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A fractional Fourier integral operator and its extension to classes of function spaces

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Abstract

In this paper, an attempt is being made to investigate a class of fractional Fourier integral operators on classes of function spaces known as ultraBoehmians. We introduce a convolution product and establish a convolution theorem as a product of different functions. By employing the convolution theorem and making use of an appropriate class of approximating identities, we provide necessary axioms and define function spaces where the fractional Fourier integral operator is an isomorphism connecting the different spaces. Further, we provide an inversion formula and obtain various properties of the cited integral in the generalized sense.

Keywords: Fractional Fourier integral; Shrinking support; UltraBoehmians

1 Introduction and preliminary

Generalized functions, which are continuous linear functionals over a space of smooth functions, are useful in making discontinuous functions more likely smooth and describe physical phenomena as point charges that consequently lead to an extensive use in applied physics and engineering problems. The space of recent generalized functions known as the space of Boehmians is defined by an algebraic construction which is similar to that of field of quotients. When constructions are applied to function spaces and multiplications are interpreted as convolutions, the constructions yield various spaces of generalized functions. Approximating identities or delta sequences with shrinking support to the origin are needful in constructing Boehmian spaces. This, indeed, led among other things to uniqueness theorems which are interpreted as an uncertainty principle for Boehmian spaces. However, as Boehmians have an abstract algebraic definition, they allow different interpretations of those extended operators to be isomorphisms between the different spaces of Boehmians; see [1–7] for further details.

The fractional Fourier integral operator is a generalization of the classical Fourier integral operator into the fractional domains. Although various definitions of the fractional Fourier integral operator were given in literature, the most intuitive way of defining the fractional Fourier operator has been given by generalizing the concept of rotations over an angle $\pi/2$ in the classical Fourier integral operator. As the classical Fourier integral operator corresponds to a rotation in the time frequency plane, the fractional Fourier integral operator corresponds to a rotation over an angle $\alpha = a\pi/2$, $a \in R$.

Let $K_\alpha(t,w)$ be defined by

$$K_\alpha(t,w) = \left\{ \begin{array}{l} \frac{c(\alpha)}{\sqrt{2\pi}} \exp\{ja(\alpha)((t^2 + w^2) - 2b(\alpha)wt)\}, \alpha \neq n\pi \\ e^{-jwt}/\sqrt{2\pi}, \alpha = \frac{\pi}{2} \end{array} \right\}, \tag{1}$$

then the formal and direct definition of the fractional Fourier integral operator with an angle α of a signal $\vartheta(t)$ is defined by the integral equation [8] (see also Pathak et al. [9] and Prasad and Kumar [10])

$$F_\alpha(\vartheta)(w) = \int_{-\infty}^{\infty} \vartheta(t)K_\alpha(t,w)\,dt, \tag{2}$$

where $a(\alpha) = (\cot\alpha)/2$, $b(\alpha) = \sec\alpha$, and $c(\alpha) = \sqrt{1 - \cot\alpha}$. The inversion formula has been recovered from (2) and (1) as follows:

$$\vartheta(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_\alpha(\vartheta)(w)K_{-\alpha}(t,w)\,dw. \tag{3}$$

However, for certain values of α, the fractional Fourier integral operator satisfies $F_0(\vartheta)(w) = \vartheta(w)$, $F_\pi(\vartheta)(w) = \vartheta(-w)$, and $\frac{\pi}{2}(\vartheta)(w) = F(\vartheta)(w)$ where F denotes the ordinary Fourier integral of ϑ. The fractional Fourier integral operator has various applications in literature. Indeed, it has applications in the solution of ordinary differential equations, quantum optics (Garcia et al. [11]), quantum mechanics (Andez [12]), optical systems (Narayanana and Prabhu [13]), time filtering (Narayanana and Prabhu [13]), and some pattern recognitions (see Zayed and Garcia [14]) and Zayed [15, 16]) as well). Various properties of such a remarkable integral, such as linearity, index additivity $F_{\alpha_1}F_{\alpha_2} = F_{\alpha_1+\alpha_2}$, commutativity $F_{\alpha_1}F_{\alpha_2} = F_{\alpha_2}F_{\alpha_1}$, and associativity $(F_{\alpha_1}F_{\alpha_2})F_{\alpha_3} = F_{\alpha_1}(F_{\alpha_2}F_{\alpha_3})$ of the fractional Fourier integral operator, have been obtained in the literature; see Barbu [17] for some details.

As part of long-term research, the purpose of this article is to give an extension of the fractional Fourier integral operator into appropriately defined spaces of Boehmians and define an isomorphism between the different spaces of Boehmians. We divide this article into four sections. In Sect. 2 we define convolution products and prove a convolution theorem. In Sect. 3 we establish a class of approximating identities and derive spaces of ultra-Boehmians and, further, we extend the fractional integral operator to the defined spaces. In Sect. 4 we prove linearity and discuss an inversion formula.

2 Convolution product and convolution theorem

To establish a convolution theorem for the fractional Fourier integral operator, we introduce a convolution product as follows:

$$\bigl(\vartheta *^\alpha \vartheta_0\bigr)(t) = \int_{-\infty}^{\infty} \vartheta(z)\vartheta_0(t-z)\psi(t,z)\,dz, \tag{4}$$

where $\psi(t,z) = e^{2jz(z-t)a(\alpha)}$.

The convolution theorem for the fractional Fourier integral operator can be drawn as follows; see [7, 15] for similar discussion.

Theorem 1 *Let* $a(\alpha) = (\cot\alpha)/2$, $b(\alpha) = \sec\alpha$, $c(\alpha) = \sqrt{1-\cot\alpha}$, *and* $*^{\alpha}$ *be defined by* (4), *then we have*

$$F_{\alpha}\big(\vartheta *^{\alpha} \vartheta_0\big)(w) = \frac{\sqrt{2\pi}}{c(\alpha)} \exp\left(-j\frac{\cot\alpha}{2}w^2\right)(F_{\alpha}\vartheta)(w)(F_{\alpha}\vartheta_0)(w). \tag{5}$$

Proof By aid of (2) and (4), we are led to writing

$$\begin{aligned} F_{\alpha}\big(\vartheta *^{\alpha} \vartheta_0\big)(w) = {}& \frac{c(\alpha)}{\sqrt{2\pi}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \vartheta(z)\vartheta_0(t-z)\exp\left(j\frac{\cot\alpha}{2}t^2\right) \\ & \times \exp\left(j\frac{\cot\alpha}{2}w^2\right)\exp(-j\csc wt)\exp\big(j(z-t)\cot\alpha\big)\,dt\,dz. \end{aligned} \tag{6}$$

The change of variables $t-z=v$ gives

$$\begin{aligned} F_{\alpha}\big(\vartheta *^{\alpha} \vartheta_0\big)(w) = {}& \frac{c(\alpha)}{\sqrt{2\pi}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \vartheta(z)\vartheta_0(v)\exp\left(j\frac{\cot\alpha}{2}(v+z)^2\right) \\ & \times \exp\left(\frac{\cot\alpha}{2}w^2\right)\exp -j\big(w(v+z)\csc\alpha\big)\exp(-jzv\cot)\,dz\,dv. \end{aligned} \tag{7}$$

Hence, by rearranging the products in (7), we obtain

$$\begin{aligned} F_{\alpha}\big(\vartheta *^{\alpha} \vartheta_0\big)(w) = {}& \frac{c(\alpha)}{\sqrt{2\pi}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \vartheta(z)\exp\big(j\cot\alpha z^2\big)\exp\left(j\frac{\cot\alpha}{2}w^2\right) \\ & \times \exp(-jwz\csc\alpha)\,dz\vartheta_0(v)\exp\left(j\frac{\cot\alpha}{2}v^2\right)\exp(-j\cot\alpha wv)\,dv. \end{aligned} \tag{8}$$

Multiplying (8) by $\frac{c(\alpha)}{\sqrt{2\pi}}\exp(j\frac{\cot\alpha}{2}w^2)$ yields

$$F_{\alpha}\big(\vartheta *^{\alpha} \vartheta_0\big)(w)\frac{c(\alpha)}{\sqrt{2\pi}} \exp\left(j\frac{\cot\alpha}{2}w^2\right) = (F_{\alpha}\vartheta)(w)(F_{\alpha}\vartheta_0)(w).$$

That is,

$$F_{\alpha}\big(\vartheta *^{\alpha} \vartheta_0\big)(w) = \frac{\sqrt{2\pi}}{c(\alpha)} \exp\left(-j\frac{\cot\alpha}{2}w^2\right)(F_{\alpha}\vartheta)(w)(F_{\alpha}\vartheta_0)(w).$$

The proof is finished. □

3 UltraBoehmians

With the help of the convolution theorem, we introduce a generalized convolution product and generate a class of ultraBoehmians. The spaces under construction generalize those obtained by Prasad and Kumar [10] and Pathak [9]. Due to Prasad and Kumar [9, (5)] and Zemanian [18] as well, an infinitely differentiable complex-valued function ϑ is said to be in $S_{\beta}^{\tilde{\beta}}(R)$ if and only if

$$\gamma_{\beta,\tilde{\beta}}(\vartheta) = \sup_{t\in R}\big|t^{\beta}D^{\tilde{\beta}}\vartheta(t)\big| < \infty \tag{9}$$

for every choice of constants β, $\tilde{\beta}$. Alternatively, (9) can often be written as

$$J_{m,\tilde{\beta}}(\vartheta) = \sup_{t\in R}\left|\left(1+|t|^2\right)^{\frac{m}{2}} D^{\tilde{\beta}}\vartheta(t)\right| < \infty \quad (m,\tilde{\beta}\in N).$$

The dense subspace of $S_{\beta}^{\tilde{\beta}}(R)$ of smooth (C^{∞}) functions of compact supports over R is denoted by $D_{\beta}^{\tilde{\beta}}(R)$.

The following theorem is very needful in the sequel; see [9].

Theorem 2 *The fractional Fourier integral operator is a continuous linear mapping from $S_{\beta}^{\tilde{\beta}}(R)$ into $S_{\beta}^{\tilde{\beta}}(R)$.*

Parseval's identity of F_{α} is given as

$$\int_{-\infty}^{\infty} \vartheta(t)\overline{\psi(t)}\,dt = \int_{-\infty}^{\infty} \hat{\vartheta}_{\alpha}(\xi)\overline{\hat{\psi}_{\alpha}(\xi)}\,d\xi,$$

and hence we have

$$\int_{-\infty}^{\infty} \left|\vartheta(t)\right|^2 dt = \int_{-\infty}^{\infty} \left|\hat{\vartheta}_{\alpha}(\xi)\right|^2 d\xi,$$

where $\hat{\psi}_{\alpha}$, $\hat{\vartheta}_{\alpha}$ are the fractional Fourier integral operators of ψ_{α} and ϑ_{α}, and $\bar{\psi}$ is the conjugate of ψ, see Pathak et al. [9, Theorem 3.1, p. 243] *for some details.*

We state without proof the following two theorems as the proofs are a straightforward consequence of (4).

Theorem 3 *The product $*^{\alpha}$ is commutative in $S_{\beta}^{\tilde{\beta}}(R)$, i.e., $\vartheta *^{\alpha} \vartheta_0 = \vartheta_0 *^{\alpha} \vartheta$.*

Theorem 4 *Let $\vartheta, \vartheta_0, h \in S_{\beta}^{\tilde{\beta}}(R)$, then $\vartheta *^{\alpha} (\vartheta_0 *^{\alpha} h) = (\vartheta *^{\alpha} \vartheta_0) *^{\alpha} h$.*

Now we introduce a class of approximating identities as follows.

Definition 5 Let $\Delta_{\beta}^{\tilde{\beta}}$ be the set of sequences $(\delta_n)_1^{\infty}$ of $D_{\beta}^{\tilde{\beta}}(R)$ where the following formulas hold:

$$\int_{-\infty}^{\infty} \delta_n(t)e^{ja(\alpha)t^2}\,dt = 1 \quad (\forall n\in N), \tag{10}$$

$$\int_{-\infty}^{\infty} \left|\delta_n(t)\right|dt < M \quad (M\in R, n\in N), \tag{11}$$

$$\max_{|t|\geq\delta}\left|\delta_n(t)\right| \to 0 \tag{12}$$

as $n\to\infty$, $\forall\delta>0$. We claim that $\Delta_{\beta}^{\tilde{\beta}}$ with the convolution product $*^{\alpha}$ represents a class of approximating identities.

We establish the following theorem.

Theorem 6 *The class $(\Delta_{\beta}^{\tilde{\beta}}, *^{\alpha})$ forms a class of approximating identities.*

Proof We show $(\delta_n *^{\alpha} \gamma_n) \in \Delta_{\beta}^{\tilde{\beta}}$ when $(\delta_n), (\gamma_n) \in \Delta_{\beta}^{\tilde{\beta}}$. As the proofs of (11) and (12) are straightforward, it suffices to show that (10) holds. By (4) we write

$$\begin{aligned}\int_{-\infty}^{\infty} (\delta_n *^{\alpha} \gamma_n)(t) e^{ja(\alpha)t^2}\,dt &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta_n(z)\gamma_n(t-z)\psi(t,z)dz e^{ja(\alpha)t^2}\,dt \\ &= \int_{-\infty}^{\infty} \delta_n(z)\int_{-\infty}^{\infty} \gamma_n(t-z)\psi(t,z)e^{ja(\alpha)t^2}\,dt\,dz.\end{aligned}$$

Let $t = z + w$, then the preceding equation gives

$$\begin{aligned}\int_{-\infty}^{\infty} \delta_n(z)\int_{-\infty}^{\infty} \gamma_n(w)\psi(z+w,z)e^{ja(\alpha)(z+w)^2}\,dw\,dz = {}& \int_{-\infty}^{\infty} \delta_n(z)e^{ja(\alpha)z^2}\,dz \\ & \times \int_{-\infty}^{\infty} \gamma_n(w)e^{ja(\alpha)w^2}\,dw.\end{aligned}$$

Hence the following is obtained by simple computations:

$$\int_{-\infty}^{\infty} (\delta_n *^{\alpha} \gamma_n)(t) e^{ja(\alpha)t^2}\,dt = 1.$$

The proof is completely finished. □

Theorem 7 *If $\psi \in D_{\beta}^{\tilde{\beta}}(R)$ and $\vartheta, \vartheta_n, \vartheta_0 \in S_{\beta}^{\tilde{\beta}}(R)$, $\vartheta_n \to \vartheta$, as $n \to \infty$, then we have*

(i) $(\vartheta + \vartheta_0) *^{\alpha} \psi = \vartheta *^{\alpha} \psi + \vartheta_0 *^{\alpha} \psi$.
(ii) $\vartheta_n *^{\alpha} \psi \to \vartheta *^{\alpha} \psi$ *as* $n \to \infty$.
(iii) $\lambda(\vartheta *^{\alpha} \psi) = (\lambda\vartheta *^{\alpha} \psi)$ *for some* $\lambda \in C$.

The proof of this theorem can be obtained by easy computations.

Theorem 8 *If $\vartheta \in S_{\beta}^{\tilde{\beta}}(R)$ and $\vartheta_0 \in D_{\beta}^{\tilde{\beta}}(R)$, then $\vartheta *^{\alpha} \vartheta_0 \in S_{\beta}^{\tilde{\beta}}(R)$.*

Proof Indeed, by (9), we get

$$\gamma_{\alpha,\beta}(\vartheta *^{\alpha} \vartheta_0) \le \sup_{t\in R}\left|t^{\alpha} D^{\beta}(\vartheta * \vartheta_0)(t)\right| < \infty, \tag{13}$$

where $*$ denotes the usual Fourier convolution product. The last inequality follows from the fact that $\vartheta * \vartheta_0 \in S_{\beta}^{\tilde{\beta}}(R)$ and that $|\psi| < 1$, see Zemanian [18].

The proof is finished. □

Theorem 9 *Let $(\delta_n) \in \Delta_{\beta}^{\tilde{\beta}}$, $\vartheta \in S_{\beta}^{\tilde{\beta}}(R)$, then $\vartheta *^{\alpha} \delta_n \to \vartheta$ as $n \to \infty$.*

Proof By aid of (10), we get

$$\begin{aligned}\left|t^{\beta} D^{\tilde{\beta}}(\vartheta *^{\alpha} \delta_n - \vartheta)(t)\right| &= \left|t^{\beta} D_t^{\tilde{\beta}}\left(\int_{-\infty}^{\infty} \vartheta_z(t) - \vartheta(t)\right)\delta_n(z)\,dz\right| \\ &\le \int_K M\left|t^{\beta} D_t^{\tilde{\beta}}(\vartheta_z(t) - \vartheta(t))\right|\,dz \to 0\end{aligned}$$

as $n \to \infty$, where $\vartheta(t-z) = \vartheta_z(t)$, M is a constant, and K is a compact subset of R satisfying $|\delta_n| \le M$ and $K \supseteq \operatorname{supp}\delta_n$, $n \in N$. The proof is finished. □

The space β_1 ($\beta_1 \equiv \beta_1(S_\beta^{\tilde{\beta}}(R),(D_\beta^{\tilde{\beta}}(R),*^\alpha),*^\alpha,\Delta_\beta^{\tilde{\beta}})$) of Boehmians is defined. (ϑ_n,δ_n) and (θ_n,t_n) in β_1 are equivalent, $(\vartheta_n,\delta_n) \sim (\theta_m,t_m)$, if $\vartheta_n *^\alpha t_m = \theta_m *^\alpha \delta_n$ ($\forall m,n \in N$). Indeed, $\sim$ defines an equivalence relation on β_1. The equivalence class in β_1 containing (ϑ_n,δ_n) is denoted as $((\vartheta_n)/(\delta_n))$ and is called Boehmian. An embedding between $S_\beta^{\tilde{\beta}}(R)$ and β_1 is expressed as $x \to x *^\alpha \delta_n/\delta_n$ ($\forall m,n \in N$). If $((\vartheta_n)/(\delta_n)) \in \beta_1$ and $z \in \beta_1$, then

$$\big((\vartheta_n)/(\delta_n)\big) *^\alpha \varrho = \big(\vartheta_n *^\alpha \varrho\big)/(\delta_n).$$

To define the space of ultraBoehmians, let S_α and D_α be the fraction spaces of F_α of all members of $S_\beta^{\tilde{\beta}}(R)$ and $D_\beta^{\tilde{\beta}}(R)$, respectively, and, similarly, let Δ_α be the fractional set of F_α of all sequences in Δ^α. Then, we introduce a product on S_α as follows:

$$(F \times G)(w) = \frac{\sqrt{2\pi}}{c(\alpha)} \exp\left(-j\frac{\cot\alpha}{2}w^2\right)F(w)G(w). \tag{14}$$

Then we have the following theorem.

Theorem 10 *Let $F,F_n,H,G \in S_\alpha$, $F_n \to F$ as $n \to \infty$ and $\Psi \in D_\alpha$, then the following identities hold*:

(i) $(F + G) \times \Psi = F \times \Psi + G \times \Psi$.
(ii) $F_n \times \Psi \to F \times \Psi$ *as* $F_n \to F$ *as* $n \to \infty$.
(iii) $F \times G = G \times F$.
(iv) $F \times (G \times H) = (F \times G) \times H$.
(v) $\eta(F \times G) = (\eta F \times G)$, $\eta \in C$.

Proof Proofs of (i) and (ii) are straightforward being similar to the proofs assigned to the space $\beta(S_\beta^{\tilde{\beta}}(R),(D_\beta^{\tilde{\beta}}(R),*^\alpha),*^\alpha,\Delta_\beta^{\tilde{\beta}})$.

Proof of (iii) Let $\vartheta,\vartheta_0 \in S_\beta^{\tilde{\beta}}(R)$ be such that $F = F_\alpha\vartheta$ and $G = F_\alpha\vartheta_0$, then by (14) we have

$$\begin{aligned}(F \times G)(w) &= \frac{\sqrt{2\pi}}{c(\alpha)} \exp\left(-j\frac{\cot\alpha}{2}w^2\right)F(w)G(w)\\ &= \frac{\sqrt{2\pi}}{c(\alpha)} \exp\left(-j\frac{\cot\alpha}{2}w^2\right)(F_\alpha\vartheta)(w)(F_\alpha\vartheta_0)(w)\\ &= F_\alpha\big(\vartheta *^\alpha \vartheta_0\big)(w) \in S_\alpha.\end{aligned}$$

Since $\vartheta *^\alpha \vartheta_0 = \vartheta_0 *^\alpha \vartheta$, it follows from (14) that

$$(F \times G)(w) = F_\alpha\big(\vartheta_0 *^\alpha \vartheta\big)(w) = (G \times F)(w) \in S_\alpha. \tag{15}$$

Proof of (iv) is similar to that of (iii), whereas the proof of (v) is straightforward.

The proof is finished. □

Theorem 11 *Let $(\theta_n),(\varphi_n) \in \Delta_\alpha$ and $F \in S_\alpha$, then $(\theta_n \times \varphi_n) \in \Delta_\alpha$ and* $\lim_{n\to\infty} F \times \theta_n = F$.

Proof Let $(\delta_n),(\psi_n) \in \Delta_\beta^{\tilde{\beta}}$ be such that $F_\alpha\delta_n = \theta_n$ and $F_\alpha\psi_n = \varphi_n$, $\forall n \in N$. Then by (14) we have

$$(\theta_n \times \varphi_n)(w) = \frac{\sqrt{2\pi}}{c(\alpha)} \exp\left(-j\frac{\cot\alpha}{2}w^2\right)\theta_n(w)\varphi_n(w) = F_\alpha\big(\delta_n *^\alpha \psi_n\big)(w).$$

Hence $(\theta_n \times \varphi_n)$ belongs to Δ_α since $(\delta_n *^\alpha \psi_n)$ belongs to Δ^α. The proof of the second part of the theorem can similarly be drawn.

The proof of the theorem is therefore finished. □

The space β_2 ($\beta_2 \equiv \beta_2(S_\alpha, (D_\alpha, \times), \times, \Delta_\alpha)$) of ultraBoehmians is obtained. An ultra-Boehmian in β_2 is written as

$$(F_\alpha \vartheta_n)/(F_\alpha \theta_n),$$

where $(F_\alpha \vartheta_n) \in S_\alpha$ and $(F_\alpha \theta_n) \in \Delta_\alpha$. For similar concepts of addition, convergence, and scalar multiplication in β_1 and β_2, we refer to Omari [19, 20].

Definition 12 Let $(\delta_n) \in \Delta_\beta^{\tilde{\beta}}$ and $(\vartheta_n) \in S_\beta^{\tilde{\beta}}(R)$, then the fractional Fourier operator of a Boehmian in β_1 can be given as

$$\hat{F}_\alpha\big((\vartheta_n)/(\delta_n)\big) = (F_\alpha \vartheta_n)/(F_\alpha \delta_n), \tag{16}$$

which indeed is a member of β_2.

4 Characteristics and an inversion formula

To show that $\hat{F}_\alpha$ is well defined, we assume $(\vartheta_n)/(\delta_n) = (\theta_n)/(\varepsilon_n) \in \beta_1$, then the idea of quotients of sequences in β_1 suggests to have $\vartheta_n *^\alpha \varepsilon_m = \theta_m *^\alpha \delta_n$ $(m, n \in N)$. Hence, applying F_α gives

$$F_\alpha\big(\vartheta_n *^\alpha \varepsilon_m\big) = F_\alpha\big(\theta_m *^\alpha \delta_n\big) \quad (m, n \in N).$$

Therefore, the convolution theorem reads as follows:

$$\frac{\sqrt{2\pi}}{c(\alpha)} \exp\left(-j\frac{\cot\alpha}{2}w^2\right)(F_\alpha \vartheta_n)(F_\alpha \varepsilon_m) = \frac{\sqrt{2\pi}}{c(\alpha)} \exp\left(-j\frac{\cot\alpha}{2}w^2\right)(F_\alpha \theta_m)(F_\alpha \delta_n).$$

This equivalently can be written as

$$(F_\alpha \vartheta_n) \times (F_\alpha \varepsilon_m) = (F_\alpha \theta_m) \times (F_\alpha \delta_n).$$

Hence, the idea of quotients and equivalent classes of β_2 and the preceding equation suggest to have

$$(F_\alpha \vartheta_n)/(F_\alpha \delta_n) = (F_\alpha \theta_n)/(F_\alpha \varepsilon_n) \quad (m, n \in N).$$

Thus,we obtain that

$$\hat{F}_\alpha\big((\vartheta_n)/(\delta_n)\big) = \hat{F}_\alpha\big((\theta_n)/(\varepsilon_n)\big) \quad (m, n \in N).$$

The proof is completely finished.

Theorem 13 *The integral operator $\hat{F}_\alpha : \beta_1 \to \beta_2$ is linear.*

Proof of this theorem follows from the concept of addition of Boehmian spaces. Hence it has been deleted.

Theorem 14 *Let* $\varrho \in \beta_1$, $\varrho = 0$, *then* $\hat{F}_\alpha(\varrho) = 0$.

Proof of this theorem is straightforward. Details are, therefore, omitted.

Theorem 15 *Let* $\varrho_0, \varrho \in \beta_1$, *then we have*

$$\hat{F}_\alpha\left(\varrho_0 *^\alpha \varrho\right)(w) = \frac{c(\alpha)}{\sqrt{2\pi}} \exp\left(j\frac{\cot\alpha}{2}w^2\right)\hat{F}_\alpha(\varrho_0)\hat{F}_\alpha(\varrho).$$

Proof Let $\varrho_0 = (\vartheta_n)/(\delta_n)$, $\varrho = (\theta_n)/(\varepsilon_n) \in \beta_1$ be given. Then, by employing $*^\alpha$, we get

$$\hat{F}_\alpha\left(\varrho_0 *^\alpha \varrho\right) = \hat{F}_\alpha\left(\left((\vartheta_n) *^\alpha (\theta_n)\right)/\left((\delta_n) *^\alpha (\varepsilon_n)\right)\right).$$

Hence, the convolution theorem reveals

$$\hat{F}_\alpha\left(\varrho_0 *^\alpha \varrho\right)(w) = \frac{c(\alpha)}{\sqrt{2\pi}} \exp\left(j\frac{\cot\alpha}{2}w^2\right)\hat{F}_\alpha(\varrho_0)\hat{F}_\alpha(\varrho).$$

The proof is completely finished. □

Definition 16 Let $\varrho_0 \in \beta_2$, $\varrho_0 = (F_\alpha\vartheta_n)/(F_\alpha\delta_n)$, then we introduce the inverse operator of $\hat{F}_\alpha$ as the mapping $(\hat{F}_\alpha)^{-1} : \beta_2 \to \beta_1$ given by

$$(\hat{F}_\alpha)^{-1}(\varrho_0) = (\vartheta_n)/(\delta_n)$$

for each $(\delta_n) \in \Delta_\beta^{\tilde{\beta}}$.

Theorem 17 *The inverse operator* $(\hat{F}_\alpha)^{-1} : \beta_2 \to \beta_1$ *is well defined and linear.*

Proof Let $\varrho_0 = \varrho$ in $\beta_2, \varrho_0 = (F_\alpha\vartheta_n)/(F_\alpha\delta_n)$, $\varrho = (F_\alpha\theta_n)/(F_\alpha\varepsilon_n)$. Then

$$F_\alpha\vartheta_n \times F_\alpha\varepsilon_m = F_\alpha\theta_m \times F_\alpha\delta_n$$

for some (θ_n), (ϑ_n) in $S_\beta^{\tilde{\beta}}(R)$. By using the convolution theorem, we obtain

$$\hat{F}_\alpha\left(\vartheta_n *^\alpha \varepsilon_m\right) = \hat{F}_\alpha\left(\theta_m *^\alpha \delta_n\right) \quad (m, n \in N).$$

Therefore, $\vartheta_n *^\alpha \varepsilon_m = \theta_m *^\alpha \delta_n$ $(m, n \in N)$. Hence, we have

$$(\vartheta_n)/(\delta_n) = (\theta_n)/(\varepsilon_n).$$

To show that $(\hat{F}_\alpha)^{-1}$ is linear, let $\varrho_0 = (F_\alpha\vartheta_n)/(F_\alpha\delta_n)$, $\varrho = (F_\alpha\theta_n)/(F_\alpha\varepsilon_n)$ be members in β_2, then by addition of β_2 and the convolution theorem, we write

$$(\hat{F}_\alpha)^{-1}(\varrho_0 + \varrho) = (\hat{F}_\alpha)^{-1}\left(F_\alpha\left((\vartheta_n) *^\alpha (\varepsilon_n)\right) + \left((\theta_n) *^\alpha (\delta_n)\right)\right)/\left(F_\alpha\left(\delta_n *^\alpha \varepsilon_n\right)\right).$$

Therefore,

$$(\hat{F}_\alpha)^{-1}(\varrho_0 + \varrho) = \big((\vartheta_n) *^\alpha (\varepsilon_n) + (\theta_n) *^\alpha (\delta_n)\big)/\big((\delta_n) *^\alpha (\varepsilon_n)\big).$$

Hence addition in β_1 finishes the proof of the theorem. □

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