpha}} - \alpha h \sum_{k=a/h}^{t/h-1} \big(t-\rho_h(kh)\big)_h^{\overline{-\alpha-1}}\right)
\end{aligned}$$

$$
\begin{aligned}
&= y(t-h)\Big(h_h^{\overline{-\alpha}} - \alpha h\Big[\big(t-\rho_h(a)\big)_h^{\overline{-\alpha-1}} + \big(t-\rho_h(a+h)\big)_h^{\overline{-\alpha-1}} \\
&\quad + \cdots + \big(t-\rho_h(t-h)\big)_h^{\overline{-\alpha-1}}\Big]\Big) \\
&= y(t-h)\Big(h_h^{\overline{-\alpha}} - \alpha h\Big[(t-a+h)_h^{\overline{-\alpha-1}} + (t-a)_h^{\overline{-\alpha-1}} + \cdots + (2h)_h^{\overline{-\alpha-1}}\Big]\Big) \\
&= y(t-h)h^{-\alpha}\left[\frac{\Gamma(1-\alpha)}{\Gamma(1)} - \alpha\frac{\Gamma(\frac{t-a}{h}-\alpha)}{\Gamma(\frac{t-a}{h}+1)}) - \alpha\frac{\Gamma(\frac{t-a}{h}-\alpha-1)}{\Gamma(\frac{t-a}{h})}) - \cdots - \alpha\frac{\Gamma(1-\alpha)}{\Gamma(2)}\right] \\
&= y(t-h)h^{-\alpha} \\
&\qquad \times \left[\frac{\Gamma(1-\alpha)}{\Gamma(1)} - \alpha\frac{\Gamma(1-\alpha)}{\Gamma(2)} - \alpha\frac{\Gamma(2-\alpha)}{\Gamma(3)} - \alpha\frac{\Gamma(3-\alpha)}{\Gamma(4)} - \cdots - \alpha\frac{\Gamma(\frac{t-a}{h}-\alpha)}{\Gamma(\frac{t-a}{h}+1)}\right] \\
&= y(t-h)h^{-\alpha}\left[\frac{\Gamma(1-\alpha)}{1\Gamma(1)}(1-\alpha) - \alpha\frac{\Gamma(2-\alpha)}{\Gamma(3)} - \alpha\frac{\Gamma(3-\alpha)}{\Gamma(4)} - \cdots - \alpha\frac{\Gamma(\frac{t-a}{h}-\alpha)}{\Gamma(\frac{t-a}{h}+1)}\right] \\
&= y(t-h)h^{-\alpha}\left[\frac{2\Gamma(2-\alpha)}{2\Gamma(2)} - \alpha\frac{\Gamma(2-\alpha)}{\Gamma(3)} - \alpha\frac{\Gamma(3-\alpha)}{\Gamma(4)} - \cdots\alpha\frac{\Gamma(\frac{t-a}{h}-\alpha)}{\Gamma(\frac{t-a}{h}+1)}\right] \\
&= y(t-h)h^{-\alpha}\left[\frac{\Gamma(2-\alpha)}{\Gamma(3)}(2-\alpha) - \alpha\frac{\Gamma(3-\alpha)}{\Gamma(4)} - \cdots - \alpha\frac{\Gamma(\frac{t-a}{h}-\alpha)}{\Gamma(\frac{t-a}{h}+1)}\right] \\
&= y(t-h)h^{-\alpha}\left[\frac{3\Gamma(3-\alpha)}{3\Gamma(3)} - \alpha\frac{\Gamma(3-\alpha)}{\Gamma(4)} - \cdots - \alpha\frac{\Gamma(\frac{t-a}{h}-\alpha)}{\Gamma(\frac{t-a}{h}+1)}\right].
\end{aligned}
$$

If we continue in the same manner, we conclude that

$$
\begin{aligned}
\nabla_h S(t) &\geq y(t-h)h^{-\alpha}\left(\frac{\Gamma(\frac{t-a}{h}-\alpha)}{\Gamma(\frac{t-a}{h})} - \alpha\frac{\Gamma(\frac{t-a}{h}-\alpha)}{\Gamma(\frac{t-a}{h}+1)}\right) \\
&= y(t-h)h^{-\alpha}\frac{\Gamma(\frac{t-a}{h}-\alpha)}{\Gamma(\frac{t-a}{h})}\left(1-\alpha\frac{1}{(\frac{t-a}{h})}\right) \\
&= y(t-h)h^{-\alpha}\frac{(\frac{t-a}{h}-\alpha)\Gamma(\frac{t-a}{h}-\alpha)}{(\frac{t-a}{h})\Gamma(\frac{t-a}{h})} = y(t-h)\left(h^{-\alpha}\frac{\Gamma(\frac{t-a+h}{h}-\alpha)}{\Gamma(\frac{t-a+h}{h})}\right) \\
&= y(t-h)(t-a+h)_h^{\overline{-\alpha}} \geq 0, \quad \text{which completes the proof.}
\end{aligned}
$$

□

The proof of the following theorem is similar to that in Theorem 3.3.

Theorem 3.4 *Let a function $y : \mathbb{N}_{a-h,h} \to \mathbb{R}$ satisfy $y(a) > 0$ and be strictly increasing on $\mathbb{N}_{a,h}$, where $0 < \alpha \leq 1$ and $0 < h \leq 1$. Then*

$$\big({}_{a-h}\nabla_h^{\alpha} y\big)(t) > 0.$$

Theorem 3.5 *Let $y : \mathbb{N}_{a-h,h} \to \mathbb{R}$, and suppose that $({}_{a-h}\nabla_h^{\alpha} y)\ (t) \leq 0$ for $0 < \alpha \leq 1$, and $0 < h \leq 1$, $t \in \mathbb{N}_{a-h,h}$. Then $y(t)$ is α-decreasing.*

Proof Let $g : \mathbb{N}_{a-h,h} \to \mathbb{R}$ be a function such that $g(t) = -y(t)$, hence

$$\big({}_{a-h}\nabla_h^{\alpha} g\big)(t) = \big({}_{a-h}\nabla_h^{\alpha}(-y)\big)(t) = -\big({}_{a-h}\nabla_h^{\alpha} y\big)(t) \geq 0.$$

Then the proof follows by applying Theorem 3.1 to $g(t)$. □

Theorem 3.6 *Let a function* $y : \mathbb{N}_{a-h,h} \to \mathbb{R}$ *satisfy* $y(a) \leq 0$ *and be decreasing on* $\mathbb{N}_{a,h}$. *Then, for* $0 < \alpha \leq 1$ *and* $0 < h \leq 1$, *we have*

$$\left({}_{a-h}\nabla_h^{\alpha} y\right)(t) \leq 0, \quad \forall t \in \mathbb{N}_{a-h,h}.$$

Proof The proof follows by applying Theorem 3.3 to $g(t) = -y(t)$. □

4 Riemann-type fractional difference initial value problem

The following results are essential to proceed for the mean value theorem.

Lemma 4.1 *For any* $0 < \alpha \leq 1$, $0 < h \leq 1$, *and* $f : \mathbb{N}_{a+h,h} \to \mathbb{R}$, *the following equality holds*:

$${}_a\nabla_h^{-\alpha}\nabla_h f(t) = \nabla_h\, {}_a\nabla_h^{-\alpha} f(t) - \frac{(t-a)_h^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(a).$$

Proof Recalling that

$${}_a\nabla_h^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{k=a/h+1}^{t/h} \left(t - \rho_h(kh)\right)_h^{\overline{\alpha-1}} f(kh)h,$$

we have

$$\begin{aligned}
&\nabla_h\, {}_a\nabla_h^{-\alpha} f(t) \\
&= \frac{1}{h}\left(\frac{1}{\Gamma(\alpha)} \sum_{k=a/h+1}^{t/h} \left(t - \rho_h(kh)\right)_h^{\overline{\alpha-1}} f(kh)h - \frac{1}{\Gamma(\alpha)} \sum_{k=a/h+1}^{(t-h)/h} \left(t - h - \rho_h(kh)\right)_h^{\overline{\alpha-1}} f(kh)h\right) \\
&= \frac{h}{\Gamma(\alpha)} \sum_{k=a/h+1}^{t/h} \frac{(t-\rho_h(kh))_h^{\overline{\alpha-1}}}{h} f(kh) - \frac{h}{\Gamma(\alpha)} \sum_{k=a/h+1}^{t/h} \frac{(t-h-\rho_h(kh))_h^{\overline{\alpha-1}}}{h} f(kh) \\
&\quad + \frac{h}{\Gamma(\alpha)} \left(t - h - \rho_h(t)\right)_h^{\overline{\alpha-1}} f(t) \\
&= \frac{h}{\Gamma(\alpha)} \sum_{k=a/h+1}^{t/h} f(kh) \nabla_h \left(t - \rho_h(kh)\right)_h^{\overline{\alpha-1}}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&{}_a\nabla_h^{-\alpha}\nabla_h f(t) \\
&= \frac{1}{\Gamma(\alpha)} \sum_{k=a/h+1}^{t/h} \left(t - \rho_h(kh)\right)_h^{\overline{\alpha-1}} \nabla_h f(kh)h \\
&= \frac{1}{\Gamma(\alpha)} \sum_{k=a/h+1}^{t/h} \left(t - \rho_h(kh)\right)_h^{\overline{\alpha-1}} \frac{f(kh) - f(kh-h)}{h} h \\
&= \frac{1}{\Gamma(\alpha)} \sum_{k=a/h+1}^{t/h} \left(t - \rho_h(kh)\right)_h^{\overline{\alpha-1}} f(kh) - \frac{1}{\Gamma(\alpha)} \sum_{k=a/h+1}^{t/h} \left(t - \rho_h(kh)\right)_h^{\overline{\alpha-1}} f(kh-h) \\
&= \frac{1}{\Gamma(\alpha)} \sum_{k=a/h+1}^{t/h} \left(t - \rho_h(kh)\right)_h^{\overline{\alpha-1}} f(kh) - \frac{1}{\Gamma(\alpha)} \sum_{k=a/h}^{t/h-1} \left(t - \rho_h(kh+h)\right)_h^{\overline{\alpha-1}} f(kh-h+h)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(\alpha)} \sum_{k=a/h+1}^{t/h} \big(t-\rho_h(kh)\big)_h^{\overline{\alpha-1}} f(kh) - \frac{1}{\Gamma(\alpha)} \sum_{k=a/h+1}^{t/h} \big(t-h-\rho_h(kh)\big)_h^{\overline{\alpha-1}} f(kh) \\
&\quad + \frac{1}{\Gamma(\alpha)} \big(t-h-\rho_h(t)\big)_h^{\overline{\alpha-1}} f(t) - \frac{1}{\Gamma(\alpha)} \big(t-h-\rho_h(a)\big)_h^{\overline{\alpha-1}} f(a) \\
&= \frac{h}{\Gamma(\alpha)} \sum_{k=a/h+1}^{t/h} \frac{(t-\rho_h(kh))_h^{\overline{\alpha-1}} - (t-h-\rho_h(kh))_h^{\overline{\alpha-1}}}{h} f(kh) - \frac{1}{\Gamma(\alpha)} (t-a)_h^{\overline{\alpha-1}} f(a) \\
&= \frac{h}{\Gamma(\alpha)} \sum_{k=a/h+1}^{t/h} \nabla_h \big(t-\rho_h(kh)\big)_h^{\overline{\alpha-1}} f(kh) - \frac{(t-a)_h^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(a) \\
&= \nabla_h {}_a\nabla_h^{-\alpha} f(t) - \frac{(t-a)_h^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(a). \qquad \square
\end{aligned}
$$

Lemma 4.2 *For any $0 < \alpha \le 1$, $0 < h \le 1$, and $y : \mathbb{N}_{a+h,h} \to \mathbb{R}$, the following equality holds*:

$$
{}_{a-h}\nabla_h^{\alpha} y(t) = {}_a\nabla_h^{\alpha} y(t) + \frac{(t-a+h)_h^{\overline{-\alpha-1}}}{\Gamma(-\alpha)} y(a)h.
$$

Proof From the definition and the proof of Lemma 2.1, we have

$$
\begin{aligned}
{}_{a-h}\nabla_h^{\alpha} y(t) &= \nabla_h {}_{a-h}\nabla_h^{-(1-\alpha)} y(t) \\
&= \nabla_h \left(\frac{1}{\Gamma(1-\alpha)} \sum_{k=a/h}^{t/h} \big(t-\rho_h(kh)\big)_h^{\overline{-\alpha}} y(kh)h \right) \\
&= \frac{1}{h\Gamma(1-\alpha)} \left(\sum_{k=a/h}^{t/h} \big(t-\rho_h(kh)\big)_h^{\overline{-\alpha}} y(kh)h - \sum_{k=a/h}^{t/h-1} \big(t-h-\rho_h(kh)\big)_h^{\overline{-\alpha}} y(kh)h \right) \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{k=a/h}^{t/h} \nabla_h \big(t-\rho_h(kh)\big)_h^{\overline{-\alpha}} y(kh)h + \frac{1}{\Gamma(1-\alpha)} (t-h-t+h)_h^{\overline{-\alpha}} y(t) \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{k=a/h}^{t/h} (-\alpha) \big(t-\rho_h(kh)\big)_h^{\overline{-\alpha-1}} y(kh)h \\
&= \frac{-\alpha}{-\alpha\Gamma(-\alpha)} \sum_{k=a/h}^{t/h} \big(t-\rho_h(kh)\big)_h^{\overline{-\alpha-1}} y(kh)h \\
&= \frac{1}{\Gamma(-\alpha)} \sum_{k=a/h+1}^{t/h} \big(t-\rho_h(kh)\big)_h^{\overline{-\alpha-1}} y(kh)h + \frac{1}{\Gamma(-\alpha)} \big(t-\rho_h(a)\big)_h^{\overline{-\alpha-1}} y(a)h \\
&= {}_a\nabla_h^{\alpha} y(t) + \frac{(t-a+h)_h^{\overline{-\alpha-1}}}{\Gamma(-\alpha)} y(a)h. \qquad (5)
\end{aligned}
$$

$\square$

Theorem 4.3 *For any $0 < \alpha \le 1$, $0 < h \le 1$, and $y : \mathbb{N}_{a+h,h} \to \mathbb{R}$, the following equality holds*:

$$
{}_a\nabla_h^{-\alpha} {}_{a-h}\nabla_h^{\alpha} y(t) = y(t) - \frac{h^{1-\alpha}}{\Gamma(\alpha)} (t-a+h)_h^{\overline{\alpha-1}} y(a).
$$

Proof By the help of Lemma 4.2, we have

$$\begin{aligned}
{}_a\nabla_h^{-\alpha}\big({}_{a-h}\nabla_h^{\alpha}y(t)\big) &= {}_a\nabla_h^{-\alpha}\left({}_a\nabla_h^{\alpha}y(t) + \frac{(t-a+h)_h^{\overline{-\alpha-1}}}{\Gamma(-\alpha)}y(a)h\right) \\
&= {}_a\nabla_h^{-\alpha}\,{}_a\nabla_h^{\alpha}y(t) + {}_a\nabla_h^{-\alpha}\left(\frac{(t-a+h)_h^{\overline{-\alpha-1}}}{\Gamma(-\alpha)}y(a)h\right) \\
&= y(t) + {}_a\nabla_h^{-\alpha}\left(\frac{(t-a+h)_h^{\overline{-\alpha-1}}}{\Gamma(-\alpha)}y(a)h\right) \\
&= y(t) + {}_a\nabla_h^{-\alpha}\nabla_h\left(\frac{(t-a+h)_h^{\overline{-\alpha}}}{\Gamma(1-\alpha)}\right)y(a)h.
\end{aligned} \tag{6}$$

Moreover, by the help of Lemma 4.1 and that $h_h^{\overline{-\alpha}} = h^{-\alpha}\Gamma(1-\alpha)$, we have

$$\begin{aligned}
&{}_a\nabla_h^{-\alpha}\nabla_h\left(\frac{(t-a+h)_h^{\overline{-\alpha}}}{\Gamma(1-\alpha)}y(a)h\right) \\
&\quad = \nabla_h\,{}_a\nabla_h^{-\alpha}\left(\frac{(t-a+h)_h^{\overline{-\alpha}}}{\Gamma(1-\alpha)}y(a)h\right) - \frac{(t-a)_h^{\overline{\alpha-1}}}{\Gamma(\alpha)}y(a)h^{1-\alpha}.
\end{aligned} \tag{7}$$

Applying the identity

$${}_a\nabla_h^{-\alpha}g(t) = {}_{a-h}\nabla_h^{-\alpha}g(t) - \frac{(t-a+h)_h^{\overline{\alpha-1}}g(a)h}{\Gamma(\alpha)}$$

to the function $g(t) = \frac{(t-a+h)_h^{\overline{-\alpha}}}{\Gamma(1-\alpha)}y(a)h$, we obtain

$$\begin{aligned}
&{}_a\nabla_h^{-\alpha}\left(\frac{(t-a+h)_h^{\overline{-\alpha}}}{\Gamma(1-\alpha)}y(a)h\right) \\
&\quad = {}_{a-h}\nabla_h^{-\alpha}\left(\frac{(t-a+h)_h^{\overline{-\alpha}}}{\Gamma(1-\alpha)}y(a)h\right) - \frac{(t-a+h)_h^{\overline{\alpha-1}}}{\Gamma(\alpha)}y(a)h^{2-\alpha}.
\end{aligned} \tag{8}$$

Hence, by substituting (8) in (7) and by making use of Lemma 2.3 with $\mu = -\alpha$, we obtain

$$\begin{aligned}
&{}_a\nabla_h^{-\alpha}\nabla_h\left(\frac{(t-a+h)_h^{\overline{-\alpha}}}{\Gamma(1-\alpha)}y(a)h\right) \\
&\quad = \nabla_h\left\{{}_a\nabla_h^{-\alpha}\left(\frac{(t-(a-h))_h^{\overline{-\alpha}}}{\Gamma(1-\alpha)}y(a)h\right)\right\} - \frac{(t-a)_h^{\overline{\alpha-1}}}{\Gamma(\alpha)}y(a)h^{1-\alpha} \\
&\quad = \nabla_h\left\{{}_{a-h}\nabla_h^{-\alpha}\left(\frac{(t-(a-h))_h^{\overline{-\alpha}}}{\Gamma(1-\alpha)}y(a)h\right) - \frac{(t-(a-h))_h^{\overline{\alpha-1}}}{\Gamma(\alpha)}y(a)h^{2-\alpha}\right\} \\
&\qquad - \frac{(t-a)_h^{\overline{\alpha-1}}}{\Gamma(\alpha)}y(a)h^{1-\alpha} \\
&\quad = \nabla_h\big(y(a)h\big) - \frac{h^{1-\alpha}}{\Gamma(\alpha)}(t-a+h)_h^{\overline{\alpha-1}}y(a) = -\frac{h^{1-\alpha}}{\Gamma(\alpha)}(t-a+h)_h^{\overline{\alpha-1}}y(a).
\end{aligned}$$

We have used that

$$\nabla_h\frac{(t-(a-h))_h^{\overline{\alpha-1}}}{\Gamma(\alpha)}y(a)h^{2-\alpha} = \frac{h^{1-\alpha}}{\Gamma(\alpha)}(t-a+h)_h^{\overline{\alpha-1}}y(a) - \frac{(t-a)_h^{\overline{\alpha-1}}}{\Gamma(\alpha)}y(a)h^{1-\alpha}.$$

Hence,

$$ {}_{a}\nabla_h^{-\alpha}\, {}_{a-h}\nabla_h^{\alpha} y(t) = y(t) - \frac{h^{1-\alpha}}{\Gamma(\alpha)}(t-a+h)_h^{\overline{\alpha-1}} y(a). \qquad \square $$

Consider the following initial fractional difference equation:

$$ {}_{a-h}\nabla_h^{\alpha} y(t) = f\bigl(t, y(t)\bigr) \quad \text{for } t = a+h, a+2h, \ldots, \tag{9} $$

$$ {}_{a-h}\nabla_h^{-(1-\alpha)} y(t)\big|_{t=a} = h^{1-\alpha} y(a) = c, \tag{10} $$

where $0 < \alpha, h < 1$ and a is any real number.

By means of Theorem 4.3, we can state the following theorem.

Theorem 4.4 *y is a solution of the initial value problem, (9), (10) if and only if it has the representation*

$$ y(t) = \frac{(t-a+h)_h^{\overline{\alpha-1}}}{\Gamma(\alpha)} c + {}_{a}\nabla_h^{-\alpha} f\bigl(t, y(t)\bigr). \tag{11} $$

5 Application: Mean Value Theorem (MVT)

First, for the sake of simplification, depending on Theorem 4.3, we shall write

$$ {}_{a}\nabla_h^{-\alpha}\, {}_{a-h}\nabla_h^{\alpha} y(t) = y(t) - R_h(\alpha, t, a) y(a), \tag{12} $$

where $R_h(\alpha, t, a) = \frac{h^{1-\alpha}}{\Gamma(\alpha)}(t-a+h)_h^{\overline{\alpha-1}}$.

Theorem 5.1 (The h-fractional difference MVT) *Let f and g be functions defined on $\mathbb{N}_{a,h} \cap {}_{b,h}\mathbb{N} = \{a, a+h, a+2h, \ldots, b-2h, b-h, b\}$, where $b = a+kh$ for some $k \in \mathbb{N}$. Assume that g is strictly increasing, $g(a) > 0$, and $0 < \alpha < 1$, $0 < h \leq 1$. Then there exist $s_1, s_2 \in \mathbb{N}_{a,h} \cap {}_{b,h}\mathbb{N}$ such that*

$$ \frac{({}_{a-h}\nabla_h^{\alpha} f)(s_1)}{({}_{a-h}\nabla_h^{\alpha} g)(s_1)} \leq \frac{f(b) - R_h(\alpha, b, a) f(a)}{g(b) - R_h(\alpha, b, a) g(a)} \leq \frac{({}_{a-h}^{R}\nabla_h^{\alpha} f)(s_2)}{({}_{a-h}^{R}\nabla_h^{\alpha} g)(s_2)}. \tag{13} $$

Proof First we need to show that $g(b) - R_h(\alpha, b, a) g(a) > 0$. Since g is strictly increasing, then by Theorem 3.4 we have

$$ \bigl({}_{a-h}\nabla_h^{\alpha} g\bigr)(t) > 0 \quad \forall t \in \mathbb{N}_{a,h} \cap {}_{b,h}\mathbb{N}. $$

Applying the fractional sum operator on both sides of the inequality, by means of (12), we get

$$ {}_{a}\nabla_h^{-\alpha}\bigl({}_{a-h}\nabla_h^{\alpha} g\bigr)(t) > {}_{a}\nabla_h^{-\alpha}(0) \quad \forall t \in \mathbb{N}_{a,h} \cap {}_{b,h}\mathbb{N}, $$

or by means of (12) we have

$$ g(t) - R_h(\alpha, t, a) g(a) > 0 \quad \forall t \in \mathbb{N}_{a,h} \cap {}_{b,h}\mathbb{N}. $$

For $t = b$, we get

$$g(b) - R_h(\alpha, b, a)g(a) > 0.$$

To prove the theorem, we use contradiction. Assume that (13) is not true, then either

$$\frac{f(b) - R_h(\alpha, b, a)f(a)}{g(b) - R_h(\alpha, b, a)g(a)} < \frac{({}_{a-h}\nabla^{\alpha}_{h}f)(t)}{({}_{a-h}\nabla^{\alpha}_{h}g)(t)}, \quad \forall t \in \mathbb{N}_{a,h} \cap {}_{b,h}\mathbb{N}, \tag{14}$$

or

$$\frac{f(b) - R_h(\alpha, b, a)f(a)}{g(b) - R_h(\alpha, b, a)g(a)} > \frac{({}_{a-h}\nabla^{\alpha}_{h}f)(t)}{({}^{R}_{a-h}\nabla^{\alpha}_{h}g)(t)}, \quad \forall t \in \mathbb{N}_{a,h} \cap {}_{b,h}\mathbb{N}. \tag{15}$$

Again, since g is strictly increasing, then by Theorem 3.4 we conclude that

$$\big({}_{a-h}\nabla^{\alpha}_{h}g\big)(t) > 0 \quad \forall t \in \mathbb{N}_{a,h} \cap {}_{b,h}\mathbb{N}.$$

Hence (14) becomes

$$\frac{f(b) - R_h(\alpha, b, a)f(a)}{g(b) - R_h(\alpha, b, a)g(a)} \big({}_{a-h}\nabla^{\alpha}_{h}g\big)(t) < \big({}_{a-h}\nabla^{\alpha}_{h}f\big)(t), \quad \forall t \in \mathbb{N}_{a,h} \cap {}_{b,h}\mathbb{N}.$$

Applying the fractional sum operator on both sides of the inequality at $t = b$ and by making use of (12), we see that

$$\frac{f(b) - R_h(\alpha, b, a)f(a)}{g(b) - R_h(\alpha, b, a)g(a)} \big(g(b) - R_h(\alpha, b, a)g(a)\big) < \big(f(b) - R_h(\alpha, b, a)f(a)\big),$$

and hence $f(b) < f(b)$, which is a contradiction. In a similar way, (15) will lead to contradiction. □

Remark 5.1 If we let $h = 1$ in Theorem 5.1, then we reobtain the results in [34] via using the dual identities presented in [30, 31], or else we refer to [37].

6 Conclusions

The contributions of this paper can be concluded as follows:

1. Nabla fractional sums and differences of order $0 < \alpha \leq 1$ on the time scale $h\mathbb{Z}$ have been formulated.
2. Riemann–Liouville and Caputo discrete fractional operators on the time scale $h\mathbb{Z}$ have been defined.
3. The relation between nabla h-RL and h-Caputo fractional differences has been detected.
4. If $({}_{a}\nabla^{\alpha}_{h}y)(t) > 0$, then $y(t)$ is α-increasing.
5. If $y(t)$ is α-increasing and $y(a) > 0$, then $({}_{a}\nabla^{\alpha}_{h}y)(t) > 0$.
6. The monotonicity factor, which is α, has not been affected by the discretization step h.
7. A Riemann-type fractional difference initial value problem has been formulated and solved, and hence we generalize the representation obtained in [20].

8. A monotonicity result for the nabla h-Caputo fractional difference operator has been proved as well.
9. As an application, a fractional difference version of the Mean Value Theorem on $h\mathbb{Z}$ has been proved.
10. Working on $h\mathbb{Z}$, $h \in (0,1)$ rather than on $\mathbb{Z}$ makes it possible to guarantee the convergence of solutions for a larger class of fractional difference initial value problems.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All the authors participated in obtaining the main results of this manuscript and drafted the manuscript. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics and Statistics, Arab American University, Zababdeh, Palestine. [2]Department of Mathematics and General Sciences, Prince Sultan University, Riyadh, Saudi Arabia.

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